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SHORT COMMUNICATION

Mathematical Model to Simulate the Trajectory Elements of an Artillery Projectile Proof Shot

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ABSTRACT

In external ballistics of a conventional spin-stabilised artillery projectile, there are a number of trajectory models developed for computing trajectory elements having varying degrees of complexity. The present study attempts to propose a single mathematical model, viz., simplified point-mass/simple particle trajectory model to simulate the trajectory elements of a typical spin-stabilised flat-head artillery projectile proof shot. Due to difficulties in the projectile shape and size, and the complicated nature of air resistance, an accurate mathematical prediction of the trajectory is difficult. To simplify the computations, the governing equations of motion of the projectile have been simplified and assumed that the projectile is a particle and the only forces acting on the projectile are drag and gravity. With this model, trajectory elements have been generated and compared with experimental results obtained in the field test. The measuring instrument used in this case is a Doppler radar.

Keywords: Simulation, artillery, projectile proof shot, trajectory models, range table, trajectory elements, drag force, mathematical model

NOMENCLATURE

ρ	Average density of the air
S	Reference area/maximum cross-sectional area of the projectile = $\pi r^2 = \pi d^2/4$
m	Mass of the projectile
l	Length of the projectile
d	Diameter of the projectile
C_D	Drag coefficient (dimensionless number)
g	Acceleration due to gravity at sea level
h	Step size time
R	Horizontal range of the projectile
V	Velocity of the projectile wrt the ground coordinate system at any time t
V_0	Initial/muzzle velocity of the projectile
u	Horizontal component of the velocity
v	Vertical component of the velocity
θ	Angle of inclination of trajectory to horizontal at time t
θ_0	Initial angle of inclination of trajectory
a	Velocity of sound at sea level
ε	Angle of sight (= $\tan^{-1}(y/x)$)
ω	Angle of fall
α	Angle of yaw

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x, y — Coordinates of the CG of the projectile at time t

x_0, y_0 — Coordinates of the origin of the trajectory, i.e., at $t = 0$

X — Range

Y — Maximum vertex altitude/height

Ω — Point of fall, i.e., the end-point of trajectory

P — Position of the projectile at any time t

t — Flight time to any point along the trajectory from (x_0,y_0)

T — Total time of flight

D — Drag force on the projectile = $\frac{1}{2} \rho S V^2 C_D$

D_{ft} — Drift of the projectile = $K.t^2.\cos \varepsilon$, where K is a constant

1. INTRODUCTION

External ballistics deals with the part of motion of the projectile through the external medium and its behaviour during flight, i.e., from the muzzle of the weapon to the impact or burst point. There are a number of trajectory models like point-mass (PM), modified point-mass (MPM) and six degrees-of-freedom (6-DOFs) which have been developed for computing trajectory elements and each of which has varying degree of complexity[1,2]. Computation of trajectory elements and generation of range tables (RTs) for conventional artillery projectiles are some of the essential tasks in many theoretical and practical applications. The principal problem of external ballistics is the computation of the trajectory traced by the centre of gravity (CG) of the projectile, along with the prediction of the expected point of impact with given characteristics, such as initial/muzzle velocity, angle of projection/inclination, and also along with the prediction of associated quantities such as range, deviation, time-of-flight (TOF), angle of fall, velocity at impact, vertical height and its different probable errors.

Though the general equations of motion of a symmetrical projectile are quite complicated, however for computational work, it is found that the solution of a relatively simple system of equations gives an excellent approximation to the actual motion of a projectile[3]. The trajectory elements of the projectiles like shells, bombs, and unguided missiles are estimated from the respective range table. But in case of any typical nonconventional artillery flat-head projectile like proof shot, there are no range tables readily available to estimate the trajectory elements due to its shape and size. The shot is required to be dynamically evaluated for its strength of design by post-firing observation. The impact points are difficult for their location in the absence of range tables. So, there is a need for theoretical estimation of suitable range location for safe deployment of recovery and observatory teams and easy recovery of the shots after the firing. Also, the trajectory behaviour of a flat-head projectile does not match with the conventional artillery projectiles.

The objective of the present study is to propose a single mathematical model to estimate the trajectory elements of a typical 105 mm axi-symmetric flat-head proof shot for the generation of appropriate range table parameters for regular firing, using concept of simplified point-mass model, which is validated through the Doppler DR-5000 measurements. A sketch of a typical 105 mm projectile shot is shown in Fig. 1.

