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Prioritising Emergency Bridgeworks Assessment under Military Consideration using an Enhanced Fuzzy Weighted Average Approach

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ABSTRACT

Prioritising emergency bridgeworks assessment has been a key to winning battles in combat circumstances because of soldier safety, attack or defence tactics, and logistic supply ability. However, an imprecise or vague satisfaction level of importance of criteria may also affect the prioritising evaluation of bridgeworks under military consideration. In this paper, the fuzzy set theory is employed to treat this aspect. With linguistic variables, fuzzy numbers and an enhanced fuzzy weighted average approach will be used. The proposed approach is used to investigate an example to illustrate its applications in emergency bridgeworks assessment. The approach is shown to be useful and effective. In order to make computing and ranking results easier and to increase recruiting productivity, a computer-based decision support system has been developed, which may help the commander make decisions more efficiently.

Keywords: Fuzzy weighted average, fuzzy multi-criteria decision making (FMCDM), bridgeworks assessment, military circumstance

1. INTRODUCTION

Under harsh combat circumstances, ground forces are usually restricted by special terrain or bridges damaged in warfare. To make ground forces proceed in a viable and timely manner, a theatre of operations (TO) commander needs to make a decision whether to build a bridge or repair a demolished one. Either way, it is time-consuming because of limited engineering forces and their dispersion. The priority of which bridge to build or repair may greatly affect the leverage of the combat initiative to both sides in time-pressing combat situations, which in turn may affect the outcome of the war. Thus, how to prioritise the bridgeworks is really one of the crucial decision-making issues that may affect whether a TO commander makes an attack or takes defence measures.

Furthermore, the decision-making on the priority of restoring the function of damaged bridges is not only derived from the safety factors of the target bridges but from others, such as operational environment, the speed of bridgework, or combat mission considerations. Currently, some researchers[1-5] have contributed much to the risk management or assessment of bridges, but these researchers involved in this issue have not yet found a satisfactory solution.

In addition, to the priority of bridges needing emergency repair under military consideration, often in the decision processes, there are certain forms of imprecision that may be identified, e.g., incompleteness where insufficient data occurs, or fuzziness where there are difficulties in obtaining the precise features, attributes, or criteria[6]. To deal with this problem, a good approach is to apply modelling using linguistic variables and fuzzy set theory. Fuzzy set theory is one of the most important approaches in addition to the probability theory[7-9]. Fuzzy set theory has been utilised in almost all areas of applications[10]. Among these, one of the most important applications happens to be in the decision analysis or alternative evaluation. Since the fuzzy set theory can manage a great deal of imprecision, it has contributed to the richness of the decision-making[10-11]. In effect, almost all measurements in all problems can be found to have a certain vagueness and uncertainties. When the fuzzy set theory is applied, fuzzy measures from different criteria may be defined and correspondingly weighted with fuzzy importance. Fuzzy numbers can be manipulated through arithmetic. The processes can produce more credible results, and the results can be more informative. The fuzzy weighted average (FWA) approach may also provide such informative results. This approach is adopted here for evaluating the priority of emergency bridgeworks, and an enhanced FWA algorithm is adopted.

2. FUZZY SETS AND FUZZY WEIGHTED AVERAGE

In this section, some definitions and properties of the fuzzy sets are discussed. Some further details of fuzzy sets, are given by Zimmerman[10].

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2.1 Basic Concepts of Fuzzy Sets

2.1.1 Fuzzy Set

A fuzzy set may be defined as, $A = \{(x, \mu_A(x)), x \in U, \mu_A(x) \in [0, 1]\}$, where $x \in U$ is the universe of discourse and $\mu_A(x) \in [0, 1]$ denotes the membership function or degree of x belonging to A.

2.1.2 Fuzzy Number

A fuzzy number (FN) is a fuzzy set defined on the real line R and has the properties of convexity and normality of fuzzy sets. Moreover, a FN can be written as $A = (a^L, a^M, a^R)$, where a^L and a^R denote the left and right bounds, respectively, and a^M represents the mode of A. (a^L, a^R) is called the support of A. The special cases of FNs may include the crisp real numbers and intervals of real numbers. For instance, triangular FNs (TFNs) may be defined with the triangular membership functions as

$$\mu_A(x) = \begin{cases} 0, & x < a^L, \\ (x - a^L)/(a^M - a^L), & a^L \le x \le a^M, \\ (a^R - x)/(a^R - a^M), & a^M \le x \le a^R, \\ 0, & x > a^R, \end{cases} \quad (1)$$

and the α-cuts are therefore continuous closed bounded intervals.

