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A New C^1 Eight-Noded Plate Element for Static & Dynamic Analyses of Composite Laminates

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ABSTRACT

This paper deals with a new 48-degrees-of-freedom rectangular finite element for analysing moderately thick, multilayered composite plates. The formulation is based on a kinematics which allows one to exactly ensure the continuity conditions for the displacements, and the transverse stresses at the interfaces between the layers of a laminated structure and zero stress conditions at the top and bottom surfaces of the plate. The shear correction factors are not required in the formulation, as the transverse shear deformations are defined using trigonometric functions that are of higher order. The effectiveness of the element is tested against standard problems concerning statics, vibration and buckling, for which exact three-dimensional/numerical solutions are available.

1. INTRODUCTION

The increased use of fibre-reinforced composites as structural members in aerospace, nuclear and marine engineering has resulted in several studies, such as structural modelling, failure and damage assessment of composite materials. The research in these areas is increasing for the past few decades. Owing to the difficulties in obtaining the solution of three-dimensional problems with general boundary conditions, there is a growing appreciation of the importance of developing some approximate two-dimensional theories and the solutions. Since the solutions based on analytical methods are not always feasible, numerical procedures like finite element method are preferred.

Many of the existing methods of analyses for multilayered anisotropic plates are direct extensions of those developed earlier for homogeneous isotropic and orthotropic plates and employ a displacement field which does not satisfy continuity requirements at the interfaces of the composite laminates. Exhaustive overviews on this topic can be found from the work of Noor and Buron[1], and Kapania and Racita[2]. A refined computational model has been presented by Reddy[3]. Many higher order shear deformation theories[4-6] have been proposed for achieving the continuity requirement at the interfaces and satisfying the stress conditions at the top and bottom surfaces. Recently, an efficient plate theory based on a new kinematics utilising trigonometric functions has been outlined by Touratier[7].

The aim of the present work is to develop a new eight-noded rectangular plate element for the analysis of composite laminates based on the theory proposed by Touratier, incorporating the

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following features

(a) A higher-order shear deformation distributions using trigonometric functions without shear correction factors

(b) Inter-layer continuity conditions on displacement and transverse stresses, and vanishing transverse shear stresses at the top and bottom surfaces of the laminate

(c) Finite element model utilising five-independent-generalised displacements (three displacements and two rotations)

The shape functions employed here are Hermitian cubic interpolation for the transverse normal displacement and Serendipity quadratic functions for in-plane displacements and rotations. The performance of the element is evaluated through numerical experiments considering several problems related to statics, and vibration and buckling for which exact three-dimensional analytical solutions are available in the literature.

2. LAMINATED PLATE THEORY

A laminated composite plate is considered with the coordinates x, y along the in-plane directions and z along the thickness direction, respectively. Using formulation based on shear flexible theory, the displacements in k^{th} layer $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ at a point (x, y, z) from the median surface are expressed as functions of mid-plane displacement u, v, w and independent rotation θ_x and θ_y of normal in xz and yz planes, respectively as

$$u^{(k)}(x,y,z,t) = u\,(x,y,t) - z\partial w/\partial x + [f_1(z) + g_1^{(k)}(z)]\{\partial w/\partial x+q_x\} + g_2^{(k)}(z)\{\partial w/\partial y+\theta_y\}$$

$$v^{(k)}(x,y,z,t) = v\,(x,y,t) - z\partial w/\partial y + g_3^{(k)}(z)[\partial w/\partial x+\theta_x]+[f_2(z)+g_4^{(k)}(z)][\partial w/\partial y+\theta_y]$$

$$w^{(k)}\,(x,y,z,t) = w\,(x,y,t) \quad (1$$

where t is the time. The functions involved in Eqn (1) for defining the kinematics are as follows:

$$f_1(z) = h/\pi\, sin\,(\pi z/h) - (h/\pi)b_{55}\, cos\,(\pi z/h)$$

$$f_2(z) = h/\pi\, sin\,(\pi z/h) - (h/\pi)b_{44}\, cos\,(\pi z/h)$$

$$g_i^{(k)}=a_i^{(k)}\, z+d_i^{(k)}, i=1,2,3,4; k=1,2,3,\ldots,N \quad (2)$$

where N is the number of layers of the multilayered structure, h is the total thickness of the laminate, π is equal to 3.141592, and b_{44}, b_{55}, $a_i^{(k)}$, $d_i^{(k)}$ are coefficients to be determined from contact conditions for displacements and stresses between the layers and from the boundary conditions on the top and bottom surfaces of the plate. The details of these coefficients can be found from the work of Touratier[7], and Beakou and Touratier[8].

