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Estimation of Circular Error Probability of Strapped Down Inertial Navigation System by Propagation of Error Covariance Matrix

S Vathsal

Defence Research and Development Laboratory, Hyderabad-500 258

ABSTRACT

This paper provides an error model of the strapped down inertial navigation system in the state space format. A method to estimate the circular error probability is presented using time propagation of error covariance matrix. Numerical results have been obtained for a typical flight trajectory. Sensitivity studies have also been conducted for variation of sensor noise covariances and initial state uncertainty. This methodology seems to work in all the practical cases considered so far. Software has been tested for both the local vertical frame and the inertial frame. The covariance propagation technique provides accurate estimation of dispersions of position at impact. This in turn enables to estimate the circular error probability (CEP) very accurately.

1. INTRODUCTION

Strapdown systems are of interest to almost all aerospace missions employing inertial navigation to achieve high performance. These systems eliminate the gimbals normally employed in a stabilized platform resulting in easier maintenance, less cost and perhaps improved reliability[1,2]. The sensors employed by them are gyros and accelerometers. These are mounted on the vehicle body which is subject to fast changing environmental disturbances during flight causing motion induced errors in the system. The sensor errors could be classified into a deterministic and a random part. The deterministic errors could be compensated in the navigation computer whereas the random part results in a circular error probability (CEP) of the

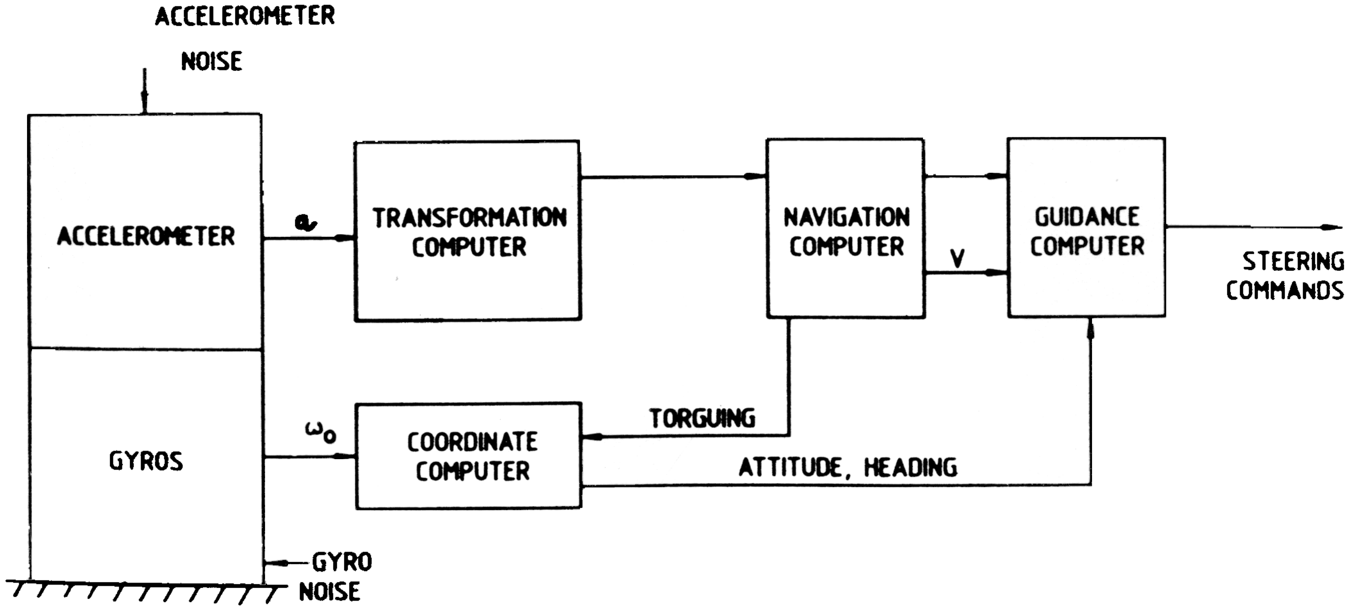

Figure 1. Block diagram of strapdown inertial navigation and guidance system.

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system[4]. Estimation of CEP of the system calls for a stochastic error analysis of the inertial navigation system. Figure 1 shows the schematic system.

2. OVERVIEW OF TECHNIQUE

Friedland has presented the theoretical analysis of strapdown navigation system using quaternions[3]. The covariance propagation equations have been derived for attitude estimation using KALMAN and non linear filters[5,6]. While the error equations of reference 3 are mainly applicable to inertial frame of reference, recently Minoru Shibata has derived error equations for terrestrial applications using the local vertical frame of reference based on a quaternion relation between body fixed coordinates and navigation coordinates[7]. This error model is highly suitable to missile and aircraft navigation. The error covariance propagation studies using this error model of terrestrial navigation employing the quaternion parametrization for attitude have not been reported so far. Shibata's model employs the relative quaternion between body fixed coordinates and local vertical coordinates[7]. The tilt errors and quaternion errors are related by a matrix transformation[7]. The potential of this transformation for stable numerical computation of the error covariance matrix has been fully utilised.