2.1.3 The α-cuts

For a fuzzy set A on a universe of discourse U and $\alpha \in (0, 1]$, the α-cuts denoted as $(A)_\alpha$ of A can be defined as

$$(A)_\alpha = \{x \in U \mid \mu_A(x) \ge \alpha\}. \quad (2)$$

The α-cut fuzzy arithmetic is important for the FNs. It can be defined as follows. For instance, for a general function $f(A_1, A_2, \ldots, A_n)$ representing an arithmetic and the α-cuts of A_i, $(A_i)_\alpha$, also denoted as $[(a_i^L)_\alpha, (a_i^R)_\alpha]$ for $i = 1, \ldots, n$, the α-cuts of the fuzzy image Y through the function f from $A_1, A_2, \ldots, A_n$ can be defined as $(Y)_\alpha = [(y^L)_\alpha, (y^R)_\alpha]$ and with

$$\begin{aligned}(Y)_\alpha &= [(y^L)_\alpha, (y^R)_\alpha] = f\left((A_1)_\alpha, \ldots, (A_n)_\alpha\right) \\ &= f\left([(a_1^L)_\alpha, (a_1^R)_\alpha], \ldots, [(a_n^L)_\alpha, (a_n^R)_\alpha]\right).\end{aligned} \quad (3)$$

2.1.4 Arithmetic Operations of Fuzzy Numbers

By employing the concept of α-cuts, the fuzzy arithmetic of FNs can be defined by interval arithmetic on the closed intervals on R. For instance, for a two fuzzy-number arithmetic operation as follows,

Addition:

$$\begin{aligned}(A_1 + A_2)_\alpha &= (A_1)_\alpha + (A_2)_\alpha \\ &= \left[(a_1^L)_\alpha + (a_2^L)_\alpha, (a_1^R)_\alpha + (a_2^R)_\alpha\right]\end{aligned} \quad (4)$$

Subtraction:

$$\begin{aligned}(A_1 - A_2)_\alpha &= (A_1)_\alpha - (A_2)_\alpha \\ &= \left[(a_1^L)_\alpha - (a_2^R)_\alpha, (a_1^R)_\alpha - (a_2^L)_\alpha\right],\end{aligned} \quad (5)$$

Multiplication:

$$\begin{aligned}&(A_1 \cdot A_2)_\alpha = (A_1)_\alpha \cdot (A_2)_\alpha \\ &= \Big[\min\left\{(a_1^L)_\alpha \cdot (a_2^L)_\alpha,\ (a_1^L)_\alpha \cdot (a_2^R)_\alpha, (a_1^R)_\alpha \cdot (a_2^L)_\alpha,\ (a_1^R)_\alpha \cdot (a_2^R)_\alpha\right\}, \\ &\max\left\{(a_1^L)_\alpha \cdot (a_2^L)_\alpha,\ (a_1^L)_\alpha \cdot (a_2^R)_\alpha, (a_1^R)_\alpha \cdot (a_2^L)_\alpha,\ (a_1^R)_\alpha \cdot (a_2^R)_\alpha\right\}\Big],\end{aligned} \quad (6)$$

Division:

$$\begin{aligned}&(A_1/A_2)_\alpha = (A_1)_\alpha/(A_2)_\alpha \\ &= \Big[\min\left\{(a_1^L)_\alpha/(a_2^L)_\alpha,\ (a_1^L)_\alpha/(a_2^R)_\alpha,\ (a_1^R)_\alpha/(a_2^L)_\alpha,\ (a_1^R)_\alpha/(a_2^R)_\alpha\right\}, \\ &\max\left\{(a_1^L)_\alpha/(a_2^L)_\alpha,\ (a_1^L)_\alpha/(a_2^R)_\alpha,\ (a_1^R)_\alpha/(a_2^L)_\alpha,\ (a_1^R)_\alpha/(a_2^R)_\alpha\right\}\Big], \\ &0 \notin \left[(a_2^L)_\alpha, (a_2^R)_\alpha\right],\end{aligned} \quad (7)$$

$\forall\ \alpha \in [0, 1]$. The results of fuzzy arithmetic are obtainable by recomposing the α-cuts into the fuzzy numbers.

2.1.5 Linguistic Variable

A linguistic variable can also be defined with the fuzzy sets. A linguistic variable is such that the possible states are fuzzy sets or FNs that are assigned to relevant linguistic terms (e.g., "important", "unimportant", etc. as used here).

In this paper, the appropriate triangular fuzzy numbers are defined to capture the linguistic variables of criteria ratings and importance weighting rating. The relative importance of each criterion is distinguished by seven levels, shown in Table 1.

