Defence Science Journal, Vol. 60, No. 6, November 2010, pp. 606-613

The Evolution of AUSM Schemes

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ABSTRACT

This paper focuses on the evolution of advection upstream splitting method(AUSM) schemes. The main ingredients that have led to the development of modern computational fluid dynamics (CFD) methods have been reviewed, thus the ideas behind AUSM. First and foremost is the concept of upwinding. Second, the use of Riemann problem in constructing the numerical flux in the finite-volume setting. Third, the necessity of including all physical processes, as characterised by the linear (convection) and nonlinear (acoustic) fields. Fourth, the realisation of separating the flux into convection and pressure fluxes. The rest of this review briefly outlines the technical evolution of AUSM and more details can be found in the cited references.

Keywords: Computational fluid dynamics methods, hyperbolic systems, advection upstream splitting method, conservation laws, upwinding, CFD

1. INTRODUCTION

With the concept of upwinding laid out by Courant, Isaacson and Rees (CIR)[1] in 1952 for a scalar linear equation, what is to be further followed in this vein is the ability to compute discontinuities (shock and contact) from a system of nonlinear equations in multiple space dimensions with high-order accuracy. For this, the criteria that a useful numerical method should have are: (i) accurate and monotonic resolution of discontinuities, (ii) entropy satisfying, (iii) positivity preserving, and (iv) generality for other conservation laws.

We will focus on the development of methods that find their roots in the concept of upwinding are robust and reliable to meet some, if not all, of the above properties. In particular, having a monotonic and sharp resolution of shocks is the aim. First, the connection of employing characteristics in the space-time dimensions and upwinding in space dimension alone is elaborated. The latter is the basis of Eulerian formulation at a fixed time level.

Prominent upwind methods developed in the period of over half a century, including the methods by Godunov, Roe, and van Leer, laid the groundwork for describing the development of the advection upstream splitting method (AUSM) have been reviewed. We are interested in its evolution beyond the realm of single phase compressible aerodynamics, into low speed flow and multiphase flow. It has been noted that during this evolution, the fundamental gist still remains, requiring only slight expansions to handle mathematical properties unique to the new flows.

2. CASE FOR UPWINDING

As a preface to the topic of this paper, we begin by considering the scalar hyperbolic equation written in conservation form:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1}$$

For our purpose, it is instructive to consider its nonconservative version,

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0 \tag{2}$$

where $a(u)= df/du$, is called the advection speed since a equation of this form can be used to describe the transport of a material.

In the first upwind method known in the literature, Courant, Isaacson and Rees[1] took advantage of the fact that the solution u is constant along the characteristic $dx/dt = a$ and obtained the solution by tracing backward from the grid point (x_j, t^{n+1}) along the characteristic line to find the intersection x_{ch} at the time line of t^n. Since $a(u)$ is constant on the characteristic, the characteristic is thus a straightline, and we have an exact solution of the form

$$u(x_j,t^{n+1})=u_{ch}(x_{ch},t^n)=u(x_j \mp a\Delta t,t^n), \quad \text{if} \ \pm a \geq 0 \tag{3}$$

Since $x_{ch} = x_j \mp a\Delta t$ in general does not coincide with the existing grid points, the function values $u_{ch} = u^n\left(x_j \mp a\Delta t\right)$ can be obtained only from the grid values $\{\ldots,u^n_{j-1},u^n_j,u^n_{j+1},\ldots\}$ through interpolation. For example, a linear interpolation of functions at grids that sandwich x_{ch} gives, assuming here a is a constant,

$$u_{ch}=\left(1-\frac{|a|\Delta t}{\Delta x}\right)u^n_j+\frac{|a|\Delta t}{\Delta x}u^n_{j\mp1}, \ \text{and} \ 0<\frac{\pm a\Delta t}{\Delta x}\leq 1 \tag{4}$$

