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Postbuckling Behaviour of Anisotropic Laminated Composite Plates due to Shear Loading

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ABSTRACT

This study investigates postbuckling behaviour of laminated composite plates using a nine-noded shear flexible quadrilateral plate element. The formulation includes nonlinear strain-displacement relation based on von Karman's assumption. The nonlinear governing equations are solved through iteration. A detailed parametric study is carried out to bring out the influence of ply-angle, aspect ratio and material properties on the postbuckling strength of laminates due to in-plane shear loads.

1. INTRODUCTION

In recent years, fibre-reinforced laminated composites have found increasing application in many engineering fields, such as aircraft, missile, hydrospace, automobiles, etc. This is mainly due to their high specific strength and specific stiffness. These structures are often subjected to in-plane loadings which may cause buckling. The accurate knowledge of critical buckling loads, mode shapes, and the subsequent postbuckling behaviour is essential for reliable and lightweight design of such structures. Experimental difficulties in obtaining accurate buckling loads even for isotropic homogeneous plates, are well known. These are mainly associated with obtaining desired conditions at the plate edges to achieve the desired in-plane loading conditions (*e.g.* uniform stress) and boundary conditions (*e.g.* simply-supported or free). Additional difficulties arise for composite plates. Physical discontinuities and exposed fibers at the plate edges make the desired loadings and boundary conditions even more difficult to achieve and internal discontinuities (*e.g.* delamination or debonding) degrade the reliability and reproducibility of results.

Hence, analytical or numerical approaches only can predict the buckling behaviour. Among the numerical approaches, the finite element procedure is ideally suited for analysis of structures because of its flexibility in accounting for arbitrary geometries, boundary conditions, loading and material property variations. It also has advantage

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over the analytical method because there is no need for *apriori* assumption of mode shape, and the solution itself predicts the mode shape based on chosen boundary conditions.

Postbuckling behaviour of composite plate[1] due to uniaxial, biaxial, compressive or compressive tensile loading has been investigated at large, but only a few investigated the postbuckling behaviour of composite plates due to shear loading experimentally [2,3] and analytically[4]. But no work has been reported on the postbuckling behaviour of composite plates due to shear loading using finite element method. Finite elements based on field-consistency principle have been developed recently for the structural analysis of thick as well as thin plates/shells, and these elements do not exhibit membrane or shear locking and so do not require *ad hoc* techniques like reduced/selective integration[5]. They also give accurate results for linear/nonlinear dynamic analysis of plates[6,7]. In the present investigation, a nine-noded field consistent plate/shell element has been used for predicting the postbuckling behaviour of laminated composite plates.

2. FORMULATION

The laminated composite plate element used in the study is of C^0 continuous shear flexible type. x and y coordinates are along the in-plane directions and z along the thickness direction. Using Mindlin theory, the displacements u, v and w at a point (x, y and z) from the median surface are expressed as the functions of mid-plane displacements u^o, v^o, and w, and independent rotations φ_x and φ_y of normal in xz and yz planes, respectively as

$$u(x, y, z, t) = u^o(x, y, t) + z\,\varphi_x(x, y, t)$$

$$v(x, y, z, t) = v^o(x, y, t) + z\,\varphi_y(x, y, t)$$

$$w(x, y, z, t) = w(x, y, t) \tag{1}$$

Using Von-Karman's assumption for moderately large deformation which imply that derivatives of u and v with respect to x, y and z are small and noting that w is independent of z, Green's strains can be written in terms of mid-plane deformations as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_p^L \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\varepsilon_b \\ \varepsilon_s \end{Bmatrix} + \begin{Bmatrix} \varepsilon_p^{NL} \\ 0 \end{Bmatrix} \tag{2}$$

where, the membrane strains $\{\varepsilon_p^L\}$, bending strains $\{\varepsilon_b\}$, shear strains $\{\varepsilon_s\}$ and nonlinear in-plane strains $\{\varepsilon_p^{NL}\}$ in the Eqn (2) are written as

$$\{\varepsilon_p^L\} = \begin{Bmatrix} u^o_{,x} \\ v^o_{,y} \\ u^o_{,y} + v^o_{,x} \end{Bmatrix}$$

$$\{\varepsilon_b\} = \begin{Bmatrix} \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{x,y} + \varphi_{y,x} \end{Bmatrix}$$

$$\{\varepsilon_s\} = \begin{Bmatrix} \varphi_x - w_{,x} \\ \varphi_y - w_{,y} \end{Bmatrix}$$

$$\{\varepsilon_p^{NL}\} = \begin{Bmatrix} \frac{1}{2} w_{,x}^2 \\ \frac{1}{2} w_{,y}^2 \\ w_{,x} w_{,y} \end{Bmatrix} \tag{3d}$$

