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# Steady Laminar Second Grade Fluid Flow Between Two Rotating Porous Discs with Moderate Rotation

P.D Verma and P.R. Sharma
*University of Rajasthan, Jaipur–302 004*

**ABSTRACT**

The flow of a second grade fluid between two rotating porous discs is investigated. The equations of motion are solved by a regular perturbation technique for small cross-flow Reynolds number. The effects of viscoelastic parameter-$\alpha$, suction/injection parameter-$R$ and rotation parameter-$\beta$ on the velocity components, pressure distribution and skin friction have been discussed numerically and the results are compared to those of a Newtonian fluid case.

## 1. INTRODUCTION

The study of fluid flowing between parallel porous/non-porous discs is of practical importance in the design of thrust bearings, radial diffusers etc. The viscous laminar flow between porous discs has recently been studied by several authors. Elkouh[1-3] obtained the solutions of laminar flow between non-rotating and rotating porous discs with equal suction/injection through porous discs. Gaur[4] has discussed the viscous incompressible fluid flow between two infinite porous rotating discs. Narayana[5] has considered the steady flow of a Newtonian fluid between two infinite parallel discs when one disc (upper) is rotating and other disc (lower) is at rest with uniform suction at the stationary disc. Rudraiah[6] *et al.* have studied a singular perturbation problem of non-Newtonian fluid flow between porous discs. The authors[6] obtained the solutions for both small and large values of cross-flow Reynolds number by regular perturbation and matched asymptotic expansions technique, respectively. Sacheti and Bhatt[7] have discussed the steady laminar flow of a non-Newtonian fluid with suction/injection through discs and heat transfer through porous parallel discs. The authors[7] have considered the flow entirely due to either suction or injection

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through discs and the equations of momentum and energy have been solved by using the perturbation method. Wang[8] has studied symmetric viscous flow between two rotating porous discs with moderate rotation. In the paper[8], he has considered the two co-axial porous discs rotating with small and large cross-flow Reynolds number by perturbation and asymptotic expansions techniques, respectively; and the numerical results for both are compared with numerical integration and reported that for small Reynolds number the perturbation method is good i.e. when $|R| \leqslant 1$; but the asymptotic large Reynolds number expansion is good if $|R|$ is above 20.

In the present analysis, the steady laminar flow of an incompressible second grade fluid between two rotating porous discs is considered. The governing equations of motion have been solved by perturbation technique, in which the suction/injection parameter $R$ is taken as the small perturbation parameter. Also, in the present investigation two discs rotating with the same angular velocity in the same sense have been considered.

Following Fosdick and Rajagopal[9] the Cauchy stress $T$ in a homogeneous incompressible fluid of second grade is related to the fluid motion in the following form:

$$T = -pI + \mu A_1 + a_1 A_2 + a_2 A_1^2 \tag{1}$$

where $-pI$ denotes the indeterminate spherical stress, $\mu$ is the coefficient of viscosity, $a_1$ and $a_2$ are material moduli and $A_1$ and $A_2$ stand for the first two Rivlin-Ericksen tensors[10], defined by

$$A_1 = \text{grad } v + (\text{grad } v)^{\mathrm{T}} \tag{2}$$

and

$$A_2 = \dot{A}_1 + A_1 \text{ grad } v + (\text{grad } v)^T A_1 \tag{3}$$

where $v$ denotes the velocity field and the dot represents material time differentiation.

In this analysis, the model represented by Eqn. (1) shall be considered as an exact model. This model was shown to be a second order approximate to the response functional of a simple fluid in the sense of retardation by Coleman and Noll[11]. Recently, Dunn and Fosdick[12], and Fosdick and Rajagopal[9,13] have reported that if the model in Eqn. (1) is required to be compatible with thermodynamics in the sense that all motions satisfy the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is minimum in equilibrium (at constant temperature), then the material moduli must satisfy the following restrictions:

$$\mu \geqslant 0, \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0 \tag{4}$$

The third condition in Eqn. (4) is a consequence of the Clausius-Duhem inequality while the second follows from the requirement that the specific Helmholtz free energy is a minimum in equilibrium.

The present investigation can be made use of in porous bearings and self-impregnated bearings used in defence equipments.