Received 13 April 2010, Revised 02 August 2010

Substituting into Eqn. (3), the solution at the new time level is obtained and written in the following form:

$$u\left(x_j, t_{n+1}\right) = u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} a \begin{cases} \left(u_j^n - u_{j-1}^n\right), \text{if } a \geq 0 \\ \left(u_{j+1}^n - u_j^n\right), \text{if } a < 0 \end{cases} \text{ and } \frac{\Delta t}{\Delta x}|a| \leq 1 \tag{5}$$

The CIR method Eqn. (4) is upwinding since it takes information only in the direction from which the characteristic propagates. This formula turns out to be identical with that resulting from the first-order explicit approximation of time derivative and one-sided approximation of space derivative. One-sided differencing of u_x by taking in the upwind direction yields a stable scheme, whereas the central-differencing is known to result in an unstable scheme unless a different time stepping is employed. Moreover, if there is discontinuity initially (such as a jump in concentration), the upwind scheme can yield an exact solution if the CFL number $\left(|a|\frac{\Delta t}{\Delta x}\right)$ is unity, while a centred scheme smears out the discontinuity and produces a non-monotone profile even if it is stable.

The CIR method, as presented in Eqn. (3), traces the solution upwindly in both space and time directions, while the upwind method, as formulated in the framework of fixed grid [Eqn. (5)] considers solely in the space direction. The former is natural and intuitive. The characteristic approach has some serious technical drwabacks. The first one concerns with solving flows with shock waves since the interpolation itself does not guarantee satisfaction of conservation laws. The second concerns with the practical implementation for a system of equations in several space dimensions. In this case, there are multiple characteristics with mixed signs and one can no longer form a decoupled family of characteristics so that a constant set of variables can be obtained. Hence, upwinding by viewing only in the space direction has become the prevailing approach because it can easily alleviate the above drawbacks. In the following decades, further developments have been made by incorporating the idea of spatial upwinding and solid evidences of its superior performance over directional-neutral methods. For example, the upwind scheme applied to Burger's equation, is known to yield a sharper representation of a shock wave than the centred Lax-Friderichs scheme[2].

2.1 Resolution of Shock Waves

Pursuit of accurate computation of shock waves is as old as CFD and still remains a challenge. The main difficulty with computing shocks has its roots in that the grid stencils desired for the smooth regions are in conflict with those for providing the sharpest representation of a discontinuity. The method of characteristics, as proposed in the CIR method, is not capable of dealing with discontinuities, but modifications are needed when free boundaries are to be determined in the problem (e.g. shock, contact discontinuities, etc.[1]). These modifications turn out to be no small feats, as feasible procedures would not be available for more than 20 years, until the work by Moretti[3], *et al*. The only challenge left then was the satisfaction of the Rankine-Hugoniot jump relations across a shock. The procedure required a robust way of defining, tracking the shock location, and implementing precisely the jump condition there. Appropriately, the approach is called shock-fitting. The result is the shock wave resolved with high accuracy, having no internal point at the shock, as it should be for a true discontinuity. The coding logic demanded within this framework is daunting, in situations where shocks are embedded in the flow or with multiple discontinuities interacting with each other; it took them several years to build a logic[4] to track and fit the shocks. This is the reason for the least use of shock fitting for real-life multi-dimensional problems; however, such computations appeared in the1990s by Nasuti and Onofari[5]. It is interesting to note that the equations were formulated in nonconservative form written for some unconventional and nonconservative variables such as speed of sound and entropy.

Clearly the mystique of conservation equations[4] is dispelled in computing shocks, as long as the jumps are properly incorporated, since in the smooth region, there is no preference of one form over the other. Nevertheless, the benefit of having accurate shock representation with neither non-physical oscillations or non-physical smoothing may compensate for the extensive and complicated programming effort.

