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SHORT COMMUNICATION

Creep Transition in a Thin Rotating Disc with Rigid Inclusion

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ABSTRACT

Creep stresses and strain rates have been obtained for a thin rotating disc with inclusion using Seth's transition theory. Results have been discussed numerically and depicted graphically. It has been observed that radial stress has maximum value at the internal surface of the rotating disc made of incompressible material as compared to circumferential stress and this value of radial stress further increases with the increase in angular speed. Strain rates have maximum values at the internal surface for compressible material. Rotating disc is likely to fracture by cleavage close to the inclusion at the bore

Keywords: Creep stress, thin rotating disc, stress, strain, rotors, turbines, creep transition, solid mechanics

NOMENCLATURE

e_{ii}^{A}	Principal finite strain components
a, b	Internal and external radii of the disc
u, v, w	Displacement components
r, θ, z	Radial, circumferential and axial directions
ω	Angular velocity of rotation
δ_{ij}	Kronecker's delta
ρ	Density of material
C	Compressibility factor
$\dot{e}_{ij}$	Strain rate tensor
Y	Yield stress
ε_{ij}	Swainger strain components
ν	Poisson's ratio
Ω^2	$\rho\omega^2 b^2/E$ (Speed factor); $R=r/b$; $R_0=a/b$
σ_r	Radial stress component (T_{rr}/E)
σ_θ	Circumferential stress component $(T_{\theta\theta}/E)$

1. INTRODUCTION

Rotating disc forms an essential part of the design of rotating machinery namely, rotors, turbines, compressors, fly wheels and computer disc drives, etc. The analytical procedures presently available are restricted to problems with simplest configurations. The use of rotating disc in machinery and structural applications has generated considerable interest in many problems in domain of solid mechanics. Solutions for thin isotropic discs can be found in most of the standard creep text books[2-7]. Wahl[1] has investigated creep deformation in rotating discs assuming small deformation, incompressibility condition, Tresc'a yield criterion, its associated flow rule and a power strain law. Seth's transition theory[8] does not acquire

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any assumptions like an yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out.

Seth[9] has defined the generalised principal strain measure as

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1-2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1-\left(1-2\overset{A}{e}_{ii}\right)^{\frac{n}{2}}\right]$$

(i,j=1,2,3) (1)

where n is the measure and is the Almansi finite strain components. In this study, creep stresses and strain rates for a thin rotating disc with rigid inclusion have been obtained using Seth's transition theory. Results have been discussed numerically and depicted graphically.

2. GOVERNING EQUATIONS

A thin disc of constant density was considered with central bore of radius a and external radius b. The annular disc was mounted on a shaft. The disc was rotating with angular speed ω about an axis perpendicular to its plane and passed through the centre as shown in Fig. 1. The thickness of disc was assumed to be constant and was taken to be sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero.

The displacement components in cylindrical polar coordinate are given by[9]

$$u = r(1-\beta),\ v = 0,\ w = dz, \tag{2}$$

where β is function of $r = (x^2 + y^2)^{1/2}$ only and d is a constant.

The finite strain components are given by Seth[9] as

$$\overset{A}{e}_{rr} = \frac{1}{2}\left[1-\left(r\beta'+\beta\right)^2\right]$$

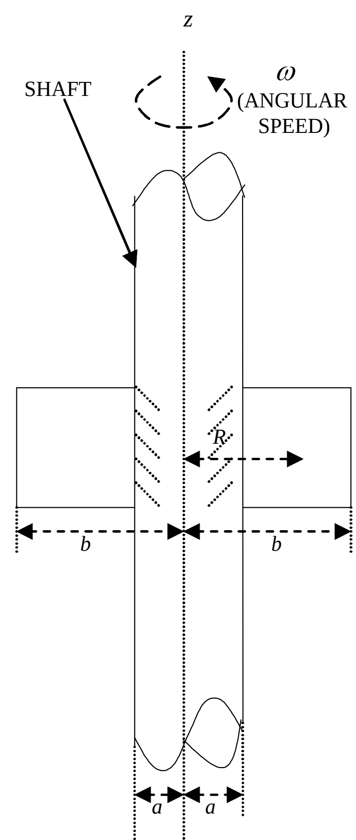

Figure 1. Geometry of rotating disc.

$$\overset{A}{e}_{\theta\theta} = \frac{1}{2}\left[1-\beta^2\right],$$

$$\overset{A}{e}_{zz} = \frac{1}{2}\left[1-(1-d)^2\right],$$

$$\overset{A}{e}_{r\theta} = \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0, \tag{3}$$

where $\beta' = d\beta/dr$.

