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Modelling a Markov Attrition Process

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ABSTRACT

This study proposes another computational approach to solve a stochastic attrition model. The initial contact forces for both sides can be treated as a random variable. The approach is manipulated in a matrix form, and on account of the special form of its infinitesimal generator, some recursive algorithms are derived to compute the intended results. Numerical results to illustrate the differences between the proposed model and the stochastic model with known initial contact forces are presented.

1. INTRODUCTION

In order to understand more about the combat dynamics, several authors such as Jennings[1,2], Bhat[3], Weale[4], Karmeshu and Jaiswal[5], and Jaiswal[6] have paid attention to the developing of a stochastic combat model recently. Although the stochastic attrition is better in representing the reality of combat attrition phenomena, it is considerably less convenient to handle and compute.

In this study, we propose another computational approach to solve the stochastic attrition model. The approach is based upon defining a Markov attrition process and applying the concept of matrix-geometric computational algorithms by Neuts[7]. Since the stochastic model is manipulated in the matrix form and on account of the special form of its infinitesimal generator, some efficient recursive algorithms can be derived. Numerical results by taking the example from Jaiswal[6] are presented.

2. MARKOV ATTRITION PROCESS

Let us consider a homogeneous combat between two forces, Red and Blue (for the heterogeneous case, a transforming model in Jaiswal[6] can be used to derive an equivalent homogeneous model). We first define

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B_U = the maximum possible number of Blue combatants at $t = 0$,
R_U = the maximum possible number of Red combatants at $t = 0$,
B_E = the surviving Blue combatants when Blue surrenders, and
R_E = the surviving Red combatants when Red surrenders.

All of these are random variables for each t. The states of the Markov process are denoted by (i,j), $B_E \leqslant i \leqslant B_U$ and $R_E \leqslant j \leqslant R_U$. Let the state space of the Markov process by E which can be regarded as a model of attrition in combat between a homogeneous Blue side and a homogeneous Red side, provided that the paths $t \rightarrow R_t$ be nonincreasing, where B_t = number of surviving Blue combatants at time t and R_t = number of surviving Red combatants at time t.

For the purpose of simplicity, we let* $B_U = R_U = m$, and $B_E = R_E = 0$. We will arrange the state space E in the lexicographic order, that is, $(m,m),\ldots,(m,1),(m-1,m),\ldots,(m-1,1),\ldots,(1,m),\ldots,(1,1),(0,0)$**. The set of states $\{(i,m),\ldots,(i,1)\}$, $0 < i \leqslant m$, will be called the level i. The infinitesimal generator of the Markov attrition process can be constructed as

$$Q = \begin{vmatrix} T T^{o} \\ 0\ 0 \end{vmatrix}$$

where T is an m^2 square matrix, T^o is an $m^2 \times 1$ column vector, 0 is an $1 \times m^2$ row vector, and o is a scaler.

Let $T_{(i,j)(l,k)}$ denote the element of T in row (i,j) and column (l,k), and let $T^o_{(i,j)}$ denote the element of T^o in row (i,j), then

$$T^0_{(i,j)} \geqslant 0,\ 0 < i \leqslant m,\ 0 < j < m$$

$$T_{(i,j)(i,j)} < 0, 0 < i \leqslant m,\ 0 < j \leqslant m$$

$$T_{(i,j)(l,k)} \geqslant 0,\ l \neq i,\ k \neq j$$

such as $Te + T^o = 0$. The initial probability vector of Q is given by $(a, a_{(0,0)})$, as $ae + a_{(0,0)} = 1$.

In this modelling, the (a,T) representation satisfies the definition of phase-type probability distribution (Neuts[8]). Then, we can decide the elements of T and T^o according to the combat situation by applying linear or square laws. To illustrate the results, we assume that the initial contact forces are $m = 3$ for both sides and combat behaviour can be described by square law (for example, both sides taking attack strategy). Table 1 gives the infinitesimal generator with $k_1 = k_2 = 0.5$ where k_1 and k_2 are the attrition rates of Blue and Red respectively.

