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Free Vibrations Analysis of Laminated Composite Rotating Beam using C^1 Shear Flexible Element

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ABSTRACT

The free flexural vibrations of rotating beam made of anisotropic laminated composite beam are investigated using a new three-noded finite element. The governing equations for the free vibration of rotating beam are derived using Lagrange's equation of motion. The element employed is based on shear flexible theory. It also includes inplane and rotary inertia terms. The formulation takes care of continuity conditions for stresses and displacements at the interfaces between the layers of a laminated beam. Numerical results for uniform rotating cantilever beam are presented by considering various parameters like slenderness ratio, modular ratio and rotational speed, etc.

1. INTRODUCTION

The dynamic analysis of structural elements like rotating blades in the design of rotorcraft, gas turbine and many other systems is important in evaluating the resonant/forced vibration characteristics and flutter behaviour with acceptable accuracy. By idealising the blades as cantilever beams, vibration study of rotating beam has received considerable attention. Analytical methods, such as Rayleigh-Ritz method[1-3], Galerkin procedure[4] and perturbation technique[5,6] are employed, whereas numerical solution like finite element method is used[7,8]. Beam made of different isotropic materials is analysed[9]. All these works are limited to the study of rotating isotropic beams.

The recent advances in design and manufacturing technologies have greatly enhanced the use of advanced fibre-reinforced composite for marine, aircraft and aerospace applications. Hence, studies related to rotating elements made of composite materials are the order of the day. Dynamic response analysis of composite laminated beam with rotational speed is carried out by Chen[10], *et al* in time domain. However, to the authors' knowledge, there is no work available in the literature, except the work by Chen, *et al.* dealing

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with dynamic behaviour of rotating laminated/sandwich beam.

In the present study, the efficacy of a three-noded shear flexible beam element, developed recently[11] for dynamic analysis is investigated by considering free vibration behaviour of single layered orthotropic and laminated anisotropic cross-ply/sandwich composite rotating beams. This problem is analysed by retaining the degrees of freedom (DOFs) of the element pertaining to the flexural and axial deformations of the beam. The formulation includes rotary and inplane inertia terms. Since the transverse shear deformation is represented by cosine function, which is of higher order, there is no shear correction factor required. Detailed numerical result are presented considering different parameters.

2. FORMULATION

A laminated three-layered symmetric composite beam of constant rectangular cross-section is considered with the coordinates x along the axis of the beam and z along the thickness direction. The displacements in k^{th} layer u^k, w at point (x, z) from the median surface are expressed as functions of midplane displacement u_o and w and independent rotation θ of the normal in xz plane, as

$$u^k(x,z,t) = u_o(x,t) - zw_{,x}(x,t) + [f(z)+g^k(z)]\,(w_{,x}(x,t)+\theta(x,t))$$
$$w(x,z,t) = w(x,t) \tag{1}$$

where t is the time, and the functions $f(z)$ and $g^k(z)$ are defined as

$$f(z) = h/\pi \ \sin\ (\pi z/h) - h/\pi\ b_{55} \cos(\pi z/h) \tag{2a}$$

$$g^k(z) = a^k z + b^k \tag{2b}$$

In Eqn (2), coefficients b^k are determined such that displacement component u^k is continuous at the interface of adjacent layers and the coefficient b_{55} allows the term $\{f(z) + g^k(z)\}$ to vanish at the mid-plane. Finally coefficients a^k in Eqn (2) are computed from the requirement that the transverse shear stress σ_{31}^k is continuous at interface of the adjacent layers and vanish at the top and bottom surfaces of the beam. The details of derivations of constants b_{55}, a^k and b^k are available[11,12]. The linear strains in terms of mid-plane deformation for k^{th} layer can be written as

$$\{\varepsilon^k\} = \begin{Bmatrix} \varepsilon_p \\ 0 \end{Bmatrix} + \begin{Bmatrix} \varepsilon_b^k \\ \varepsilon_s^k \end{Bmatrix} \tag{3}$$