The other approach for handling shock waves is known as shock capturing, by which the shock wave automatically appears in the solution without resorting to extraneous devices. Lax[6] in 1957 laid out the theoretical foundation of the shock-capturing approach and the entropy conditions that guarantee the existence of a weak solution with correct jump conditions. The Rankine-Hugoniot relations are typically satisfing over several cells; multiple discontinuities and the interactions among them are accommodated properly as part of the computed solution. Several questions however have not been addressed by Lax[6], such as (i) what would be a proper discretisation method, and (ii) what would be the quality of a captured shock profile: how sharp is the profile and is it monotone in one and multiple space dimensions.

The search for answers to these questions consumed the CFD community in the following decades. In fact, the centred schemes, such as by Lax-Wendroff[7] and Lax-Friedrichs[6] or their variants, have the favourable attribute of being simple and not requiring sophisticated Riemann solution; hence these were considered the mainstream for capturing shock up to the 1970s. However, a breakthrough took place in the Soviet Union more than half century ago, when Godunov in 1959[8] proposed an ingenious way of defining the interface flux, via solving an initial-value Riemann problem. While it was introduced to the Western world in a textbook[9], the Godunov method had not gained recognition in practice until two decades later by van Leer[10]. Arguably the introduction of Godunov's method began the era of modern CFD and the shock-capturing approach took a giant leap forward. In the light of voluminous and extensive body of research works in the decades after Godunov's method, we shall focused only on representative efforts, those increasing efficiency, accuracy, and generality.

3. CONSERVATIVE UPWIND METHODS FOR HYPERBOLIC SYSTEMS

The following system of conservation laws in one dimension is considered,

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=0 \tag{6}$$

The finite volume approach is adopted because it fits most naturally the description of conservation laws of fluids within a control volume and it is consistent with the weak solution formulation in the sense of Lax[11]. For this reason, the finite-volume framework is adopted by almost all modern schemes developed for compressible flows.

After an explicit time integration, the finite-volume version of the above system gives the following generic equation

$$\mathbf{u}_j^{n+1}-\mathbf{u}_j^n+\frac{\Delta t}{\Delta x}\left[\mathbf{f}\left(x_{j+1/2},t_n\right)-\mathbf{f}\left(x_{j-1/2},t_n\right)\right]=0 \tag{7}$$

where $\mathbf{u}_j$ is a vector of cell-average values of U and f is a vector of numerical fluxes evaluated at the cell faces $x_{j\pm1/2}$.

For a hyperbolic system of equations, there are a multitude of waves propagating with positive and negative velocities. Owing to the upwinding concept, the numerical flux $f_{j+1/2}$ at the cell face $x_{j+1/2}$ should receive contributions originating from the cells to its left and right (u_L, u_R). Hence,

$$\mathbf{f}_{j+1/2}\equiv\mathbf{f}\left(x_{j+1/2},t^n\right)=\mathbf{f}\left(\mathbf{u}_j^n,\mathbf{u}_{j+1}^n\right)=\mathbf{f}\left(\mathbf{u}_L,\mathbf{u}_R\right) \tag{8}$$

where the subscripts L and R refer to the values associated with the left and right cells.

The epoch-making work by Godunov[8] lays out a solid foundation for constructing the numerical flux function. Godunov regards the solutions (u_L, u_R) at each time level t^n as the initial-value pair of the Riemann problem. Clearly, the Riemann solution is the heart of the entire method. Since there is no characteristic time and length in the equations under consideration, a similarity solution $w(\xi)$ can be formulated in terms of the similarity variable $\xi=(x-x_{j+1/2})/(t-t^n)$ and the solution is constant along each ray of constant value of ξ. The cell interface coincides with the ray on which $\xi = 0$, hence the numerical flux is simply

$$\mathbf{f}_{j+1/2}=\mathbf{f}\left(\mathbf{u}_L,\mathbf{u}_R\right)=\mathbf{f}\left(\mathbf{w}\left(\xi=0;\mathbf{u}_L,\mathbf{u}_R\right)\right) \tag{9}$$