* For practical application, B_U and R_U are not necessarily equal to each other, and B_E and R_E can be any predetermined integer values such that $0 \leqslant B_E \leqslant B_U$, and $0 \leqslant R_E \leqslant R_U$.

** (0,0) is an absorption state which stands for the ending condition of the process. If B_E and R_E are not equal to 0, then the state is a set of $\{(B_U, R_E), (B_U-1, R_E),\ldots,(B_E+1, R_E), (B_E, R_U),\ldots,(B_E, R_E+1)\}$

3. ANALYSIS OF THE MARKOV ATTRITION PROCESS

As the Markov attrition model is defined in the previous section, we can proceed to develop some useful computational algorithms. The α vector specifies the initial contact forces for both sides. Thus, the confronting enemy forces can be viewed as a random variable instead of a constant. Without loss of generality, we let $a_{(0,0)}=0$; namely, the probability that either side surrenders without any engagement in the beginning of combat is zero.

3.1 The Distribution of the Time Until Absorption (Combat Terminated)

From Neuts[9], the distribution of the time until absorption in state (0,0) given the initial probability vector $(a, a_{(0,0)})$ is

$$F(x) = 1 - \alpha \exp(Tx)\, e, \; x \geqslant 0$$

There are several approaches to evaluate $\exp(Tx)$. We use the method of spectrum decomposition of an exponential matrix, by taking advantage of the special structure of T, to develop the computational algorithm. By letting

$\lambda_{(i,j)} = T_{(i,j)(i,j)}$,

$x^{(l,k)}_{(i,j)}$ = the (l,k) element of the orthogonal right Eigen vector of T corresponding to the Eigen value of $\lambda_{(i,j)}$,

$y^{(l,k)}_{(i,j)}$ = the (l,k) element of the orthogonal left Eigen vector of T corresponding to the Eigen value of $\lambda_{(i,j)}$, we can express the $F(x)$ as

$$F(x) = 1 - \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ \left[\sum_{p=i}^{m} \sum_{q=j}^{m} \alpha_{(p,q)} \cdot x^{(p,q)}_{(i,j)} \right] \left[\sum_{r=1}^{i} \sum_{s=1}^{j} y^{(r,s)}_{(i,j)} \right] \cdot \exp\left[T_{(i,j)(i,j)} \cdot x \right] \right\},$$
$$\text{for } x \geqslant 0 \tag{2}$$

The mathematical structure of the recursive formula is given in Appendix A.

3.2 The Expected Value and Variance

From Neuts[10], the expected value and variance of the time until absorption can be computed as

$$\mu_1' = E(x) = -\alpha T^{-1} e \tag{3}$$

$$\mu_2' = E(x^2) = 2!\alpha T^{-2} e \tag{4}$$

$$\text{Var}(x) = E(x^2) - E(x)^2 \tag{5}$$

To take advantage of the special structure of T, we develop two recursive formulae for computing both $E(x)$ and $\text{Var}(x)$. The developing of the formula is given in Appendix B. The results are

$$\mu_1' = E(x) = -\left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\sum_{s=i}^{m} \sum_{t=j}^{m} \alpha_{(s,t)} \cdot T^{-1}_{(s,t)(i,j)} \right] \right\} \tag{6}$$

$$\mu_2' = E(x^2) = 2 \cdot \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\sum_{s=i}^{m} \sum_{t=j}^{m} \alpha_{(s,t)} \cdot T^{-1}_{(s,t)(i,j)} \right] \cdot \left[\sum_{p=1}^{i} \sum_{q=1}^{j} T^{-1}_{(i,j)(p,q)} \right]$$