Assuming initial membrane in-plane stress field with σ_x^o, σ_y^o and τ_{xy}^o, the final state of stress t_{ij} can be written as

$$\begin{Bmatrix} t_x \\ t_y \\ t_{xy} \\ t_{xz} \\ t_{yz} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ t_{yz} \end{Bmatrix} + \begin{Bmatrix} \sigma_x^o \\ \sigma_y^o \\ \tau_{xy}^o \\ 0 \\ 0 \end{Bmatrix} \tag{4}$$

or abbreviated as $\{t\} = \{\sigma\} + \{\sigma^o\}$

The strain energy can be expressed as

$$U_1 = \{\sigma^o\}^T \{\varepsilon\} + ½ \{\sigma\}^T \{\varepsilon\} \tag{5}$$

Neglecting third and higher order terms for displacement gradient, U_1 can be rewritten as

$$U_1 = \{\sigma^o\}^T\{\varepsilon^L\} + \{\sigma^o\}^T\{\varepsilon^{NL}\} + \tfrac{1}{2}\{\sigma\}^T\{\varepsilon^L\} \qquad (6)$$

The incremental strain energy per unit volume is expressed as

$$\Delta U_1 = U_1 - \{\sigma^o\}^T\{\varepsilon^L\} = \{\sigma^o\}^T\{\varepsilon^{NL}\} + \tfrac{1}{2}\{\sigma\}^T\{\varepsilon^L\} \qquad (7)$$

Performing integration wrt z from $-h/2$ to $+h/2$, the incremental strain energy per unit area is

$$\Delta U_{1A} = \frac{1}{2}\int_{-h/2}^{+h/2}\{\sigma\}^T\{\varepsilon^L\}dz + \int_{-h/2}^{+h/2}\{\sigma^o\}^T\{\varepsilon^{NL}\}dz \qquad (8\text{ a})$$

and can be abbreviated as

$$\Delta U_{1A} = V_1(\delta) + V_2(\delta) \qquad (8\text{ b})$$

where, V_1 is the elastic strain energy (for a linear elastic solid) and V_2 is the potential energy of the in-plane loads due to transverse deflection.

The structural behaviour is modelled based on shear flexible lamination. If $\{N\}$ represents the membrane stress resultants and $\{M\}$ the bending stress resultants, we can relate these to membrane strains, $\{\varepsilon_p^L + \varepsilon_p^{NL}\}$ and bending strains (curvature) $\{\varepsilon_b\}$, through the constitutive relation as

$$\{N\} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{+h/2}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = [A]\{\varepsilon_p\} + [B]\{\varepsilon \qquad (9\text{ a})$$

$$M\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{+h/2}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z dz = [B]\{\varepsilon_p\} + [D]\{\varepsilon_b\} \qquad (9\text{ b})$$

where A_{ij}, D_{ij} and B_{ij} $(i, j = 1, 2, 6)$ are extensional, bending and bending-extensional stiffness coefficients of the composite laminate given by

$$[A],[B],[D]\,) = \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}[\overline{Q}](1, z, z \quad dz \qquad (10)$$

where $[\overline{Q}]$ is the matrix of reduced stiffness coefficients.

If $\{Q\}$ represents transverse shear stress resultant then it is related to the transverse shear strain through the constitutive relation:

$$\{Q\} = \begin{Bmatrix} Q_{xz} \\ Q_{yz} \end{Bmatrix} = \int_{-h/2}^{+h/2}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz = [E]\{\varepsilon_s\} \qquad (11)$$

where E_{ij} $(i, j = 4, 5)$ are transverse shear stiffness coefficients of the laminate given by

$$[E] = \sum_{k=1}^{N} k_i k_j \int_{h_{k-1}}^{h_k}[\overline{Q}]_k\, dz \qquad i, j = 4,5 \qquad (12)$$