A description of its practical implementation is available[10,12,13]. Unfortunately, the details of the exact solution are not preserved after one time step, instead it is averaged out according to Eqn. (7). Because of this approximation, seeking for an exact solution in each time step might not be necessary. Also, approximate solutions may be essential because of the following situations: (i) the Riemann solution may require substantial computational efforts, or (ii) a similarity solution may not exist, for example, when a source term exists. Moreover, the exact solution of a multi-dimensional Riemann solution is still unknown. Thus, it is useful to devise other methods that are not based on the restrictive requirement of solving the Riemann problem exactly.

Several approximation procedures for solving the Riemann problem have been proposed, by Roe[14], Colella and Glaz[12], Osher-Solomon[15], HLLE[16], etc. The main difference among these lies in the way these approximate the exact wave structure by more tractable wave configurations; for example, either all shocks[12], all rarefaction/compression waves[15], or a gross description of wave envelope[16]. Consequently, how the waves structure is represented, has direct implications on their performance of the waves and potential for further extensions to other conservation laws.

The Roe splitting is perhaps the most popular approximate Riemann solver in this class. Instead of attempting to solve the nonlinear equations, Roe suggested to replace it with a locally frozen coefficient linear system such that certain features of nonlinear problems are still retained. As in Godunov's Riemann problem, Roe considers the difference of two neighbouring fluxes as a disturbance that is to be propagated in both + and – directions, based on the signs of eigenvalues of the Jacobian matrix A, evaluated at a special state $\hat{u}=\hat{u}(\mathbf{u}_L,\mathbf{u}_R)$ known as the Roe-average state.

$$\mathbf{F}\left(\mathbf{u}_R\right)-\mathbf{F}\left(\mathbf{u}_L\right)=A(\hat{u})\left(\mathbf{u}_R-\mathbf{u}_L\right) \tag{10}$$

$$=A^+(\hat{u})\left(\mathbf{u}_R-\mathbf{u}_L\right)+A^-(\hat{u})\left(\mathbf{u}_R-\mathbf{u}_L\right) \tag{11}$$

With this decomposition of A, an easily facilitates the definition of flux at $\xi = 0$ by either accounting for the negative waves (to the left of $\xi = 0$) from the left flux or the positive waves from the right flux. Thus,

$$\mathbf{f}_{j+1/2}=\mathbf{F}(\mathbf{u}_L)+A^-(\hat{\mathbf{u}})\left(\mathbf{u}_R-\mathbf{u}_L\right) \tag{12}$$

$$=\mathbf{F}(\mathbf{u}_R)+A^+(\hat{\mathbf{u}})\left(\mathbf{u}_R-\mathbf{u}_L\right) \tag{13}$$

$$=\frac{1}{2}\left[\mathbf{F}\left(\mathbf{u}_L\right)+\mathbf{F}\left(\mathbf{u}_R\right)-\left(A^+(\hat{\mathbf{u}})-A^-(\hat{\mathbf{u}})\right)\left(\mathbf{u}_R-\mathbf{u}_L\right)\right] \tag{14}$$

Because of the splitting in flux difference [Eqn. (11)], this type of approach is often referred to as flux difference splitting (FDS). By Roe's construction, the Rankine-Hugoniot condition, is satisfied if two states $\mathbf{u}_L$ and $\mathbf{u}_R$ are connected by a single discontinuity (shock or contact) moving at a speed coinciding with an eigenvalue of $A(\hat{u})$.

A downside of this exact property for the Roe splitting is that it can not distinguish between a stationary shock and an expansion shock. That is, the entropy condition may be violated, and while this situation appears, the expansion shock manifests itself as a jump in the profile.

To correct this problem, an eigenvalue associated with the nonlinear fields (either $u + a$ or $u - a$) is modified when it is near-zero. This procedure is generally known as entropy fix, first suggested by Harten and Hyman[17] and later by others with variations.