Substituting Eqn (3) in Eqn (1), the generalised components of strain are:

$$e_{rr} = \frac{1}{n}\left[1-\left(r\beta'+\beta\right)^n\right],\ e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right],$$

$$e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right], e_{r\theta} = e_{\theta z} = e_{zr} = 0 \tag{4}$$

where $\beta' = d\beta/dr$

The stress-strain relations for isotropic material are given by

$$T_{ij} = \lambda\delta_{ij} I_1 + 2\mu e_{ij}\,,\ (i, j = 1, 2, 3), \tag{5}$$

where λ and μ are lame's constants and e_{kk} is the first strain invariant. δ_{ij} is the Kronecker's delta. Equation (5) for this problem becomes

$$T_{rr} = \frac{2\lambda\mu}{\lambda + 2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2\mu e_{rr}\,,$$

$$T_{\theta\theta} = \frac{2\lambda\mu}{\lambda + 2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2e_{\theta\theta}\,,$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \tag{6}$$

where $\beta' = d\beta/dr$

Substituting Eqn (4) in Eqn (6), one gets the stresses as

$$T_{rr} = \frac{2\mu}{n}\left[3 - 2C - \beta^n\left\{1 - C + (2 - C)(P+1)^n\right\}\right],$$

$$T_{\theta\theta} = \frac{2\mu}{n}\left[3 - 2C - \beta^n\left\{2 - C + (1 - C)(P+1)^n\right\}\right],$$

(7)

where $r\beta' = \beta p$ and .

Equations of equilibrium are all satisfied except

(8)

where ρ is the density of the material of the disc.

Using Eqn (7) in Eqn (8), one gets a nonlinear differential equation in β as

$$(2 - C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta}$$
$$= \frac{n\rho\omega^2 r^2}{2\mu} + \beta^n\left[\begin{array}{l}1 - (P+1)^n - \\ nP\left\{1 - C + (2 - C)(P+1)^n\right\}\end{array}\right]. \tag{9}$$

Transition points of β in Eqn (9) are

$P \to -1$ and $P \to \pm\infty$

The boundary conditions are

$$u = 0 \text{ at } r = a \text{ and } T_{rr} = 0 \text{ at } r = b \tag{10}$$

3. SOLUTION THROUGH PRINCIPAL STRESS DIFFERENCE

For finding the creep stresses, the transition function through principal stress difference[10-19] at the transition point $P \to -1$ leads to the creep state. The transition function R is defined as

$$R = T_{rr} - T_{\theta\theta} = \frac{2\mu\beta^n}{n}\left[1 - (P+1)^n\right] \tag{11}$$

Taking the logarithmic differentiating of Eqn (11) wrt r, one gets

$$\frac{d}{dr}(\log R) = \frac{nP}{r[1 - (P+1)^n]}\left\{\begin{array}{l}1 - (P+1)^n - \\ \beta(P+1)^{n-1}\dfrac{dP}{d\beta}\end{array}\right\} \tag{12}$$

Substituting the value of $\dfrac{dP}{d\beta}$ from Eqn (9) in Eqn (12) and taking asymptotic value $P \to -1$, one gets

$$\frac{d}{dr}(\log R) = -\frac{1}{r(2 - C)}\left[n(3 - 2C) + 1 + \frac{n\rho\omega^2 r^{2+n}}{2\mu D^n}\right] \tag{13}$$

Asymptotic value of β as $P \to -1$ is D/r; D being a constant.

Integrating Eqn (13) wrt r, one gets

$$R = T_{rr} - T_{\theta\theta} = Ar^k \exp\left(Fr^{n+2}\right) \tag{14}$$

where A is a constant of integration,

$$k = -\left[\frac{n(3-2C)+1}{(2-C)}\right]$$

and $$F = -\frac{n\omega^2\rho}{2\mu D^n (2-C)(n+2)}$$

$$= -\left[\frac{n\omega^2\rho(3-2C)}{ED^n(2-C)^2(n+2)}\right]$$

From Eqns (11) and (14), one gets

$$T_{rr} - T_{\theta\theta} = Ar^k \exp\left(Fr^{n+2}\right) \tag{15}$$

Substituting Eqn (15) in Eqn (8), one gets

$$T_{rr} = -A\int r^{k-1} \exp\left(Fr^{n+2}\right)dr - \frac{\rho\omega^2 r^2}{2} + B \tag{16}$$

where B is a constant of integration.