The conventional method of error analysis is using Monte Carlo simulation in which the system errors are

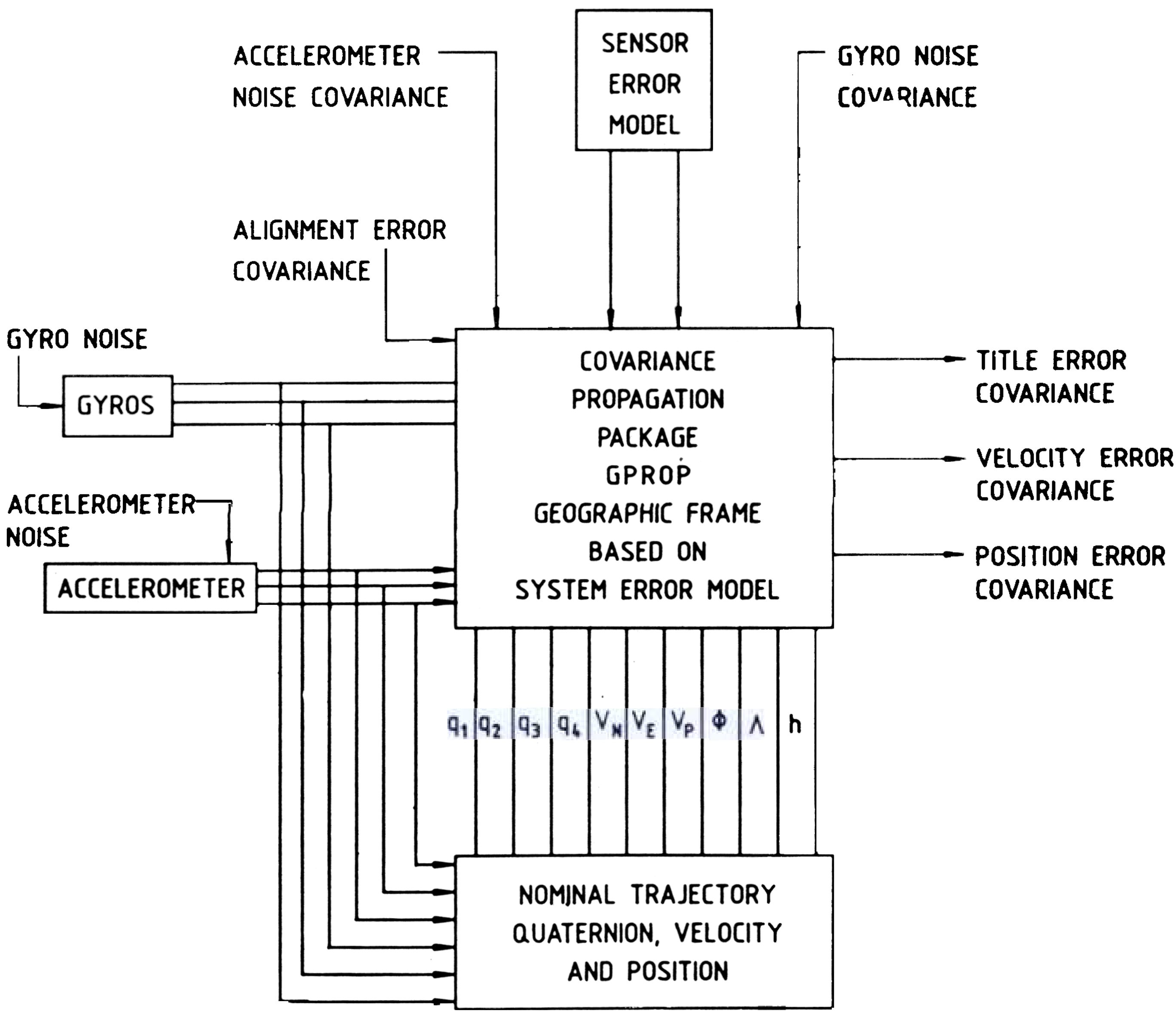

Figure 2. Schematic system of INS error propagation

obtained by root sum square (RSS) of errors due to individual component errors. It has been found that conservative estimates of errors are obtained by the RSS technique. On the other hand the covariance propagation technique provides a lower bound of errors. Since all the errors are treated in the error covariance matrix simultaneously, the cancellation of certain errors in the final output of the navigation system is automatically considered in this approach of error analysis. For estimation of circular error probability (CEP), the one sigma dispersions of position are necessary which can be directly obtained from the error covariance matrix. In the case of Monte Carlo approach, it is a time consuming process to obtain the one sigma values.

