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Elastic-plastic Transition of Transversely Isotropic Thick-walled Rotating Cylinder under Internal Pressure

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ABSTRACT

Elastic-plastic stresses for a transversely isotropic thick-walled rotating cylinder under internal pressure have been obtained by using Seth's transition theory. It has been observed that a thick-walled circular cylinder made of isotropic material yields at the internal surface at a high pressure as compared to cylinder made of transversely isotropic material. With the increase in angular speed, much less pressure is required for initial yielding at the internal surface for transversely isotropic material as compared to isotropic material. For fully-plastic state, circumferential stress is maximum at the external surface. Thick-walled circular cylinder made of transversely isotropic material requires high percentage increase in pressure to become fully plastic as compared to isotropic cylinder. Therefore, circular cylinder made of transversely isotropic material is on the safer side of the design as compared to cylinder made of isotropic material.

Keywords: Elastic-plastic transition, transversely isotropic cylinder, isotropic, rotating cylinder.

NOMENCLATURE

a, b	Internal and external radii of disc
D	a constant
R	Radial distance
u,v,w	Displacement components
x,y,z	Cartesian coordinates
r,θ,z	Cylindrical polar coordinates
e_{ij},T_{ij}	Strain and stress tensor
e_{ii}	First strain invariant
β	Function of r only
Y	Yield stress
P	Function of βonly
C_{ij}	Material constants
λ,μ	Lame's constants
$R=(r/b)$	$R_0=(a/b)$
$\Omega^2=\rho b^2\omega^2/Y$,	
$\sigma_r=(T_{rr}/Y)$	Radial stress component
$\sigma_\theta=(T_{\theta\theta}/Y)$	Circumferential stress component
$\sigma_z=(T_{zz}/Y)$	Axial stress component

1. INTRODUCTION

The constantly increasing demand for axisymmetrical cylindrical and spherical components or elements of these by the industry has drawn the attention of designers and scientists on this particular area of activity. The progressive, worldwide scarcity of materials, combined with their consequently higher cost, makes it increasingly less attractive to confine design to the customary elastic regime only. Thick-walled cylinders of circular cross-section are commonly used either as pressure vessels intended for storage of industrial gases or as medium for transportation of high pressurised fluids. A thick-walled cylinder is also widely used as a structural component in oil refineries, power industries, atomic power plants, etc. Problems of thick-walled cylinder under internal pressure have been analysed[1-4] for isotropic homogeneous elastic-plastic states. In their treatment, the following assumptions were made:

(i) the incompressibility conditions,
(ii) the deformation is small enough to make infinitesimal strain theory applicable,and
(iii) the yield criterion,

Incompressibility of the material in plasticity is one of the most important assumption that simplifies the problem. Infact, in most of the cases, it is not possible to find a solution in closed form without this assumption. Transition theory[4] does not require any of the above assumptions, and thus solves a more general problem. This theory utilizes the concept of generalised strain measure, which not only gives the well-known strain measures, but can also be used to find the stresses in plasticity and creep problems by determining the asymptotic solution at the transition points of the governing differential equations.

In this paper, elastic-plastic stresses for a transversely isotropic thick-walled rotating cylinder under internal pressure

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have been obtained using transition theory. Results obtained have been discussed numerically and depicted graphically.

2. GOVERNING EQUATIONS

Consider a thick-walled circular cylinder of internal and external radii a and b, subjected to internal pressure applied at the internal surface and rotating with an angular velocity ω.

The generalized principal components of strain[4] is defined as,

$$e_{ii} = \int_0^{e_{ii}^A} [1-2e_{ii}^A]^{\frac{n}{2}-1} de_{ij}^A = \frac{1}{n}[1-(1-2e_{ii}^A)^{\frac{n}{2}}], \qquad (i,j=1,2,3) \quad (1)$$

where n is the measure and e_{ij}^A is the principal Almansi finite strain components[4].

The displacement components in cylindrical polar co-ordinates are given by,

$$u = r(1-\beta),\ v=0,\ w=dz \quad (2)$$

where β is a function of $r=\sqrt{x^2+y^2}$ only and d is a constant.