The mid-plane strains $\{\varepsilon_p\}$, bending strains $\{\varepsilon_b^k\}$ and shear strains $\{\varepsilon_s^k\}$ in Eqn (3) are written as

$$\{\varepsilon_p\} = \{u_{o,x}\} \tag{4a}$$

$$\{\varepsilon_b^k\} = \left\{ -z\,w_{,xx} + (f(z)+g^k(z))\,(w_{,xx}+\theta_{,x}) \right\} \tag{4b}$$

$$\{\varepsilon_s^k\} = \left\{ (f_{,z} + g_{,z}^k)\,(w_{,x}+\theta) \right\} \tag{4c}$$

where the subscript comma denotes the partial derivative wrt spatial coordinate succeeding it.

For a composite laminated beam of thickness h_k $(k=1,2,3..)$, and the ply-angle ϕ_k $(k = 1,2,3)$, the necessary expressions for computing the stiffness coefffcients, available in literature[13], are used. The stress-strain relations for k^{th} layer is written as

$$\{\sigma^k\} = \begin{bmatrix} Q_{11}^k & Q_{16}^k \\ Q_{16}^k & Q_{66}^k \end{bmatrix} \{\varepsilon^k\} \tag{5}$$

where Q_{ij}^k $(i, j = 1,6)$ are the reduced stiffness coefficients of k^{th} layer.

The strain energy functional U for a rotating beam element of length, l is given as

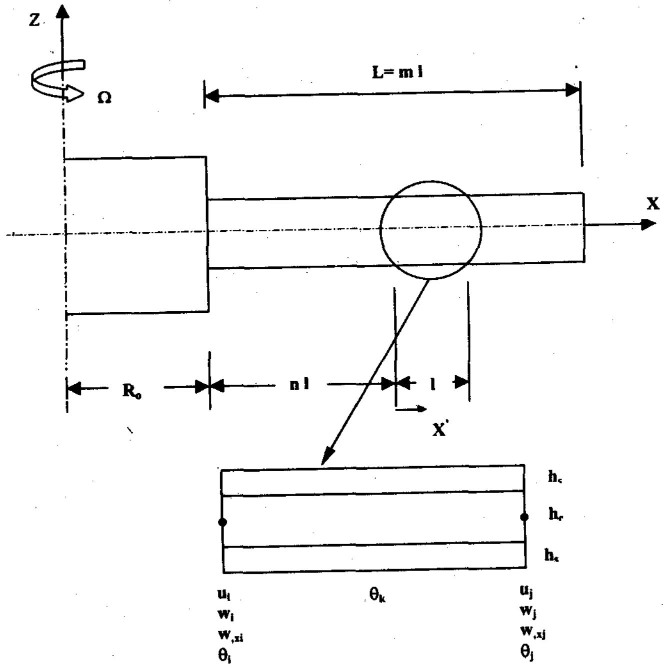

Figure 1. Structural idealisation and element description

$$U(\delta) = (1/2) \int_{o}^{l} \sum_{k} \int_{h_k}^{h_{k+1}} \{\sigma^k\}^T \{\varepsilon^k\}\, dz\, dx'$$
$$+(1/2) \int_{0}^{l} F_{x'} \left(\frac{dw}{dx'}\right)^2 dx' \qquad (6)$$

where δ is the vector of DOFs associated with the displacement field in a finite element discretisation. $F_{x'}$ is the centrifugal force due to rotation which is defined as

$$F_{x'} = A \int_{x'}^{l} \rho \Omega^2 (R_0 + nl + x')\, dx'$$
$$+ A \int_{(n+1)l}^{ml} \rho \Omega^2 (R_0 + x)\, dx \qquad (7)$$

where x' represents the local coordinate for the element. R_0, m and n are hub radius and element numbers as denoted in Fig. 1.

