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Angular Steering for Proportional Navigation-commanded Surface-to-air Guided Missile

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ABSTRACT

The paper briefly reviews the guidance laws and their implementation in surface-to-air missiles. The trajectories for the line-of-sight and proportional navigation guidance laws are discussed and the effect of steering on demand for increased lateral acceleration is appreciated. The mathematical model is then evolved to estimate the launch angle of the missile, ie, bearing and elevation, in the direction of the future position of the moving air target as well as the steering commands in pitch and yaw planes in accordance with the proportional navigation guidance law, to enable collision with the target.

Keywords: Surface-to-air missile trajectories, surface-to-air missile launch equations, proportional navigation guidance

NOMENCLATURE

λ_m	LOS angle of the missile from the tracker
λ_t	LOS angle of the target from the tracker
$\dot{S}_{tm}$	Steering rate of the missile
$\dot{\lambda}_{tm}$	LOS angular rate of the target from the missile
N	Navigation constant
V_m	Missile velocity (| Vm | = constant)
A_{mn}	Normal acceleration perpendicular to the instantaneous missile velocity
X_t, Y_t, Z_t	Component of instantaneous position of the target along X,Y,Z-axes, respectively
V_{tx}, V_{ty}, V_{tz}	Component of instantaneous velocity of the target along X,Y,Z-axes, respectively
A_{tx}, A_{ty}, A_{tz}	Component of instantaneous acceleration of the target along X,Y,Z-axes, respectively
X_m, Y_m, Z_m	Component of instantaneous position of the missile along X,Y,Z-axes, respectively
V_{mx}, V_{my}, V_{mz}	Component of instantaneous velocity of the missile along X,Y,Z-axes, respectively
t	Time
b_t	Instantaneous bearing of the target
e_t	Instantaneous elevation of the target
r_t	Instantaneous range of the target
b_m	Instantaneous bearing of the missile
e_m	Instantaneous elevation of the missile
r_m	Instantaneous range of the missile

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$B_{tm}(n)$ Bearing aspect of LOS angle of the target from the missile, at t(n) instance of time

$E_{tm}(n)$ Elevation aspect of LOS angle of the target from the missile, at $t(n)$ instance of time

$\dot{B}_{tm}(n)$ Rate of change of $B_{tm}(n)$

$\dot{E}_{tm}(n)$ Rate of change of $E_{tm}(n)$

1. INTRODUCTION

All the ground-based air defence systems used the worldover for bringing down any moving target in the air, ie, aircraft, missile, etc, use anti-aircraft guns and/or surface-to-air missiles. Although anti-aircraft guns are a more economical option, the ability of a guided missile to pursue an evading target as well as larger effective range present a more effective way of neutralising a moving air target. To achieve this, the missile must follow the guidance laws.

2. GUIDANCE LAWS

A guidance law is a relationship between the missile and the target motion used to generate missile steering commands, which if executed by the missile, will enable it to hit the target[1]. These guidance laws are:

(a) The missile position is controlled so that it always lies on the imaginary line between the reference point and the target.

(b) The missile velocity vector is controlled so that it always points directly at the target.

(c) The missile velocity vector is controlled so that the rotational rate of the imaginary line between the missile and the target is nullified.

The first guidance law statement is that of line-of-sight (LOS) type of guidance where the LOS angle from the ground-based tracker to the missile is equal to the LOS angle from the ground-based tracker to the target[1,2]. This may be expressed mathematically as

$$\lambda_m = \lambda_t \tag{1}$$

The second and the third guidance law statements are those of proportional-navigation (PN) type of guidance where missile steering rate is proportional to the rotational rate of the imaginary line between the missile and the target. The second statement is a special case of the third statement where the constant of proportionality[1,2] is 1. This may be expressed mathematically as

$$\dot{S}_m = N.\dot{\lambda}_{tm} \tag{2}$$

The implementation of the guidance is by homing, beam riding, and command methods. In the homing guidance implementation, the sensors in the missile measure the relative kinematical quantities between the missile and the target, using which steering commands are generated by the guidance computer in the missile by carrying out the PN guidance law. It is mostly used in the terminal phase of the missile trajectory.

In the beam-riding guidance implementation, a ground-based tracker maintains its beam of radiation, ie, radar, laser, etc on the target and the sensors in the missile measure its deviation relative to the beam using which the guidance computer in the missile generates steering commands to steer the missile towards the centre of the beam by carrying out LOS guidance law. Since the missile travels along the LOS from the tracker to the target, it is called line-of-sight-beam-riding (LOSBR) guidance. It is mostly used in the mid-course phase of the missile trajectory. In the command guidance implementation, a ground-based tracker follows the target and another ground- based tracker follows the missile. The steering commands are then generated in the guidance computer of the ground-based tracker and transmitted to the missile. The command guidance implementation carries out LOS, PN as well as other special- purpose complex optimal guidance laws. It is mostly used in the mid-course phase of the missile trajectory.

It is thus clear that the missile steering commands for guiding a surface-to-air missile in the mid-course phase of the missile trajectory are generated in the guidance computer of a ground-based tracker only in command guidance implementation of PN as well as LOS guidance laws.

The missile steering rate is controlled by control of the rotational rate of missile velocity vector. The rotational rate of missile velocity vector is controlled by the normal (lateral) acceleration of the missile, which is the rate of change of missile velocity vector perpendicular to the instantaneous velocity vector[2]. The normal acceleration in turn is controlled by the aerodynamic control surfaces ie, canards, wings, and tail fins. The magnitude of normal acceleration, which the missile is subjected to, depends on how much the control surfaces are moved, and is specific to each missile type. Also, the direction of the normal acceleration of the missile is controlled by two independent channels of pitch, ie, up/down and yaw, ie, left/right.