The finite components of strain[4-8] are defined as,

$$e_{rr}^A = \frac{1}{2}[1-(r\beta'+\beta)^2], \quad e_{\theta\theta}^A = \frac{1}{2}[1-\beta^2]$$
$$e_{zz}^A = \frac{1}{2}[1-(1-d)^2], \quad e_{r\theta}^A = e_{\theta z}^A = e_{zr}^A = 0 \quad (3)$$

where $\beta' = d\beta/dr$

Substituting Eqn (3) in Eqn (1) one gets the generalised components of strain as

$$e_{rr} = \frac{1}{n}[1-(r\beta'+\beta)^n], \quad e_{\theta\theta} = \frac{1}{n}[1-\beta^n]$$
$$e_{zz} = \frac{1}{n}[1-(1-d)^n], \quad e_{r\theta} = e_{\theta z} = e_{zr} = 0 \quad (4)$$

The stress-strain relations for transversely isotropic material are

$$T_{rr} = C_{11}e_{rr} + (C_{11}-2C_{66})e_{\theta\theta} + C_{13}e_{zz}$$
$$T_{\theta\theta} = (C_{11}-2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz}$$
$$T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz}$$
$$T_{zr} = T_{\theta z} = T_{r\theta} = 0 \quad (5)$$

where C_{ij}'s are material constants.

Using Eqs (4) in Eqn (5), one gets

$$T_{rr} = (C_{11}/n)\left[1-(\beta+r\beta')^n\right] + \left[(C_{11}-2C_{66})/n\right]\left[1-\beta^n\right] + C_{13}e_{zz}$$

$$T_{\theta\theta} = \left[(C_{11}-2C_{66})/n\right]\left[1-(\beta+r\beta')^n\right] + (C_{11}/n)\left[1-\beta^n\right] + C_{13}e_{zz}$$

$$T_{zz} = (C_{13}/n)\left[1-(\beta+r\beta')^n\right] + (C_{13}/n)\left[1-\beta^n\right] + C_{33}e_{zz}$$

$$T_{r\theta} = T_{\theta z} = T_{rz} = 0 \quad (6)$$

Equations of equilibrium are all satisfied except,

$$\frac{d}{dr}(T_{rr}) + \left(\frac{T_{rr}-T_{\theta\theta}}{r}\right) + \rho r\omega^2 = 0 \quad (7)$$

where ρ is the density of the material.

Substituting Eqn (6) in Eqn (7), one gets a nonlinear differential equation in β as

$$nPC_{11}\beta^{n+1}(1+P)^{n-1}\frac{dP}{d\beta}$$
$$= -nPC_{11}\beta^n(1+P)^n - (C_{11}-2C_{66})nP\beta^n$$
$$+2C_{66}\left[1-\beta^n(1+P)^n\right] - 2C_{66}(1-\beta^n) + \rho n r^2\omega^2 \quad (8)$$

where $r\beta' = \beta P$

The transitional points of P in equation (8) are $P \to -1$ and $P \to \pm\infty$

The boundary conditions are

$$T_{rr} = -p \quad \text{at} \quad r = a$$
$$T_{rr} = 0 \quad \text{at} \quad r = b \quad (9)$$

The resultant force normally applied to the ends of cylinder is

$$2\pi\int_a^b rT_{zz}\,dr = \pi a^2 p \quad (10)$$

3. SOLUTION THROUGH THE PRINCIPAL STRESSES

It has been shown[4-8] that the asymptotic solution through the principal stress leads from elastic-to-plastic state at the transition point $P \to \pm\infty$. For finding the plastic stress at the transition point $P \to \pm\infty$, the transition function R is defined as

$$R = 2(C_{11}-C_{66}) + nC_{13}e_{zz} - nT_{rr}$$
$$= \beta^n\left[C_{11}-2C_{66}+C_{11}(1+P)^n\right] \quad (11)$$

Taking the logarithmic differentiation of Eqn (11) with respect to r, and taking the asymptotic value as $P \to \pm\infty$, and integrating, we get

$$R = A_1 r^{-C_1} \quad (12)$$

where A_1 is a constant of integration and $C_1 = 2C_{66}/C_{11}$.