Another useful class of numerical flux functions for inviscid flows is known as flux-vector splitting (FVS). The basic idea of the flux-vector splitting is that the full flux at the interface collects contributions from its neighbours on the basis of upwinding, in a similar spirit of the CIR method[1] by tracing backward to the source of signals.

$$\mathbf{f}\left(\mathbf{u}_L,\mathbf{u}_R\right)=\mathbf{F}^{+}\left(\mathbf{u}_L\right)+\mathbf{F}^{-}\left(\mathbf{u}_R\right) \tag{15}$$

where the Jacobian matrices of the split functions F^{+} and F^{-} are required to have non-negative and non-positive eigenvalues, respectively. Hence, proper upwind states, either $\mathbf{u}_L$ or $\mathbf{u}_R$ are taken in the respective fluxes.

Various decomposition can be devised, but they all can be interpreted as consisting of two streams of particles travelling to the interface, with certain distribution functions and physical contents (mass, momentum, and energy)[18,19]. Two flux-vector splittings are well-known, proposed by van Leer[20] and Steger and Warming[21], respectively. The former has several advantages over the latter as follows: (i) it is smooth at sonic points, (ii) it does not involve derivatives of flux functions with respect to u, and (iii) it remains identical for a fluid having different equation of state (EOS). As a result of (i), the van Leer splitting in general also performs better than the Steger-Warming splitting. However, excessive numerical diffusion, in describing a contact discontinuity in particular, is the common deficiency of this class of methods; this error cannot be simply reduced by reducing grid size and/or using higher-order differencing. This fact had largely gone unnoticed until 1987 when van Leer again brought attention to this deficiency[22].

The ability to resolve a contact discontinuity is very important for both inviscid and viscous calculations. In the inviscid calculations, especially in multi-dimensions, varieties of discontinuities exist and interact; excessive numerical diffusion in contact discontinuity can lead to an erroneous solution.

However, the Van Leer splitting has many desirable features. It is algorithmically simple, has the same form for fluids with complex EOS, satisfies entropy condition, and is operationally robust for a wide variety of problems.

Despite its superior capability in computing a contact discontinuity, the Roe splitting is complicated and is limited to fluids with simple EOS. In fact, the extension to non-ideal gases is not unique[23,24].

Based on the above comparison, it has been concluded that the FVS and FDS approaches have their own distinct advantages and drawbacks, but these clearly compensate each other. This observation has led to the development of a new type of numerical flux, the AUSM, originally by Liou and Steffen[25], and the subsequent improvements[26, 27].

4. EVOLUTION OF ADVECTION UPSTREAM SPLITTING METHODS

4.1 The Beginning of AUSMs: The Era of Interest in Shock and Contact Capturing

The pursuit of AUSM is based on the premise that the mathematical details similar to the FVS can be exploited but the linear field must be recognised and incorporated as part of the complete splitting. The aim is to combine the desirable attributes belonging to both FDS and FVS splittings and simultaneously eliminating their weaknesses.

For a scalar hyperbolic equation, [Eqn. (2)], the contact discontinuity is inherently associated with the convection process, and the quantity transported by the convection process is the material property, namely the density (or species concentration). For a system of conservation laws, examining the continuity equation gives the first clue about the roles of flow convection velocity and the density; the density must follow with the flow from upstream. Hence, once the convection velocity at a certain location is defined, the source of the density is known from its upstream side and the mass flux is determined accordingly. Examination of the other conservation equations gives the second clue that this mass flux also appears in the momentum and energy equations in the form of convective fluxes. This is the missing link in the van Leer method.