The normal acceleration experienced by the missile is sensed by the yaw and pitch accelerometers/ rate gyros and compared by the missile's lateral autopilots to that commanded by the guidance computer of the ground-based tracker to achieve the desired steering rate. The manner in which the magnitude and direction of the steering rate is controlled in the yaw and pitch planes is decided by the guidance law being implemented. The missile steering rate may be expressed mathematically[1] as

$$\dot{S}_m = A_{mn} / V_m \tag{3}$$

Irrespective of the functioning of the aerodynamic control surfaces and the autopilots of the missile, the ground-based trackers view the missile and the target as point objects in space, and the data that the trackers collect are their range, elevation, and bearing, using which the missile steering commands are generated for the yaw and pitch channels of the missile in accordance with the guidance law being implemented. To decide which guidance law to implement, one needs to deliberate qualitatively on the effect of LOS and PN guidance law on the missile trajectory wrt the target trajectory.

3. EFFECT OF LOS AND PN GUIDANCE LAW ON MISSILE TRAJECTORY

Consider a target such as an enemy aircraft in the 3-D air space, flying straight with some constant velocity, being observed from a ground-based platform with a target-tracking device as well as a missile-tracking device, such as a tracking radar, mounted on it while the surface-to-air missile is mounted on another platform nearby, as shown in Fig. 1. The target will be at a certain elevation, bearing, and range wrt the platform at different instance of time. Depending on the target's velocity, the target may approach the platform (approacher),

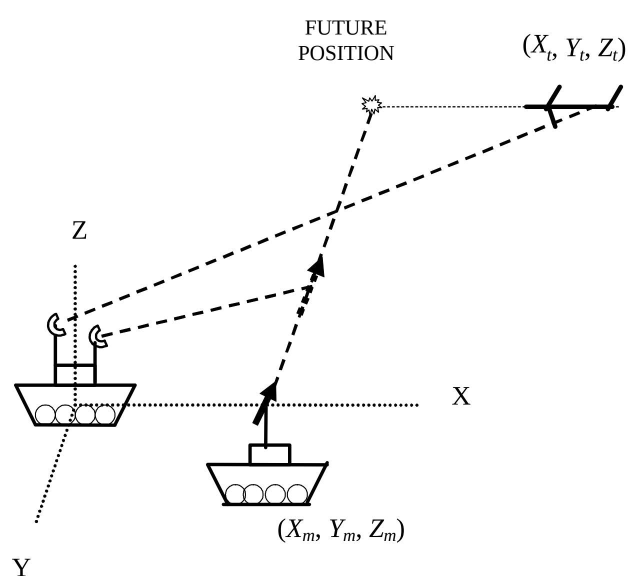

Figure 1. Two-platform system in the stabilised reference frame

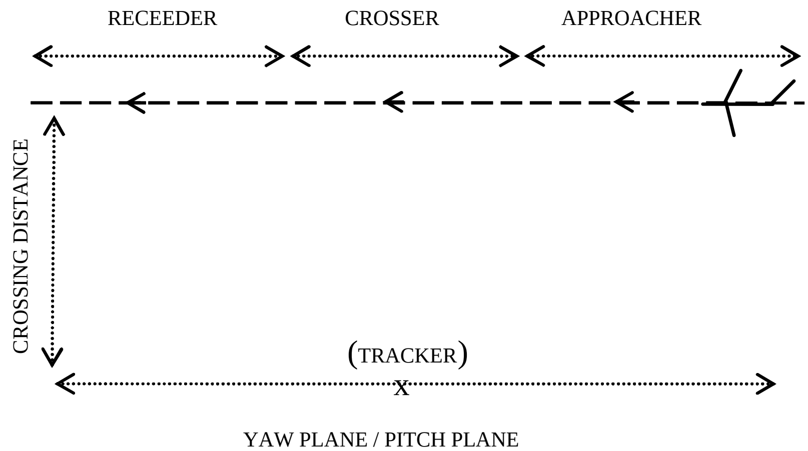

Figure 2. Approacher/crosser/receder target in yaw plane/pitch plane

fly past the platform (crosser), or recede away from the platform (receder). This 3-D scenario may be viewed in the horizontal yaw plane, ie, left/right, disregarding the elevation aspect and also in the vertical pitch plane, ie, up/down, disregarding the bearing aspect. The target, in either of the two planes, can be depicted as shown in Fig. 2. To prevent the target, ie, enemy aircraft, from causing damage in the area where the missile platform is located on ground, the target has to be hit by the missile while it is an approacher. At the appropriate time, the missile is launched in appropriate direction, at a speed which is always greater than that of the target, and it flies towards the target by controlling its trajectory in pitch and yaw planes. During this time, the target will approach closer to tracker platform and will change from an approacher to a crosser. It may be appreciated that the angular change in the LOS between the platform and the target due to move of the target over a certain distance is less when the target is an approacher as compared to when it is a crosser.

For implementing the LOS guidance law as defined in Eqn (1), the missile must always be on the LOS between the ground-based tracker platform and the target. Such a trajectory is a curved one, and for an approaching target, the curvature becomes increasingly severe towards the end of the trajectory, as shown in Fig. 3. To execute such a severely-curved trajectory, the normal (lateral) acceleration required will be of a high magnitude[2].

For implementing the PN guidance law as defined in Eqn (2), the missile must steer at a rate proportional to the rate of change of LOS between the missile and the target. For the missile to be able to hit the target, the missile must steer towards the target at least as much as the angular rate of change of LOS between the two for which the constant of proportionality must be at least 1. Since the target is changing from an approacher to a crosser not only wrt the tracking platform on ground but also wrt the missile itself, the angular rate of change of LOS between the missile and the target will be greater at the end of the trajectory. Thus for a missile aimed at the target in the first place, the trajectory will once again be curved one, though not as much as for LOS guidance, becoming increasingly severe towards the end of the trajectory, as shown in Fig. 4. The normal (lateral) acceleration required to execute such a trajectory will once again be of a high magnitude.