3.3 The Probability of Winning the Battle

Let $W_B(W_R)$ be the probability of Blue (Red) wins, then

$$W_R = \sum_{j=2}^{m} \Pi_{(1,j)} \cdot \frac{-T^0_{(1,j)}}{T_{(1,j)(1,j)}} + \Pi_{(1,1)} \cdot P_{(1,1)(0,1)} + \sum_{j=1}^{m} \alpha_{(0,j)}$$

$$W_B = \sum_{i=2}^{m} \Pi_{(i,1)} \cdot \frac{-T^0_{(i,1)}}{T_{(i,1)(i,1)}} + \Pi_{(1,1)} \cdot \left[1 - P_{(1,1)(0,1)} \right] + \sum_{i=1}^{m} \alpha_{(i,0)} \tag{9}$$

where $P_{(1,1)(0,1)}$ is the transition probability from state (1,1) to state (0,1); for example, $P_{(1,1)(0,1)} = \frac{k_1}{k_1 + k_2}$ in square law, and $P_{(1,1)(0,1)} = \frac{c_1}{c_1 + c_2}$ in linear law (c_1 and c_2 are the attrition rates for Blue and Red), $\Pi_{(i,j)}$ is the probability that the process reaches state (i,j). The mathematical structure of $\Pi_{(i,j)}$ is presented in Appendix C. If we assume that both sides start at m combatants, i.e., $a_{(m,m)} = 1$, then the derived probability of winning is the same as that of Jaiswal[6].

3.4 The Expected Survivors when Red (Blue) Wins

Let S_R (S_B) be the expected survivors when Red (Blue) wins, then

$$S_R = \left[\sum_{j=2}^{m} \Pi_{(1,j)} \cdot \frac{-T^0_{(1,j)}}{T_{(1,j)(1,j)}} \cdot j + \Pi_{(1,1)} \cdot P_{(1,1)(0,1)} + \sum_{j=1}^{m} \alpha_{(0,j)} \cdot j \right] \Big/ W_R$$

and

$$S_B = \left[\sum_{i=2}^{m} \Pi_{(i,1)} \cdot \frac{-T^0_{(i,1)}}{T_{(i,1)(i,1)}} \cdot i + \Pi_{(1,1)} \cdot \left[1 - P_{(1,1)(0,1)} \right] + \sum_{i=1}^{m} \alpha_{(i,0)} \cdot i \right] \Big/ W_B$$

If we let $a_{(m,m)} = 1$, then the expected number of survivors of Red force can be expressed as $S_R . W_R + R_E . W_B$ which is the same as Jaiswal's result.

4. NUMERICAL RESULTS

To demonstrate the results, an example taking from Jaiswal[6] is considered to show the computation. We divide the example into two different cases. In case 1, we assume that the initial forces of both sides are certain at m = 15, i.e., $a_{(15,15)} = 1$. Figure 1 shows the same results as those of Jaiswal[6]. Some performance measures of interest are summarized in Table 2. In case 2, we assume that the tactical decision maker of Blue side knows their own starting force at m = 15. But the initial force of the Red side is uncertain and is uniformly distributed between 11 and 15, i.e., $a_{(15,j)} =$ 1/5, j=11,...,15. The computed results are shown in Table 3 and Figure 2.

Table 1. The elements of the infinitesimal generator of the Markov attrition process by using parameters ($k_1=k_2=0.5$, $B_U=R_U=3$, $B_E=R_E=0$) of square law for both sides