At the core of the AUSM family of schemes is the realisation that the convection and acoustic waves ought to be treated as two physically distinct processes[25]. Hence the inviscid flux at the continuum level is expressed as a sum of the convective and pressure terms:

$$\mathbf{F}=\mathbf{F}^{(c)}+\mathbf{F}^{(P)} \tag{16}$$

where

$$\mathbf{F}^{(c)}=\dot{m}\Psi,\dot{m}=\rho u,\Psi=\begin{pmatrix}1\\u\\H\end{pmatrix} \tag{17}$$

and

$$\mathbf{F}^{(p)}=\begin{pmatrix}0\\p\\0\end{pmatrix} \tag{18}$$

Here, the convective flux $F^{(c)}$ contains the convective mass flow rate $\dot{m}$ and the corresponding passive scalar quantities in Ψ. The pressure flux $F^{(p)}$ contains nothing but the pressure term.

Examination of Eqn. (17) term by term gives a clue to how the numerical convective flux $\mathbf{f}^{(c)}_{1/2}$ at the interface is to be constructed in terms of the L and R states

$$\mathbf{f}^{(c)}_{1/2}(\mathbf{u}_L,\mathbf{u}_R)=\dot{m}_{1/2}\left(\mathbf{u}_L,\mathbf{u}_R\right)\Psi_{1/2}\left(\mathbf{u}_L,\mathbf{u}_R\right)$$

$$\Psi_{1/2}\left(\mathbf{u}_L,\mathbf{u}_R\right)=\begin{cases}\Psi\left(\mathbf{u}_L\right), \text{ if } \dot{m}_{1/2}\geq 0\\ \Psi\left(\mathbf{u}_R\right), \text{ otherwise}\end{cases} \tag{19}$$

Here, the linear field, to which the contact discontinuity belongs, is characterised by the mass flux, and the mass flux selects its content ρ from the upwind source based on the sign of $u_{1/2}$:

$$\dot{m}_{1/2} = u_{1/2}\rho_{L/R} = \begin{cases} \rho_L, \text{ if } u_{1/2} \geq 0 \\ \rho_R, \text{ otherwise} \end{cases} \quad (20)$$

Clearly the definition of the interface convective velocity $u_{1/2}$ is a critical step, for which we employ two functions, respectively expressed in terms of eigenvalues $u \pm a$ when $|u| \leq a$, as envisioned in Van Leer's flux-vector splitting. Note that using the eigenvalues as a basis for expressing the numerical fluxes is quite common in the upwind formulation, easily identifiable in all flux schemes mentioned above. Here, we write in terms of split Mach-number functions

$$u_{1/2} = a_{1/2}\left[M^{+}_{(m)}(M_L) + M^{-}_{(m)}(M_R)\right], M_{L/R} = \frac{u_{L/R}}{a_{1/2}} \quad (21)$$

where the subscript m refers to the degree of polynomial used in $M^{\pm}$ specific definitions[26].

The algorithm is an improved version from the original one[25], hence referred to as AUSM^{+}[26]. A major concept introduced in AUSM^{+} is the single speed of sound $a_{1/2}$ commonly used for defining both M_L and M_R, rather than separate a_L and a_R. Another modification is the use of higher degree polynomials, as specified by the subscript m.

The pressure flux $\mathbf{f}^{(p)}_{1/2}$ involves only $p_{1/2}$, for which we again apply the same concept used in defining $u_{1/2}$, consisting of contributions from the left and right cells in accordance with waves $u \pm a$,

$$p_{1/2} = P^{+}_{(n)}(M_L)p_L + P^{-}_{(n)}(M_R)p_R \quad (22)$$

Again the subscript n refers to the degree of polynomials[26].

4.1.1 Remarks

(i) It is evident that the upwinding idea prevails throughout the construction of AUSM, [Eqns (19) to (22)]. Moreover, the definition of $u_{1/2}$ and $p_{1/2}$ can be interpreted from the viewpoint of following characteristics originating from L and R cells at time level t^n to the cell face, in the spirit of CIR.

(ii) Unlike Roe's FDS, the numerical dissipation in the AUSM family is merely a scalar, not of a matrix type. As a result, the system is decoupled, and hence, requires only $O(n)$ operations, n being the number of unknowns. Moreover, the same formula is easily extendable to include other conservation laws, or to fluids with general EOS, as in the case of a multiphase flow.