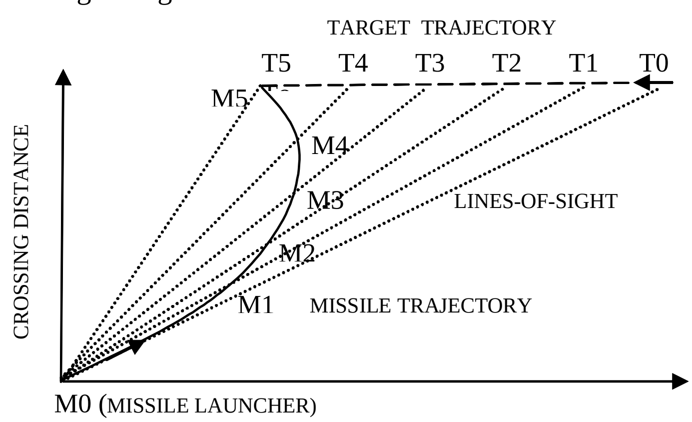

Figure 3. LOS guidance law missile trajectory

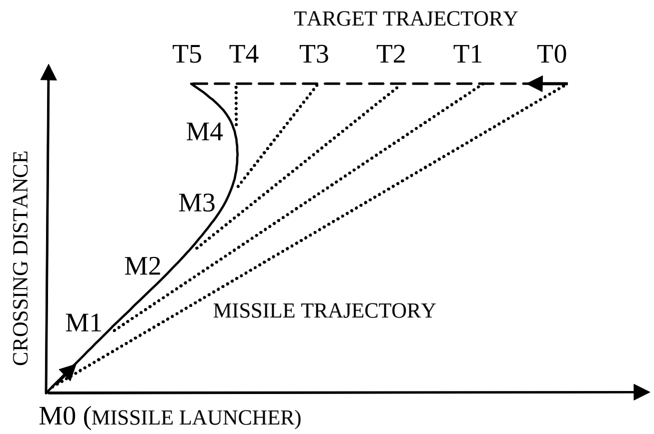

Figure 4. PN guidance law missile trajectory (*N*=1)

However, if the constant of proportionality is more than 1, say 4, initially the LOS rate between the missile and the target will be the same as for proportionality constant equal to 1, but the steering commands are 4-times stronger. As a result, the missile veers off much more towards the future position of the target by establishing a lead angle in the early phase of the trajectory, and thus requiring very little steering effort at the end of the trajectory. Such a trajectory is more curved than that for a proportionality constant equal to 1 in the early phase of the trajectory and becomes almost straight in the terminal phase of the trajectory, as shown in Fig. 5.

However, too high a value of the proportionality constant will steer the missile on a course which may not lead to a collision with the target, especially in the end-phase of the trajectory just before impact

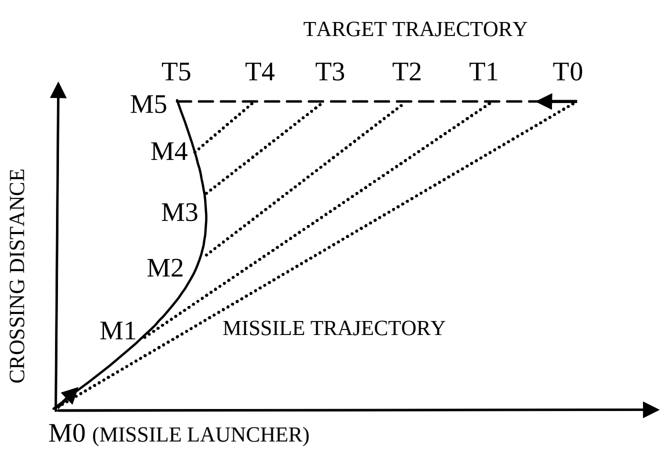

Figure 5. PN guidance law missile trajectory (*N*=4)

when not much time is left for repeated steering corrections for the excessive steering executed previously. Although most researchers agree on the value of proportionality constant between 3 and 6, but no detailed estimation has been offered by anyone. If the aim is to establish a lead angle to minimise the steering effort by the missile, especially at the end of the trajectory, one may aim off in the first place in the direction of the future position of the target[2]. The trajectory of the missile for such a case will be absolutely straight, from the start to the end, unless the target manoeuvres to evade the missile, which it will, in any case. The steering effort required by the missile will then be only to compensate for the manoeuvres of the evading target, unlike that for LOS guidance and PN guidance without aim off, where steering effort is required even for a target flying straight.

To justify the use of PN guidance with aim off, let the effect of a curved trajectory be analysed for requirement of increased lateral acceleration. If a missile is traversing a curved trajectory, which forms an arc of a circle with radius R from its centre and with an angular speed (speed divided by distance R from the centre) W, then the missile will experience a centrifugal acceleration $W^2 R$ directed radially outwards[2,4]. This centrifugal acceleration will throw the missile radially outwards, to compensate which increased lateral acceleration will be required by the missile just like in case of a motor vehicle negotiating a curved road. The missile would be designed to be able to generate a certain amount of lateral acceleration, at least as much as any modern-day aircraft, if not more. However, if this lateral acceleration capability is used for negotiating an already curved trajectory, like in case of LOS guidance or PN guidance (N = 1) without aim off, then any further requirement of lateral acceleration arising due to evasive target manoeuvre would be beyond the capability of the missile. It is for this reason that PN guidance with $N > 1$ is invariably used in mid-course phase of the missile trajectory.

Thus, the major advantages that accrue for a missile aimed off towards the future position of the moving target while using PN guidance with N = 1 are:

(a) Straightline trajectory of the missile, requiring minimal steering effort, and thus, minimum steering errors.

(b) Straightline trajectory is the shortest path to the target for collision, giving minimum time to the target for evasive manoeuvres.

(c) Aim off permits command implementation, in the mid-course phase of the missile trajectory, of only PN guidance law since the missile is not on the LOS between the tracker and the target, and thereby facilitates smoother switchover to homing implementation in the terminal phase of the missile trajectory when PN guidance law is invariably used.