## 2. EQUATIONS OF MOTION

Consider two co-axial porous discs situated at $z = \pm L$, rotating with the same angular velocity $\Omega$ and fluid is withdrawn from both discs with velocity $W$ (Fig. 1).

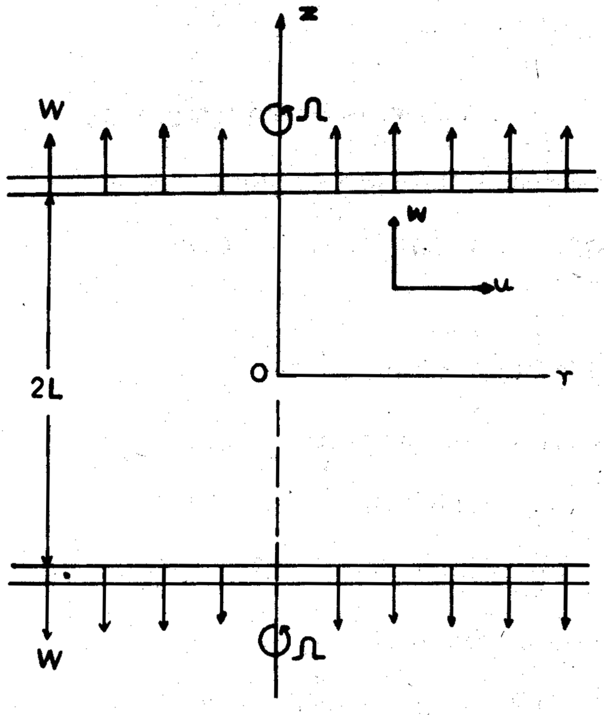


**Figure 1.** The Physical model

Assuming the gap with $2L$ is small compared to the diameter of the discs, so that the end effects are neglected. The flow field is symmetric about $z = 0$ plane and about the $z$-axis.

The steady incompressible axisymmetric equations of motion and continuity in cylindrical polar coordinates are

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\delta \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} \tag{5}$$

$$\rho\left(u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = \frac{\delta \tau_{r\theta}}{\partial r} + \frac{\delta \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} \tag{6}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{\delta \tau_{rz}}{\partial r} + \frac{\delta \tau_{zz}}{\delta z} + \frac{\tau_{rz}}{r} \tag{7}$$

and $$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

where $\tau_{ij}$ are the Cauchy stress components, $\rho$ is the density and $u$, $v$, $w$ are the velocity components in the directions of $r$, $\theta$, $z$-axis respectively.

The boundary conditions are

$$\text{at } z = \pm L, \quad u = 0, \quad v = r\Omega, \quad w = \pm W \tag{9}$$

## 3. SOLUTION OF THE PROBLEM

Utilizing the symmetry of the problem, we define

$$u = r\dot{f}(\eta)W/L, \quad v = rg(\eta)W/L, \quad w = -2f(\eta)W$$

and

$$p = -\rho r^2 A W^2/2L^2 + \rho r^2 Q(\eta) + \rho P(\eta) \tag{10}$$

where $\eta = z/L$ and $A$ is a constant to be determined.

Using Eqn. (10) into Eqns. (5) – (8), we have

$$f^{\cdots} - R(f^{\cdot 2} - 2ff^{\cdot\cdot} - g^2) - \alpha R(f^{\cdot\cdot 2} + 2ff^{iv} - 2f^{\cdot}f^{\cdots} + g^{\cdot 2}) + AR = 0 \tag{11}$$

or, after differentiating once, we get

$$f^{iv} + 2R(ff^{\cdots} + gg^{\cdot}) - 2\alpha R(ff^{v} + g^{\cdot}g^{\cdot\cdot}) = 0 \tag{12}$$

$$g^{\cdot\cdot} - 2R(f^{\cdot}g - fg^{\cdot}) + 2\alpha R(f^{\cdot}.g^{\cdot\cdot} - fg^{\cdots}) = 0 \tag{13}$$

$$P(\eta) = -2W^2f^2 - 2\nu Wf^{\cdot}/L + 2\nu_1 W^2(f^{\cdot 2} + 2ff^{\cdot\cdot})/L^2 + B \tag{14}$$

and

$$Q(\eta) = \nu_1 W^2(f^{\cdot\cdot 2} + g^{\cdot 2})/L^4. \tag{15}$$

where $\nu = \mu/\rho$, $\nu_1 = a_1/\rho$, dot denotes differentiation with respect to $\eta$,

$R = \rho WL/\mu$ is the cross-flow Reynolds number, and

$a = a_1/\rho L^2$ is the dimensionless viscoelasticity parameter.