For a composite laminate consisting of N layers with stacking angle θ_k $(k = 1,\ldots,N)$ and layer thickness h_k $(k = 1, \ldots, N)$, the stiffness coefficients are computed as per the details available in literature[8]. The strain energy functional V_1 (δ) of Eqn. (8b) is given by

$$V_1(\delta) = \frac{1}{2}\int_A\Big[\{\varepsilon_p\}^T[A]\{\varepsilon_p\} + \{\varepsilon_p\}^T[B]\{\varepsilon_b\} + \{\varepsilon_b\}^T[B]\{\varepsilon_p\} + \{\varepsilon_b\}^T[D]\{\varepsilon_b\} + \{\varepsilon_s\}^T[E]\{\varepsilon_s\}\Big]dA \qquad (13)$$

where δ is the vector of degrees-of-freedom (DOFs). Following the procedure of Rajasekaran and Murray[9], the strain energy functional V_1 is rewritten as

$$V_1(\delta) = \{\delta\}^T \left[\frac{1}{2}[k] + \frac{1}{6}[N_1 + \frac{1}{12}[N_2 \right] \{\delta\}$$
$$+ \frac{1}{2}\{\delta\}^T [N_3]\{\delta\} \quad (14)$$

where $[k]$ is the linear stiffness matrix, $[N_1]$ and $[N_2]$ are nonlinear stiffness matrices and $[N_3]$ is the shear stiffness matrix.

The potential energy of in-plane load due to transverse deflection , V_2 (δ) is given as

$$V_2(\delta) = \int_{-h/2}^{h/2} \{\sigma^o\}^T \{\varepsilon^{NL}\}\, dz$$
$$= \frac{h}{2}[\sigma_x^o w_{,x}^2 + \sigma_y^o w_{,y}^2 + 2\,\tau_{xy}^o w_{,x} w_{,y}] \quad (15)$$

Let

$$\left.\begin{matrix} P_x \\ P_y \\ P_{xy} \end{matrix}\right\} = \frac{1}{2}\int_{-h/2}^{+h/2} \left\{\begin{matrix} \sigma_x^o \\ \sigma_y^o \\ \tau_{xy}^o \end{matrix}\right\} dz \quad (16\text{ a})$$

Then

$$V_2(\delta) = \frac{1}{2}[P_x w_{,x}^2 + P_y w_{,y}^2 + 2P_{xy} w_{,x} w_{,y}] \quad (16\text{ b})$$

If P_{cr} is the critical buckling load, then let, $P_x = P_{cr}\, n_x$, $P_y = P_{cr}\, n_y$ & $P_{xy} = P_{cr}\, n_{xy}$.

$$V_2(\delta) = \frac{1}{2} P_{cr} \{w_{,x} \;\; w_{,y}\} \begin{bmatrix} n_x & n_{xy} \\ n_{xy} & n_y \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} \quad (16\text{ c})$$

For pure shear buckling, $n_{xy} = -1$ and $n_x = n_y = 0$. Now, the total potential energy can be obtained by substituting Eqns (14) and (16) into Eqns (8) and integrating through the neutral surface as

$$\Delta U_1 = \int \Delta U_{1A}\, dA = \int (V_1(\delta) + V_2(\delta))\, dA$$

Equating the variation of the total potential ΔU_{1c} to zero, one can obtain the governing equation for the plate as

$$\left[[K] + \frac{1}{2}[N_1 + \frac{1}{3}[N_2] + [N_3] + P_{cr}[K_G] \right] \{\delta\} = 0 \quad (18)$$

where $[K_G]$ is the geometric stiffness and P_{cr} is the critical buckling load. The above nonlinear eigen system is solved for P_{cr} for various values of amplitude-to-thickness ratio.

3. ELEMENT DESCRIPTION

The laminated plate element employed here is C^0 continuous shear flexible element with five-DOF at nine nodes in QUAD-9 element.

If the interpolations for QUAD-9 are used directly to interpolate the five field variables u to φ_y in deriving the shear strains, the element will lock and show oscillations in the shear stresses. Field consistency requires that the transverse shear strains must be interpolated in a consistent manner. Thus u, v, φ_x and φ_y terms in the expression for ε_s have to be consistent with field function $w_{,x}$ and $w_{,y}$. This is achieved by using field redistributed substitute shape functions to interpolate those specific terms, which must be consistent[5].

4. RESULTS & DISCUSSION

The nonlinear governing equations are solved using direct iteration technique with a linear solution as the starting one. The eigen system is solved using Lanczas scheme[10]. Since pure shear buckling analysis needs finer mesh, 6 x 6 uniform mesh of nine-noded quadrilateral plate element have been used based on progressive mesh refinement, for both cross-ply and angle-ply laminated plates. The shear correction factor is taken as 5/6.