Q =

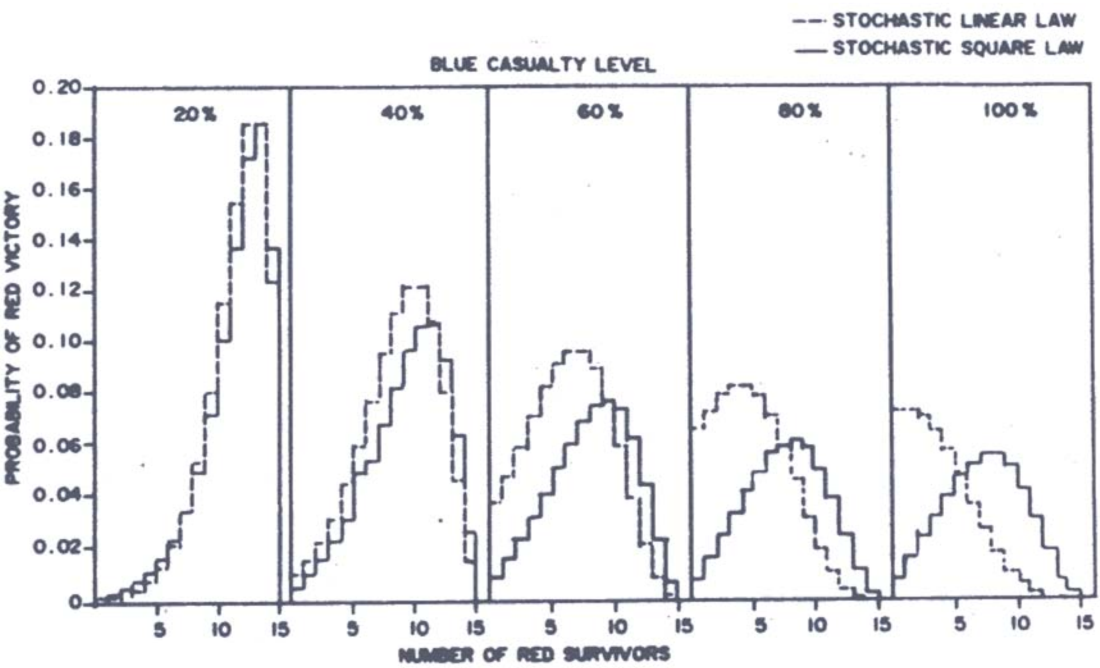
	(3,3)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,3)	(1,2)	(1,1)	(0,0)
(3,3)	–3.0	1.5	0	1.5	0	0	0	0	0	0
(3,2)	0	–2.5	1.5	0	1.0	0	0	0	0	0
(3,1)	0	0	–2.0	0	0	0.5	0	0	0	1.5
(2,3)	0	0	0	–2.5	1.0	0	1.5	0	0	0
(2.2)	0	0	0	0	–2.0	1.0	0	1.0	0	0
(2,1)	0	0	0	0	0	–1.5	0	0	0.5	1.0
(1,3)	0	0	0	0	0	0	–2.0	0.5	0	1.5
(1,2)	0	0	0	0	0	0	0	–1.5	0.5	1.0
(1,1)	0	0	0	0	0	0	0	0	–1.0	1.0
(0,0)	0	0	0	0	0	0	0	0	0	0

Table 2. The computation results in case 1

Case 1 : $a_{(15,15)}=1$, $B_U=R_U=15$, $R_E=0$, $k_1=k_2=c_1=c_2=0.5$

Ending condition	$E(x)$		Var (x)		Prob. of Red victory 1 unit of Red survivors		S_R		Total CPU time* (s)	
	Linear	Square	Linear	Square	Linear	Square	Linear	Square	Linear	Square
Blue casualty level=20% ($B_E=12$)	0.487989	0.034808	0.099854	0.000633	0.000915	0.001512	12.015583	11.965897	2.32	2.31
Blue casualty level=40% ($B_E=9$)	1.092066	0.099694	0.150965	0.001143	0.011089	0.005261	9.226476	9.861500	7.24	7.51
Blue casualty level=60% ($B_E=6$)	1.705840	0.230298	1.100335	0.004987	0.038119	0.007672	6.979632	8.659484	15.70	16.26
Blue casualty level=80% ($B_E=3$)	2.265921	0.486774	2.511399	0.043831	0.066420	0.008323	5.393860	8.032256	27.66	28.57
Blue casualty level=100% ($B_E = 0$)	2.783177	1.194916	4.153284	0.120960	0.074727	0.008368	4.333934	7.798670	43.18	44.60