The constant $B$ is determined from the pressure at the disc.

The boundary conditions in Eqn. (9) are reduced to

$$f(0) = 0 = f^{\cdot}(1) = f^{\cdot\cdot}(0), \quad f(1) = -\frac{1}{2} \tag{16}$$

and

$$g^{\cdot}(0) = 0, \quad g(1) = \Omega L/W = \beta \tag{17}$$

The Eqns. (12) and (13) are the two simultaneous non-linear coupled differential equations with the boundary conditions of Eqns. (16) and (17). We assume that the suction/injection parameter $R$ is small. Then $f(\eta)$ and $g(\eta)$ can be expanded in the powers of $R$ :

$$f(\eta) = f_0 + Rf_1 + R^2 f_2 + \ldots \tag{18}$$

and

$$g(\eta) = g_0 + Rg_1 + R^2 g_2 + \ldots \tag{19}$$

Using Eqns. (18) and (19) in Eqns. (12) and (13) and equating coefficients of like powers of $R$, we have the following set of equations :

$$f_0^{iv} = 0 \tag{20}$$

$$f_1^{iv} + 2(f_0 f_0''' + g_0 g_0') - 2\alpha(f_0 f_0^{v} + g_0' g_0'') = 0 \tag{21}$$

$$f_2^{iv} + 2(f_1 f_0''' + f_0 f_1''' + g_0 g_1' + g_1 g_0') - 2\alpha(f_0 f_1^{v} + f_1 f_0^{v}$$

$$+ g_0' g_1'' + g_1' g_0'') = 0 \tag{22}$$

$$g_0'' = 0 \tag{23}$$

$$g_1'' - 2(f_0' g_0 - f_0 g_0') + 2\alpha(f_0' g_0'' - f_0 g_0''') = 0 \tag{24}$$

$$g_2'' - 2(f_0' g_1 + f_1' g_0 - f_0 g_1' - f_1 g_0') + 2\alpha(f_0' g_1'' + f_1' g_0'' - f_0 g_1'''$$

$$- f_1 g_0''') = 0 \tag{25}$$

The corresponding boundary conditions of Eqns. (16) and (17) are reduced to :

$$f_0(0) = 0 = f_0''(0) = f_0'(1),\ f_0(1) = -\frac{1}{2};\ f_1(0) = 0 = f_1''(0) = f_1'(1) = f_1(1)$$

and

$$f_2(0) = 0 = f_2''(0) = f_2'(1) = f_2(1) \tag{26}$$

$$g_0(0) = 0,\ g_0(1) = \beta;\ g_1(0) = 0 = g_1(1);\ \text{and}\ \ g_2(0) = 0 = g_2(1) \tag{27}$$

After making use of Eqns. (26) and (27) through straight forward algebra, the solutions of $f_0$, $f_1$, $f_2$, $g_0$, $g_1$ and $g_2$ are

$$f_0 = \eta^3/4 - 3\eta/4 \tag{28}$$

$$f_1 = \eta^7/1120 + 3\eta^5/160 - 39\eta^3/1120 + 19\eta/1120 \tag{29}$$

$$\begin{aligned} f_2 = {} & 3\eta^{11}/246400 - \eta^9/3360 + 531\eta^7/23520 - 51\eta^5/16800 \\ & +443\eta^3/1034880 + 137\eta/215600 + \beta^2(-\eta^7//840 + \eta^5/40 \\ & -13\eta^3/280 + 19\eta/840) + \alpha(-11\eta^9/56448 + 315\eta^7/117600 \\ & -63\eta^5/2240 + 3457\eta^3/70560 - 2197\eta/94080) \end{aligned} \tag{30}$$

$$g_0 = \beta \tag{31}$$

$$g_1 = \beta(\eta^4/8 - 3\eta^2/4 + 5/8) \tag{32}$$

$$\begin{aligned} g_2 = {} & \beta[-3\eta^8/2240 + \eta^6/80 - 111\eta^4/3360 - 253\eta^2/560 \\ & +3183/6720 + \alpha(-\eta^6/40 - 9\eta^2/8 + 23/20)] \end{aligned} \tag{33}$$