The material used for the study has the following elastic properties unless otherwise specified:

$E_x/E_y = 40.0$, $G_{xy}/E_y = 0.6$, $G_{yz}/E_y = 0.5$, $\nu_{xy} = 0.25$

Before proceeding for the detailed analysis, the present formulation is validated for buckling against available results[11,12] by considering an isotropic plate subjected to shear load. It is seen from Table 1 that the present results are in good

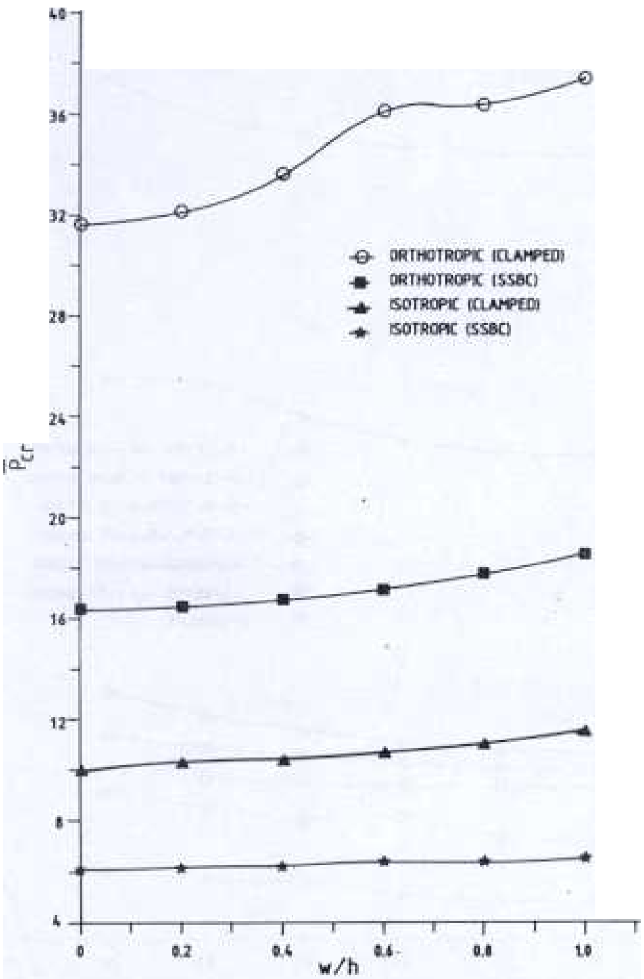

Figure 1. Variation of postbuckling shear load $\overline{P}_{cr}$ *vs* amplitude (w/h) for isotropic and orthotropic cases (a/b=2).

agreement with the available solution. The critical buckling load is nondimensionalised as

$$\overline{P}_{cr} = P_{cr} b^2 / E_T h^3$$

Next, the postbuckled paths due to shear load for a single-layered orthtropic and symmetric composite laminates (cross-ply and angle-ply) are given in Figs. 1-3. It was observed from Fig. 1 that the orthotropic plates have pronounced effect on postbuckling behaviour and they enhance the load-carrying capacity with increase in amplitude, compared to the isotropic case. Furthermore, the clamped support case increases the critical load in comparison to the simply-supported case, as expected. Also, the postbuckling path does not