(iii) As in van Leer's splitting, the AUSM family does not require differentiation or the flux Jacobian matrix, for the evaluation of $f_{1/2}$; they always involve only the common term $\dot{m}_{1/2}$ for any additional conservation law.

(iv) A general procedure of constructing the mass flux within the framework of AUSM by encompassing other formulae[28]. It also expounds the impact of the flux formula on the occurrence of shock instability, the so-called 'Carbuncle' phenomena.

(v) There is a great deal of freedom for defining the interface speed of sound $a_{1/2}$. A peculiar one is proposed that gives an exact capturing of a 1-D stationary entropy-satisfying shock wave[29,27].

(vi) Note that when $\dot{m}_{1/2}$ is equal to zero, the sign of $\dot{m}_{1/2}$ is immaterial since the convective flux vanishes with $\dot{m}_{1/2}$. This also means the switching is continuous.

4.2 The Era of Interest in Low Mach-number Flows

As the upwind flux functions have, by and large, overcome the challenges for computing compressible flow phenomena reliably with satisfactory accuracy, it is only logical to evaluate how they would perform in an incompressible (or more precisely, low Mach number) flow regime in which the scales between the speed of sound and flow convective speed are disparate, that is, $u \ll a$. This results in an inordinate amount of numerical dissipation that worsens as $u \to 0$, and it manifests itself in a slow or stalling convergence and large numerical disturbances overwhelming discretisation errors. To rectify both the problems, a proper rescaling must be introduced so that the resulting speed of sound and convective speed become of equal order. For schemes utilising the Jacobian matrix, such as Roe's FDS, this rescaling can be accomplished by introducing a pre-conditioning matrix so that the eigenvalues of the resulting matrix are of the same order[30]. For the AUSM family, the modification to equalise scales is simple, requiring only a rescaling of the speed of sound by a scalar factor so that it is diminishing as M is decreasing. However, this leaves with insufficient numerical dissipation for stability, hence, additional couplings between velocity and pressure fields are introduced in the convective (via mass) and pressure fluxes. The final results are:

$$M_{1/2} \leftarrow M_{1/2} - K_p \max\left(1 - \rho\bar{M}^2, 0\right)\frac{p_R - p_L}{\rho_{1/2}\left(f_a a^2_{1/2}\right)}$$

$$\rho_{1/2} = \frac{\rho_L + \rho_L}{2} \quad (23)$$

where

$$\bar{M}^2 = \frac{u_L^2 + u_R^2}{2a^2_{1/2}} \quad (24)$$

Similarly,

$$f_a(M_o) = M_o(2 - M_o) \in [0,1]$$

$$M_o^2 = \min\left(1, \max\left(M^2, M_\infty^2\right)\right) \quad (25)$$

Similarly,

$$p_{1/2} \leftarrow p_{1/2} - 2K_u P^{+}_{(n)}P^{-}_{(n)}\rho_{1/2}\left(f_a a^2_{1/2}\right)(M_R - M_L) \quad (26)$$

It is noted that the factor $f_a a_{1/2}^2$ produces a factor of the order of $u_{1/2}$. The effectiveness of this rescaling for low-Mach number flows is documented[27].