2.2 Fuzzy Weighted Average

Let A_j, $j = 1, 2, \ldots, m$ denote objective (alternative) with respect to a set of criteria, attributes or factors i as, C_{ji}, $i \in \{1, 2, \ldots, n\}$, and relative importance weights for each criterion as, W_i, $i \in \{1, 2, \ldots, n\}$. Finally, through using the FWA approach, it reaches the objective function that aggregates the fuzzy criteria ratings and weights into the FNs Y_j for the objects. Thus, it consists of the fuzzy addition, fuzzy multiplication, and fuzzy division and can be defined by

$$\begin{aligned}Y_j &= f(C_{j1}, \ldots C_{ji}, \ldots, C_{in}, W_1, \ldots, W_i, \ldots, W_n) \\ &= \frac{W_1 C_{j1} + W_2 C_{j2} + \cdots + W_n C_{jn}}{W_1 + W_2 + \cdots + W_n} \\ &= \sum_{i=1}^{n} W_i \cdot C_{ji} \Big/ \sum_{i=1}^{n} W_i .\end{aligned} \quad 8(a)$$

By the extension principle of Zadeh[12], the membership function of Y_j can be defined as

$$\mu_{Y_j}(y) := \sup_{\substack{y = f(c_{j1}, \ldots, c_{jn}, w_1, \ldots, w_n), \\ c_{ji} \in C_{ji}, w_i \in W_i, i = 1, \ldots, n}} \min\left\{\mu_{C_{ji}}(c_{ji}), \mu_{W_i}(w_i), i = 1, 2, \ldots, n \,\middle|\, y = \sum_{i=1}^{n} w_i \cdot c_{ji} \Big/ \sum_{i=1}^{n} w_i\right\}. \quad 8(b)$$

In order to find the FWA membership function $\mu_{Y_j}(y)$, a number of researchers have proposed appropriate methods[13-21]. By denoting the α-cuts of the fuzzy weights W_i and the

Table 1. Linguistic terms for criteria rating and importance weighting

Linguistic terms		
Criteria rating	Importance	Triangular fuzzy numbers
Very good (VG)	Very important (VI)	(0.833, 1.0, 1.0)
Good (G)	Important (I)	(0.667, 0.833, 1.0)
Medium good (MG)	Medium important (MI)	(0.5, 0.667, 0.833)
Medium (M)	Medium (M)	(0.333, 0.5, 0.667)
Medium poor (MP)	Medium unimportant (MU)	(0.167, 0.333, 0.5)
Poor (P)	Unimportant (U)	(0, 0.167, 0.333)
Very poor (VP)	Very unimportant (VU)	(0, 0, 0.167)

fuzzy criteria ratings C_{ji} as:

$$(W_i)_\alpha = \left[(w_i^L)_\alpha, (w_i^R)_\alpha\right],\ (C_{ji})_\alpha = \left[(c_{ji}^L)_\alpha, (c_{ji}^R)_\alpha\right] \quad (9)$$

where, $(c_{ji}^L)_\alpha$ and $(w_i^L)_\alpha$ represent the left end-points and $(c_{ji}^R)_\alpha$ and $(w_i^R)_\alpha$ the right end-points of $(C_{ji})_\alpha$ and $(W_i)_\alpha$, respectively. The α-cuts $(Y_j)_\alpha$ of the FWA for Eqn (8) is obtainable as:

$$(Y_j)_\alpha = \left[(y_j^L)_\alpha, (y_j^R)_\alpha\right] := \left[\min_{\substack{(c_{ji}^L)_\alpha \le C_{ji} \le (c_{ji}^R)_\alpha, (w_i^L)_\alpha \le W_i \le (w_i^R)_\alpha,\\ i=1,\dots,n}} f\left(C_{j1},\dots,C_{jn},W_1,\dots,W_n\right),\ \max_{\substack{(c_{ji}^L)_\alpha \le C_{ji} \le (c_{ji}^R)_\alpha, (w_i^L)_\alpha \le W_i \le (w_i^R)_\alpha,\\ i=1,\dots,n}} f\left(C_{j1},\dots,C_{jn},W_1,\dots,W_n\right)\right],\ \forall\alpha \in (0,1]. \quad (10)$$

For Eqn (10), the following can also be obtained. The proof can be found in Liou and Wang[15] and Chang[21], *et al*. due to the monotonicity of f wrt all supports of C_{ji}.

$$\min_{(c_{ji}^L)_\alpha \le C_{ji} \le (c_{ji}^R)_\alpha, (w_i^L)_\alpha \le W_i \le (w_i^R)_\alpha, i=1,\dots,n} f(C_{j1},\dots,C_{jn},W_1,\dots,W_n) = \min_{W_i \in \{(w_i^L)_\alpha, (w_i^R)_\alpha\}, i=1,\dots,n} f_{\mathrm{L}}(W_1,W_2,\dots,W_n), \quad 11(a)$$

$$\max_{(c_{ji}^L)_\alpha \le C_{ji} \le (c_{ji}^R)_\alpha, (w_i^L)_\alpha \le W_i \le (w_i^R)_\alpha, i=1,\dots,n} f(C_{j1},\dots,C_{jn},W_1,\dots,W_n) = \max_{W_i \in \{(w_i^L)_\alpha, (w_i^R)_\alpha\}, i=1,\dots,n} f_{\mathrm{R}}(W_1,W_2,\dots,W_n), \quad 11(b)$$