4. FRAME OF REFERENCE AND ANGULAR REFERENCES

Before evolving the mathematical model for the missile launch angle and missile steering commands, the stabilised frame of reference and the angular references for measurement of the variables involved are defined.

In the 3-D space, a frame of reference is required as datum wrt which the angular measurements of the variables bearing and elevation of the target and the missile are made[5]. The frame of reference will have three mutually perpendicular axes, named arbitrarily as X, Y, and Z. Let the XY plane of this frame be always perpendicular to the direction of the gravitational force and parallel to perfectly horizontal surface of earth. The XZ and the YZ plane are perpendicular to XY plane as well as each other. However, the reference frame has no particular orientation wrt the earth's north. For a ground-based missile and ground-based target- tracking platform, the X, Y, and Z axes of the reference frame aligned themselves along the longitudinal, lateral, and vertical axes of the platform and the centre of gravity coincident with the origin, as shown in Fig. 1. Such an imaginary reference frame, as described here, is the stabilised reference frame.

The angular references are required for quantifying the angular measurements of the variables, ie, bearing and elevation of the target and the missile, in the stabilised frame of reference under consideration. Bearing is measured in XY plane positively from +X-axis towards +Y-axis and elevation is measured positively from XY plane towards +Z-axis.

5. MISSILE LAUNCH ANGLE MODEL

To evolve the generalised mathematical model, consider again a twin platform system with the target/missile-tracking device such as the tracking radar on one platform and the surface-to-air missile on the other platform, and also with the stabilised frame of reference aligned with the platform mounted with the target/missile tracking device, as shown in Fig. 1. To enable collision of the missile with the moving air target, the equations of their motion along the X, Y, and Z axes may be equated similarly as for estimation of the angular orientation of anti-aircraft gun in the direction of future position of a moving air target except for two terms pertaining to gravity and viscous friction of the atmospheric medium, which are dispensed with from the right side of the equations. Missiles fly at a constant velocity due to sustained propulsion till the end of the trajectory, and thus, viscous friction does not change its velocity. Also, any lateral acceleration experienced by the missile due to gravity is automatically negated by the lateral autopilot through the control surfaces since the same is not commanded by the tracking platform on the ground.

$$X_t + V_{tx}.t + ½ A_{tx}.t^2 = X_m + V_{mx}.t \qquad (4)$$

$$Y_t + V_{ty}.t + ½ A_{ty}.t^2 = Y_m + V_{my}.t \qquad (5)$$

$$Z_t + V_{tz}.t + ½ A_{tz}.t^2 = Z_m + V_{mz}.t \qquad (6)$$

Substituting the values of V_{mx} , V_{my}, and V_{mz} in Eqns (4) to (6), one gets

$$X_t + V_{tx}.t + ½ A_{tx}.t^2 = X_m + V_m.\cos e_m.\cos b_m.t \qquad (7)$$

$$Y_t + V_{ty}.t + ½ A_{ty}.t^2 = Y_m + V_m.\cos e_m.\sin b_m.t \qquad (8)$$

$$Z_t + V_{tz}.t + ½ A_{tz}.t^2 = Z_m + V_m.\sin e_m.t \qquad (9)$$

Rewriting Eqns (7) to (9), one gets

$$(X_t – X_m) + V_{tx}.t + ½ A_{tx}.t^2 = V_m.t.\cos e_m.\cos b_m \qquad (10)$$

$$(Y_t - Y_m) + V_{ty}.t + ½\, A_{ty}.t^2 = V_m.t.\cos e_m.\sin b_m \qquad (11)$$

$$(Z_t - Z_m) + V_{tz}.t + ½\, A_{tz}.t^2 = V_m.t.\sin e_m \qquad (12)$$

Squaring Eqns (10) and (11) and adding these, one gets

$$\begin{aligned} &[(X_t - X_m) + V_{tx}.t + ½\, A_{tx}.t^2]^2 \\ +\ &[(Y_t - Y_m) + V_{ty}.t + ½\, A_{ty}.t^2]^2 \\ =\ &V_m^{\,2}.t^2.\text{Cos}^2\, e_m \end{aligned} \qquad (13)$$

Squaring Eqn (12), one gets

$$[(Z_t - Z_m) + V_{tz}.t + ½\, A_{tz}.t^2]^2 = V_m^{\,2}.t^2.\sin^2 e_m \qquad (14)$$

Adding Eqns (13) and (14), one gets

$$\begin{aligned} &[(X_t - X_m) + V_{tx}.t + ½\, A_{tx}.t^2]^2 \\ +\ &[(Y_t - Y_m) + V_{ty}.t + ½\, A_{ty}.t^2]^2 \\ +\ &[(Z_t - Z_m) + V_{tz}.t + ½\, A_{tz}.t^2]^2 \\ =\ &V_m^{\,2}.t^2 \end{aligned} \qquad (15)$$

Expanding Eqn (15) and collecting terms with same power of variable time t, one gets

$$\begin{aligned} &¼\, [A_{tx}^{\,2} + A_{ty}^{\,2} + A_{tz}^{\,2}]\, t^4 + [A_{tx}.V_{tx} + A_{ty}.V_{ty} + A_{tz}.V_{tz}]t^3 \\ +\ &[(X_t - X_m).A_{tx} + V_{tx}^{\,2} + (Y_t - Y_m).A_{ty} \\ +\ &V_{ty}^{\,2} + (Z_t - Z_m).\, A_{tz} + V_{tz}^{\,2} - V_m^{\,2}]t^2 \\ +\ &2[(X_t - X_m).V_{tx} + (Y_t - Y_m).V_{ty} + (Z_t - Z_m).V_{tz}]t \\ +\ &(X_t - X_m)^2 + (Y_t - Y_m)^2 + (Z_t - Z_m)^2 = 0 \end{aligned} \qquad (16)$$

The Eqn (16) above is a fourth-order equation for the variable time t containing terms for t^4, t^3, t^2, t and a constant. The formulae for finding roots of second-, third-, and fourth-order equations are available above which one must resort to numerical method. The formula for solving the quadratic is easily applied, but the cubic solution is rather long and for quartic, very complicated. It turns out that one usually uses a numerical method for all equations above quadratic. The Lin's method and the Newton's method are two well-documented methods[6]. Although either of these two methods or any similar method will give the roots, however for the model under consideration, a better way of finding the value of the variable time t would be as under.