Using boundary condition Eqn (10) in equation Eqn (16), one gets

$$B = A\int_{r=b} r^{k-1} \exp\left(Fr^{n+2}\right)dr + \frac{\rho\omega^2 b^2}{2}$$

Substituting the value of B in Eqn (16), one gets

$$T_{rr} = A\int_r^b r^{k-1} \exp\left(Fr^{n+2}\right)dr + \frac{\rho\omega^2\left(b^2 - r^2\right)}{2} \tag{17}$$

From Eqns (15) and (17), one gets

$$T_{\theta\theta} = A\left\{\int_r^b r^{k-1} \exp\left(Fr^{n+2}\right)dr - r^k \exp\left(Fr^{n+2}\right)\right\}$$

$$+ \frac{\rho\omega^2\left(b^2 - r^2\right)}{2} \tag{18}$$

From Eqns (11) and (15), taking asymptotic value P$\to -1$, one gets

$$\beta = \left[\left(\frac{n}{2\mu}\right)\left[T_{rr} - T_{\theta\theta}\right]\right]^{\frac{1}{n}}$$

$$= \left[\frac{n(3-2C)}{E(2-C)} Ar^k \exp\left(Fr^{n+2}\right)\right]^{\frac{1}{n}} \tag{19}$$

Substituting Eqn (19) in Eqn (2), one gets

$$u = r - r\left[\frac{n(3-2C)}{E(2-C)} Ar^k \exp\left(Fr^{n+2}\right)\right]^{\frac{1}{n}} \tag{20}$$

Using boundary condition [Eqn (10)] in Eqn (20), one gets

$$A = \frac{E(2-C)}{n(3-2C)a^k \exp\left(Fa^{n+2}\right)}$$

Substituting the value of A in Eqns (17), (18) and (20), one gets

$$T_{rr} = \left[\frac{E(2-C)}{n(3-2C)a^k \exp\left(Fa^{n+2}\right)}\left\{\int_r^b r^{k-1} \exp\left(Fr^{n+2}\right)dr\right\}\right]$$

(21)

$$T_{\theta\theta} = \left[\frac{E(2-C)}{n(3-2C)a^k \exp\left(Fa^{n+2}\right)}\left\{\begin{array}{l}\int_r^b r^{k-1} \exp\left(Fr^{n+2}\right)dr \\ - r^k \exp\left(Fr^{n+2}\right)\end{array}\right\}\right]$$

(22)

$$u = r - r\left[\frac{r^k \exp\left(Fr^{n+2}\right)}{a^k \exp\left(Fa^{n+2}\right)}\right]^{\frac{1}{n}} \tag{23}$$

The following nondimensional components are introduced as

$$R = \frac{r}{b}, R_0 = \frac{a}{b}, \sigma_r = \frac{T_{rr}}{E}, \sigma_\theta = \frac{T_{\theta\theta}}{E}, \Omega^2 = \frac{\rho\omega^2 b^2}{E}$$

and $\bar{u} = \dfrac{u}{b}$.

Equations (21) to (23) in nondimensional form become:

(24)

$$\sigma_\theta = \left\{ \begin{aligned} &\left[\frac{(2-C)}{n(3-2C)R_0^k \exp\left(F_1 R_0^k\right)} \left\{ \begin{aligned} &\int_R^1 R^{k-1} \exp\left(F_1 R^{n+2}\right) dR \\ &- R^k \exp\left(F_1 R^{n+2}\right) \end{aligned} \right\} \right] \\ &+ \frac{\Omega^2}{2}\left(1 - R^2\right) \end{aligned} \right\} \tag{25}$$

$$\bar{u} = R - R\left[\frac{R^k \exp\left(F_1 R^{n+2}\right)}{R_0^k \exp\left(F_1 R_0^{n+2}\right)}\right]^{\frac{1}{n}} \tag{26}$$

where $F_1 = -\dfrac{n\Omega^2 (3-2C) b^n}{(2-C)^2 D^n (n+2)}$

and $k = -\left[\dfrac{n(3-2C)+1}{(2-C)}\right]$

For a disc made of incompressible material (C $\rightarrow$ 0), Eqns (24) to (26) become

(27)

(28)

$$\bar{u} = R - R\left[\frac{R^{k_1} \exp\left(F_2 R^{n+2}\right)}{R_0^{k_1} \exp\left(F_2 R_0^{n+2}\right)}\right]^{\frac{1}{n}} \tag{29}$$

where and

4. STRAIN RATES

When creep sets in, the strains should be replaced by strain rate. The stress-strain relations [Eqn (5)] become

$$\dot{e}_{ij} = \frac{1+\nu}{E} T_{ij} - \frac{\nu}{E} \delta_{ij} \Theta \tag{30}$$

where $\dot{e}_{ij}$ is the strain rate tensor wrt flow parameter t and $\Theta = T_{11} + T_{22} + T_{33}$.