The linear strains in terms of mid-plane deformation can be written as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon^o \\ \chi \\ \omega \\ \gamma^o \end{Bmatrix} \quad (3)$$

The mid-plane strains $\{\varepsilon^o\}$, bending strains $\{\chi\},\{\omega\}$ {due to lower and higher order terms involved in defining the kinematics, Eqn (1)}, and shear strains $\{\gamma^o\}$ in Eqn (3) are written as

$$\{\varepsilon^o\} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}; \{\chi\} = -\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x dy \end{Bmatrix}$$

$$\{\omega\} = \begin{Bmatrix} \partial\gamma_1^0/\partial x \\ \partial\gamma_2^0/\partial y \\ \partial\gamma_1^0/\partial y \\ \partial\gamma_2^0/\partial x \end{Bmatrix}; \{\gamma^0\} = \begin{Bmatrix} \gamma_1^0 \\ \gamma_2^0 \end{Bmatrix} + \begin{Bmatrix} \partial w/\partial x + \theta_x \\ \partial w/\partial y + \theta_y \end{Bmatrix} \quad (4)$$

If $\{N\}$ represents the membrane stress resultants (N_{xx}, N_{yy}, N_{xy}) and $\{M\},\{\tilde{M}\}$ represent the bending stress resultants due to lower and higher-order terms involved in defining the kinematics $[(M_{xx},M_{yy},M_{xy}),(\tilde{M}_{xx}, \tilde{M}_{yy}, \tilde{M}_{xy})]$, one can relate these to membrane strains $\{\varepsilon^o\}$ and bending strains $\{\chi\},\{\omega\}$ through the constitutive relations as

$$\{N\} = [A]\{\varepsilon^o\} + [B]\{\chi\} + [E]\{\omega\}$$

$$\{M\}=[B]^T\{\varepsilon^o\}+[D]\{\chi\}+[\bar{B}]\{\omega\} \tag{5}$$
$$\{\tilde{M}\}=[E]^T\{\varepsilon^o\}+[B]^T\{\chi\}+\tilde{D}\{\omega\}$$

Similarly, the transverse shear stress resultants $\{Q\}$ representing the quantities (Q_{xz}, Q_{yz}) are related to the transverse $\{\gamma^o\}$ strains through the constitutive relation as

$$\{Q\}=[\tilde{A}]\{\gamma^o\} \tag{6}$$

The different matrices involved in Eqns (5) and (6) are defined as follows:

$$\left.\begin{aligned}
\{A\}&=\int_{-h/2}^{h/2}[Q_p]\,dz\\
\{B\}&=\int_{-h/2}^{h/2}z[Q_p]\,dz\\
\{E\}&=\int_{-h/2}^{h/2}[Z(z)]^T[Q_p]\,dz\\
\{D\}&=\int_{-h/2}^{h/2}z^2[Q_p]\,dz\\
\{\bar{B}\}&=\int_{-h/2}^{h/2}z[Z(z)][Q_p]\,dz\\
\{\tilde{D}\}&=\int_{-h/2}^{h/2}[Z(z)]^T[Q_p][Z(z)]\,dz\\
\{\tilde{A}\}&=\int_{-h/2}^{h/2}[Y(z)]^T[Q_t][Y(z)]\,dz
\end{aligned}\right\} \tag{7}$$

The matrices $Y(z)$ and $Z(z)$ are given as

$$Y(z)=\begin{bmatrix}\partial f_1/\partial z+\partial g_1^{(k)}/\partial z & \partial g_2^{(k)}/\partial z\\ \partial g_3^{(k)}/\partial z & \partial f_2/\partial z+\partial g_4^{(k)}/\partial z\end{bmatrix}$$

$$Z(z)=\begin{matrix}f_1+g_1^{(k)} & 0 & 0 & g_2^{(k)}\\ 0 & f_2+g_4^{(k)} & g_3^{(k)} & 0\\ g_3^{(k)} & g_2^{(k)} & f_1+g_1^{(k)} & f_2+g_4^{(k)}\end{matrix} \tag{8}$$

For a composite laminate of thickness (h), consisting of N layers with stacking angle $\phi_k(k=1,2,3....,N)$ the layer thickness (h_k) the necessary expressions for computing the reduced stiffness coefficients of ($[Q_p]$, $[Q_t]$) available in the literature[9], are used here.