Covariance. propagation results are given for terrestrial navigation using a set of flight data which are obtained from a strapdown inertial navigation system. The CEP of the system has also been estimated using a proven formula. Figure 2. presents the block diagram of covariance propagation package which has been developed and tested.

3. ERROR MODELS OF STRAPDOWN INERTIAL NAVIGATION SYSTEM

The error models normally employ the quaternion errors, velocity errors and position errors as the state variables. Let $x_I = [q_1, q_2, q_3, q_4, V_x, V_y, V_z, X, Y, Z]^T$ denote the navigation state of the system with reference to an inertial coordinate reference frame in Fig. 3 The navigation equations are given below.

3.1 Inertial Frame

$$q(t) = \frac{1}{2}\,\Omega[\omega(t)]\,q(t);\; q(t_o) = q_o \tag{1}$$

$$V(t) \quad C_B^I\,[q(t)]\,a(t) + g[r(t)];\; V(t_o) = V_o \tag{2}$$

$$(t) = V(t);\; \gamma(t_o) = \gamma_o \tag{3}$$

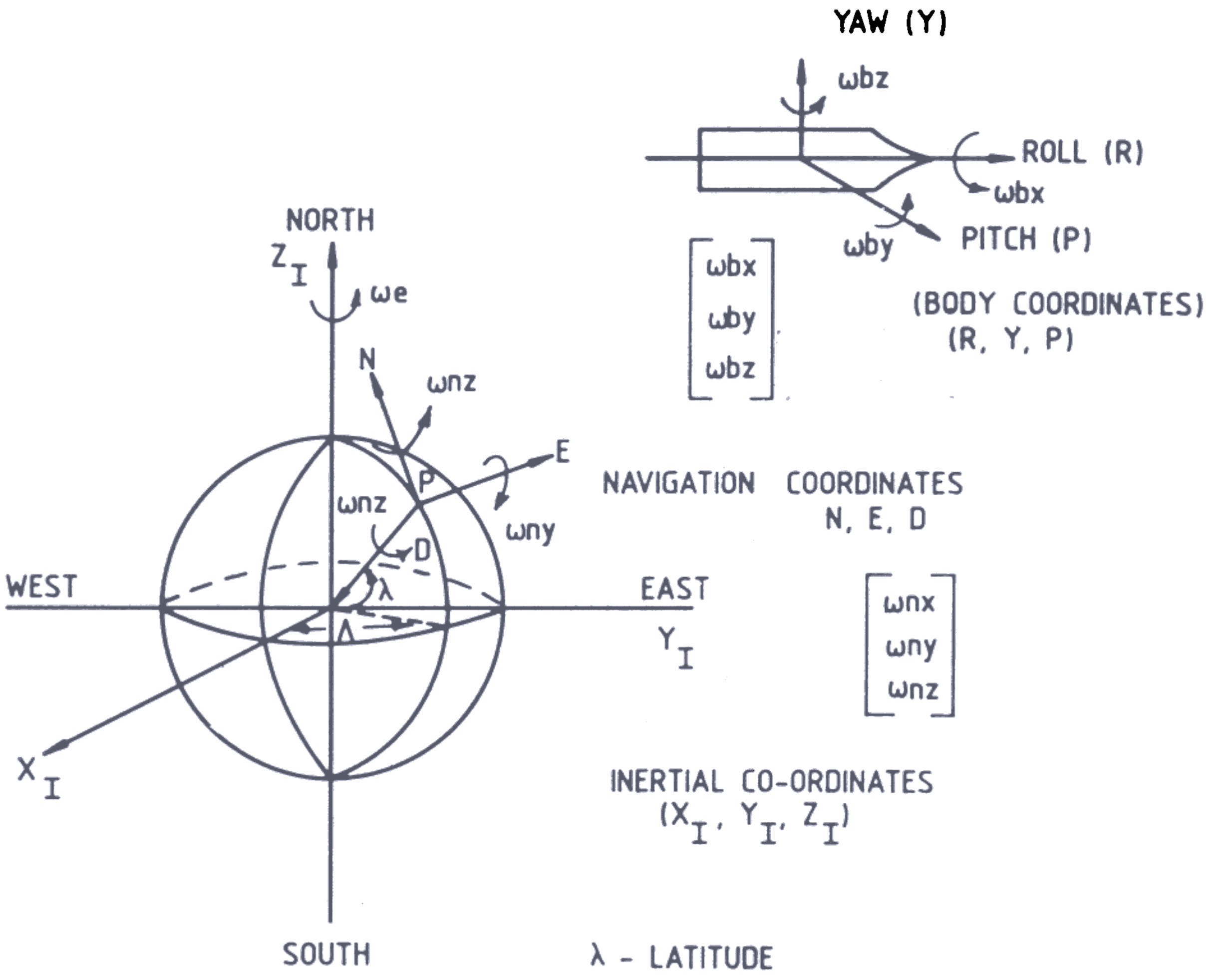

Figure 3. Inertial coordinate reference frame (1).