The kinetic energy of the plate is written as

$$T(\delta) = (1/2) \int_{o}^{L} \sum_{k} \int_{h_k}^{h_{k+1}} \rho\left(\dot{u}_o + w + z^2 \theta^2\right) dx\, dz \qquad (8)$$

where the dot over the variable denotes the partial derivative wrt time and ρ the mass density. Substituting Eqns (6) and (8) in Lagrange's equation of motion, one obtains the governing equation for the free flexural vibration of rotating beam structure as

$$[M]\,\{\delta\} + [[K] + [K_f]]\,\{\delta\} = \{o\} \qquad (9)$$

where $[M]$ is the consistent mass matrix; $[K]$ the structural stiffness matrix, and $[K_f]$ is the stiffness due centrifugal force, resulting from rotation of the beam.

Free vibration frequencies and modeshapes are extracted from Eqn (9) by employing the standard Eigen value approach.

3. DESCRIPTION OF THE ELEMENT

The element used is based on Hermite cubic function for transverse displacement, w according to the C^1 continuity requirement, quadratic function for rotation, θ and linear function for inplane displacement, u. Further, the element needs four nodal DOFs u, w, $w_{,x}$ and θ at both ends of the three-noded beam element, whereas the centre node has one DOF θ, shown in Fig. 1.

The above choice of the functions allows to have the same order of interpolation for both $w_{,x}$ and θ in the definition of transverse shear-strain and permits to avoid transverse shear locking phenomenon. The element behaves very well for both thick and thin situations. It has no spurious mode and is represented by correct rigid body modes.

4. RESULTS & DISCUSSION

In this section, the above formulation and element have been used to investigate the effects of slenderness ratio and orthotropicity on the free vibration behaviour of laminated cross-ply/sandwich cantilever beam with rotational speed. Since the finite element used here is derived

Table 1. Natural frequencies of isotropic rotating cantilever beam (thickness = 0.1 m)

α	L/r_g = 1000			L/r_g = 10		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	3.5156 (3.516)	22.0332	61.6895	3.2585	15.2235	34.0138
2	4.1373 (4.137)	22.6137	62.2655	3.8805	15.9632	34.9930
5	6.4495	25.4449	65.1975	6.1338	19.3465	39.5654
8	9.2568 (9.257)	29.9942	70.2857	8.8337	24.2824	46.2902
10	11.2022 (11.203)	33.6392	74.6422	10.7083	27.9610	51.0625
10+	---	---	---	23.2997	48.6171	65.4918

* Results from Ref. [8]

\+ Results with hub radius (Ro = R/L) = 3

based on consistency approach, an exact integration is employed to evaluate all the strain energy terms. Also, due to the representation of higher order function for the transverse shear deformation, no shear correction factor is introduced. Based on progressive refinement, 20 elements idealisation is found to be adequate to model the full beam for the present analysis. The material properties, unless specified otherwise, assumed in the present analysis are:

$$E_1 = 3.4156 \text{ GPa}, \ E_2 = 1.7931 \text{ GPa}, \ G_{12} = 1.0 \text{ GPa},$$
$$G_{13} = 0.608 \text{ GPa}, G_{23} = 1.015 \text{ GPa}, \nu = 0.44$$

where E, G, and ν are Young's modulus, shear modulus and Poisson's ratio, respectively. Subscripts 1 and 2 are the longitudinal and transverse directions, respectively wrt the fibres. All the layers are of equal thickness. The ply-angle is measured wrt the x-axis.

To start with, efficacy of the present formulation was evaluated by considering isotropic rotating cantilever beam for which results are available. The non-dimensional natural frequencies λ^2 $(= \omega^2(\rho A L^4 / EI))$, calculated for different rotational speeds α^2 $[=\Omega^2(\rho A L^4 / EI)]$ were compared with values given by Yokoyama (Table 1) and these were in good agreement. Furthermore, it could be noticed from Table 1 that the frequency increased with increase in rotational speeds. Also, one could see that the variation in the rate of increase in frequency was very high at lower speeds whereas it was negligible at higher speeds. This behaviour is because of the inclusion of stiffness contribution due to rotation, which is proportional to the square of the rotational speed, i.e., stiffness contribution due to rotational speed is dominant at higher speed compared to structural stiffness. Also, it was observed that the influence of rotational speed was more on the lower modes compared to the higher ones. For short beams $(L/r_g \leq 10)$, the behaviour was qualitatively similar to that of slender beams $(L/r_g \geq 100)$. The increase in the natural frequencies due to the presence of the hub is significant as can be seen from Table 1.