In this study, the influence parameters such as muzzle velocity, angle of inclination, drag coefficient, atmospheric conditions, and projectile shape and size have been considered for estimation of trajectory elements. The trajectory is considered as stable and it describes the position of the mass centre of the projectile as function of time and external forces. As initial elevation angle is high, it results into a

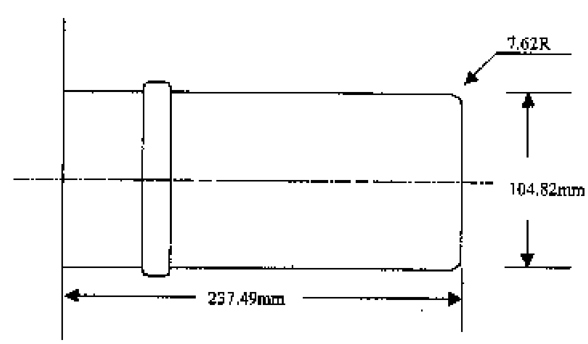


Figure 1. A typical 105 mm artillery projectile proof shot.

more drift, and a parabolic trajectory. In this case, the predominant aerodynamic force, besides the force of gravity, is an axial drag force that acts in the direction of the longitudinal axis (opposite of the projectile axis) and opposes to projectile movement, which has a profound effect on the range performance of a projectile. The components of air resistance is the crosswind force in directions at right angles to the drag force due to the yaw, which cause a drift to the right. Yaw is the angle between the longitudinal axis of the projectile and the tangent to the trajectory. Drift is a component or part of the deviation not due to the wind, which is assumed as the product of square of time of flight and cosine of angle of sight, and estimated experimentally as a variation from the plane (2-D) trajectory. But the effect of the rotation of the projectile, in any case is difficult to account for, so it is ignored[4-6].

The resulting governing planer equations of motion of the mass centre of the projectile were written using Newton's second law of motion and then the numerical integration was carried out using fourth-order Runge-Kutta (R-K) algorithm. Results of these simulations have been compared with the measurements obtained in field tests through a Doppler radar (DR-5000) for a few seconds time of flight (up to 10 s). The results have been found in reasonably good agreement with the measured values and a sensitivity analysis was performed by varying different parameters. The advantages of the point-mass model is that it requires only small amount of data and the governing planer equation of motion can be solved fast using computer.

2. MATHEMATICAL MODEL

2.1 Assumptions

Because of lack of reliable experimental aerodynamic coefficients data, it is difficult to directly estimate the trajectory elements of a proof shot. Therefore, a simplified 2-D point-mass model ($\rho/d < 3$) was considered. Following are the assumptions considered to setup the model[1-3]:

- The earth is flat
- The projectile is in planer motion, i.e., non-rolling axes.
- There is no wind speed.
- The rotation of the earth is ignored.
- The drag force is proportional to the square of the instantenous velocity.
- The drag coefficient and density of air are assumed to be constant during firing.
- The force of gravity is independent of the altitude, i.e., the gravity force is constant.
- All types of forces, like centrifugal, coriolis, magnus forces and its cross-effects are negligible.
- The drift is proportional to the product of square of time of flight and cosine of angle of sight.

2.2 Governing Equations of Motion

With the simplifying assumptions mentioned above, the projectile is regarded as a material particle acted by the force of gravity and by tangential retarding force due to the resistance of air. The acceleration due to gravity is considered to be constant in magnitude and direction, which means that the earth is taken to be flat and the height reached by the projectile is small compared to the radius of the earth[2,5]. Figure 2 shows the references system and the external forces that act on the projectile during the flight. Figure 3 shows the range and deviation of a projectile for a typical flight situation.

A mathematical model with 3-DOFs, viz., simplified point-mass/simple particle trajectory model was proposed to simulate a planer projectile trajectory. It was assumed that the trajectory has high initial angle of inclination so that the projectile is stabilised by rotation and there are no forces acting outside the plane of the figure.

Therefore, the trajectory problem consists in the numerical integration of equations of motion the projectile to find the velocity and the position of mass centre at each instant. A parabolic trajectory and the variation of the density of the air with altitude were assumed to be constant for a few