The total potential energy functional (U) consisting of energy stored in the plate is given by:

$$U(\delta)=\frac{1}{2}\int\Big([N]^T\{\varepsilon^o\}+[M]^T\{\chi\}+[\tilde{M}]^T\{\omega\}+[Q]^T\{\gamma^o\}\Big)dA+\int(w^Tf)dA \tag{9}$$

where δ is the vector of the degrees-of-freedom (DOFs) associated with the displacement field in a finite element discretisation. f is the distributed force.

The kinetic energy of the plate is written as

$$T(\delta)=\frac{1}{2}\int\left(\int_{-h/2}^{h/2}\rho\left[\dot{u}^{(k)}\dot{v}^{(k)}\dot{w}^{(k)}\right]^T\left[\dot{u}^{(k)}\dot{v}^{(k)}\dot{w}^{(k)}\right]dz\right)dA \tag{10}$$

where the dot over the variable denotes the partial derivative wrt time and ρ is the mass density.

The potential energy due to external in-plane force, N_x^o in x direction is written as

$$W(\delta)=\frac{1}{2}\int N_x^o[\partial w/\partial x]^2dA \tag{11}$$

Substituting the Eqns (9) - (11) in Lagrange's equation of motion, one obtains the governing equation of the plate as

$$M]\{\ddot{\delta}\}+([K]+[K_G])\{\delta\} \quad \{F \qquad 2$$

where $[M]$ is the consistent mass matrix; $[K]$ and $[K_G]$ are the structural stiffness and the geometric stiffness matrices, respectively; and $\{F\}$ is the load vector.

The coefficient in the stiffness and mass matrices can be rewritten as the product of term having thickness coordinate z alone and the term containing x and y. In the present study, while performing the integration for the evaluation of the stiffness and mass coefficients, terms having thickness coordinate z are explicitly integrated, whereas the terms containing x and y are evaluated using full integration with 4 x 4 points Gauss integration rule.

3 ELEMENT DESCRIPTION

The eight-noded rectangular element used here is based on Hermite cubic function for transverse displacement, w according to the C^1 continuity requirement, Serendipity quadratic function for the in-plane displacements u, v and rotations θ_x, θ_y. Further, the element needs eight nodal DOFs (u, v, w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x \partial y$, θ_x, θ_y) for all corner nodes and 4-DOFs (u, v, θ_x, θ_y) for the mid-node of all four sides.

4 RESULTS & DISCUSSION

Before proceeding for the detailed numerical computations, the element developed here has been

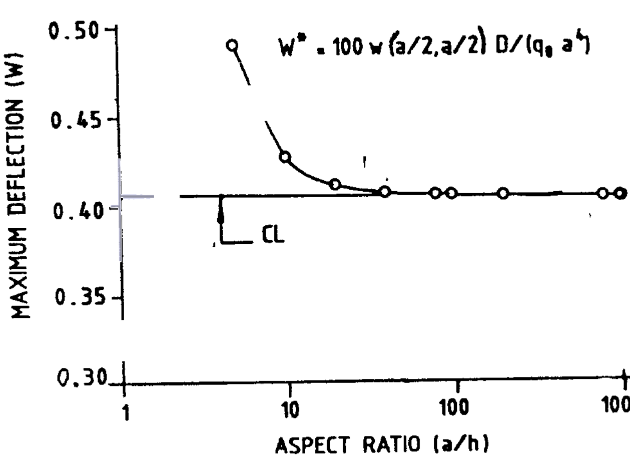

Figure 1. Variation of maximum deflection with aspect ratio for the simply supported isotropic square plate under uniform load (q_0).