Using Eqn (12) in Eqn (11), we get

$$T_{rr} = C_3 - (A_1/n)r^{-C_1} \quad (13)$$

where $C_3 = \left[2(C_{11}-C_{66}) + nC_{13}e_{zz}\right]/n$

Using boundary condition of Eqn (9) in Eqn (13), one gets

$$A_1 = nb^{C_1}\left[\frac{p}{(b/a)^{C_1}-1}\right] \quad C_3 = \left[\frac{p}{(b/a)^{C_1}-1}\right] \quad (14)$$

Substituting the value of A_1 and C_3 in Eqn (13), one gets

$$T_{rr} = \left[\frac{p}{(b/a)^{C_1} - 1}\right]\left[1 - \left(\frac{b}{r}\right)^{C_1}\right] \tag{15}$$

Using Eqn (15) in Eqn (7), one gets

$$T_{\theta\theta} = \left[\frac{p}{(b/a)^{C_1} - 1}\right]\left[1 - (1 - C_1)\left(\frac{b}{r}\right)^{C_1}\right] + \rho r^2 \omega^2 \tag{16}$$

The axial stress is obtained from Eqn (5) as

$$T_{zz} = \frac{C_{13}}{2(C_{11} - C_{66})}\left[T_{rr} + T_{\theta\theta}\right] + \left[\frac{C_{33}(C_{11} - C_{66}) - C_{13}^2}{C_{11} - C_{66}}\right] e_{zz} \tag{17}$$

Applying the end condition (10) in Eqn (17), the axial strain is given by

$$e_{zz} = \frac{(C_{11} - C_{66})}{\left[C_{33}(C_{11} - C_{66}) - C_{13}^2\right]} \left[\frac{a^2 p}{b^2 - a^2} - \frac{a^2 C_{13} p}{(C_{11} - C_{66})(b^2 - a^2)} - \frac{C_{13}}{4(C_{11} - C_{66})}\rho\omega^2 (b^2 + a^2)\right] \tag{18}$$

Substituting Eqn (18) in Eqn (17), one gets

$$\begin{aligned} T_{zz} = {} & \frac{C_{13}}{C_{11}(2 - C_1)}\left[\left(\frac{p}{(b/a)^{C_1} - 1}\right)\left(2 - (2 - C_1)\left(\frac{b}{r}\right)^{C_1}\right)\right] \\ & + \frac{a^2 p}{b^2 - a^2} - \frac{2a^2 p C_{13}}{C_{11}(2 - C_1)(b^2 - a^2)} \\ & + \frac{C_{13}\rho r^2 \omega^2}{C_{11}(2 - C_1)} - \frac{C_{13}\rho\omega^2 (b^2 + a^2)}{2C_{11}(2 - C_1)} \end{aligned} \tag{19}$$

From Eqns (15) and (16), one gets

$$T_{\theta\theta} - T_{rr} = \left[\frac{p}{(b/a)^{C_1} - 1}\right] C_1 \left(\frac{b}{r}\right)^{C_1} + \rho r^2 \omega^2 \tag{20}$$

It is found that the value of $\left|T_{\theta\theta} - T_{rr}\right|$ is maximum at $r = a$, which means yielding of the cylinder will take place at the internal surface. Therefore, one has

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=a} = \left|\left[\frac{p}{(b/a)^{C_1} - 1}\right] C_1 \left(\frac{b}{a}\right)^{C_1} + \rho a^2 \omega^2\right| \equiv Y(say) \tag{21}$$

The pressure required for initial yielding is given by,

$$P_i = \frac{p}{Y} = \left(\frac{1 - (\rho a^2 \omega^2 / Y)}{C_1 (b/a)^{C_1}}\right)\left[\left(\frac{b}{a}\right)^{C_1} - 1\right] \tag{22}$$