It may be appreciated that the value of the variable time t cannot be imaginary or complex and can have only a positive real value. Further aim is to find only that one particular root amongst the possible four, that would result in correct estimation of the launch angle of the missile, which will enable collision of the missile with the target while maintaining a straightline trajectory. This value of the variable time t when substituted in the left side of the Eqn (13) will equate with the right side of the equation, ie, zero.

To determine the value of the variable time t which is of interest, the target is considered to be stationary at its instantaneous position. The distance of the target from the initial position of the missile will be

$$D_1 = [(X_t - X_m)^2 + (Y_t - Y_m)^2 + (Z_t - Z_m)^2]^{1/2}$$

and the time of flight of the missile will be

$$T_1 = D_1 / V_m$$

The value that will satisfy Eqn (16) will not be exactly T_1 but in its vicinity, either less or more than T_1, depending on whether the target is approaching or receding. To determine in which direction the value will lie, the position of the moving target is considered now after T_1 seconds. The distance will be

$$\begin{aligned} D_2 = &[(X_t + V_{tx}.T_1 + ½A_{tx}.T_1^{\,2} - X_m)^2 \\ +\ &(Y_t + V_{ty}.T_1 + ½A_{ty}.T_1^{\,2} - Y_m)^2 \\ +\ &(Z_t + V_{tz}.T_1 + ½A_{tz}.T_1^{\,2} - Z_m)^2]^{1/2} \end{aligned}$$

If $D_2 > D_1$, the target is a receding one and consequently, the value of variable time t that will satisfy Eqn (16), will be $> T_1$. If $D_2 < D_1$, the target is an approaching one, and consequently, the value of variable time t that will satisfy Eqn (16), will be $< T_1$.

To justify the use of variables T_1, D_1 and D_2 in choosing the direction of search, consider the target/missile fly plane as shown in Fig. 6. In the 3-D stabilised reference frame, the plane containing the straightline trajectory of the target, which also passes through the initial position of the missile on the ground, is the target/missile fly plane. For a target initially acquired at point A at a distance D_1, the corresponding value time T_1 is the most obvious reference in whose vicinity the actual value of the variable time t lies. Let A' be the point ahead on the target trajectory which is at the same distance D_1 and for which the missile would take the same time T_1 to reach. One may now let the missile travel from its initial position to the point A' and simultaneously the target from point A towards point A'. By the time the missile reaches point A', if the target reaches any point B which is short of A', then the distance D_2 will be $< D_1$. It can thus be inferred that if the missile was to be launched in the correct direction, the time taken to intercept the target will be $< T_1$. Similarly, if the target was to reach some point C ahead of point A', then the distance D_2 will be $> D_1$ and the time taken by the missile to intercept the target will be $> T_1$.

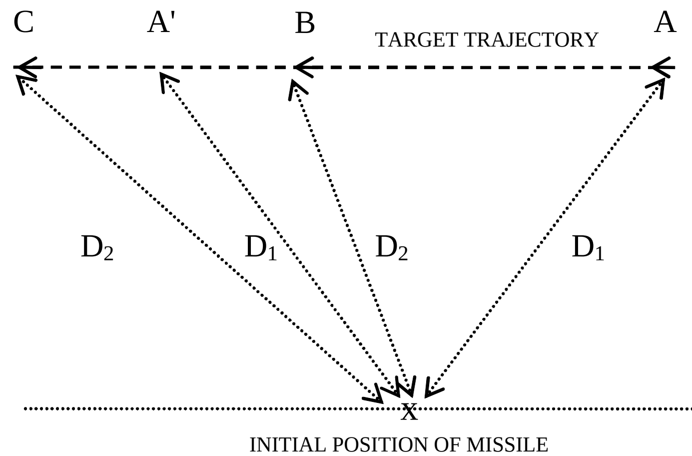

Figure 6. Target/missile fly plane

Having ascertained the direction in which the value of variable time t lies, the expression in Eqn (16) is calculated successively starting with value of $t = T_1$, and thereafter, increasing or decreasing it, as determined earlier, in practically useful steps of say 0.01 s or as per the requirement of accuracy.

As these calculations are done successively, the value of the expression will approach zero. Once that value of the variable time t has been reached for which the value of the expression on the left side of Eqn (16) changes from a positive value to a negative value (or vice versa) or is within certain + limit of zero or say within 1 per cent of the value of the expression with $t = T_1$, the value of the variable time t arrived at will be the one that will nearly satisfy Eqn (16).

Mathematically, one may be tempted to find the value of the variable time t for which the slope of Eqn (16) will change by taking the derivative of Eqn (16) and for which Eqn (16) would have been satisfied. However, this may not render the value of variable time t for two reasons. Firstly, taking derivative of Eqn (16) will give a third-order equation with three roots, and where once again, a numerical method will have to be applied. Secondly, the nature of Eqn (16) will depend on its constants and may be such that a change of slope may not exist.

The method described above may perhaps be the most workable way of finding the solution of Eqn (16). Having found the desired value of variable time t, the same is substituted in Eqn (12) to get the value of the variable elevation e_m. Substituting the value of the variable time t and elevation e_m in either Eqn (10) or (11) will give the value of the variable bearing b_m.

The Eqns (10), (11), (12), and (16) above are launch angle equations for surface-to-air missile whose problem-specific numerical solution is as elucidated above.