Differentiating Eqn (4) wrt t, one gets

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta} \tag{31}$$

For SWAINGER measure (n = 1), one has from Eqn (4.2)

$$\dot{\varepsilon}_{\theta\theta} = \dot{\beta} \tag{32}$$

The transition value of Eqn (12) at $P \rightarrow -1$, gives

$$\beta = \left[\frac{n(3-2C)}{(2-C)}\right]^{\frac{1}{n}} \left(\sigma_r - \sigma_\theta\right)^{\frac{1}{n}} \tag{33}$$

Using Eqns (31), (32) and (33) in Eqn (30), one gets

$$\dot{\varepsilon}_{rr} = \left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{(2-C)}\right]^{\frac{1}{n}-1} \left(\sigma_r - \nu\sigma_\theta\right),$$

$$\dot{\varepsilon}_{\theta\theta} = \left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{(2-C)}\right]^{\frac{1}{n}-1} \left(\sigma_\theta - \nu\sigma_r\right),$$

$$\dot{\varepsilon}_{zz} = -\left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{(2-C)}\right]^{\frac{1}{n}-1}\left[\nu(\sigma_r + \sigma_\theta)\right] \tag{34}$$

For incompressible material ($C \rightarrow 0$), Eqn (34) becomes

$$\dot{\varepsilon}_{rr} = \left[\frac{3n(\sigma_r - \sigma_\theta)}{2}\right]^{\frac{1}{n}-1}\left(\frac{2\sigma_r - \sigma_\theta}{2}\right)$$

$$\dot{\varepsilon}_{\theta\theta} = \left[\frac{3n(\sigma_r - \sigma_\theta)}{2}\right]^{\frac{1}{n}-1}\left(\frac{2\sigma_\theta - \sigma_r}{2}\right)$$

$$\dot{\varepsilon}_{zz} = -\left[\frac{3n(\sigma_r - \sigma_\theta)}{2}\right]^{\frac{1}{n}-1}\left[\frac{1}{2}(\sigma_r + \sigma_\theta)\right] \tag{35}$$

These constitutive equations are the same as obtained by Odquist[6] provided one takes $n = 1/N$.

5. DISCUSSION

For calculating stresses, strain rates and displacement based on the above analysis, the following values have been taken

$$\Omega^2 = \frac{\rho\omega^2 b^2}{E} = 50,\ 75\ ;$$

C = 0.00, 0.25, 0.5;

n = 1/3, 1/5, 1/7 (i.e. N = 3, 5, 7) and D = 1.

In classical theory, measure N is equal to $1/n$. Definite integrals in the Eqns (24) to (25) have been solved using Simpson's rule.

Curves have been drawn in Figs 2(a) to Fig 2(c) between stresses and radii ratio $R = r/b$ for a rotating disc made of compressible/ incompressible material at different angular speeds. It is seen from Figs 2(a) to Fig 2(c) that the radial stress has maximum value at the internal surface of disc as compare to circumferential stress. It is also observed