As $f(\eta)$ is known, the constant $A$ is determined by

$$A = f^{\cdot\cdot}(1) - \beta^2 - f^{\cdot\cdot\cdot}(1)/R + \alpha[(f_{(1)})^2 - f^{iv}(1) + (g_{(1)})^2] \tag{34}$$

### 3.1 Pressure Distribution

The pressure on either of disc is

$$p(r, 1) - p(r_0, 1) = \frac{\rho W^2}{2L^2}[-A + 2\alpha(f_{(1)}^{\cdot\cdot 2} + g_{(1)}^{\cdot 2})](r^2 - r_0^2) \tag{35}$$

where $r_0$ is a certain distance in radial direction.

The dimensionless pressure coefficient is

$$p^* = \left\{\frac{p(r,1) - p(r_0, 1)}{\frac{1}{2}\rho W^2}\right\}\left(\frac{L}{r_0}\right)^2 \tag{36}$$

$$\begin{aligned} = & \left[\left(\frac{27}{35}\right) - \beta^2 - \left(\frac{3}{2R}\right) - \left\{\left(\frac{151}{5390}\right) + \left(\frac{34}{35}\right)\beta^2\right\}R + \left\{\left(\frac{1341}{80850}\right) + \left(\frac{6}{35}\right)\beta^2\right\}R^2 \right. \\ & \left. -\alpha\left\{\left(\frac{9}{4}\right) + \left(\frac{703}{735}\right)R + R^2\left\{\left(\frac{87073}{80850}\right) - \left(\frac{327}{147}\right)\alpha + \left(\frac{123}{35}\right)\beta^2\right\}\right\}\right](1 - \xi^2/\xi_0^2) \end{aligned} \tag{37}$$

where $\xi = r/L$ and $\xi_0 = r_0/L$

### 3.2 Skin Friction

The dimensionless skin friction coefficient is

$$\tau^* = \frac{T_1}{\frac{1}{2}\rho W^2}\left(\frac{L}{r_0}\right) \tag{38}$$

where $T_1$ is the shear stress at the upper/lower disc.

Hence, the skin friction coefficient at the upper disc

$$= -[2f''/R + 4\alpha\,(2f'f'' - ff''')]\left(\frac{r}{r_0}\right) \tag{39}$$

$$= -\left[\left(\frac{9}{35}\right) + \left(\frac{3}{R}\right) + \left\{\left(\frac{1341}{40425}\right) + \left(\frac{12}{35}\right)\beta^2\right\}R + \alpha\left\{3 + \left(\frac{821}{735}\right)R + \left(\frac{69}{224}\right) - \left(\frac{1366}{735}\right)\alpha + \left(\frac{68}{35}\right)\beta^2\right)R^2\right\}\right](\xi/\xi_0) \tag{40}$$

## 4. PARTICULAR CASES

(i) When $\alpha = 0$, the expressions of $f(\eta)$, $g(\eta)$, $p^*$ and $\tau^*$ reduce for Newtonian fluid case. And $f(\eta)$ and $g(\eta)$ expressions reduces to those obtained by Wang[8], who studied the symmetric viscous flow between two rotating porous discs with moderate rotation.

(ii) When $\alpha = 0 = \beta$, the results of the present investigation reduces to that of steady laminar viscous incompressible fluid flow between two porous discs with equal suction/injection through the discs.

## 5. RESULTS AND DISCUSSIONS

Figs. 2–7 show the velocity distribution i.e. $f(\eta)$ (normal velocity), $g(\eta)$ (azimuthal velocity) and $f'(\eta)$ (radial velocity); and Figs. 8 & 9 show the pressure distribution and skin friction, respectively, for various values of cross-flow Reynolds number $R$ (= –0.5, –0.25, 0.25, 0.5), viscoelasticity parameter $\alpha$ (= 0, 0.2) and rotation parameter $\beta$ (= 0.5, 1.0).