where $q(t) = [q_1, q_2, q_3, q_4]^T$; $V(t) = [V_x, V_y, V_z]^T$

$\gamma(t) = [X, Y, Z]^T$

$$\Omega[\omega(t)] = \left[\begin{array}{ccc|c} 0 & \omega_{bz} & -\omega_{by} & \omega_{bx} \\ -\omega_{bz} & 0 & \omega_{bx} & \omega_{by} \\ \omega_{by} & -\omega_{bx} & 0 & \omega_{bz} \\ \hline -\omega_{bx} & -\omega_{by} & -\omega_{bz} & 0 \end{array}\right] \qquad C_B^I$$

$$\left[\begin{array}{c|c|c} q_4^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ \hline 2(q_1q_2-q_3q_4) & q_4^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1q_4) \\ \hline 2(q_1q_3+q_2q_4) & 2(q_2q_3-q_1q_4) & q_4^2-q_1^2-q_2^2+q_3^2 \end{array}\right]$$

$$g[r(t)] = \frac{\mu r}{|r|} \qquad |\gamma|^2 = x^2+y^2+z^2$$

Linearizing Eqns (1) to (3) over nominal values of (q, V, r), an error model in inertial frame can be derived which can be expressed in compact form as

$$\delta X_I = F_I\, \delta X_I + G_I\, w_I;\ \delta X(t_o) = \delta X_o \tag{4}$$

For details of derivation of Eqn (4) reference has to be made to Appendix 'A'. The vector w_I represents the random errors of sensors.

3.2 Local Vertical Frame

Inertially referenced navigation systems are widely used for spacecraft applications where geographic navigation information is not needed. For terrestrial navigation, the time varying relationships between the inertial and geographic frames complicate the system design. Then the local vertical mechanization is a better choice for terrestrial applications[7]. With reference to local vertical frame shown in Fig. 2, the navigation equations are presented below

$$q = \frac{1}{2}\,\Omega[\omega_b]\, q - \frac{}{2}\,\Omega\,[\omega_n]\, q \tag{5}$$

$$V_N = C_B^N\, ab - [(2\Omega)\omega e) + (\rho)]V_N + g(R)_n \tag{6}$$

$$\lambda = V_{Nx}/(R_N + h) \tag{7}$$

$$\Lambda = V_{Ny}/(R_E + h)\cos\lambda] \tag{8}$$

$$h = V_{Nz} \tag{9}$$

where $V_N = [V_{Nx}, V_{Ny}, V_{Nz}]^T$; $R = [\lambda, \Lambda, h]^T$

$\rho = [\dot{\Lambda}\ Cos\ \lambda \quad -\dot{\lambda} \quad \dot{\Lambda}\ sin\ \lambda]^T$

$R_E = R_o(1+E\,sin^2\lambda)$; $R_N = R_o(1-2\,E + 3E\,sin^2\lambda)$

$E = \dfrac{1}{294.978613}$; $R_o = 6378163\ m$

The state vector $X_V = (q_1, q_2, q_3, q_4, V_{Nx}, V_{Ny}, V_{Nz}, \lambda, \Sigma, h)^T$

Linearizing Eqns (4) to (8) over nominal values of $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4, V_{Nx}, V_{Ny}, V_{Nz}, \bar{\lambda}, \bar{\Sigma}, \bar{h})^T$

an error model in local vertical frame or the geographic frame can be derived which can be expressed in vector matrix notation as

$$\delta X_V = F_V\, \delta X_V + G_V\, w_V;\ \delta X_V(t_o) = \delta X_{V_o} \tag{10}$$

For details of derivation of Eqn (10), one can refer to Appendix 'B' The vector w_v represents the random errors of sensors namely the gyros and accelerometers.