4.3 The Era of Interest in Multiphase Flows

Computation of nonequilibrium multiphase flows is confronted with two major difficulties, one is the closure modelling of the effective field (multi-fluid) equations, and the other is the design of a robust and accurate numerical procedure for these systems. The difficulty with the latter stems from terms in nonconservative form, which often make the system nonhyperbolic[31,32]. Moreover, the flow usually contains disparate fluids properties, severely restricting stability. The contact discontinuities are of major importance and often the centre of flow phenomenon. Hence, the ability to, or inability to, accurately compute the contact discontinuities plays a pivotal role in determining the reliability in correctly simulating flow phenomenon. Unfortunately, excessive smearing is typically inherent in any capturing scheme, even for the successful upwind schemes described so far, unless a special tracking or fitting approach (much like the shock fitting) is enacted. The property of preserving positivity of scalar quantities (e.g., volume fraction) is critical for stability. It is well known that the lack of hyperbolicity property poses numerical difficulties as to instability and convergence; what is not known is how to handle the situations when presented. Finally, treating nonconservative terms is still an open problem; spurious oscillations can appear if they are not properly discretised[33].

Liou and Chang[33,34] continue to expand the basic framework of single-phase AUSM to solve multi-fluid equations. In this setting, it is conceptually and procedurally identical to extend the single-phase AUSM to each individual phase since it is described by equations consisting of the same convective and pressure fluxes, except additional nonconservative terms. It is noted that the Jacobian matrix of this equations system can become exceedingly complicated, making inaccessible an analytical form for the eigen system. Hence, the FDS-type schemes are not feasible to be used. The AUSM framework, not based on the Jacobian matrix, is not restricted by this complexity; its advantage in this regard is distinct.

What is further to be addressed is the nonconservative terms, which typically characterise the action of pressure force at a phase interface. These must be addressed with care to ensure there is no imbalance with the conservative terms; otherwise, spurious oscillations[35] are often encountered at phase boundaries. An effective balance at the discrete level can be found[33,34].

4.4 The Era of Interest in High-order Schemes

Godunov's order barrier theorem[8] (Any linear monotone scheme for solving partial differential equations can be at most first-order accurate.) stood to crush any hope to achieve high-order monotone solutions for even a linear problem. This barrier had not been circumvented until the concept of nonlinearity in terms of limiter functions of variations was introduced by Boris[36] and van Leer[37]. This nonlinearity concept has been the only way around the barrier ever since. Once successfully applied to various realistic problems, interest in achieving accuracy beyond second or third order naturally arose. Harten developed a systematic approach to construct high-order essentially non-oscillatory (ENO) interpolations [38]. These schemes are based on a comparative procedure to select the discrete stencil that yields the smoothest interpolant. A later version with a weighted combination of all stencils under consideration gives a much easier implementation and improved computational results; the method is hence called WENO[39].

Interest in efficient utilisation of discrete data to achieve high accuracy and flow resolution has led to the study of combining very high-order accurate interpolations into the AUSM, especially with emphasis on flows with shocks. This effort is documented in a recent paper[40] in which a detailed comparison of very high-order interpolations (up to 13th order) based on WENO and a monotonicity preserving scheme[41] is shown for some problems with complicated shock waves. This work is the recent addition to the evolution of AUSM, the only known publication on AUSM-based high-order solutions. Further research on this topic is their future plan.

5. CONCLUSIONS

In conlusion, we summarise a few key lessons drawn from the discussion as follows:

(i) Upwinding embodies physics into the mathematical procedure. It has been the fundamental underlying concept for nearly all modern CFD methods.
(ii) The accuracy of the shock fitting method for resolving shock waves is superior to the shock capturing method.
(iii) The shock-capturing framework is easy to implement and to include weak solutions; hence, it is the only framework in use today. Practicality (in implementation) determines acceptability. The lack of sharpness in computing shocks, in comparison to the shock-fitting methods, is compensated for by deploying the grid adaptation and high-order interpolations; both are again relatively easy to be incorporated.
(iv) Searching approximate procedures for solving the Riemann problem produces numerous interesting and useful works. Perhaps, the search isn't yet exhausted.
(v) The framework of AUSM proves to be quite fruitful and broad, to be easily adaptable to various types/ regimes of flows, with only small modifications.

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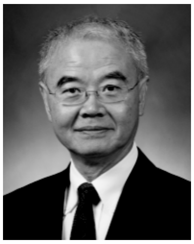

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