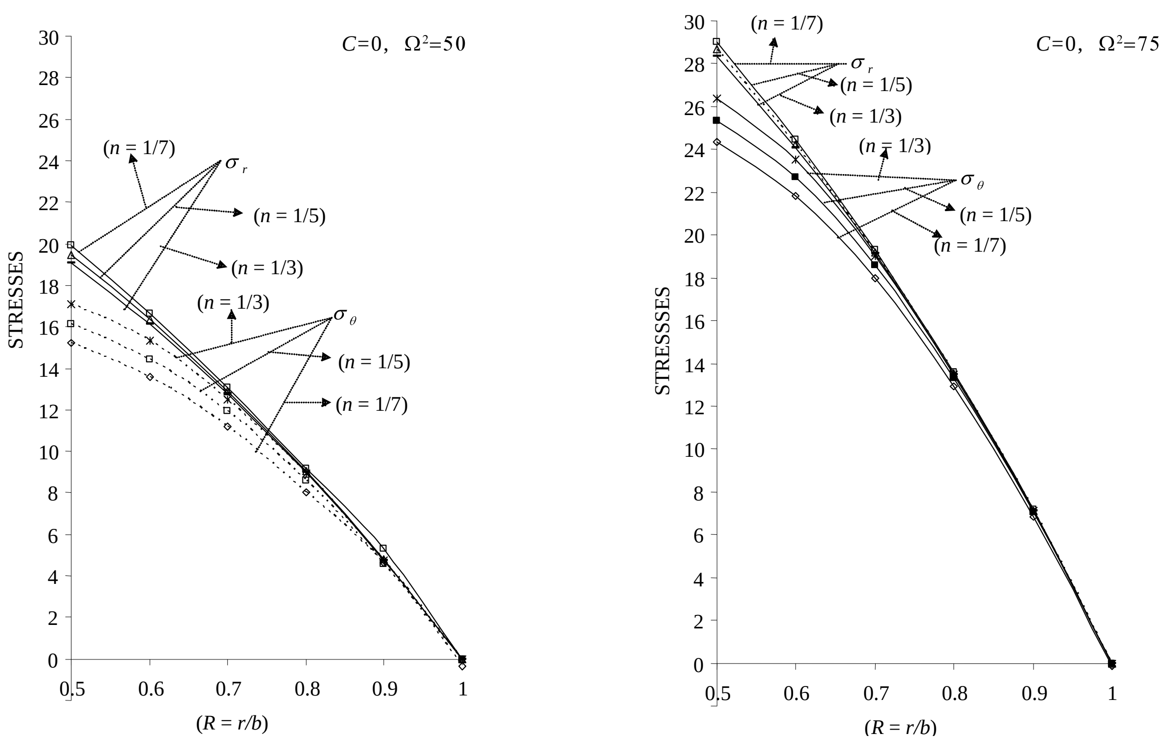

Figure 2(a). Creep stresses in a thin rotating disc with inclusion for incompressible material at different angular speeds along the radius ($R = r/b$).

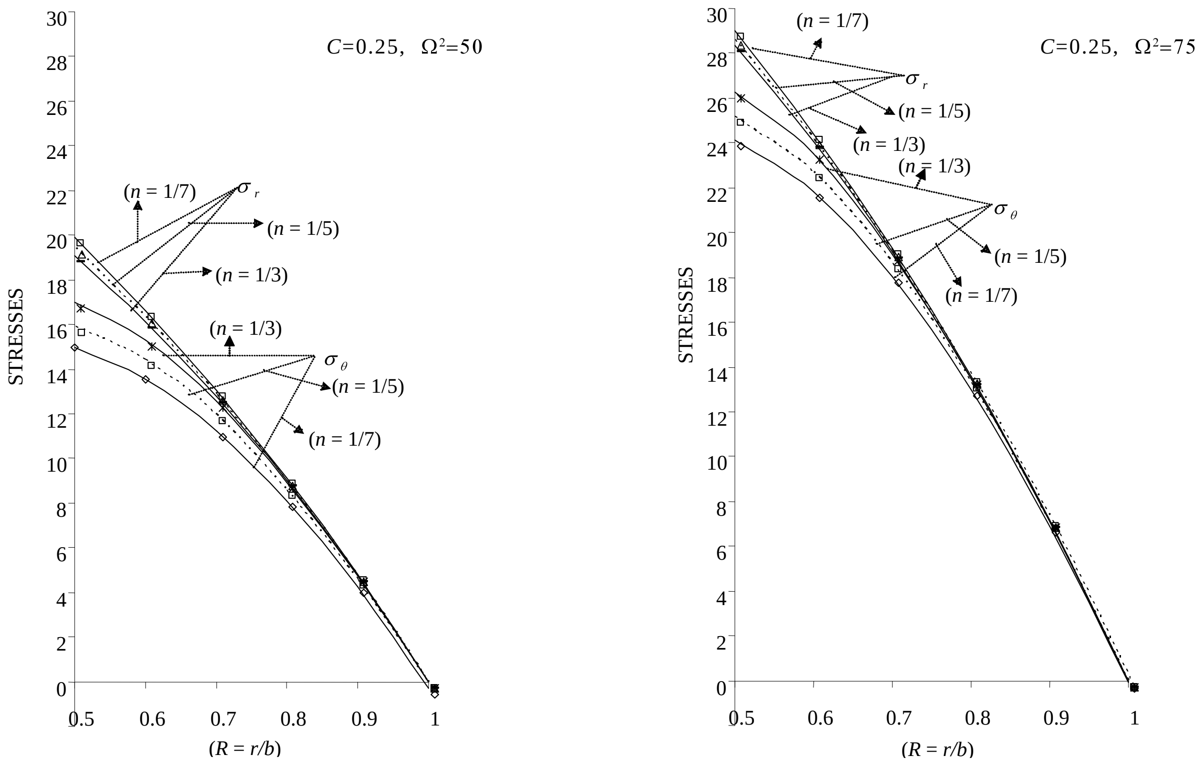

Figure 2(b). Creep stresses in a thin rotating disc with inclusion for compressible material at different angular speeds along the radius ($R = r/b$).