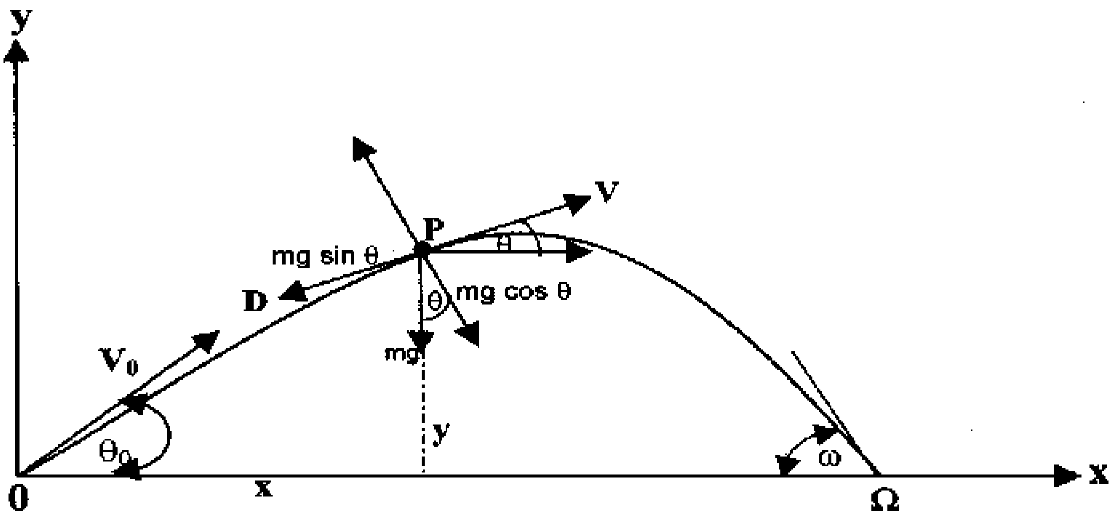


Figure 2. References system and external forces acting on a projectile for a typical flight.

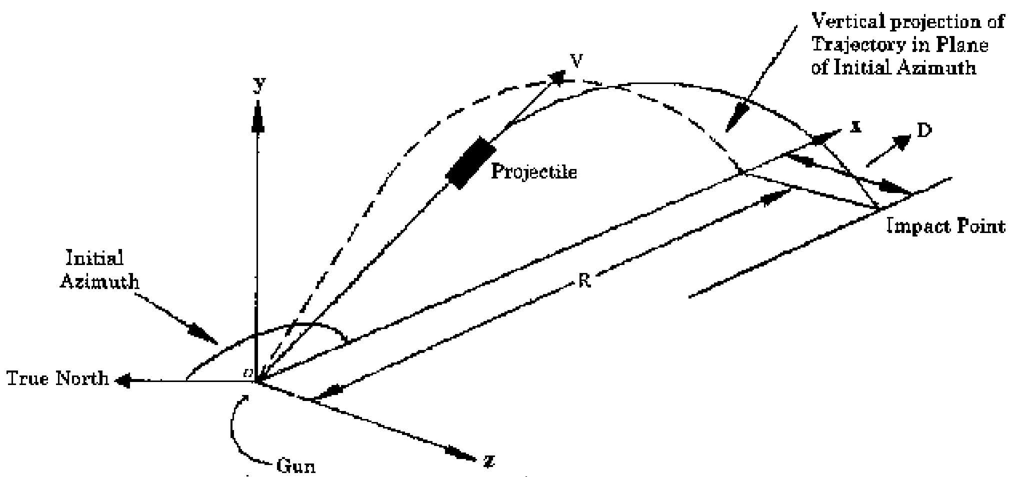


Figure 3. Range and deviation of a projectile for a typical flight.

seconds time of flight. Also, earth rotation can be neglected without loss of precision.

Let X-axis be taken horizontally in the direction of firing and Y-axis vertically upwards through the point of projection. Using the procedure–resolutions along the tangent and normal to the trajectory, and applying Newton's second law of motion and neglecting the rotation of the projectile in the plane of motion, the system of differential equations for the motion of the centre of mass of the projectile in air was obtained in terms of the projectile velocity V and θ, the inclination of the tangent to the trajectory to the horizontal and which can be put in the following scalar form[2,4,7]:

$$m \times dV/dt + D + mg\sin\theta = 0$$

$$mV \times d\theta/dt + mg\cos\theta = 0$$

$$dx/dt - V\cos\theta = 0 \qquad (1)$$

$$dy/dt - V\sin\theta = 0$$

subject to initial conditions of position and velocity.

The above equations were solved subject to the following initial conditions:

$$u = dx/dt = V_0\cos\theta_0, v = dy/dt = V_0\sin\theta_0,$$
$$V = V_o, \quad x = y = 0 \quad \text{at} \quad t = 0 \qquad (2)$$

The drag coefficient C_D in $D = \rho S C_D V^2/2$ usually well approximated[2] by: $C_D = C_{D_0} + C_{D_{\alpha^2}} \alpha^2$, where C_{D_0} = zero yaw drag coefficient and $C_{D_{\alpha^2}}$ = yaw drag coefficient and was assumed to be constant and estimated from the curve of the 1940 Resistance law[2,7,8].

Similarly, the drift $D_{ft} = K.t^2.\cos\varepsilon$, where the constant K depends on the shape and size of the projectile and can be approximated with the value of C_D. The angle of sight (ε) was computed from tan ε = (y/x) at each instant of position (x, y) at time t.