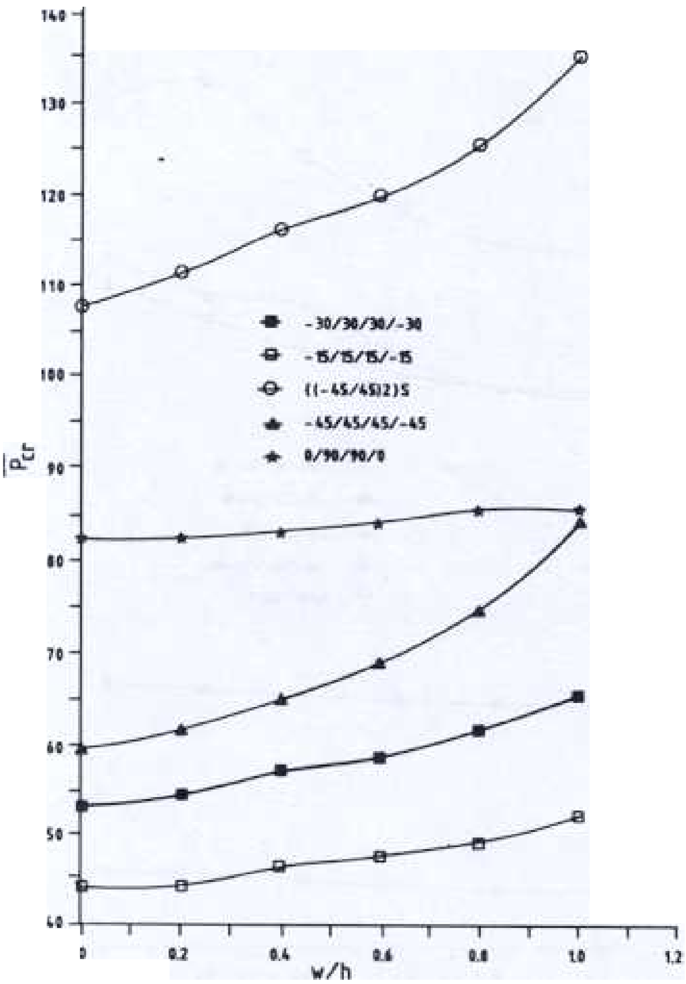

Figure 2 (a). Variation of postbuckling shear load $\overline{P}_{cr}$ *vs* amplitude (w/h) for simply-supported laminated cases (a/b=1) corresponding to positive shear.

depend on the direction of applied shear load, irrespective of isotropic or orthotropic laminates.

For the symmetric angle-ply laminates, the results due to positive and negative shear loads are depicted in Fig. 2. It was observed from these figures that the buckling load is different for different directions of applied shear load for the angle-ply case, whereas it does not vary for cross-ply case wrt the direction of the applied shear load. Also the critical loads for the angle-ply cases are symmetric about 45° case, the maximum being with 45° laminate and the negative applied shear load predicts more buckling load, compared to positive load. The influence of increase in the

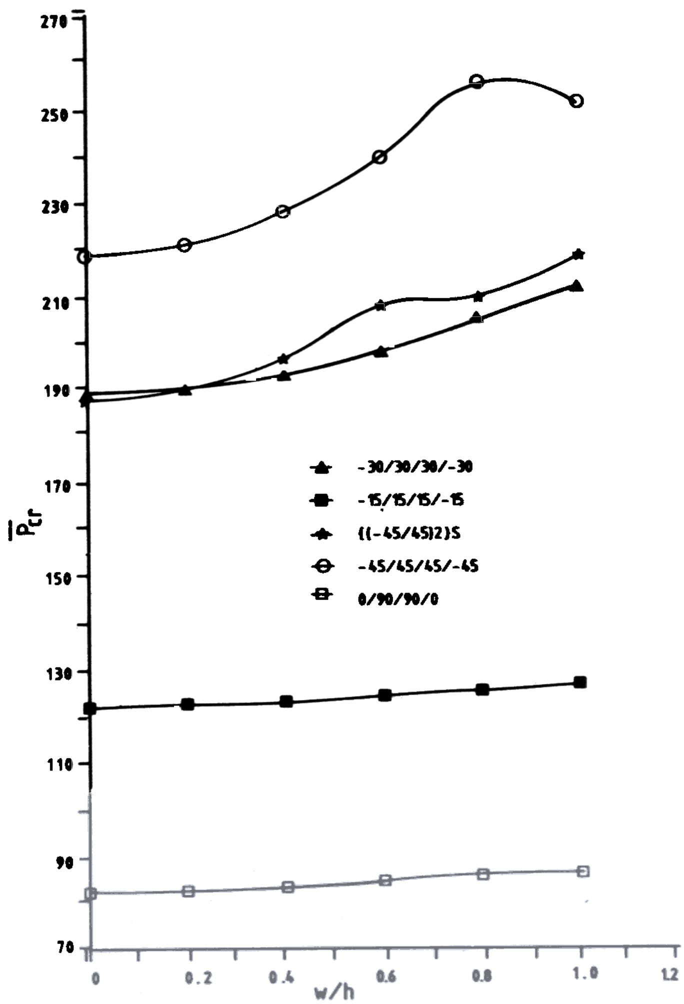

Figure 2 (b). Variation of postbuckling shear load $\bar{P}_{cr}$ *vs* amplitude (w/h) for simply-supported laminated cases (a/b=1) corresponding to negative shear.