where one can define

$$f_{\mathrm{L}}(W_1,W_2,\dots,W_n) := f((c_{j1}^L)_\alpha,\dots,(c_{jn}^L)_\alpha,W_1,\dots,W_n) = \frac{W_1(c_{j1}^L)_\alpha + W_2(c_{j2}^L)_\alpha + \cdots + W_n(c_{jn}^L)_\alpha}{W_1 + W_2 + \cdots + W_n} \quad 12(a)$$

$$f_{\mathrm{R}}(W_1,W_2,\dots,W_n) := f((c_{j1}^R)_\alpha,\dots,(c_{jn}^R)_\alpha,W_1,\dots,W_n) = \frac{W_1(c_{j1}^R)_\alpha + W_2(c_{j2}^R)_\alpha + \cdots + W_n(c_{jn}^R)_\alpha}{W_1 + W_2 + \cdots + W_n} \quad 12(b)$$

For f_{L}, $C_{ji} = (c_{ji}^L)_\alpha$ for all i = 1, …, n and for f_{R}, $C_{ji} = (c_{ji}^R)_\alpha$ for all i = 1, …, n can be used in the correct results of the $(Y_j)_\alpha$.

Further, if one define the initial evaluations for min $\{f_{\mathrm{L}}\}$ and max $\{f_{\mathrm{R}}\}$ in Eqn (12) as:

$$\ell_0' := f_{\mathrm{L}}\left(W_1 = (w_1^L)_\alpha, W_2 = (w_2^L)_\alpha, \dots, W_n = (w_n^L)_\alpha\right) = \frac{(w_1^L)_\alpha (c_{j1}^L)_\alpha + (w_2^L)_\alpha (c_{j2}^L)_\alpha + \cdots + (w_n^L)_\alpha (c_{jn}^L)_\alpha}{(w_1^L)_\alpha + (w_2^L)_\alpha + \cdots + (w_n^L)_\alpha} \quad 13(a)$$

$$\rho_0' := f_{\mathrm{R}}\left(W_1 = (w_1^L)_\alpha, W_2 = (w_2^L)_\alpha, \dots, W_n = (w_n^L)_\alpha\right) = \frac{(w_1^L)_\alpha (c_{j1}^R)_\alpha + (w_2^L)_\alpha (c_{j2}^R)_\alpha + \cdots + (w_n^L)_\alpha (c_{jn}^R)_\alpha}{(w_1^L)_\alpha + (w_2^L)_\alpha + \cdots + (w_n^L)_\alpha}, \quad 13(b)$$

It should be clear now that the solution concept of $(Y_j)_\alpha$ of the FWA may turn to the evaluations, in which $W_i = (w_i^L)_\alpha$ should be substituted by $(w_i^R)_\alpha$ for improving ℓ_0' and ρ_0' to $\min\{f_{\mathrm{L}}\}$ and $\max\{f_{\mathrm{R}}\}$. Based on this concept, several approaches[14-16, 21-23] have been proposed for the correct FWA solution.

3. PROPOSED ALGORITHM

The purpose of the FWA algorithms is to facilitate operations and to increase the computational efficiency of the FWAs. With this objective, an enhanced fuzzy weighted average approach and its complexity are introduced.

3.1 An Enhanced Fuzzy Weighted Average Approach

According Guh[16], *et al*., two important observations hold on f_L and f_R for Eqns 12(a) and 12(b). They are: (i) for a higher criterion rating, the higher the corresponding weighting, the higher the calculated weighted average and (ii) for a lower criterion rating, the higher the corresponding weighting, the lower the calculated weighted average. Therefore, from these observations, which are also observed in the present research, it is obvious that

(a) for $\min\{f_{\mathrm{L}}\}$, if $(c_{jz}^L)_\alpha = \max_{\forall i}((c_{ji}^L)_\alpha)$, it should be determined having the highest weight (in $(W_{i=z})_\alpha = \left[(w_{i=z}^L)_\alpha, (w_{i=z}^R)_\alpha\right]$, i.e., $w_{i=z} = (w_{i=z}^R)_\alpha$, and if $(c_{jg}^L)_\alpha = \max_{\forall i}((c_{ji}^L)_\alpha)$, it should be determined with the lowest weight or $(w_{i=g}^L)_\alpha$.

(b) for $\max\{f_{\mathrm{R}}\}$, if $(c_{ju}^R)_\alpha = \max_{\forall i}((c_{ji}^R)_\alpha)$, it should be determined with the highest weight or $(w_{i=u}^{R})_\alpha$, and if $(c_{jv}^{R})_\alpha = \min_{\forall i}((c_{ji}^{R})_\alpha)$ it should be determined with the lowest weight ($(w_{i=v}^{L})_\alpha$).