To gain faith, let one consider an example and demonstrate the efficacy/validity of Eqn (16). Based on experience of Army's Air Defence Regiments, it has been established that attacking enemy aircraft fly in at a low altitude ranging between 300 m and 800 m to avoid radar detection and at speeds no more than 1.2 Mach (Mach 1 is approximately 331 m/s, the speed of sound) to enable precision delivery of conventional explosives. Also, most surface-to -air missiles in use today fly at constant speeds ranging from 700 m/s to 900 m/s, which is higher than the maximum speed of 2.2 Mach which aircraft achieve. Accordingly, the values of variables have been selected consistent with the above facts.

The target is assumed to be initially positioned + 4000 m away along X and Y axes, approaching at a constant velocity of 100 m/s along X-axis only (approacher target with crossover distance) and with no acceleration. Further, the missile launcher is assumed to be displaced + 50 m along X and Y axes from the missile/target-tracking device which is at the origin of the reference frame. The missile is assumed to have a constant velocity of 700 m/s. With this data, the Eqn (16) was solved using software made in MATLAB 6.1, as in Table 1. The result of the iteration is given in Table 2.

At the starting point of $t = T_1 = 8.0121$ s, the value of Eqn (16) computes to $-5.6876 \times 10^{+6}$. Since the target is an approacher, the time of flight of the missile till impact will be $< T_1$. Accordingly, the search is made by reducing the value of the variable time t in steps of 0.01 s. As regards accuracy, even for an aircraft flying at 1.2 Mach (400 m/s approx.) the distance it would travel in 0.01 s will be only 4 m, sufficiently accurate for the missile's proximity fuse to activate and detonate. However, the aircraft would invariably respond with evasive manoeuvre whence the missiles guidance will come into play and the error of 4 m would be inconsequential. Between the 69th and 70th iteration, the value of Eqn (16) changes from negative ($-6.3834 \times 10^{+4}$) to positive ($1.4310 \times 10^{+4}$) and the search stops there. The value of the launch angle in terms of bearing and elevation is then calculated corresponding to the value of variable time t for the 70th iteration, as in Table 3.

The correctness of the values of the variables t, e_m, and b_m arrived at above through the numerical computation can be checked by analytical computation of the position of the target and the missile after time t. The position of the target and the missile in the 3-D stabilised reference frame at any point of time can be computed using equations as given in Table 4. The programming codes in Table 5 for these equations when used in tandem with the program in Table 1 render the position of the target and missile as tabulated in Table 6. The position of the missile is the same as that of the target, which enables one to conclude that the missile will intercept the target and that the numerically computed values of variables t, e_m, and b_m are correct.

To implement the mathematical model, the variables used in the model need to be determined from the input of the target and missile states. The instantaneous target data that any ground-based tracker acquires are target bearing $b_t(n)$, target elevation $e_t(n)$, and the target range $r_t(n)$ at any instance of time $t(n)$. Then the instantaneous components of the target position along X, Y, and Z axes at any instance of time $t(n)$ are:

$$X_t(n) = r_t(n).\cos\ e_t(n).\cos\ b_t(n) \quad (17)$$

$$Y_t(n) = r_t(n).\cos\ e_t(n).\sin\ b_t(n) \quad (18)$$

$$Z_t(n) = r_t(n).\sin\ e_t(n) \quad (19)$$

The components of the target position along the three axes at previous instances of time $t(n-1)$, $t(n-2)$ and so on can similarly be found. The instantaneous components of the target velocity along the three axes at any instance of time $t(n)$ are:

$$V_{tx}(n) = [X_t(n) - X_t(n-1)]/[t(n) - t(n-1)] \quad (20)$$

$$V_{ty}(n) = [Y_t(n) - Y_t(n-1)]/[t(n) - t(n-1)] \quad (21)$$

$$V_{tz}(n) = [Z_t(n) - Z_t(n-1)]/[t(n) - t(n-1)] \quad (22)$$

The instantaneous components of the target velocity along the three axes at previous instance of time $t(n-1)$ can be found similarly. The instantaneous components of the target acceleration along the three axes at any instance of time $t(n)$ are:

$$A_{tx}(n) = [V_{tx}(n) - V_{tx}(n-1)]/[t(n) - t(n-1)] \quad (23)$$

$$A_{ty}(n) = [V_{ty}(n) - V_{ty}(n-1)]/[t(n) - t(n-1)] \quad (24)$$

$$A_{tz}(n) = [V_{tz}(n) - V_{tz}(n-1)]/[t(n) - t(n-1)] \quad (25)$$

These instantaneous components $X_t(n)$, $Y_t(n)$, $Z_t(n)$, $V_{tx}(n)$, $V_{ty}(n)$, $V_{tz}(n)$, $A_{tx}(n)$, $A_{ty}(n)$, and $A_{tz}(n)$ at any instance of time $t(n)$ are nothing but the instantaneous values X_t, Y_t, Z_t, V_{tx}, V_{ty}, V_{tz}, A_{tx}, A_{ty}, and A_{tz} used in the missile launch angle model.