*PC/AT, 80287 Coprocessor, Language C



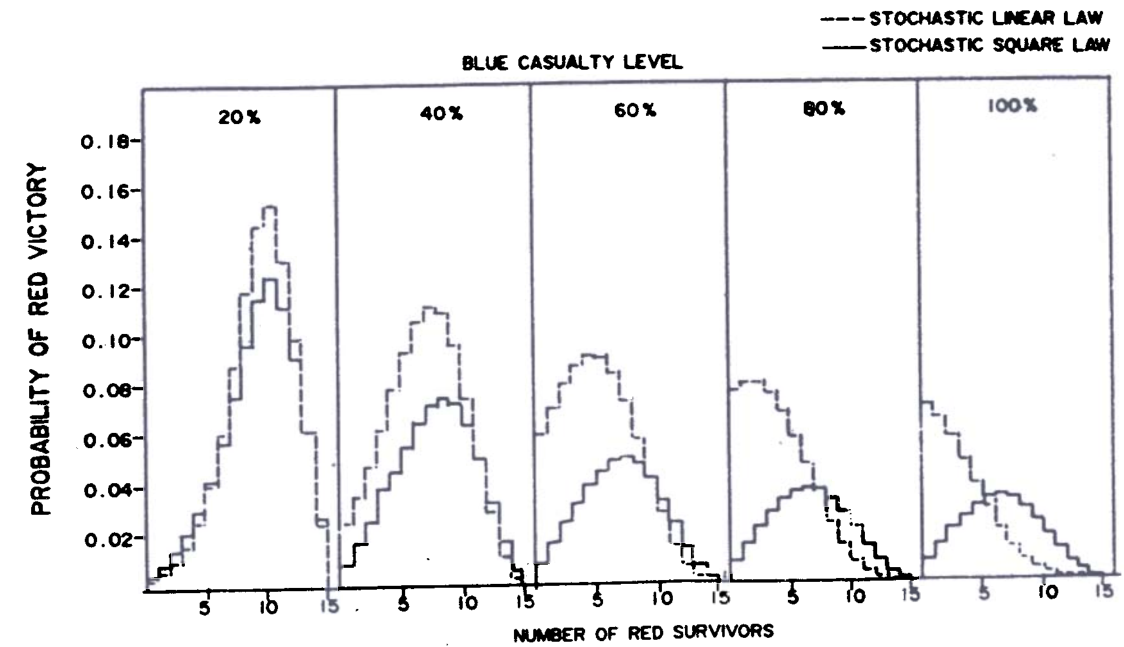
$\alpha_{(15,15)} = 1,\ k_1 = k_2 = c_1 = c_2 = 0.5$

Figure 1. The probabilities of Red victory against Red survivors in case 1.

Table 3. The computation results in case 2

Case 2 : $a_{(15,j)}=0.2$, $j=11,\ldots,15$, $B_U=15$, $R_E=0$, $k_1=k_2=c_1=c_2=0.5$

Ending condition	$E(x)$		Var (x)		Prob. of Red victory 1 unit of Red survivors		S_R		Total CPU time (s)	
	Linear	Square	Linear	Square	Linear	Square	Linear	Square	Linear	Square
Blue casualty level=20% (B_E=12)	0.529375	0.037881	0.210648	0.001648	0.003622	0.004558	10.056026	9.893386	2.30	2.35
Blue casualty level=40% (B_E=9)	1.181039	0.111131	0.159432	0.005884	0.026037	0.008574	7.522074	8.260827	7.19	
Blue casualty level=60% (B_E=6)	1.786854	0.256535	1.291746	0.004145	0.060449	0.008532	5.701286	7.446439	15.65	16.19
Blue casualty level=80% (B_E=3)	2.293159	0.522514	2.748128	0.042264	0.077918	0.007446	4.505566	7.037412	27.52	
Blue casualty level=100% (B_E = 0)	2.738072	1.189266	4.298878	0.010185	0.069255	0.006886	3.724249	7.035674	43.00	44.39



$\alpha_{(15,j)} = 0.2, j = 11, \ldots, 15, k_1 = k_2 = c_1 = c_2 = 0.5$

Figure 2. The probabilities of Red victory against Red survivors in case 2.