This additional information is proposed in this study and is also used to improve the initial evaluations ℓ'_0 and ρ'_0.

Moreover, it is realised that the searches for $\min\{f_L\}$ and $\max\{f_R\}$ in Eqn (11) can be influenced by the support length or fuzziness too of the fuzzy weights in the FWAs, as their α-cuts (endpoints) will be utilised in the $\min\{f_L\}$ and $\max\{f_R\}$. Therefore, a further improvement of the initial evaluations may be developed here again by considering the averages of $(w_i^L)_\alpha$ and $(w_i^R)_\alpha$ of $(W_i)_\alpha$. Let $(w_i^{Avg})_\alpha = ((w_i^L)_\alpha + (w_i^R)_\alpha)/2$. The initial evaluations ℓ'_0 and ρ'_0 may be further improved as:

$$\ell_0 := \frac{\left((w_z^R)_\alpha \cdot (c_{jz}^L)_\alpha + (w_g^L)_\alpha \cdot (c_{jg}^L)_\alpha + \sum_{\substack{\forall i \in \{1,2,\ldots,n\} \text{ and} \\ i \notin \{z,g\}}} (w_i^{Avg})_\alpha \cdot (c_{ji}^L)_\alpha \right)}{\left((w_z^R)_\alpha + (w_g^L)_\alpha + \sum_{\substack{\forall i \in \{1,2,\ldots,n\} \text{ and} \\ i \notin \{z,g\}}} (w_i^{Avg})_\alpha \right)} \quad 14(a)$$

$$\rho_0 := \frac{\left((w_u^R)_\alpha \cdot (c_{ju}^R)_\alpha + (w_v^L)_\alpha \cdot (c_{jv}^R)_\alpha + \sum_{\substack{\forall i \in \{1,2,\ldots,n\} \text{ and} \\ i \notin \{u,v\}}} (w_i^{Avg})_\alpha \cdot (c_{ji}^R)_\alpha \right)}{\left((w_u^R)_\alpha + (w_v^L)_\alpha + \sum_{\substack{\forall i \in \{1,2,\ldots,n\} \text{ and} \\ i \notin \{u,v\}}} (w_i^{Avg})_\alpha \right)} \quad 14(b)$$

These initial evaluations ℓ_0 and ρ_0 may provide initial solutions for searching for the $\min\{f_L\}$ and $\max\{f_R\}$ in the solution approach of the FWA, which are better than those used in other FWA algorithms.

Furthermore, among the developed algorithms, Chang[21], *et al.* have proposed a natural recursive benchmark adjusting approach based on the initial evaluations defined in Eqn (13). This approach has been shown to possess the natural convergent efficient nature and is proven to be more efficient than all other algorithms of FWAs in the general case experiment (with 4,950 randomly generated FWAs). But, theoretically in the worst case, it still appears inferior to the algorithm of Guu[23], which applies a well-known technique, median-finding technology[24-25], originally used in arrays or sets. The Guu[23] algorithm has been proven to possess the least theoretical-worst-cased computational complexity among the existing algorithms of FWAs. However, in general, it is also inferior to Chang[21], *et al.*'s algorithm. Consequently, this paper proposes a newly developed algorithm by adopting the improved initial evaluations as developed in the last section and also a two-phase concept by extending and applying the algorithms of both Chang[21], *et al.* and Guu[23]. For convenience, hereinafter the enhanced FWA algorithm will be abbreviated as 'MBMFWA', where 'MBM' stands for moved benchmark and median meaning. Using the initial evaluations as ℓ_0 and ρ_0 (Eqns 14(a) and 14(b)), for $[(y_j^L)_\alpha, (y_j^R)_\alpha]$, Chang[21], *et al.*'s algorithm may be extended and also used as follows:

First, define these index sets:

$$I_0 = \left\{ i \in I \mid (c_{ji}^L)_\alpha < \ell_0 \text{ and } i \neq z \right\}$$

$$J_0 = \left\{ i \in I \mid (c_{ji}^R)_\alpha > \rho_0 \text{ and } i \neq u \right\} \quad 15(a)$$

where $I = \{1, 2, \ldots, n\}$. Then, define the index sets

$$I_p = \left\{ i \in I_{p-1} \mid (c_{ji}^L)_\alpha < \ell_p \right\},\ J_q = \left\{ i \in J_{q-1} \mid (c_{ji}^R)_\alpha > \rho_q \right\} \quad 15(b)$$

$$\Delta I_p = I_{p-1} \setminus I_p,\ \Delta J_q = J_{q-1} \setminus J_q, \quad 15(c)$$