Using Eqn (22) in Eqns (15), (16) and (19), one gets transitional stresses as

$$\sigma_r = \frac{T_{rr}}{Y} = \left[\frac{P_i}{(b/a)^{C_1} - 1}\right]\left[1 - \left(\frac{b}{r}\right)^{C_1}\right]$$

$$\sigma_\theta = \frac{T_{\theta\theta}}{Y} = \left[\frac{P_i}{(b/a)^{C_1} - 1}\right]\left[1 - (1 - C_1)\left(\frac{b}{r}\right)^{C_1}\right] + \frac{\rho r^2 \omega^2}{Y}$$

$$\begin{aligned} \sigma_z = \frac{T_{zz}}{Y} = {} & \frac{C_{13}}{C_{11}(2 - C_1)}\left[\left(\frac{P_i}{(b/a)^{C_1} - 1}\right)\left(2 - (2 - C_1)\left(\frac{b}{r}\right)^{C_1}\right)\right] + \frac{a^2 P_i}{b^2 - a^2} + \frac{C_{13}}{C_{11}(2 - C_1)}\frac{\rho r^2 \omega^2}{Y} \\ & - \frac{2a^2 C_{13} P_i}{C_{11}(2 - C_1)(b^2 - a^2)} - \frac{C_{13}}{2C_{11}(2 - C_1)}\frac{\rho\omega^2 b^2}{Y} - . \frac{C_{13}}{2C_{11}(2 - C_1)}\frac{\rho\omega^2 a^2}{Y} \end{aligned} \tag{23}$$

Eqn (23) give elastic-plastic transitional stresses in thick-walled rotating cylinder under internal pressure.

For fully plastic state $(C_1 \to 0)$, Eqn (21) becomes

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=b} = \left|\frac{p}{\log(b/a)} + \rho b^2 \omega^2\right| \equiv Y^* (say) \tag{24}$$

From Eqn (24), one has

$$P_f = \frac{p}{Y^*} = \left(1 - \left(\rho b^2 \omega^2 / Y^*\right)\right)\log(b/a) \tag{25}$$

From Eqn (23), one gets fully plastic stresses as

$$\sigma_r^* = \frac{T_{rr}}{Y^*} = \left(1 - \frac{\rho b^2 \omega^2}{Y^*}\right)\log\left(\frac{r}{b}\right)$$

$$\sigma_\theta^* = \frac{T_{\theta\theta}}{Y^*} = \left(1 - \frac{\rho b^2 \omega^2}{Y^*}\right)\left(1 + \log\left(\frac{r}{b}\right)\right) + \frac{\rho r^2 \omega^2}{Y^*}$$

$$\begin{aligned} \sigma_z^* = \frac{T_{zz}}{Y^*} = {} & \frac{C_{13}}{2C_{11}}\left[\left(1 - \frac{\rho b^2 \omega^2}{Y^*}\right)\left(1 + 2\log\left(\frac{r}{b}\right)\right)\right] \\ & + \left(\frac{a^2}{b^2 - a^2}\right)\left[\left(1 - \frac{\rho b^2 \omega^2}{Y^*}\right)\log\left(\frac{b}{a}\right)\right] \\ & - \frac{a^2 C_{13}}{C_{11}(b^2 - a^2)}\left[\left(1 - \frac{\rho b^2 \omega^2}{Y^*}\right)\log\left(\frac{b}{a}\right)\right] \\ & + \frac{C_{13}}{2C_{11}}\frac{\rho r^2 \omega^2}{Y^*} - \frac{C_{13}}{4C_{11}}\frac{\rho\omega^2}{Y^*}(b^2 + a^2) \end{aligned} \tag{26}$$

4. ISOTROPIC CASE

For an isotropic material

$$T_{\theta\theta} - T_{rr} = \left[\frac{p}{(b/a)^c - 1}\right] c (b/r)^c + \rho r^2 \omega^2 \tag{27}$$

where $c = (2\mu/\lambda + 2\mu)$

It is found that the value of $\left|T_{\theta\theta} - T_{rr}\right|$ is maximum at $r = a$, which means that yielding of the cylinder will take place at the internal surface.