4. COVARIANCE PROPAGATION STUDIES

From Eqn (4), the error covariance propagation equations can be derived. Let

$$P_I = E\,[\delta X_I\, \delta X_I^T] \tag{11}$$

where E(.) is the expectation operator

The propagation of P_I is governed by the following matrix differential equation

$$\dot{P}_I = F_I P_I + P_I F_I^T + G_I\, Q_I\, G_I^T;\ P_I = P_{I_o} \tag{12}$$

For the local vertical frame.

$$P_V = E\,[\delta X_V\, \delta X_V^T] \tag{13}$$

and

$$\dot{P}_V = F_V\, P_V + P_V F_V^T + G_V Q_V G_V^T;\ P_V = P_{Vob} \tag{14}$$

$$Q_I = E\,[w_I w_I^T];\ Q_V = E[\,w_V w_V^T\,] \tag{15}$$

Computation of Eqns (12) and (14) is beset with numerical problems in the sense that P_I or P_V does not maintain its definite positive property. For maintaining a positive definite covariance matrix, the following transformation matrix S has been used to obtain reduced order and numerically stable covariance matrix[5]. The transformation is shown below

$$\Sigma_I = S^T(\bar{q})\, P_I\, S^T(\bar{q}) \tag{16}$$

$$\Sigma_V = S^T(\bar{q})\, P_V\, S^T(\bar{q}) \tag{17}$$

where

$$S = \begin{bmatrix} Q(\bar{q}) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{18}$$

$$O(q) = \begin{bmatrix} \bar{q}_4 & \bar{q}_3 & \bar{q}_2 \\ \bar{q}_3 & \bar{q}_4 & \bar{q}_1 \\ \bar{q}_2 & \bar{q}_1 & \bar{q}_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 \end{bmatrix} \tag{19}$$

The alignment errors can be set as initial condition in P_I (o) or P_V (o).

4.1 Numerical Results

The major inputs to the numerical studies are the body angular rates $w = [w_x,\ w_y,\ w_z]^T$ and linear acceleration

$a_b = [a_{bx},\ a_{by},\ a_{bz}]^T$

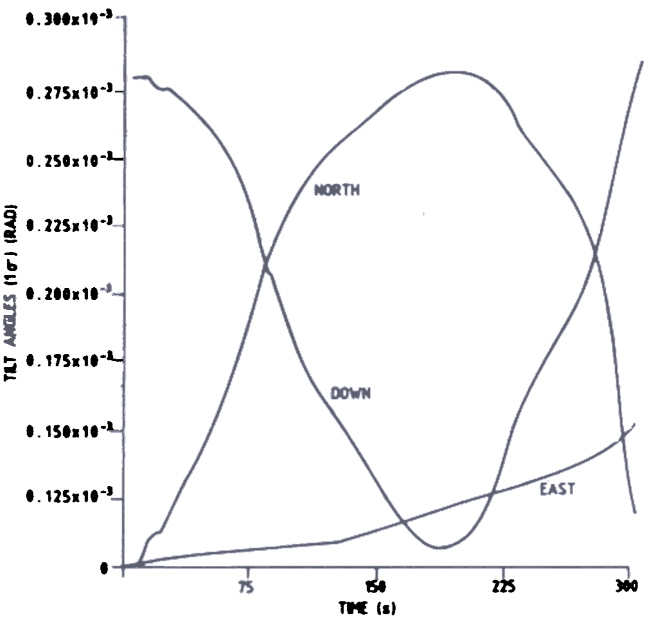

Figure 4. Propagation of tilt angle errors (1σ values).

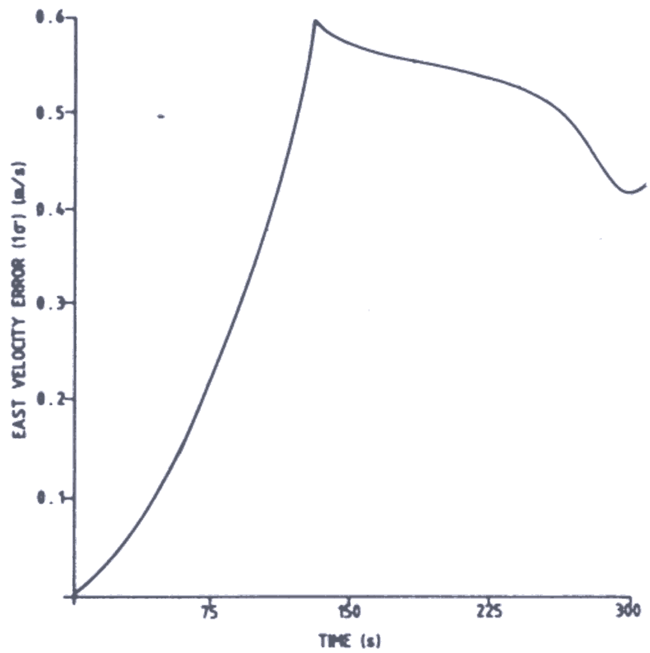

Figure 5. Propagation of velocity errors (1σ values).