As per standard atmosphere structure, density ρ was assumed to be an exponential function of the altitude and it approximates the average density structure and simplifies the computation of trajectories. The average density is reasonably good for low-level firing. It has been the practice in ballistic work, the fundamental standard structure in the following form[9]: $\rho = \rho_0 e^{-hy} = \rho_0 e^{-0.0001036y}$, where ρ_o= density at mean sea level. In this study, ρ was assumed to be constant during testing to simplify the computation of the model.

The above set of the equations, in general, are not analytically solvable since the system of equations is nonlinear due to the drag dependency on the square of the velocity and presence of angle θ as argument of trigonometric functions and so, to Runge-Kutta (R-K) algorithm[9,10] of fourth-order with time step of 10^{-2} s was resorted to. Then, for the sake of computations, the above system of equations[7] can be reduced to:

$$
\begin{aligned}
f_1 &= dV/dt = -(D/m) - g\sin\theta \\
f_2 &= d\theta/dt = -g\cos\theta/V \\
f_3 &= dx/dt = V\cos\theta \\
f_4 &= dy/dt = V\sin\theta
\end{aligned}
\tag{3}
$$

Given V_i, θ_i, x_i, and y_i, i.e., the values of V, θ, x, and y at the i^{th} stage to get the values of V_{i+1}, θ_{i+1}, x_{i+1}, and y_{i+1}. This process was repeated till the value of y_n (for some n) became negative (i.e., till the projectile reaches the ground).

3. SIMULATION MODEL AND FLOW CHART

The relatively simplified point-mass model/simple particle trajectory model assumes that the only forces on the projectile are drag and gravity. The change of state of the system is described using the simple physical laws governing motion under acceleration. The horizontal and vertical accelerations due to these forces were computed at successive points in time, and the resulting horizontal and vertical components of the projectile's velocity and position were computed for each time step. The initial states of the simulation will be the initial coordinates and velocities. If the time interval is small enough, the simulation of the trajectory can be accurate.

The fundamental equations underlying the computer program[7] presented below are:

$$
\begin{aligned}
\Delta V &= (-(D/m) - g\sin\theta)\Delta t \\
\Delta\theta &= (-g\cos\theta/V)\Delta t \\
\Delta x &= (V\cos\theta)\Delta t \\
\Delta y &= (V\sin\theta)\Delta t
\end{aligned}
\tag{4}
$$

The Runge-Kutta (R-K) algorithm of fourth-order techniques was used to improve the accuracy of the simulation. The time step for the simulation taken was 10^{-2} s. A flow chart for trajectory simulation is shown in Fig. 4.

4. RESULTS & DISCUSSION

A test firing was conducted to measure the basic parameters for estimation of the trajectory elements considering five typical 105 mm flat-head proof shots. The shots were fired at fixed charge mass (full charge) with different ranges of muzzle velocities and elevations. The projectile and other physical test data used in the simulation is given in Table 1.

