Def Sci J, Vol 39, No. 2, April 1989, pp 179-181

Minimum Wave Speed Solution of Fisher's Equation by the Method of Least Squares – A Note

K.N. Mehta

Department of Mathematics, Indian Institute of Technology, New Delhi–110 016

ABSTRACT

The paper presents a simple solution of travelling-wave type (corresponding to the minimum speed c=2) of Fisher's equation, which can be readily adapted for modelling neutron density in nuclear reactors, reaction-diffusion processes in propulsion systems and growth of new advantageous gene in one-dimensional habitat.

1. INTRODUCTION

The non-linear equation of evolution of the diffusive type (written in dimensionless units of length and time[1])

$$u_t = u_{xx} + u(1-u) \tag{1}$$

which had been originally proposed by Fisher[2] for a model in genetic population, has been the subject of study in numerous papers[2-6] published during the past few years on account of its ready adaptability for modelling neutron population[6] and reaction-diffusion processes[7]. Its study from different facets has been further motivated by the knowledge that travelling-wave profiles of Fisher's equation are identical to some of the steady-state solutions of Korteweg-de-Vries-Burgers equation that are obtained when dissipative effects are dominant over dispersive effects[2,8].

The initial-boundary value problem given by Eqn. (1) along with the

Initial data $0 \leqslant u(x,0) \leqslant 1, \quad -\infty < x < \infty$ (2)

and

Boundary conditions : $\lim_{x \to -\infty} u(x,t) = 1, \quad \lim_{x \to \infty} u(x,t) = 0$

and all x derivatives of u tend to zero as $x \to \pm\infty$ (3)

Received 30 September 1988

describes the evolution of a virile mutant in an infinitely long one-dimensional habitat which is saturated at the left and unoccupied at the right[1]. It is proved[9] that for each initial condition of the form (2), Eqn. (1) has a unique solution that is bounded for all times as the initial distribution, i.e. $0 \leqslant u(x, t) \leqslant 1, -\infty < x < \infty$. Also, Fisher[2] and KPP[9] found that Eqn. (1) has an infinite number of travelling-wave solutions of characteristic speeds $c \geqslant 2$.

The sole objective of this brief note is to demonstrate the application of the method of least squares for obtaining an explicit travelling-wave solution of Fisher's equation corresponding to the minimum wave speed $c = 2$. The solution obtained here satisfies the continuous initial data

$$u(x,0) = \frac{}{1+e^{x/2}}, \quad -\infty < x < \infty$$

which clearly belongs to the class of initial distributions characterised by Eqn. (2).

2. SOLUTION FOR MINIMUM WAVE SPEED

Seeking travelling-wave solutions of Eqn. (1) in the form (2)

$$u(x, t) = u(x - ct) = u(s) \tag{5}$$

where c denotes wave speed, we find that the wave profile $u(s)$ satisfies the non-linear equation

$$N[u] = u'' + cu' + u - u^2 = 0 \quad \left(u' = \frac{du}{ds} \right)$$

together with the boundary conditions

$$u(-\infty) = 1, \; u(\infty) = 0$$

and all s derivatives of $u(s)$ vanish as $s \to \pm\infty$ (7)

The non-linear boundary value problem on the infinite domain, described by Eqns. (6) and (7) involves wave speed c as a parameter. To solve it, we consider a solution of the form

$$\bar{u}(s) \quad \frac{1}{1+e^{as}}, \quad a > 0 \tag{8}$$

This choice obviously satisfies the boundary conditions (7) and we further require that Eqn. (8) also satisfy Eqn. (6) almost everywhere in $(-\infty, \infty)$. This we achieve by the method of least squares, that is, by requiring that the definite integral

$$I(c, a) = \int_{-\infty}^{\infty} (N[\bar{u}(s)])^2 \, ds \text{ be minimum} \tag{9}$$

Using the substitution $z = 1 + e^{as}$ in computing the integral, we find that

$$I(c, a) = \frac{1}{60a}[3(1-ca-a^2)^2 + 4(1-ca-a^2)(1-ca+a^2) + 3(1-ca+a^2)^2] \tag{10}$$

Then the equations $\frac{\partial I}{\partial c} = 0 \quad \frac{\partial I}{\partial a} = 0$ are found to yield $c = 2, \; a = \frac{1}{2}$

Finally, we have thus obtained

$$\bar{u}(s) = \bar{u}(x-ct) = \frac{1}{1+e^{\left(\frac{x}{2}-t\right)}}$$

as an explicit travelling-wave solution of Fisher's equation corresponding to the minimum wave speed $c = 2$. This solution satisfies the continuous initial data (4) in contrast to one of the discontinuous type $u(x, 0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$ considered as one of the illustrations[9].

REFERENCES

1. Canosa, J., *IBM J. Res. Develop.*, **17** (1973), 307-313.
2. Fisher, R.A., *Ann. Eugen.*, **7** (1936), 355-369.
3. Gazdag, J. & Canosa, J., *J. Appl. Prob.*, **11** (1974), 445-457.
4. Ablowitz, M.J. & Zeppetella, A., *Bull. Math. Biology,* **41** (1979), 835-840.
5. Abdelkader, M.A., *J. Math. Anal. Appl.*, **85** (1982), 287-290.
6. Canosa, J., *J. Math. Phys.*, **10** (1969), 1862-1868.
7. Fife, P.C., *Bull. Amer. Math. Soc.*, **84** (1978), 693-726.
8. Jeffrey, A. & Kakutani, T., *SIAM Rev.*, **14** (1972), 582-643.
9. Kolmogoroff, A., Petrovsky, I. & Piscounoff, N., Elude de l' equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique, *Bull-de l'univ. d'Etate a Moseou* (Ser. Intern), **AI** (1937), 1-25.