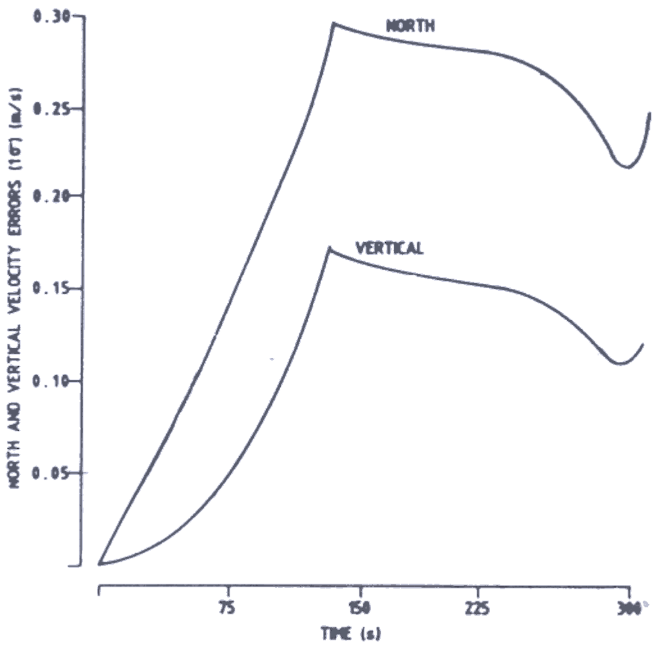

Figure 6. Propagation of velocity errors (1σ values).

The one sigma errors of the sensors are fed into the Q_I or Q_v noise covariance matrix. For initial validation of the software package the nominal body angular velocities, accelerations, quaternions, linear velocities, (north, east, down) and positions (latitude, longitude and altitude) have been generated from the available six degree of freedom simulation package. The covariance matrix P_I or P_v is symmetric and positive

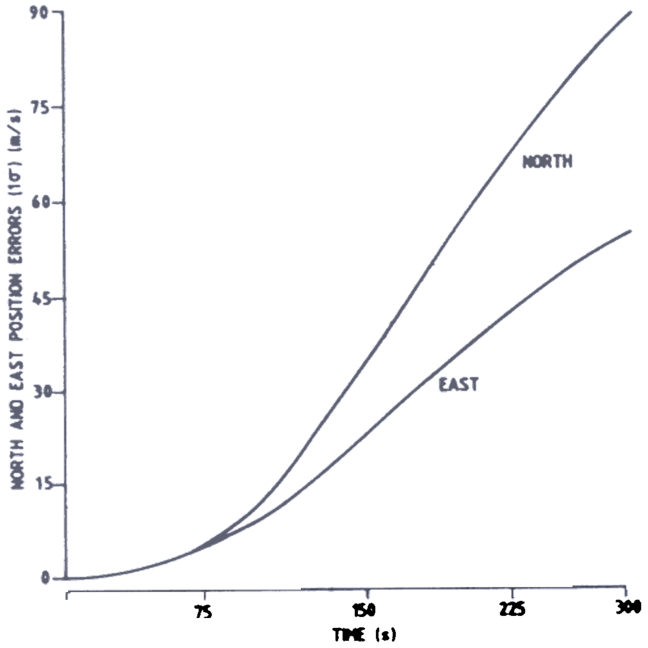

Figure 7. Propagation of position errors (North & East) (1σ).

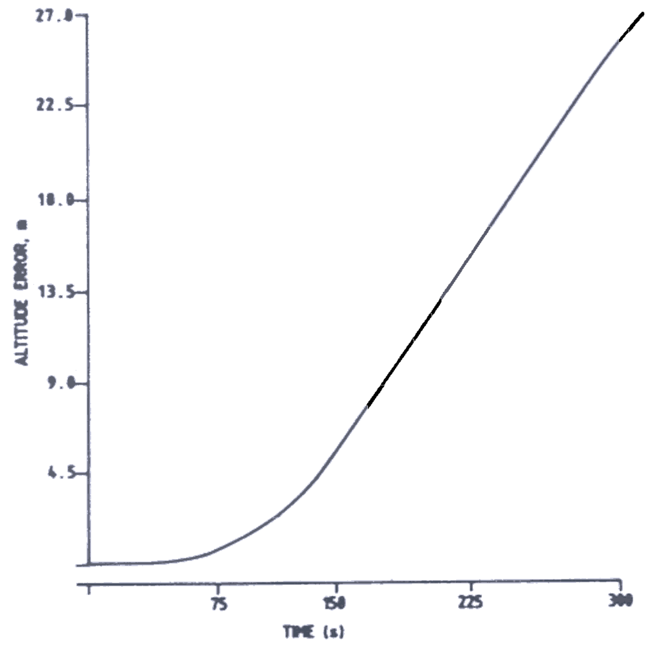

Figure 8. Propagation of altitude error (1σ values).