The normal velocity increases towards the upper disc as $\eta$ varies from 0 to 1. The normal velocity increases due to increase in fluid withdrawn through the disc when

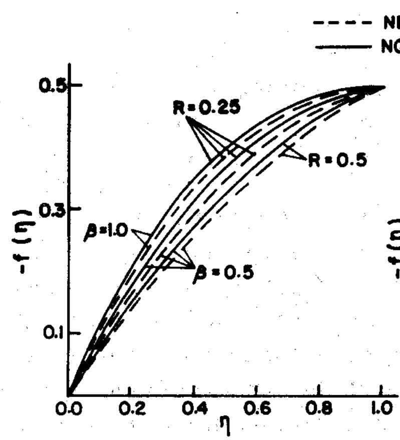


**Figure 2.** Normal velocity distribution versus $\eta$ for $R$ = 0.25, 0.5 and $\beta$ = 0.5, 1.0.

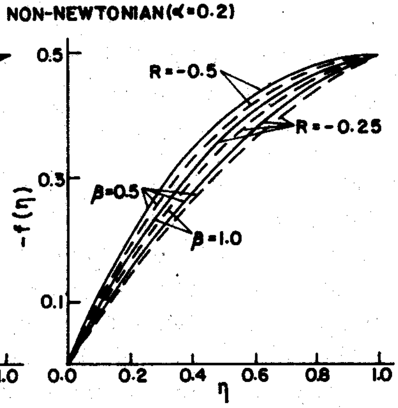


**Figure 3.** Normal velocity distribution versus $\eta$ for $R$ = –0.25, –0.5 and $\beta$ = 0.5, 1.0.

viscoelastic and rotation parameters are fixed but it decreases due to increase of either viscoelasticity parameter of the fluid or rotation of the disc when uniform fluid withdrawn through the disc (Fig. 2); while different behaviour is observed when more fluid is injected through the disc (Fig. 3).

The azimuthal velocity decreases from the centre of the discs i.e. $\eta = 0$ to the upper disc i.e. at $\eta = 1$; but opposite behaviour is found when fluid injected through the disc. The azimuthal velocity (for a given $r$) increases with the increase of any

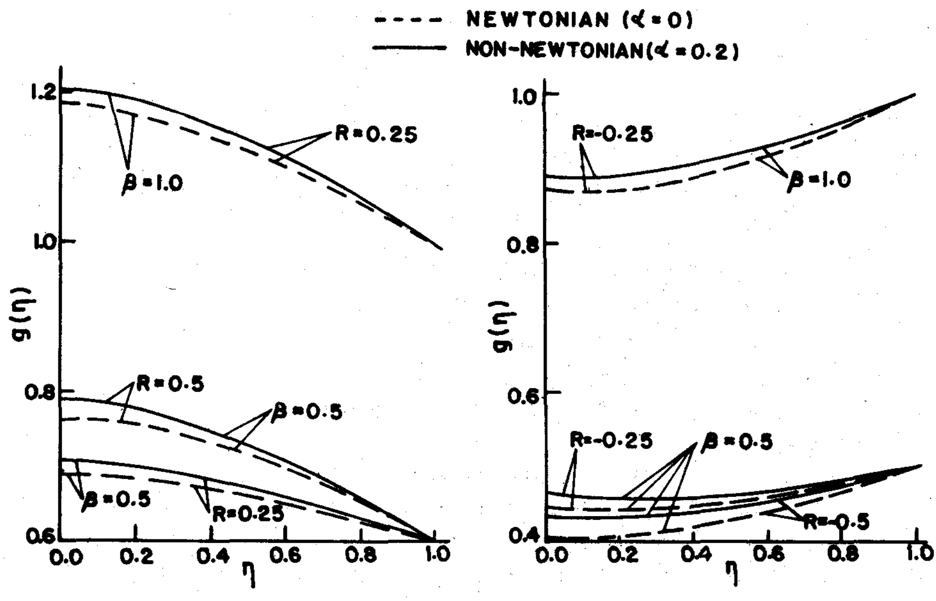


**Figure 4.** Azimuthal velocity distribution versus $\eta$ for $R = 0.25$, $0.5$ and $\beta = 0.5, 1.0$.

**Figure 5.** Azimuthal velocity distribution versus $\eta$ for $R = -0.25, -0.5$ and $\beta = 0.5, 1.0$.

either fluid withdrawn through the disc, rotation of the disc or viscoelasticity of the fluid (see Fig. 4); but it decreases due to increase of either fluid injected through the disc, rotation of the disc or viscoelasticity of the fluid (see Fig. 5).