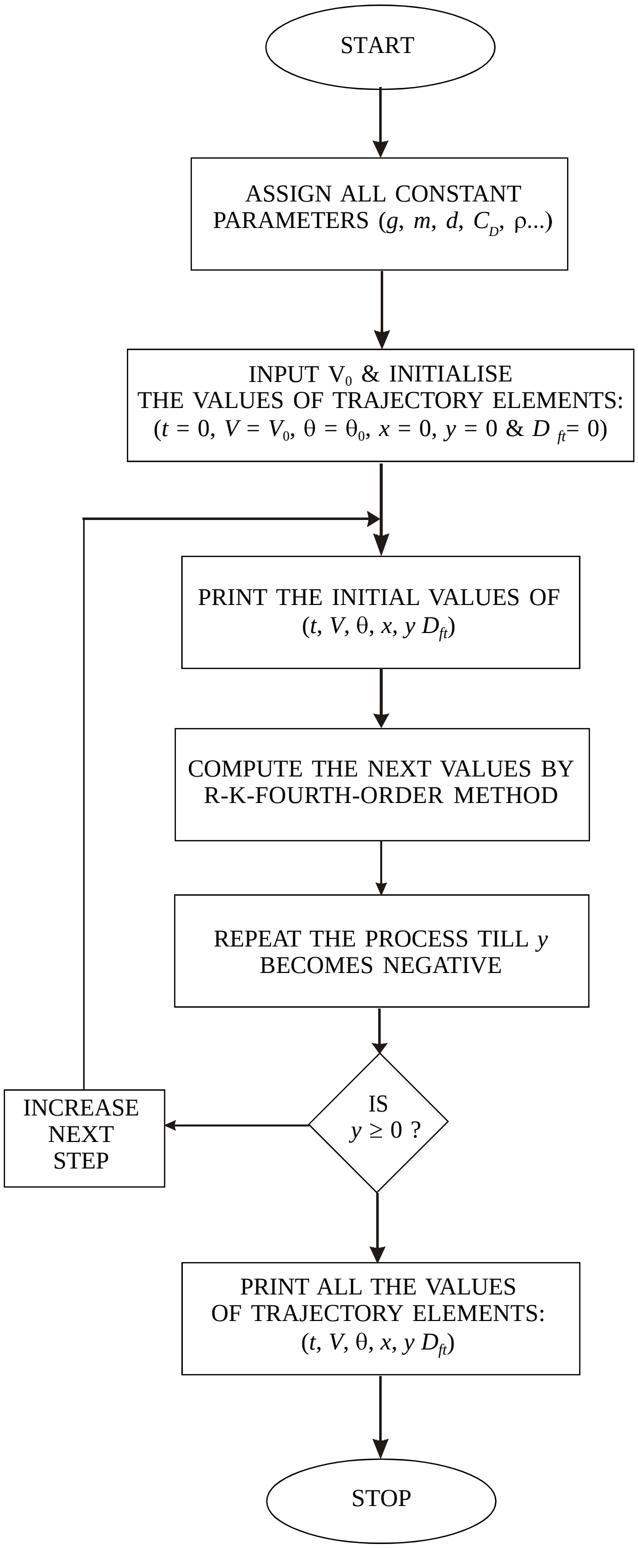


Figure 4. Flow chart for trajectory simulation.

Table 1. Physical parameters

Parameters	Numerical Values
Mass (m)	(16.556 ± 0.005) kg
Length(l)	237.49 mm
Calibre (d)	104.82 mm
Drag coefficient (C_D)	0.65
Constant(K)	0.65
Density of air (ρ)	1.225 kg/m^3
Acceleration due to gravity (g)	9.80665 m/s^2
Velocity of sound (a)	340.3 m/s
Wind velocity (W)	$W_x = W_y = W_z$ = 0.0 m/s
Ground air temperature (T)	Ambient
Ground barometric pressure (Pr)	1 atm
Initial / muzzle velocity (V_0)	(740.0 ± 5.0) m/s

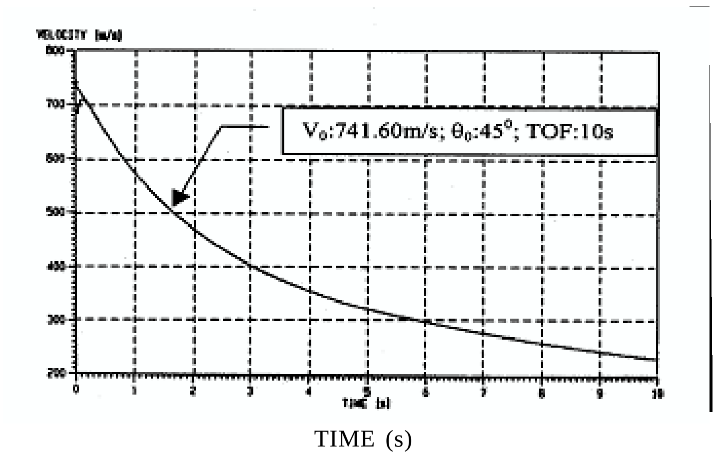


Figure 5. Velocity versuss time plot for shot-1 (S-1) by measurement.

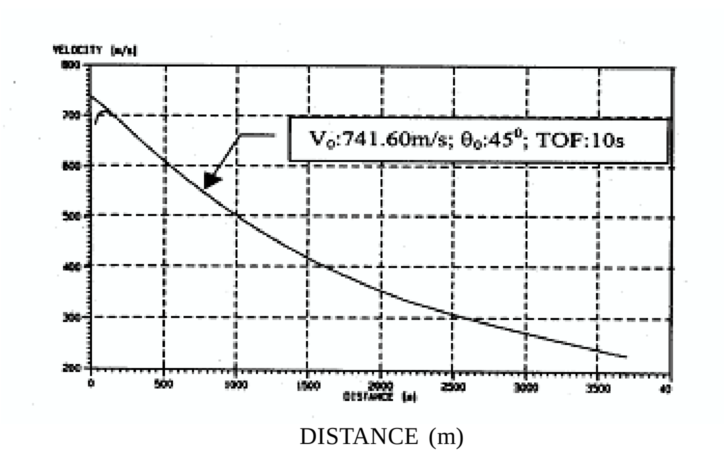


Figure 6. Velocity versus distance plot for shot-1 (S-1) by measurement.