definite. It is adequate if the lower half or the upper half of the matrix is computed numerically. The reduced order covariance matrix is 9×9 in size. For symmetry it is enough if one computers $(n\times\ (n+1)/2\ =\ 45)$ elements when compared to 81 elements when symmetry is not considered. The output $(wx, wy, wz)^T$ from gyros is not directly available but the incremental angles $(\Delta\phi_x, \Delta\phi_y, \Delta\phi_z)^T$ are available at 6 ms interval. A quadratic polynomial fit has been made to derive $(w_x, w_y, w_z)^T$ at intervals of 18 ms. The $(a_{bx}, a_{by}, a_{bz})^T$ data from accelerometers is available at 18 m sec in terms of incremental velocities $(\Delta V_x, \Delta V_y, \Delta V_z)^T$. For covariance propagation, one needs to solve 45 coupled time varing differential equations. It has been carried out using the fourth order Range-Kutta scheme of solving the differential equations at an interval of 18 ms. Many parameter sensitivity studies and performance comparisons have been made with simulated data before feeding the actual flight data into the program. Typical tilt angle errors, velocity errors and position errors are shown through Figs. 4 to 8.

4.2 Estimation of Circular Error Probability

The terminal miss distance accuracy is normally estimated using Monte Carlo simulations and CEP is derived from the analysis of simulated data. This process is time consuming and costly. From the propagated covariance, it is relatively easy to fit an error ellipsoid and approximate it to circular to represent the CEP of the system[2].

If σX_{NE} and σX_{EE} represent the north and east position errors then

$$C.E.P = 0.589\ (\sigma X_{NE} + \sigma X_{EE}) \pm 3\% \tag{20}$$

$$\text{if} \quad \frac{\sigma X_{NE}}{3} < \sigma X_{EE} < 3\ \sigma X_{NE}$$

Eqn (20) is less conservative when compared to the root sum square value. Table 1 presents a comparision of CEP values obtained by Eqn (20) and RSS techniques for different sensors used in the study.

It has been found that CEP based on formula (20) is always lower than that predicted by RSS technique, in which the square root of the sum of the squares of errors is obtained.

Computations of the system errors have been carried out for both, the inertial frame and the local vertical frame. It has been found that the error behaviour appears to be independent of the mechanisation frame. The reasons could be explained from Eqns (4) and (10) which describe the error model. The covariance propagation P has been found to be more sensitive to P_o and Q which depict the initial state of uncertainty, and gyro and accelerometer noise covariances. Propagation of P is less sensitive to F and G matrices for the flight trajectory chosen for illustration. It implies

Table 1. Performance and sensitivity studies

	Cases	σX_{NE} m	σX_{EE} m	σX_{NH} m	CEP based on (20) m	RSS criterion m
	Sagem gyro sagam accelerometer	53.248	90.06	26.35	84.40	107.89
2.	SFIM gyro Ferranti accelerometer	54.923	90.15	27.499	85.44	109.09
3.	Sagem gyro & Ferranti accelerometer	52.99	89.95	25.98	84.19	107.58

that the error behaviour is less sensitive to dynamic couplings for the present study.

5. CONCLUSION

Covariance propagation methodology has been employed successfully for estimating errors of the strapdown inertial navigation system. The error models for both the inertial frame and the local vertical frame have been derived and used in this development. Both simulated data and actual flight data of the system have been utilized to estimate CEP and there is a good match between the two. For the flight trajectory chosen, CEP appears to be insensitive to the frame of navigation namely, the local vertical or inertial. The dispersions in position expected for different combinations of gyros and accelerometers have also been presented. The CEP predicted from covariance of errors is shown to be lower than the conventional RSS technique employed in Monte Carlo approach. Further work can be done to extend this approach to a multi sensor hybrid navigation system.