The radial velocity (for a given $r$) increases with $\eta$, which varies from 0 to 1, but different pattern is observed when fluid injected through the disc. The radial velocity increases with the increase of fluid withdrawn through the disc or rotation of the disc when fluid viscoelasticity is fixed (Fig. 6); while it decreases with the increase of viscoelasticity of the fluid when rotation of the disc and fluid withdrawn through disc are uniform. The radial velocity decreases with the increase of fluid injected through disc or fluid viscoelasticity in uniform rotation case, but it increases due to increase of rotation of the disc when fluid injected through disc is uniform and fluid viscoelasticity is fixed (Fig. 7).

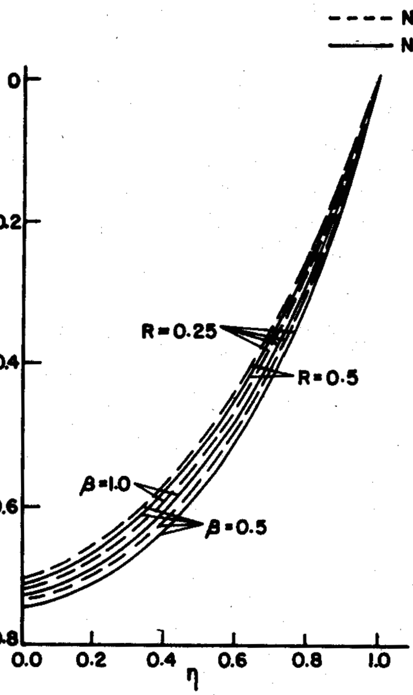


**Figure 6.** Radial velocity distribution versus $\eta$ for $R = 0.25$, $0.5$ and $\beta = 0.5, 1.0$.

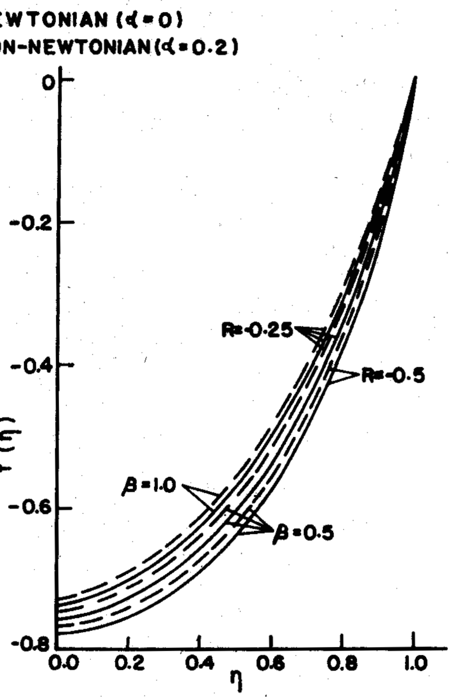


**Figure 7.** Radial velocity distribution versus $\eta$ for $R = -0.25$, $-0.5$ and $\beta = 0.5, 1.0$.

The pressure distribution on either of disc is shown in Fig. 8 and the results are compared with that of a Newtonian fluid case. It may be observed that the pressure coefficient of non-Newtonian fluid is less than that of a Newtonian fluid for particular values of a set of parameters when fluid injected through disc; while this phenomenon