and $p, q \geq 1$, recursively, where ℓ_p and ρ_q apply and update ℓ_{p-1} and ρ_{q-1} recursively as

$$\ell_p = f_L \left(\begin{array}{l} w_1, \ldots, w_i, \ldots, w_n \mid w_i = (w_i^R)_\alpha\ \forall i \in I_{p-1} \\ \text{and } w_i = (w_i^L)_\alpha\ \forall i \notin I_{p-1} \end{array} \right) \quad 16(a)$$

$$\rho_q = f_R \left(\begin{array}{l} w_1, \ldots, w_i, \ldots, w_n \mid w_i = (w_i^R)_\alpha\ \forall i \in J_{q-1} \\ \text{and } w_i = (w_i^L)_\alpha\ \forall i \notin J_{q-1} \end{array} \right) \quad 16(b)$$

and "\" stands for the element subtraction. The above equations (Eqns (15)-(16)) are performed until the natural conditions, $\Delta I_p = \varnothing$ and $\Delta J_q = \varnothing$, are reached. Therefore, the developed algorithm applies and extends the Chang[21], *et al.* algorithm by the improved initial benchmarks (evaluations) and the natural recursive benchmark adjustment at the first phase. In addition, ℓ_p and ρ_q constitute the natural improved benchmarks recursively for $(y_j^L)_\alpha$ and $(y_j^R)_\alpha$. In a certain number of iterations, if, however, (or) cannot be reached as the theoretical worst case may happen, phase 2 can be executed to improve its theoretical-worst-cased computational complexity as Guu's algorithm by switching to the Guu[23] algorithm. The algorithm of the proposed MBMFWA may be introduced as given in Section 3.2.

3.2 Analysis of the Complexity

The complexity of the proposed MBMFWA algorithm in the worst case can be proofed as shown in the *Appendix 1*. The proposed algorithm requires an $O(n)$ of complexity which is the best level achieved to date. In Table 2, the theoretical worst-case complexity and abbreviations of these algorithms[14-16,21-23] and the proposed MBMFWA, have summarised. Furthermore, more discussion through the worst cases and the general cases comparison may be provided as follows.

In the worst case, MBMFWA and MFWA[23] have the

Table 2. Complexity of each fuzzy weight average algorithm

	Dong and Wong[14]	Liou and Wang[15]	Guh[16], *et al.*	Lee and Park[22]	Guu[23]	Chang[21], *et al.*	Proposed algorithm
Method's abbreviation	VFWA	IFWA	PFWA	EFWA	MFWA	AFWA	MBMFWA
Complexity	$O(2^n)$	$O(n^2)$	$O(n^2)$	$O(n \log_2 n)$	$O(n)$	$O(n \log_2 n)$	$O(n)$

same level of complexity $O(n)$, which is better than all the other FWA algorithms. In addition, a general comparison is provided as follows:

In general cases, based on the experiment design by AFWA[21], 4,950-randomly-generated-FWA experiments were performed using the algorithms of MFWA[23], AFWA[21] and MBMFWA algorithm. Due to the huge quantities of data and computed results obtained, Figs 1 and 2 summarises the results by MFWA[23], AFWA[21] and MBMFWA algorithms. These figures depict the overall average numbers of calculations and overall average CPU time by these algorithms during the tests with different numbers of FWA-terms as shown.

These results show that the MBMFWA provides indeed an even more efficient approach for the FWAs. It is more efficient than MFWA[23] and also further improves AFWA[21].

4. ILLUSTRATIVE EXAMPLE

Sometimes, the key to winning the battle lies in how speedily the warfare commander decides the priority of the bridges to be built or repaired to effectively support the major combat forces as well as lower the military risks. Hence, this is an important issue in military operations; that is, how emergent bridgeworks are prioritised, may lead to military success.

Under combat conditions, the representations of heuristic knowledge from bridge engineers and the descriptions of the observed defects by bridge inspectors are usually in the form of natural language that contains intrinsic imprecision and uncertainty. Variable exceptional circumstances may influence a bridge commander or engineers' confidence in making decisions. Thus, the assessment of emergency bridgeworks can be characterized by imprecise or vague requirements. Fuzzy set theory and fuzzy logic have emerged as powerful ways of representing quantitatively and manipulating the imprecision in the prioritised bridgeworks. Fuzzy sets or fuzzy numbers can appropriately represent imprecise parameters, and can be manipulated through different operations of fuzzy sets or fuzzy numbers. Since imprecise parameters are treated as imprecise values instead of precise ones, the process will be more powerful and its results more credible.

This paper proposes to use MBMFWA approach that, as an aggregation, operates on the fuzzy numbers and obtains the final scores during the priority of emergency bridgework repair assessments. The entire process of assessment covers the following steps:

Step 1: Identify the criteria for emergency bridgeworks assessment

Step 2: Capture the fuzzy rating and fuzzy weighting of each criterion

Step 3: Compute the total fuzzy values of individual bridgeworks form the fuzzy weights and criteria rating matrix, and

Step 4: Perform the ranking operations and obtain the final priority.