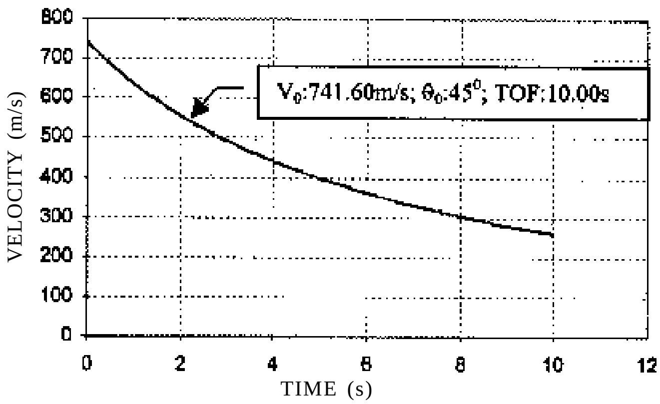


Figure 7. Velocity versus time plot for shot-1 (S-1) by computation.

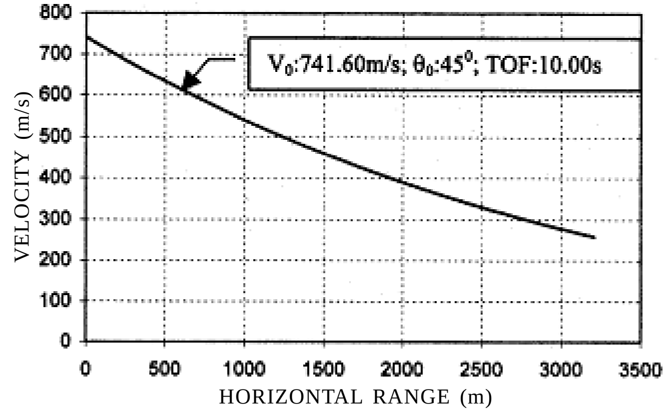


Figure 8. Velocity versus range plot for shot-1 (S-1) by computation.

A comparison of output for five proof shots: S-1 to S-5 is given in the Table 2. Figures 5 to 8 (TOF:10.0 s) show the graphs of the velocity versus time, and velocity versus range profiles for the measurements and numerical simulation results corresponding to the first shot (S-1). Similarly, Figs 9 to 13 (TOF: 53.06 s) show the various parameters corresponding to the shot (S-1) based on simulation results. Also, Figs 14 to 16 depict the various graphs of range versus altitude, range versus time, and drift versus time profiles corresponding to shots S-1, S-4, and S-5. A comparison of typical results generated by the simulation and measurements given above is also presented in Table 2. The sub-column three of the columns five and six of Table 2 show percentage deviations in simulated range considering measured range as standard. The mean deviations between the simulated and the measured range and deviations in drift between simulation and measurement are of the order of less than 5 per cent in both the cases. From the above results one can find that with higher angle of elevation,

Table 2. Numerical simulation comparison with measured values

Muzzle velocity (*MV*) (m/s)	Initial angle of elevation (θ^O)	Time of flight (s)		Range (m)			Drift (m)		
		NS	M	NS	M	Accuracy (%)	NS	M	Accuracy (%)
741.60	45.0	53.06	55.16	7997.32	7973.4	-0.30	1406.54	1356.3	-03.70
737.40	45.0	52.93	54.10	7972.93	7715.0	-3.34	1400.26	1314.9	-06.49
736.56	45.0	52.91	53.25	7968.03	7480.7	-6.51	1398.99	1304.2	-07.27
741.42	40.0	49.20	51.68	8295.12	7925.7	-4.66	1209.79	1189.2	-01.73
747.30	30.0	40.73	39.84	8511.56	7989.3	-6.54	0829.08	0806.3	-02.82

Legends: NS – Numerical Simulation result; M – Measured values; Accuracy- Accuracy level wrt measured values M.

Table 3. Comparison of different safety zones (simulated versus computed)

Zone	Range (R)		Deviation (D)	
	Simulated (m)	Computed (m)	Simulated (m)	Computed (m)
50 per cent PE zone	330.13	293.60	335.94	304.23
100 per cent PE zone	1320.51	1174.38	1343.76	1216.90
Absolute safety zone (ASZ)	2641.02	2348.77	2687.53	2433.81
Normal safety zone (NSZ)	1848.71	1644.14	1881.27	1703.67

$\bar{x}_R(S) = 8148.99m$; $\sigma_R(S) = 244.72m$; $\bar{x}_D(S) = 1248.93m$; $\sigma_D(S) = 249.03m$; S= Simulated;
$\bar{x}_R(C) = 7816.82m$; $\sigma_R(C) = 217.64m$; $\bar{x}_D(C) = 1194.14m$; $\sigma_D(C) = 225.52m$; C=Computed;