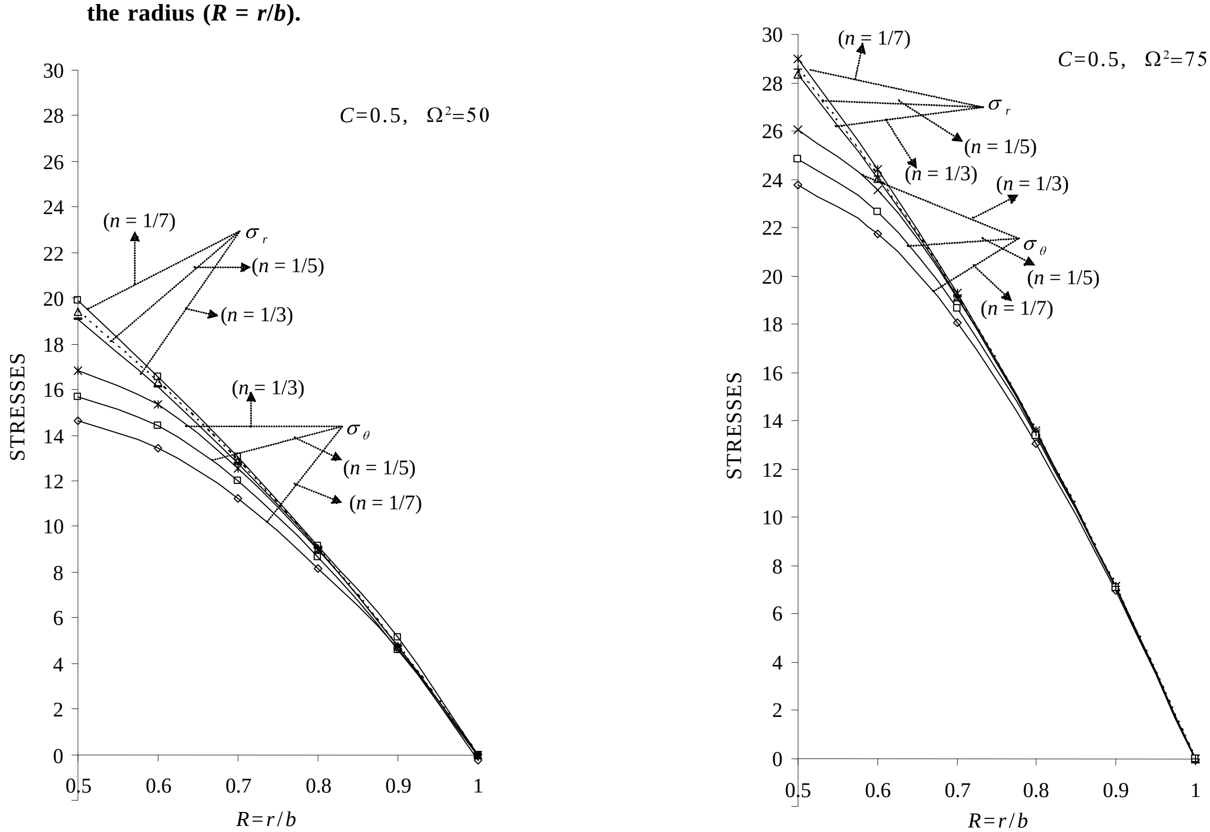

Figure 2(c). Creep stresses in a thin rotating disc with inclusion for compressible material at different angular speeds along the radius ($R = r/b$).