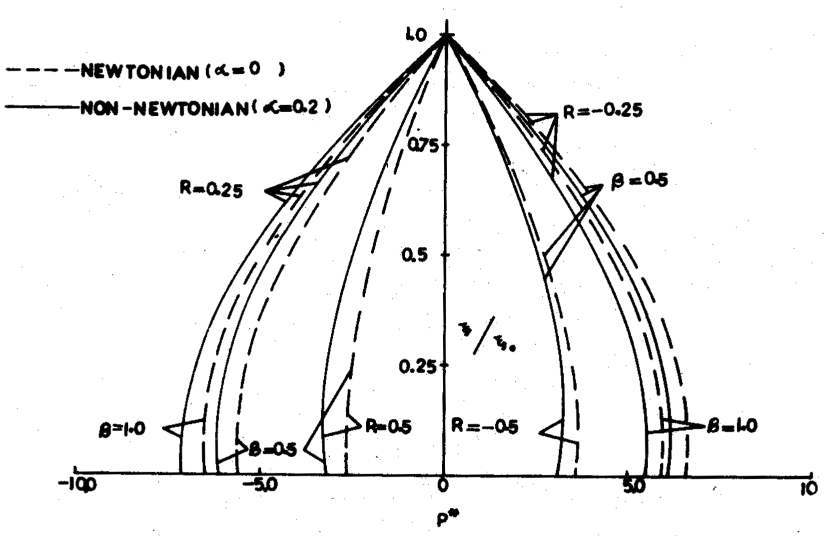


**Figure 8.** Pressure coefficient on either disc versus $\xi/\xi_0$ for $R = -0.5, -0.25, 0.25, 0.5$ and $\beta = 0.5, 1.0$.

is reversed when fluid withdrawn through the disc. The pressure decreases with the increase of any rotation of disc or viscoelasticity of fluid when fluid injected through disc is uniform; but it decreases rapidly when more fluid injected through the disc when rotation of the disc and viscoelasticity of the fluid are fixed. The pressure coefficient increases with more fluid withdrawn through the disc when rotation of disc is uniform and viscoelasticity of the fluid is fixed, but it decreases with the increase of rotation of disc or viscoelasticity of the fluid when fluid withdrawn through the disc is uniform.

The coefficient of skin friction at the upper disc is plotted versus $\xi/\xi_0$ (see Fig. 9). The skin friction coefficient is a linear function to $\xi/\xi_0$ for a fixed values

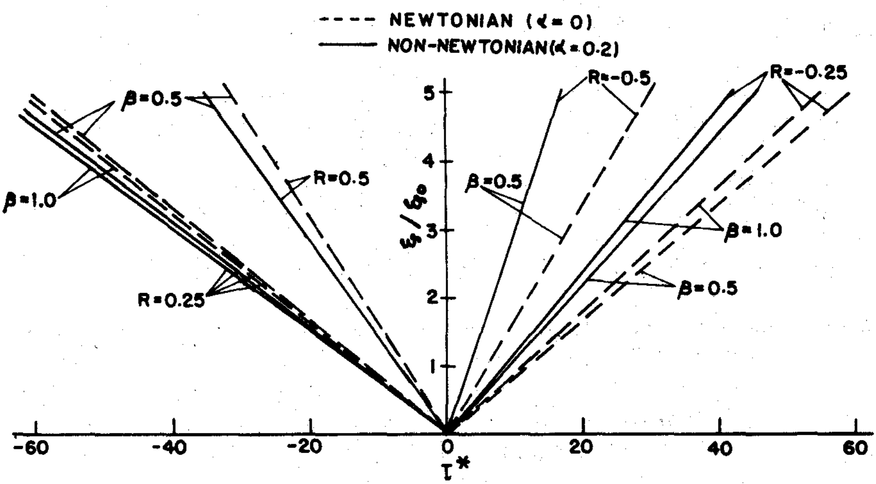


**Figure 9.** Coefficient of skin friction at the upper disc versus $\xi/\xi_0$ for $R = -0.5, -0.25, 0.25, 0.5$ and $\beta = 0.5, 1.0$.

of the set of parameters. The coefficient of skin friction is positive when fluid injected through the disc, but opposite behaviour is observed when fluid withdrawn through disc. The skin friction coefficient decreases with the increase of rotation of disc or viscoelasticity of the fluid when fluid is injected uniformly through disc, while it decreases rapidly when more fluid injected through the disc when rotation of the disc and viscoelasticity of the fluid are fixed. The skin friction coefficient increases with the increase of fluid withdrawn through disc when rotation of disc is uniform and viscoelasticity of the fluid is fixed, but it decreases with more rotation of the disc or increase in viscoelasticity of the fluid when fluid withdrawn through disc is uniform.

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