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=a} = \left|\left[\frac{p}{(b/a)^c - 1}\right] c \left(\frac{b}{a}\right)^c + \rho a^2 \omega^2\right| \equiv Y(say) \tag{28}$$

The pressure required for initial yielding is given by

$$P_i = \frac{p}{Y} = \left(\frac{1 - (\rho a^2 \omega^2 / Y)}{c (b/a)^c}\right)\left[\left(\frac{b}{a}\right)^c - 1\right] \tag{29}$$

The pressure required for fully plastic state is given as,

$$P_f = \left(1-\rho b^2\omega^2/Y^*\right)\log\left(b/a\right)$$

The stresses required for fully plastic state $\left(c \to 0\right)$ is given as,

$$\sigma_r^* = \left[1-\rho b^2\omega^2/Y^*\right]\log\left(r/b\right)$$

$$\sigma_\theta^* = \left[1-\rho b^2\omega^2/Y^*\right]\left[1+\log\left(r/b\right)\right]+\rho r^2\omega^2/Y^* \qquad (30)$$

$$\sigma_z^* = \frac{1}{2}\left[\left(1-\frac{\rho b^2\omega^2}{Y^*}\right)\left(1+2\log\left(r/b\right)\right)\right] + \frac{1}{2}\frac{\rho r^2\omega^2}{Y^*} - \frac{\rho b^2\omega^2}{Y^*}\frac{\left(1+\left(a/b\right)^2\right)}{4}$$

These equations are the same as obtained by Gupta[5].

5. NUMERICAL ILLUSTRATION AND DISCUSSIONS

As a numerical illustration, elastic constants C_{ij} are given in Table 1. for transversely isotropic material (*Mg*) and isotropic material (Brass). The pressure required for initial yielding (P_i) and for fully plastic state (P_f) at different angular speeds has is in Table 2.

It has been observed from Fig. 1 that a thick-walled circular cylinder made of isotropic material having (*a*/*b*=0.2) yields at high pressure as compared to cylinder made of transversely isotropic material whereas for a thick-walled cylinder having (*a*/*b*=0.5), yields at low pressure. With the increase in angular speed, it has been observed that much less pressure is required for initial yielding at (*a*/*b*=0.2) for

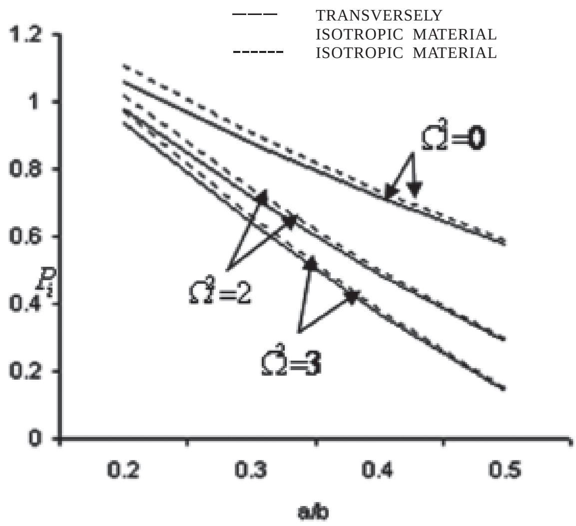

Figure 1. Pressure required for initial yielding at the internal surface of the cylinder at different angular speeds.

Table 1. Elastic constants C_{ij} used (in units of 10^{10} N/m^2)

Materials	C_{44}	C_{11}	C_{12}	C_{13}	C_{33}
Brass (Isotropic Material)	0.9999	3.0	1.0	1.0	3.0
Magnesium (Transversely Isotropic Material)	1.64	5.97	2.62	2.17	6.17

Table 2. The pressure required for initial yielding and Fully-plastic state at different angular speeds.