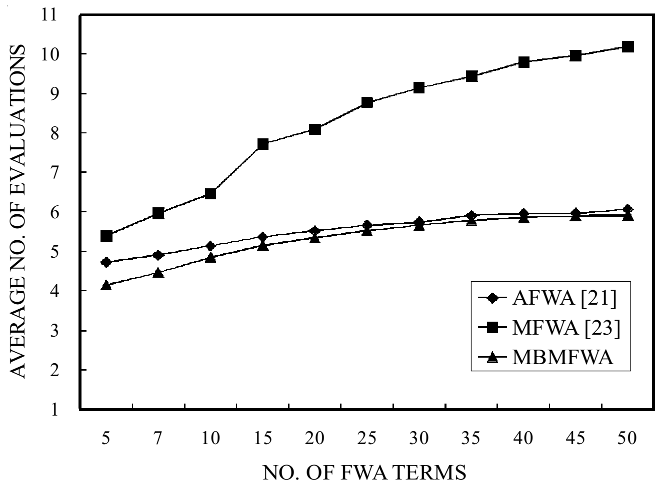

Figure 1. Overall average number of evaluations by the algorithms on different FWA terms.

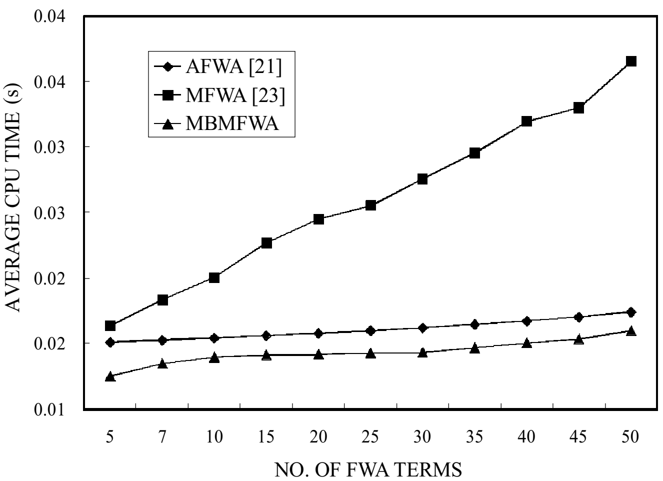

Figure 2. Overall average CPU time by the algorithms on the different FWA terms.

4.1 Criteria and Weights

The set of criteria for prioritised emergency bridgeworks have been extracted from the army tactical doctrine, the army corps of engineer operations, commanders and bridge engineering experts (with combined service in the department of engineering of over 20 years in the military) options to select the criteria of the priority emergency bridgeworks assessment, with six criteria. The relative criteria and the meanings can be defined in Table 3. Furthermore, following Table 1, the appropriate triangular fuzzy numbers are defined to capture the linguistic variables of criteria ratings and weighting rating. Thus, the levels of achievement in these six criteria rating and the relative importance of each criterion for five bridgeworks (BW_1, BW_2, BW_3, BW_4 and BW_5) are compiled as Table 4.

4.2 Aggregation using Enhanced FWA Algorithm

With the six criteria items, ratings and weights in Table 4, applying the proposed MBMFWA algorithm as an aggregated method, the FWA evaluation computation of the five bridgeworks can be performed with respect to each criterion and mapping weighting. One can obtain the entire final overall FWA scores of these bridgeworks, which are summarised in Table 5.

Table 3. Criteria for evaluating emergency bridge repairs under military consideration

Criteria	Notation	Meaning
Mission	C_1	Bridgework conducted by combat engineers may facilitate movement and logistics of friendly forces and impede that of enemies. In order to make a significant contribution to the mission, the priority of the emergency bridge repairs should be taken into account. Factors involved include maintenance of the bridges used as major supply line, attack line of the major force, attack line of the counterattack force, and attack line of the reserve force.
Defensive measures	C_2	While conducting bridgework, combat engineers are protected from enemy attack by friendly forces to keep them from injury or loss of life. Defensive measures taken include cover from fire and ground attack, air defence and nuclear biological chemical (NBC) countermeasures.
Combat environment	C_3	A composite of the conditions, circumstances, and influences of locality that affect the evaluation of emergency bridge repairs. Factors involved include geographic features, clearance of operational position, operated space, river width constraint, and the differences between both sides of the riverbank.
Time required	C_4	The amount of time from planning the emergent repairing to the completion of the bridgework. Whether the bridgework is accomplished within the time frame is affected by factors such as bridgework technique proficiency, bridgework complexity, the amount of bridgework manpower, and vehicles.
Material and equipment	C_5	Materials, special tools, or equipment requisite for bridge construction. Factors involved include standard or non-standard materials, available amount, and the availability ratio of bridgework equipment.
Requisitioning abilities	C_6	Manpower and material resources requisitioned in a certain time limit for bridgeworks. Factors involved include bridgework materials, affiliated machines, manpower for bridge engineering, and service.