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APPENDIX 'A'

Here the strapdown navigation system error model is presented in the inertial frame. Let q, V, r, w, a be the true value of quaternion, velocity, position, angular velocity and linear acceleration and $\bar{q}$, $\bar{V}$, $\bar{r}$, $\bar{w}$, $\bar{a}$ be their estimated quantities. Then

$$\delta q = q - \bar{q}; \quad \delta_V = V - \bar{V}; \delta\gamma = \gamma \qquad \text{(A1)}$$
$$\delta\omega = \omega - \bar{\omega}\,; \delta a = \dot{a} - \bar{a}$$

Substituting $A1$ into Eqns 1 to 3, expanding by Taylor series and neglecting higher order terms, one gets

$$\delta\dot{q} = \Omega(\bar{\omega})\,\delta q + Q(\bar{q})\,\delta\omega \qquad \text{A2)}$$

$$\delta\dot{v} = D(\bar{q}, \bar{a})\,\delta q + C(\bar{q})\,\delta a + G(\bar{\gamma})\delta\bar{\gamma} \qquad \text{(A3)}$$

$$\delta\gamma = \delta v \qquad \text{A4)}$$

Eqns $(A2)$ to $(A4)$ can be written in compact vector matrix notation as

$$\begin{bmatrix}\delta\dot q\\ \delta\dot v\\ \delta\dot\gamma\end{bmatrix} \begin{bmatrix}\frac{1}{2}\,\Omega(\bar\omega) & 0 & 0\\ D(\bar q,\bar a) & 0 & G(\bar r)\\ 0 & I & 0\end{bmatrix} \begin{bmatrix} b\varrho\\ \delta v\\ \delta\gamma\end{bmatrix} \begin{bmatrix}\frac{1}{2}Q(\bar q) & 0\\ 0 & C(\bar q)\\ 0 & 0\end{bmatrix} \begin{bmatrix}\delta\omega_y\\ \delta a_y\end{bmatrix}$$

The different submatrices used in the above error model have been shown in Ref. 3. It can be identified with Eqn (4) of this paper. The matrix Q is defined in Eqn (19) of this paper.

APPENDIX-B

Here the strapdown navigation system error model is presented in the local vertical frame. Using the same notation and assumptions, the error equations can be derived as provided in Ref (7).

$$\delta\dot q = \frac{1}{2}\,\Omega[\bar\omega b]\,\delta q - \frac{1}{2}\,\Omega[\bar\omega_n]\,\delta q + \frac{1}{2}\,Q(\bar q)\,\delta\bar\omega_b - \frac{1}{2}\,R\,(\bar q)\delta\bar\omega\,\delta\bar\omega_n \qquad \text{(B1)}$$

$$\delta\dot V_N = D(\bar q,\bar a)\,\delta q - \big[2\,(\omega_e) + \rho\,\big]\delta V_N - \big[2\,[\,\delta\omega_e] + [\delta\rho]\big]$$

$$\bar V_N + C_B^N(\bar q)\,\delta ab + \delta g$$

$$\delta\dot\lambda = \delta V_x/\,(R_N+\bar h) - \bar V_x\,\delta h/(R_N+\bar h)^2$$

$$V\varrho = \frac{\sec\bar\lambda}{(R\;\cdot\;\bar h)}\,\delta V_y + \bar\Lambda\;\;\tan\phi.\,\delta\lambda - \bar\Lambda\;\;\delta h\,/\,(R_E+\bar h)$$

$$\delta\dot h = -\,\delta V_z$$

Eqns (B1) to (B5) can be written in vector matrix notation as

The different submatrices used in the above error model have been shown in Reference (7). It can be identified with Eqn (10) of this paper.

$$\begin{bmatrix}\delta\dot q\\ \\ \delta V_N\\ \\ \delta\dot\lambda\\ \delta\dot\Lambda\\ \delta\dot h\end{bmatrix} \left[\begin{array}{c|ccc|ccc} \frac{1}{2}\big[\Omega(\bar\omega_b)+\Omega^T(\bar\omega_n)\big] & & 0 & & 0 & 0 & 0\\ \hline & & & & 0 & 0 & 0\\ D(\bar q,\,\bar a) & & -\big[(2\bar\omega_e+\rho)\big] & & 0 & 0 & 0\\ & & & & \frac{2g_z}{2\lambda} & 0 & \frac{2g_z}{2h}\\ \hline 0 & \frac{1}{(R_n+\bar h)} & 0 & 0 & 0 & 0 & \frac{-V_x}{R_N+\bar h}\\ 0 & 0 & \frac{Sec\phi}{(R_E+h)} & 0 & \Lambda\tan\bar\Lambda & 0 & \frac{-\lambda}{R_E+\bar h}\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix}\delta q\\ \hline \delta V_x\\ \delta V_y\\ \delta V_z\\ \hline \delta\Lambda\\ \delta\lambda\\ \delta h\end{bmatrix} \begin{bmatrix}\frac{1}{2}Q(\bar q)\,\delta\bar\omega_b\\ -\frac{1}{2}R(\bar q)\,\delta\bar\omega_n\\ \hline C_B^N(\bar q)\,\delta a_b\\ -\big[(2\delta\bar\omega_e+P)\big]\bar V_N\\ \\ \hline 0\\ 0\\ 0\end{bmatrix}$$