		Transversely isotropic material (magnesium)								**Isotropic material (brass)**							
Ω^2	P	*a/b*= 0.2	*a/b*= 0.3	*a/b*= 0.4	*a/b*= 0.5	$P=\frac{P_f-P_i}{P_i}\times 100$ is the percentage increase in pressure required from initial yielding to fully-plastic state				*a/b*= 0.2	*a/b*= 0.3	*a/b*= 0.4	*a/b*= 0.5	$P=\frac{P_f-P_i}{P_i}\times 100$ is the percentage increase in pressure required from initial yielding to fully-plastic state			
						a/b= 0.2	*a/b*= 0.3	*a/b*= 0.4	*a/b*= 0.5					*a/b*= 0.2	*a/b*= 0.3	*a/b*= 0.4	*a/b*= 0.5
0	P_i	1.06	0.88	0.72	0.57	52	38	27.9	20.7	1.11	0.9	0.74	0.59	45.6	33.1	24.6	18.3
	P_f	1.61	1.2	0.92	0.69					1.61	1.2	0.92	0.69				
2	P_i	0.98	0.72	0.49	0.29	265	268	288	341	1.02	0.74	0.5	0.29	258	262	283	337
	P_f	-1.6	-1.2	-0.9	-0.7					-1.61	-1.2	-0.9	-0.7				
3	P_i	0.93	0.64	0.37	0.14	445	477	592	1065	0.97	0.66	0.38	0.15	431	465	579	1047
	P_f	-3.2	-2.4	-1.8	-1.4					-3.22	-2.4	-1.8	-1.4				

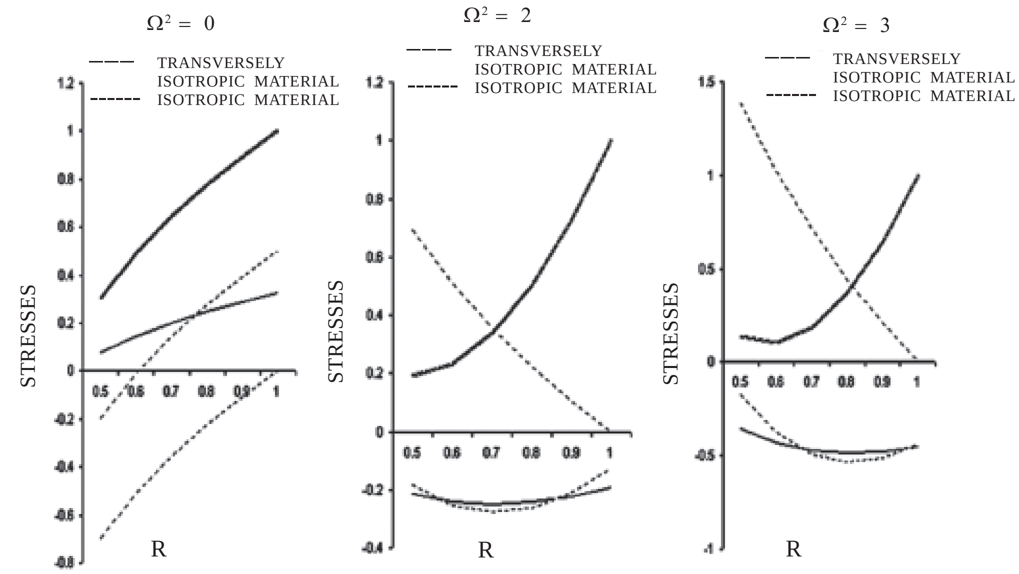

Figure 2. Fully-plastic stresses for a thick-walled cylinder under internal pressure at different angular speeds.

transversely isotropic material as compared to isotropic material. From Table 2, it has been observed that a thick-walled circular cylinder made of transversely isotropic material requires high percentage increase in pressure to become fully plastic as compared to isotropic material from its initial yielding and this percentage goes on increasing with the increase in angular speed.

In Fig. 2, curves have been drawn between stresses and radii ratio ($R=r/b$) for a fully plastic state. It has been observed that for fully plastic state, circumferential stress is maximum at external surface.

6. CONCLUSION

It can be concluded that circular cylinder under internal pressure made of transversely isotropic material is on the safer side of the design as compared to the circular cylinder under internal pressure made of an isotropic material.

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