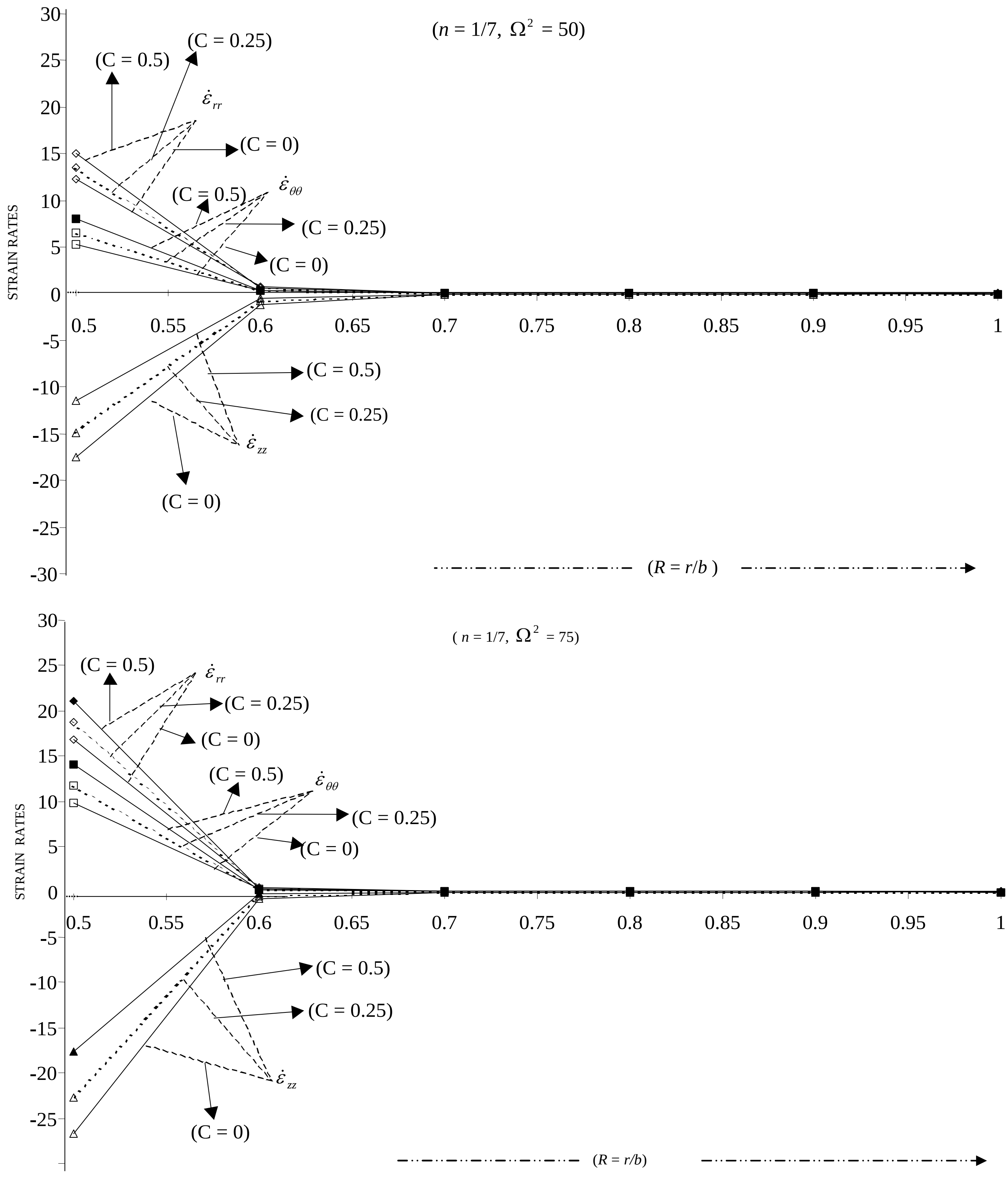

Figure 3(a). Strain rates distribution in a thin rotating disc with inclusion for measure n = 1/7 at different angular speeds = 50, 75 along the radius ($R = r/b$).