From the above two cases, we can see the corresponding probabilities of a Red victory is affected by the uncertainty of the initial force of Red. The more uncertainty, the less probability. As the percentage of casualty level of Blue increases, the expected duration time increases, the expected survivors of Red victory decreases, and the difference of expected survivors between linear and square law is gradually enlarged. In general, the expected duration time of linear law is longer than that of square law.

5. CONCLUSION

In this paper, we propose an approach to compute the expected and variance of duration time, expected survivors, and probability of victory. Since the approach is manipulated in the matrix form, the difficulties of the feasibility of numerical implementation for the Markov attrition process can be resolved. Another important feature of this approach is that it can treat the initial contact forces as random variables. Thus, we can have a more realistic stochastic model to study the combat phenomena.

Future research can be extended to the optimization of combat model and optimal strategy for using tactical reserves. Then, a decision support system may be able to help the tactical decision maker.

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APPENDIX A

Since T is an upper triangular square matrix, the Eigen values are exactly equal to its diagonal elements. By using the spectrum decomposition of an exponential matrix, we can write

$$F(x) = -\alpha \left[\sum_{i=1}^{m} \sum_{j=1}^{m} x_{(i,j)} \cdot y_{(i,j)} \cdot \exp(\lambda_{(i,j)} \cdot x)\right] e, \; x \geq 0$$

With the special structure of T, the elements of $x_{(i,j)}$ and $y_{(i,j)}$ can be derived as

$$x_{(i,j)}^{(i,j)} =$$

$$x_{(i,j)}^{(i,k)} = \frac{T_{(i,k)(i,k-1)}}{T_{(i,j)(i,j)} - T_{(i,k)(i,k)}} \cdot x_{(i,j)}^{(i,k-1)}, \qquad k = j+1 \cdot j+2, \ldots, m \tag{3}$$

$$x_{(i,j)}^{(l,j)} = \frac{T_{(l,j)(l-1,j)}}{T_{(i,j)(i,j)} - T_{(l,j)(l,j)}} \cdot x_{(i,j)}^{(l-1,j)} \qquad l = i+1, i+2, \ldots, m \tag{4}$$

$$x_{(i,j)}^{(l,k)} = \frac{T_{(l,k)(l-1,k)} . x_{(i,j)}^{(l-1,k)} + T_{(l,k)(l,k-1)} \cdot x_{(i,j)}^{(l,k-1)}}{T_{(i,j)(i,j)} - T_{(l,k)(l,k)}},$$

$$l = i+1, i+2, \ldots, m \quad \text{and} \quad k = j+1, j+2, \ldots, m$$

all other elements of $x_{(i,j)} = 0$, and

$$y^{(i,j)}_{(i,j)} = 1 \tag{6}$$

$$, \quad k = j-1, j-2, \ldots, 1 \tag{7}$$

$$y^{(l,j)}_{(i,j)} = \frac{T_{(l+1,j)(l,j)}}{T_{(i,j)(i,j)} - T_{(l,j)(l,j)}} \cdot y^{(l+1,j)}_{(i,j)} \qquad l = i-1, i-2, \ldots, 1$$

$$y^{(l,k)}_{(i,j)} = \frac{T_{(l+1,k)(l,k)} \cdot y^{(l+1,k)}_{(i,j)} + T_{(l,k+1)(l,k)} \cdot y^{(l,k+1)}_{(i,j)}}{T_{(i,j)(i,j)} - T_{(l,k)(l,k)}}$$

$$l = i-1, i-2, \ldots, 1 \qquad \text{and} \qquad k = j-1, j-2, \ldots, 1$$

all other elements of $y_{(i,j)} = 0$.