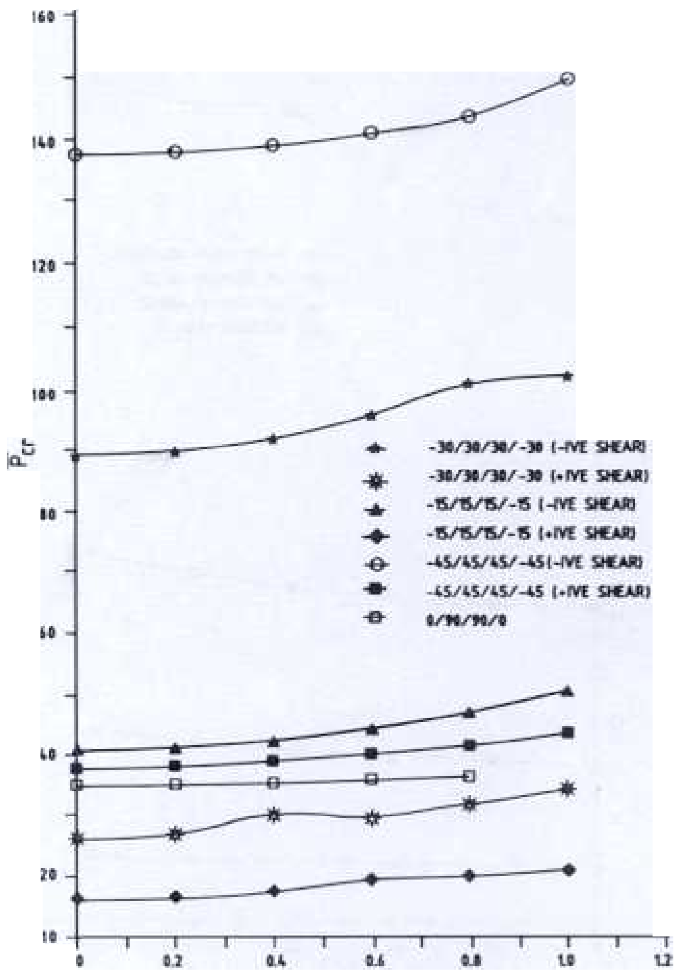

Figure 3. Variation of postbuckling shear load $\bar{P}_{cr}$ *vs* amplitude (w/h) for simply-supported laminated cases (a/b = 2).

number of layers in angle-ply case considered here, is significant on determining the critical load and it enhances the critical load when the applied load is in the positive direction.

Table 1. Nondimensional buckling load for isotropic rectangular plate for various aspect ratios due to shear loading ($\bar{P}_{cr} = P_{cr}\, b^2/\pi^2 D$)

a/b	Ref. 11	Ref. 12	Present
1.0	9.34		9.344
1.2	8.00		8.003
1.4	7.30		7.308
1.6	7.00		6.930
1.8	6.80		6.714
2.0	6.60		6.576
3.0	5.90		5.888

The effect of aspect ratio on critical is shown in Figs. 2 and 3. It can be concluded from these figures that the increase in aspect ratio decreases the critical load. The increase in modular ratio is due to increase in the critical load as shown in Fig. 4. Figure 5 shows the effects of in-plane load along with applied shear load on critical load, particularly applied to reduce the load carrying capacity of the plate.

5. CONCLUSIONS

Shear flexible QUAD-9 field consistent element has been utilised to bring out the postbuckling performance of anisotropic laminated