tested for rank deficiency, and is found to be having proper rank (five zero eigenvalues-no spurious mode). Further, based on progressive mesh refinement, 4 x 4 mesh idealisation is adequate to model the quarter plate for the analyses considered here. The element is also checked for shear locking phenomenon considering an isotropic simply supported square plate with distributed loading, and the results are shown in Fig. 1. The figure shows that the element is free from locking syndrome. The aim of the present investigation is to see the efficacy of the formulation of the new element for the statics, and vibration and buckling analyses of laminated composite plates. As such, problems for which exact solutions available in the literature are considered here. The simply supported boundary condition assumed here is given as

$$v=w=\theta_y=\partial w/\partial y=0 \ \ at \ \ x=0,a$$
$$u=w=\theta_x=\partial w/\partial x=0 \ \ at \ \ y=0,b$$

4.1 Static Analysis

4. *Simply Supported Symmetric Sandwich Square Plate under Uniform Loading*

The geometrical parameters used here are assumed as $a = b = 1$ m, thickness of the outer layer and core or middle layer are 0.01 m and 0.08 m, respectively. The material properties of the core or middle layer are:

E_1 3.4156 GPa, E_2 .7931 GPa, G_{12}=1.0 GPa, G =0.608 GPa, G, .015 GPa, γ_{12}=0.44

The properties of the skin or the outer layers can be obtained using the modular ratio C that is defined as the ratio of moduli of skin to core of the laminate. By varying the modular ratio C (=1, 10, 15, 50), one can achieve a homogenous orthotropic plate (C = 1) to a fairly sandwich-like plate (C = 50). The values for the normalised transverse displacement (W) are compared with the exact three-dimensional analytical solution of Srinivas and Rao[10] and the numerical solutions obtained using 20x20 mesh in standard software (ABAQUS, ANSYS) in Table1. It is evident that the present results are in agreement with those of the exact analytical solutions.

4.1.2 *Simply Supported Unsymmetric Sandwich Square Plate under Uniform Loading*

The geometrical parameters used here are assumed as $a = b = 1$ m; thickness of the bottom, top and middle layers are 0.03 m, 0.01 m and 0.06 m, respectively. The material properties of the core are same as above. The value of C is assumed as 10. The results evaluated for W are shown along with the exact three-dimensional analytical[10] and numerical solutions using 20 x 20 mesh in standard software (ABAQUS, ANSYS) in Table 2. It can be noted that the present results are in agreement with those of the exact analytical solutions.

Table 1. Nondimensional central deflection, W [$=w(a/2,b/2,0)G_{12}(2)/(hq_0)$] of a simply supported symmetric three-layered square sandwich plate under uniform load, q_0

	Modular ratio (C)			
	1	10	15	50
Present	181.33	41.96	32.05	16.78
Elasticity solution[7]	181.05	41.91	32.00	16.75
S8R-Elem,(ABAQUS)	181.36	35.83	24.89	7.95
STIF91-Elem.(ANSYS)	181.31	35.83	24.89	7.95

$G_{12}(2)$-Shear modulus of core

Table 2. Nondimensional central deflection, W [$=w(a/2,b/2,0)G_{12}(2)/(hq_0)$] of a simply supported unsymmetric three-layered square sandwich plate under uniform load, q_0

	Modular ratio $C=10$
Present	34.59
Elasticity solution[7]	34.55
S8R-Elem.(ABAQUS)	28.14
STIF91-Elem.(ANSYS)	25.54

$G_{12}(2)$-Shear modulus of core

Table 3. Nondimensional central deflection, W^{**} [$=w(a/2,b/2,0)h^3 E_2/(a^4 q_0)$ x 100] of a simply supported symmetric three-layered cross-ply rectangular laminate under doubly sinusiodal load, [$q_0 \sin(\pi x/a) \sin(\pi y/b)$]

	a/h = 10	20	100
Present			
Elasticity solution[8]			

4.1.3 *Simply Supported Symmetric Three-Layered Cross-Ply Rectangular Laminate* (0°/90°/0°) *under Doubly Sinusoidal Loading*

The geometrical parameters used are taken as $a = 1$ m, $b = 3$ m; thickness of the laminate is varied as 0.25 m, 0.1 m, 0.05 m and 0.01 m. All layers are assumed to be of equal thickness.

The material properties assumed here are: $E_1 = 25$ GPa, $E_2 = 1.0$ GPa, , $G_{12} = 0.5$ GPa, $G_{13} = 0.5$ GPa, $G_{23} = 0.25$ GPa, $\gamma_{12} = 0.25$

Table 3 depicts the comparison of the present results for different thickness ratios (a/h) with the exact three-dimensional solution of Pagano[11] and they are in agreement.