To combine the above formulae, we have

$$F(x) = 1 - \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ \left[\sum_{p=i}^{m} \sum_{q=j}^{m} \alpha_{(p,q)} \cdot x^{(p,q)}_{(i,j)} \right] \left[\sum_{r=1}^{i} \sum_{s=1}^{j} y^{(r,s)}_{(i,j)} \right] \cdot \exp\left[T_{(i,j)(i,j)} \cdot x \right] \right\}$$

$$\text{for } x \geqslant 0$$

APPENDIX B

The element of T^{-1} can be computed as

$$T^{-1}_{(i,j)(i,j)} = \frac{1}{T_{(i,j)(i,j)}} \quad i = 1, 2, \ldots, m \qquad \text{and} \qquad j = 1, 2, \ldots, m$$

$$T^{-1}_{(i,j)(s,j)} = \frac{-\left[T_{(i,j)(i-1,j)} \cdot T^{-1}_{(i-1,j)(s,j)}\right]}{T_{(i,j)(i,j)}} \quad i = 2, 3, \ldots, m; \qquad j = 1, 2, \ldots, m$$

and $\quad s = i-1, i-2, \ldots, 1$

$$T^{-1}_{(i,j)(i,l)} = \frac{-\left[T_{(i,j)(i,j-1)} \cdot T^{-1}_{(i,j-1)(i,l)}\right]}{T_{(i,j)(i,j)}} \quad i = 1, 2, \ldots, m; \qquad j = 1, 2, .$$

and $\quad l = 1, 2, \ldots, j-1$

$$l = 1, 2, \ldots, j \tag{4}$$

all other elements of $T^{-1} = 0$. Thus,

$$\mu_1' = E(x) = -\left\{\sum_{i=1}^{m} \sum_{j=1}^{m} \left[\sum_{s=i}^{m} \sum_{t=j}^{m} \alpha_{(s,t)} \cdot T^{-1}_{(s,t)(i,j)}\right]\right\} \tag{5}$$

$$\mu_2'' = E(x^2) = 2 \cdot \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\sum_{s=i}^{m} \sum_{t=j}^{m} \alpha_{(s,t)} \cdot T^{-1}_{(s,t)(i,j)}\right] \cdot \left[\sum_{p=1}^{i} \sum_{q=1}^{j} T^{-1}_{(i,j)(pq)}\right] \tag{6}$$

APPENDIX C

Let $\Pi_{(i,j)}$ be the probability that the process reaches state (i,j), then

$$\Pi_{(m,m)} = \alpha_{(m,m)} \tag{1}$$

$$\Pi_{(i,m)} \quad \alpha_{(i,m)} + \Pi_{(i+1,m)} \cdot \frac{-T_{(i+1,m)(i,m)}}{T_{(i+1,m)(i+1,m)}}, \qquad i \quad m - 1, m - 2, \ldots, 1 \tag{2}$$

$$\Pi_{(m,j)} = \alpha_{(m,j)} + \Pi_{(m,j+1)} \cdot \frac{-T_{(m,j+1)(m,j)}}{T_{(m,j+1)(m,j+1)}}, \qquad j = m \quad 1, m - 2, \ldots, 1 \tag{3}$$

$$\Pi_{(i,j)} = \alpha_{(i,j)} + \frac{-T_{(i+1,j)(i,j)}}{T_{(i+1,j)(i+1,j)}} \cdot \Pi_{(i+1,j)} + \frac{-T_{(i,j+1)(i,j)}}{T_{(i,j+1)(i,j+1)}} \cdot \Pi_{(i,j+1)}$$

$$i = m \quad 1, m \quad 2, \ldots, 1 \quad \text{and} \quad j = m \quad 1, m \quad 2, \ldots, \tag{4}$$