Next, single-layered orthotropic and cross-ply beams were considered for further studies. The results obtained for orthotropic and cross-ply beams are shown in Tables 2 and 3, respectively. It was inferred from these tables that the variation of

Table 2. Natural frequencies of orthotropic rotating cantilever beam (thickness = 0.1 m)

α	L/r_g = 100			L/r_g = 10		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	3.5101	21.7815	60.0578	3.0605	12.6210	26.9094
2	4.1313	22.3655	60.6451	3.7002	13.5509	28.2207
5	6.4409	25.2086	63.6295	5.9745	17.5471	33.8721
8	9.2819	29.7682	68.7899	8.6875	22.9487	40.0399
10	11.1816	33.4134	73.1926	10.5732	26.7510	50.3993

Table 3. Natural frequencies of laminated cross-ply (0/90/0, total thickness = 0.1m) rotating cantilever beam

α	$L/r_g = 100$			$L/r_g = 10$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	3.0564	19.0688	53.0688	2.8582	13.6923	30.8272
2	3.7533	19.7347	53.6834	3.5552	14.5346	31.9426
5	6.1963	22.9146	57.0383	5.9347	18.2856	37.0414
8	9.0546	27.8579	62.7437	8.6941	23.5860	44.1997
10	11.0138	31.7248	67.5354	10.5937	27.4639	48.5797

frequencies wrt speed was qualitatively similar to that of isotropic case. It was also noted that the orthotropicity and lamination scheme could affect the values of frequencies. Furthermore, it can be viewed, irrespective of material properties (i.e, isotropic or anisotropic) that fundamental natural frequency approaches to the value of rotational speed, when the rotational speed increases to a higher values.

Finally, a sandwich beam having different modular ratio, C (i.e, ratio of material properties of skin and core), is analysed for free vibration studies. Here, the material properties were kept constant for skin, whereas the core properties were varied as per the modular ratio. The results are shown in Tables 4 and 5 for two values of modular ratio (C=10, 50). It was revealed from these tables that the increase in modular ratio decreases the values of natural frequencies. For a short beam, the effect of transverse shear of beam, at lower speed, on natural frequencies is predominant for orthotropic and sandwich cases, compared to that of isotropic case, as brought out in Tables 1-4 and it decreases at higher speed.

Table 4. Natural frequencies of sandwich (skin total thickness = 0.02 m and core thickness = 0.08) rotating cantilever beam with modular ratio C = 10.

α	$L/r_g = 100$			$L/r_g = 10$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0	3.4822	20.6692	53.5583	2.0328	4.5622	6.5654
2	4.1019	21.2573	54.1663	2.7218	5.3361	7.4797
5	6.3992	24.1073	57.2353	4.8803	7.4056	10.0943
8	9.1768	28.6378	62.4736	7.1768	9.2474	15.3207
10	11.0974	32.2381	66.8873	8.5673	14.5416	17.4969

Table 5. Natural frequencies of sandwich (skin total thickness = 0.02 m and core thickness = 0.08) rotating cantilever beam with modular ratio C = 50.

α	$L/r_g = 100$		
	λ_1	λ_2	λ_3
0	3.2492	14.3519	28.0048
2	3.8455	14.8473	28.4996
5	6.0392	17.1797	30.8947
8	8.6831	20.7070	34.6522
10	10.5145	23.3872	37.5591

5. CONCLUSIONS

The effectiveness of the finite element developed by Ganapathi[11], *et al.* has been demonstrated by considering free vibration behaviour of rotating isotropic, orthotropic, laminated composite and sandwich beams. The results obtained indicate the significance of hub radius, slenderness ratio and modular ratio on the values of natural frequencies. The present formulation can be extended to study the nonuniform rotating beams with discontinuities and any combination of boundary conditions.

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