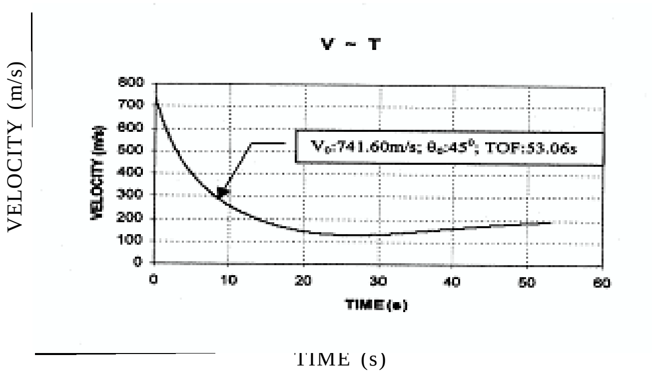


Figure 9. Velocity versus time plot for shot-1 (S-1) by computation.

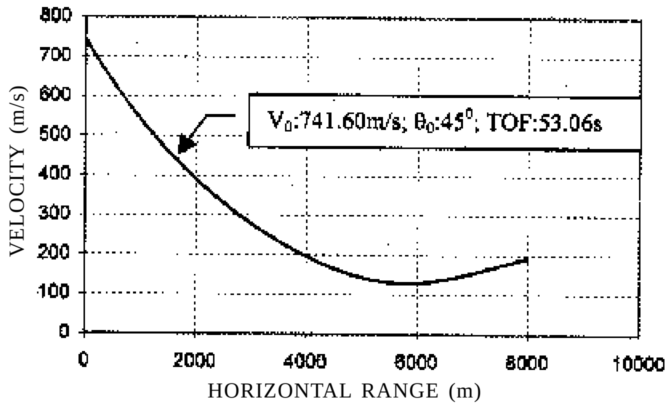


Figure 10. Velocity versus range plot for shot-1 (S-1) by computation.

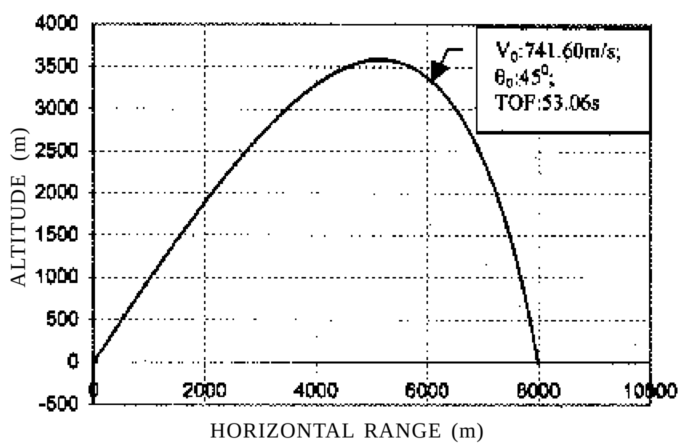


Figure 11. Range versus altitude plot for shot-1 (S-1) by computation.

the range achieves less, which is in agreement with the measured values. As 50 per cent probabble error (PE) zone forms an important part of RT, so the various safety distances like 100 per cent, absolute safety zone (ASZ) and normal safety zone (NSZ) have been evaluated through 50 per cent

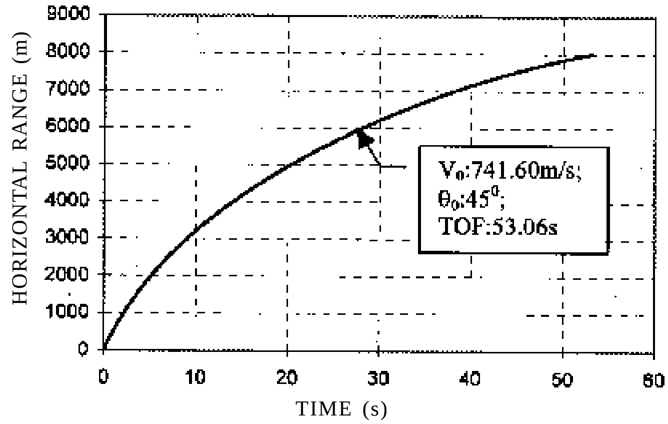


Figure 12. Range versus time plot for shot-1 (S-1) by computation.