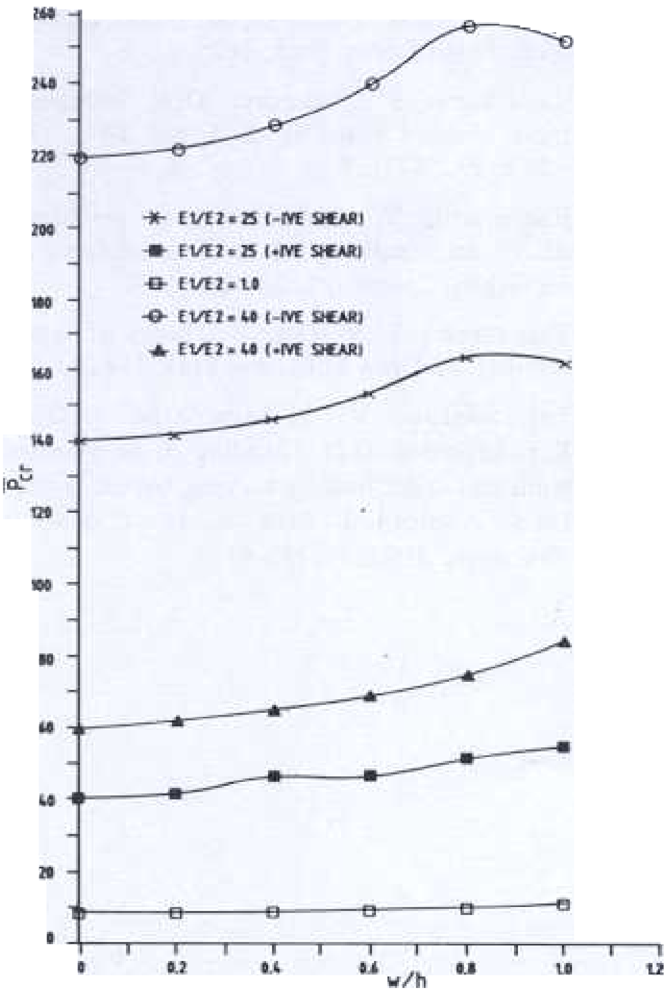

Figure 4. Effect of modular ratio (E_1/E_2) on shear postbuckling strength[(-45°/45°/45°/-45°) a/b=1, simply-supported].

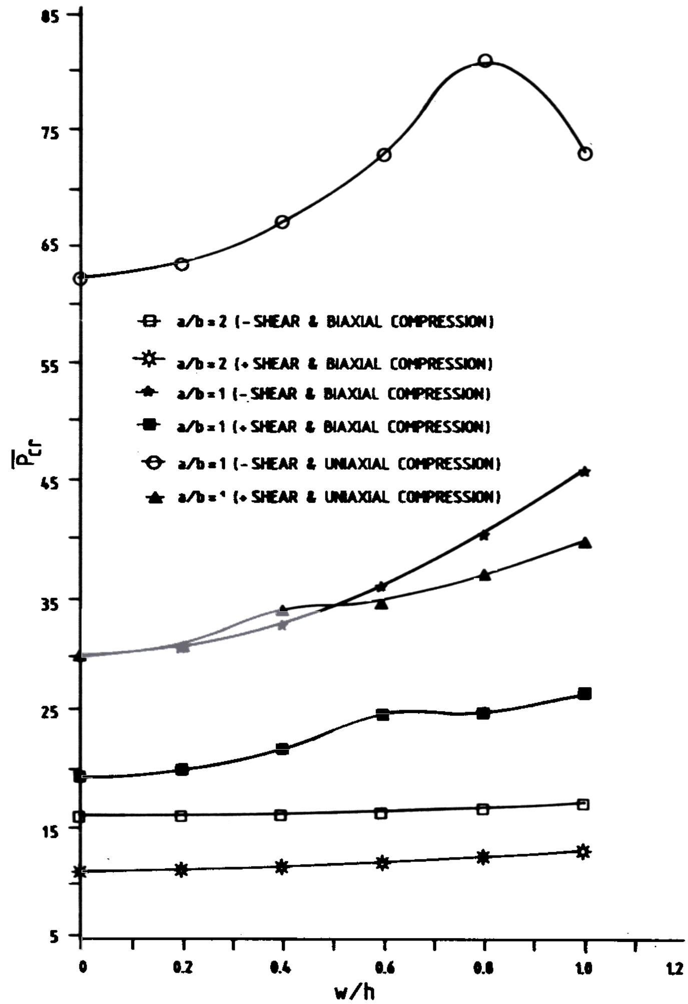

Figure 5. Combined loading effects on postbuckling shear lod [(-45°/45°/45°/-45°), a/b=1, simply-supported].

composite plates. The following conclusions are drawn:

(a) Two postbuckling paths are predicted in the case of symmetric angle-ply laminated plates. The separation of these paths is affected greatly by the ply-angle, number of layers, properties of the composites and the aspect ratio of the plates.

(b) The postbuckling performance, under combined compression and shear loading, is significantly influenced by the direction of the shear. The buckling strength is even lesser in the case of bi-directional compressive loading when compared to unidirectional compression.

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