4.2 FREE VIBRATION ANALYSIS

4.2. *Vibration of Simply Supported Square Sandwich Plates*

The data for geometry and material are assumed to be same as those for static analysis, *case* 4.1.1. The fundamental frequency parameter (Ω) calculated here by varying the modular ratio C (=1, 2, 5, 10, and 15) is reported in Table 4 along with those of exact three-dimensional values of Srinivas and Rao[10] and the numerical solutions using standard software (ABAQUS and ANSYS). This table shows that the present results are in good agreement with the exact solutions.

Table 4. Nondimensional fundamental frequency, Ω [$= \omega (\rho/h^2/C_{11}(2))^{1/2}$] of a simply supported symmetric three-layered square sandwich plate

	Modular ratio (C)				
			5	10	15
Present	0.047410	0.057031	0.077144	0.098103	0.112037
Elasticity solution[7]	0.047419	0.057041	0.077146	0.098104	0.112034
S8R-Elem,(ABAQUS)	-0.047397	0.057311	0.079578	0.106485	0.127743
STIF 91-Elem.(ANSYS)	0.047752	0.057731	0.080143	0.107226	0.128623

$C_{11}(2)$-Stiffness coefficient of core

4.2.2 *Vibration of Simply Supported Anti-Symmetric Cross-Ply Square Plates*

All layers are assumed to be of equal thickness. The material properties considered here are:

E_1=40 GPa, E_2=1.0 GPa, , G_{12}=0.6 GPa, G_{13}=0.6 GPa, G_{23}=0.5 GPa, γ_{12}=0.25

The free vibration analysis is carried out considering the different values for thickness ratio (a/h) and varying the number of layers and the results are presented in Table 5 along with the exact analytical results based on two-dimensional third-order shear deformation theory[12] (TSDT). This table shows that the frequency values obtained here are in good agreement with the existing results.

Table 5. Nondimensional fundamental frequency, Ω^* [$= \omega(\rho/E_2)^{1/2}a^2/h$] of a simply supported anti-symmetric cross-ply square laminate

a/h	Layers (N)	Present	TSDT[9]
10			
100			

* First-order theory

4.3 BUCKLING ANALYSIS

4.3.1 *Buckling of Simply Supported Square Sandwich Plates*

The data for geometry and material are the same as those given for static analysis, *case* 4.1.1. The buckling parameter, N^*_{xcr} is evaluated using the present formulation by varying the modular ratio C (=1, 2, 5, 10, and 15). The results are described in Table 6 along with those of exact three-dimensional values of Srinivas and Rao and the numerical solutions using a standard software (ANSYS). This table reveals that the present values are in agreement with the exact solutions.

Table 6. Nondimensional critical buckling load, N^*_{xcr} [$= N_{xcr}/A_{11}$) $(12/\pi^2)(b/h)^2$] of a simply supported symmetric three-layered square sandwich plate due to N_x

	Modular ratio (C)				
	1	2	5	10	15
Present	2.808	3.384	4.119	4.272	4.099
Elasticity solution[7]	2.770	3.330	4.046	4.200	4.037
STIF 91-Elem.(ANSYS)	2.807	3.418	4.391	5.052	5.356

A_{11}- Extensional stiffness coefficient

4.3.2 *Buckling of Simply Supported Anti-Symmetric Cross-Ply Square Plates*

The material properties are the same as those for dynamic analysis, *case* 4.2.2. The buckling parameters, obtained considering the different values for thickness ratio (a/h) and varying the number of layers, are compared in Table 7 along with the exact analytical results based on third-order theory[12]. The agreement between the two is good.

Table 7. Nondimensional critical buckling load, N^{**}_{xcr} [$= N_{xcr}a^2/(E_2h^3)^{1/2}$] of a simply supported anti-symmetric cross-ply square laminate due to N_x

a/h	Layers (N)	Present	TSDT[9]
10			
100			

* First-order theory

5. CONCLUSION

The capabilities and accuracy of the new eight-noded rectangular C^1 plate element based on Hermitian cubic polynomial for the transverse normal displacement, and Serendipity quadratic functions for in-plane displacements and rotations, have been demonstrated for static and dynamic analyses of laminated/sandwich composite plates.

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