Table 1. Software program to calculate the variables t, e_m and b_m

```
%- surface to air missile launch angle calculation program in matlab 6.1 -%
Xt = 4000 ; Yt = 4000 ; Zt = 500 ;
Vtx = -100 ; Vty = 0 ; Vtz = 0 ;
Atx = 0 ; Aty = 0 ; Atz = 0 ;
Xm = 50 ; Ym = 50 ; Zm = 0 ;
Vm = 700 ;
D1 = sqrt((Xt-Xm)^2+(Yt-Ym)^2+(Zt-Zm)^2) ; %- chk tgt approaching / receding -%
T1 = D1 / Vm ;
n = 0 %- set initial point of search -%
t = T1
D2 = sqrt((Xt+Vtx*t+0.5*Atx*t^2-Xm)^2+(Yt+Vty*t+0.5*Aty*t^2-Ym)^2+(Zt+Vtz*t+0.5*Atz*t^2-Zm)^2) ;
if D1 > D2 %- set direction of search -%
   dir = -1 ;
else
   dir = 1 ;
end
eqn = 0.25*(Atx^2+Aty^2+Atz^2)*t^4+ ...
        ((Atx*Vtx)+(Aty*Vty)+(Atz*Vtz))*t^3+ ...
        (((Xt-Xm)*Atx)+Vtx^2+((Yt-Ym)*Aty)+Vty^2+((Zt-Zm)*Atz)+Vtz^2-Vm^2)*t^2+ ...
        2.0*((Xt-Xm)*Vtx+(Yt-Ym)*Vty+(Zt-Zm)*Vtz)*t+ ...
        (Xt-Xm)^2+(Yt-Ym)^2+(Zt-Zm)^2
x = sign(eqn) ; %-chk initial value of eqn is +ve / -Ve -%
switch x
   case -1
      while eqn < 0 %- limit number of iteration till eqn reaches zero -%
      n = n + 1
      t = T1 + (dir*n*0.01)
      eqn = 0.25*(Atx^2+Aty^2+Atz^2)*t^4+ ...
              ((Atx*Vtx)+(Aty*Vty)+(Atz*Vtz))*t^3+ ...
              (((Xt-Xm)*Atx)+Vtx^2+((Yt-Ym)*Aty)+Vty^2+((Zt-Zm)*Atz)+Vtz^2-Vm^2)*t^2+ ...
              2.0*((Xt-Xm)*Vtx+(Yt-Ym)*Vty+(Zt-Zm)*Vtz)*t+ ...
              (Xt-Xm)^2+(Yt-Ym)^2+(Zt-Zm)^2
      end
   case +1
      while eqn > 0 %- limit number of iteration till eqn reaches zero -%
      n = n + 1
      t = T1 + (dir*n*0.01)
      eqn = 0.25*(Atx^2+Aty^2+Atz^2)*t^4+ ...
              ((Atx*Vtx)+(Aty*Vty)+(Atz*Vtz))*t^3+ ...
              (((Xt-Xm)*Atx)+Vtx^2+((Yt-Ym)*Aty)+Vty^2+((Zt-Zm)*Atz)+Vtz^2-Vm^2)*t^2+ ...
              2.0*((Xt-Xm)*Vtx+(Yt-Ym)*Vty+(Zt-Zm)*Vtz)*t+ ...
              (Xt-Xm)^2+(Yt-Ym)^2+(Zt-Zm)^2
      end
end
em = asin ((Zt-Zm+(Vtz*t)+(0.5*Atz*t^2))/(Vm*t)) %- missile launch elevation -%
bm = atan ((Yt-Ym+Vty*t+0.5*Aty*t^2)/(Xt-Xm+Vtx*t+0.5*Atx*t^2)) %- missile launch bearing -%
%- end of program -%
```

The position of the missile platform wrt the target/missile-tracking device can be measured physically for static platforms from which the components of the initial missile position along three axes can be found. However, for mobile platforms, use of global positioning system will have to be resorted to and transmitted through radio for finding instantaneous components of the initial missile position along the three axes.

6. MISSILE STEERING MODEL

The missile having been launched at a particular launch angle in bearing and elevation in the direction of the future position of the target for a straightline trajectory, the instantaneous missile state, ie, bearing $b_m(n)$, elevation $e_m(n)$, and range $r_m(n)$ will be gathered by the missile-tracking device from which

Table 2. Result of the software program showing the values of the variables *n*, *t* and equation corresponding to successive iterations

n	*t*	Equation
0	8.0121	-5.6876e+006
1	8.0021	-5.6029e+006
2	7.9921	-5.5182e+006
3	7.9821	-5.4336e+006
4	7.9721	-5.3491e+006
5	7.9621	-5.2647e+006
6	7.9521	-5.1805e+006
7	7.9421	-5.0963e+006
8	7.9321	-5.0122e+006
9	7.9221	-4.9282e+006
10	7.9121	-4.8443e+006
11	7.9021	-4.7605e+006
12	7.8921	-4.6767e+006
13	7.8821	-4.5931e+006
14	7.8721	-4.5096e+006
15	7.8621	-4.4262e+006
16	7.8521	-4.3429e+006
17	7.8421	-4.2596e+006
18	7.8321	-4.1765e+006
19	7.8221	-4.0934e+006
20	7.8121	-4.0105e+006
21	7.8021	-3.9277e+006
22	7.7921	-3.8449e+006
23	7.7821	-3.7622e+006
24	7.7721	-3.6797e+006
25	7.7621	-3.5972e+006
26	7.7521	-3.5149e+006
27	7.7421	-3.4326e+006
28	7.7321	-3.3504e+006
29	7.7221	-3.2683e+006
30	7.7121	-3.1863e+006
31	7.7021	-3.1045e+006
32	7.6921	-3.0227e+006
33	7.6821	-2.9410e+006
34	7.6721	-2.8594e+006
35	7.6621	-2.7779e+006
36	7.6521	-2.6965e+006
37	7.6421	-2.6151e+006
38	7.6321	-2.5339e+006
39	7.6221	-2.4528e+006
40	7.6121	-2.3718e+006
41	7.6021	-2.2908e+006
42	7.5921	-2.2100e+006
43	7.5821	-2.1293e+006
44	7.5721	-2.0486e+006
45	7.5621	-1.9681e+006
46	7.5521	-1.8876e+006
47	7.5421	-1.8073e+006
48	7.5321	-1.7270e+006
49	7.5221	-1.6469e+006
50	7.5121	-1.5668e+006
51	7.5021	-1.4868e+006
52	7.4921	-1.4070e+006
53	7.4821	-1.3272e+006
54	7.4721	-1.2475e+006
55	7.4621	-1.1679e+006
56	7.4521	-1.0884e+006
57	7.4421	-1.0091e+006
58	7.4321	-9.2976e+005
59	7.4221	-8.5056e+005
60	7.4121	-7.7145e+005
61	7.4021	-6.9244e+005
62	7.3921	-6.1353e+005
63	7.3821	-5.3472e+005
64	7.3721	-4.5600e+005
65	7.3621	-3.7737e+005
66	7.3521	-2.9884e+005
67	7.3421	-2.2041e+005
68	7.3321	-1.4207e+005
69	7.3221	-6.3834e+004
70	7.3121	1.4310e+004