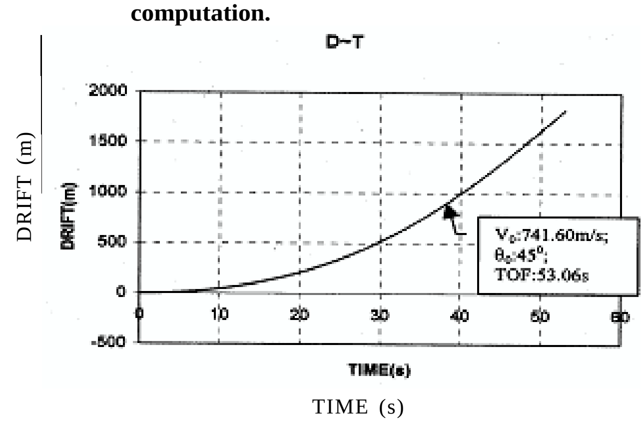
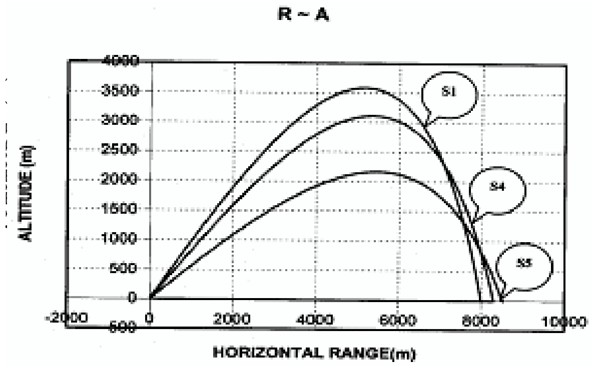


Figure 13. Drift versus time plot for shot-1 (S-1) by computation.

zone. Table 3 shows a comparison of different safety zones of range and deviation.

5. CONCLUSIONS

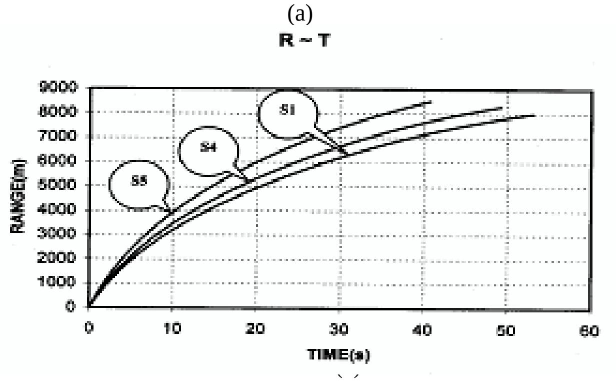
In this study, the basic objective was to estimate the various safety zones of the impact points of the shots by post-firing in the proof range in terms of the important trajectory parameters for range safety and recovery purposes. Therefore, a simplified point-mass/simple particle trajectory model has been proposed for estimating the trajectory elements of a spin-stabilised 105 mm artillery projectile proof shot as per mentioned assumptions. The model is useful for computing trajectory elements like range, deviation, time-of-flight, angle of fall, and safety zones, etc. The results generated from simulations are compared with measured values obtained in field tests through a Doppler radar (DR-5000). Computations of trajectory elements has been carried out in velocity range: (740.0 ± 5.0) m/s and elevation ranges: 30° to 45°.



HORIZONTAL RANGE (m)

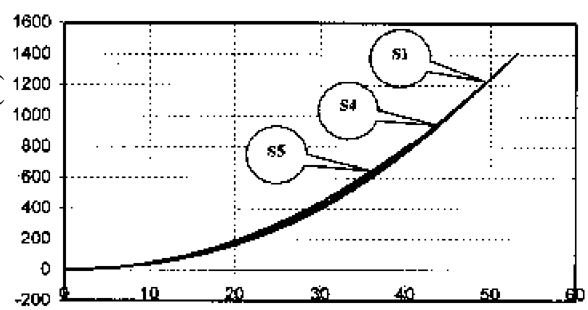
(a)

RANGE (m)



(b)

DRIFT (m)



TIME (s)

(c)

Figure 14. Various plots for shots-1, 4 and 5 by computation: (a) range *vs* altitude, (b) range vs time, and (c) drift vs time. S-1: V_0=741.60 m/s; $?_0$=45° S-4: V_0=741.42 m/s; $?_0$=40° S-5 : V_0= 747.30 m/s; $?_0$=30°

The results are encouraging and reasonably in good agreement with the measured values. The advantage of this model is that it requires less amount of data and can be easily computed. The accuracy of this model can be further improved using the concept of point mass model (PM)/modified point-mass model (MPM) to compute yaw and its rate for drift computation by incorporating the effect of wind velocity.

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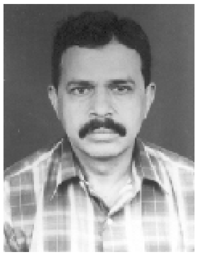

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