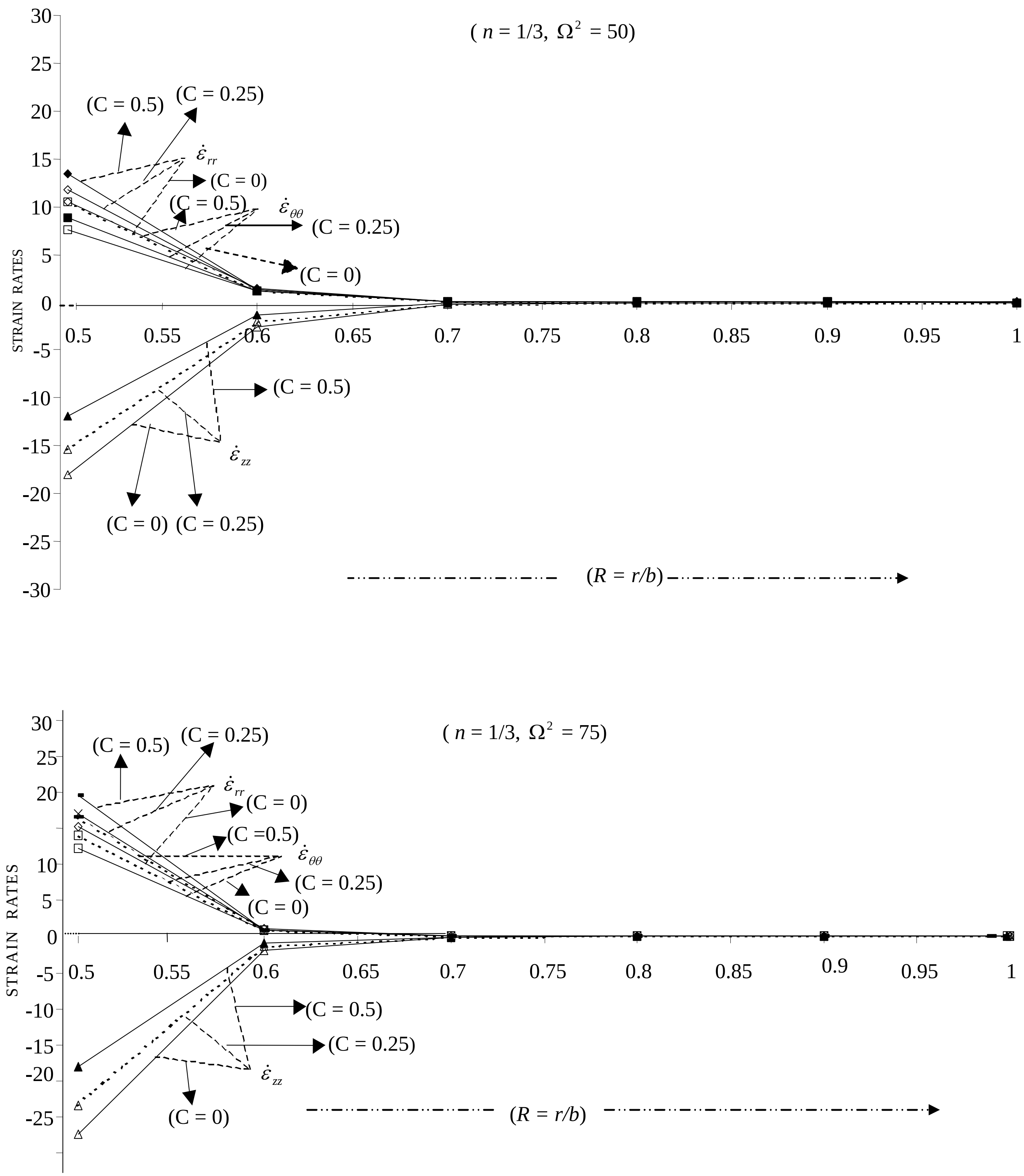

Figure 3(b). Strain rate distribution in a thin rotating disc with inclusion for measure n = 1/3 at different angular speeds = 50, 75 along the radius (R = r/b).

that the radial stress has maximum value at the internal surface of the rotating disc with inclusion made of incompressible material as compare to compressible material for measure n = 1/7 or (N = 7) at angular speed Ω^2= 50, whereas circumferential stress is maximum at the internal surface for measure n = 1/3 or (N = 3) at this angular speed. The values of radial/circumferential stress further increases at the internal surface with the increase in angular speed (Ω^2 = 75) for measure n = 1/7 or (N = 7) and n = 1/3 or (N = 3), respectively.

Curves have been drawn in Figs 3(a) and 3(b) between strain rates and radius $R = r/b$ at angular speed Ω^2= 50, 75 and measures n = 1/7, 1/3 or (N =7, 3). It has been seen from Figs 3(a) and 3(b) that rotating disc made of compressible material has maximum value at the internal surface as compared to incompressible material for measure n = 1/7 or (N = 7) and n =1/3 or (N = 3) at angular speed Ω^2= 50. The values of strain rates further increases at the internal surface with the increase in angular speed Ω^2= 75 for measure n = 1/7 or (N = 7) and 1/3 or (N = 3), respectively.

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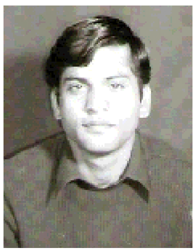

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