Table 3. Result of the software program showing the values of the variables e_m and b_m for the corresponding values of the variables *n*, *t* and equation

n	*t*	Equation	e_m (rad)	b_m (rad)
70	7.3121	1.4310e+004	0.0978	0.8870

Table 4. Equations of motion to calculate the components along the three axes of final position of the target and missile

Equations of motion for calculating the components, along the three axes, of the final position of the target ie X_{tf}, Y_{tf}, and Z_{tf}, after elapse of time *t*

$$X_{tf} = X_t + V_{tx} . t + ½ A_{tx} . t^2$$
$$Y_{tf} = Y_t + V_{ty} . t + ½ A_{ty} . t^2$$
$$Z_{tf} = Z_t + V_{tz} . t + ½ A_{tz} . t^2$$

Equations of motion for calculating the components, along the three axes, of the final position of the missile launched at elevation e_m and bearing b_m ie, X_{mf}, Y_{mf}, Z_{mf}, after elapse of time *t*

$$X_{mf} = X_m + V_m . t . \cos e_m . \cos b_m$$
$$Y_{mf} = Y_m + V_m . t . \cos e_m . \sin b_m$$
$$Z_{mf} = Z_m + V_m . t . \sin e_m$$

the instantaneous components of the missile position along the three axes are as under:

$$X_m(n) = r_m(n).\cos e_m(n).\cos b_m(n) \quad (26)$$

$$Y_m(n) = r_m(n).\cos e_m(n).\sin b_m(n) \quad (27)$$

$$Z_m(n) = r_m(n).\sin e_m(n) \quad (28)$$

To determine the LOS of the target wrt the missile, the displacement of the target wrt the missile along the three axes was found as follows:

$$X_{tm}(n) = [X_t(n) - X_m(n)] \quad (29)$$

$$Y_{tm}(n) = [Y_t(n) - Y_m(n)] \quad (30)$$

$$Z_{tm}(n) = [Z_t(n) - Z_m(n)] \quad (31)$$

from which the bearing and elevation aspects of the LOS are found as under:

$$B_{tm}(n) = \tan^{-1} \{Y_{tm}(n)/X_{tm}(n)\} \quad (32)$$

$$E_{tm}(n) = \tan^{-1}\{Z_{tm}(n) / [X_{tm}(n)^2 + Y_{tm}(n)^2]^{1/2}\} \quad (33)$$

Similarly, bearing and elevation aspects of target wrt the missile for *t*(*n*-1) instance of time can be found. The rate of change of LOS in bearing and elevation is:

$$\dot{B}_{tm}(n) = [B_{tm}(n) - B_{tm}(n-1)]/[t(n) - t(n-1)] \quad (34)$$

$$\dot{E}_{tm}(n) = [E_{tm}(n) - E_{tm}(n-1)]/[t(n) - t(n-1)] \quad (35)$$

From Eqn (1), $\dot{S}_m = N.\dot{\lambda}_{tm}$ and from Eqn (2), $\dot{S}_m = A_{mn}/V_m$. Equating these one gets,

$$A_{mn} = N . \dot{\lambda}_{tm} . V_m \quad (36)$$

Thus, the normal (lateral) acceleration required by the missile for the yaw and pitch channels to negate the effect of changing LOS can be calculated by replacing $\dot{\lambda}_{tm}$ by $\dot{B}_{tm}(n)$ and $\dot{E}_{tm}(n)$, as under:

$$A_{mny}(n) = N . \dot{B}_{tm}(n) . V_m \quad (37)$$

$$A_{mnp}(n) = N . \dot{E}_{tm}(n) . V_m \quad (38)$$

The Eqns (37) and (38) above are missile steering equations for PN commanded surface-to-air guided missile. The value of *N* should be one purely from kinematical considerations and in practice it may be kept just a little above 1, say 1.1 or 1.2, to prevent it from slipping below 1 for any reason.

Table 5. Software program (to be used in tandem with that in Table 1) to calculate the variables X_{tf}, Y_{tf}, Z_{tf}, X_{mf}, Y_{mf}, and Z_{mf}

```
%-check results-%
%- At t = ? , the position of target is at Xtf, Ytf and Ztf -%
Xtf = Xt + Vtx*t + 0.5*Atx*t^2
Ytf = Yt + Vty*t + 0.5*Aty*t^2
Ztf = Zt + Vtz*t + 0.5*Atz*t^2
%- At t = ? , em = ? and bm = ? , the position of missile is
        at Xmf, Ymf and Zmf-%
Xmf = Xm + Vm*t*cos(em)*cos(bm)
Ymf = Ym + Vm*t*cos(em)*sin(bm)
Zmf = Zm + Vm*t*sin(em)
%-end of check results-%
```

Table 6. Analytical computation results of the final position of the target and the missile after time *t* = 7.3117 s

Position of target along X, Y, and Z axes after time *t* = 7.3121 s	Position of missile along X, Y, and Z axes after time *t* = 7.3121 s
X_{tf} = 3.2688e+003	X_{mf} = 3.2679e+003
Y_{tf} = 4000	Y_{mf} = 3.9989e+003
Z_{tf} = 500	Z_{mf} = 500

7. CONCLUSION

In this study, the qualitative analysis of the LOS and PN missile trajectories reveal the advantage of employing the PN guidance with aim off. The instantaneous target and missile states that are sensed by the ground radar are invariably contaminated with random noise. The extended Kalman filter is used first as an observer to filter the random noise and to obtain the estimated values of target and missile states as well as their derivatives. These estimated values can then be used to physically implement the quantitatively estimated initial launch angle, and thereafter, the missile steering commands in pitch and yaw planes for successful and efficient destruction of the target by the missile.

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