Table 4. Evaluated characteristic capabilities of the five bridgeworks

Criteria	Weight	Bridgework's number				
		BW_1	BW_2	BW_3	BW_4	BW_5
C_1	VI (0.833, 1.0, 1.0)	M (0.333,0.5,0.667)	M 0.333,0.5,0.667)	P (0, 0.167, 0.333)	VG (0.833, 1.0, 1.0)	VG (0.833, 1.0, 1.0)
C_2	M (0.333,0.5,0.667)	MP (0.167,0.333,0.5)	M (0.333,0.5,0.667)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)
C_3	I (0.667,0.833,1.0)	MG (0.5,0.667,0.833)	M (0.333,0.5,0.667)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)
C_4	I (0.667,0.833,1.0)	VG (0833,1.0,1.0)	M (0.333,0.5,0.667)	MP (0.167,0.333,0.5)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)
C_5	MI (0.5,0.667,0.833)	VP (0, 0, 0.167)	P (0, 0.167,0.333)	MG (0.5,0.667,0.833)	P (0, 0.167,0.333)	G (0.667,0.833,1.0)
C_6	MU (0.167,0.333,0.5)	P (0, 0.167,0.333)	MG (0.5,0.667,0.833)	MG (0.5,0.667,0.833)	MP (0.167,0.333,0.5)	M (0.333,0.5,0.667)

Table 5. Overall FWA scores of the five bridgeworks

α-level	BW_1	BW_2	BW_3	BW_4	BW_5
	α -cuts of the overall FWA of the bridgeworks				
$\alpha = 1.0$	[0.5067, 0.5067]	[0.4600, 0.4600]	[0.4802, 0.4802]	[0.6402, 0.6402]	[0.7602, 0.7602]
$\alpha = 0.9$	[0.4860, 0.5267]	[0.4407, 0.4792]	[0.4599, 0.5016]	[0.6186, 0.6563]	[0.7401, 0.7752]
$\alpha = 0.8$	[0.4656, 0.5466]	[0.4212, 0.4982]	[0.4395, 0.5229]	[0.5969, 0.6723]	[0.7201, 0.7903]
$\alpha = 0.7$	[0.4453, 0.5665]	[0.4016, 0.5173]	[0.4190, 0.5442]	[0.5751, 0.6884]	[0.7001, 0.8052]
$\alpha = 0.6$	[0.4251, 0.5862]	[0.3820, 0.5362]	[0.3985, 0.5654]	[0.5533, 0.7045]	[0.6802, 0.8202]
$\alpha = 0.5$	[0.4050, 0.6059]	[0.3622, 0.5551]	[0.3778, 0.5865	[0.5313, 0.7206]	[0.6603, 0.8351]
$\alpha = 0.4$	[0.3851, 0.6255]	[0.3424, 0.5740]	[0.3570, 0.6076]	[0.5093, 0.7366]	[0.6405, 0.8499]
$\alpha = 0.3$	[0.3653, 0.6450]	[0.3224, 0.5928]	[0.3361, 0.6285]	[0.4872, 0.7527]	[0.6207, 0.8648]
$\alpha = 0.2$	[0.3456, 0.6644]	[0.3023, 0.6116]	[0.3152, 0.6495]	[0.4650, 0.7688]	[0.6010, 0.8795]
$\alpha = 0.1$	[0.3261, 0.6837]	[0.2821, 0.6303]	[0.2941, 0.6703]	[0.4427, 0.7849]	[0.5813, 0.8943]
$\alpha = 0.0$	[0.3067, 0.7029]	[0.2617, 0.6490]	[0.2728, 0.6911]	[0.4203, 0.8010]	[0.5616, 0.9089]

4.3 Ranking of the Final Results

In this study, the focus was on developing an easy and simple method. Therefore, it is proposed to use the area measurement method by Chen and Klein[26]. This method is based on an area measurement method, using α-cuts and employs a α-level fuzzy subtraction operation followed by area measurements. In this method, let Y_j denote the fuzzy priority index of j^{th} bridgework, and h denote the maximum membership height of μ_{Y_j} $j = 1, \ldots, n$. Suppose h is equally divided into m intervals such that $\alpha_r = rh/m$, $r = 0, \ldots, m$. Moreover, $(y_j^L)_{\alpha_r}$ and $(y_j^R)_{\alpha_r}$, $0 \le \alpha \le h$, denote the left and right bounds of j^{th} bridgework, respectively. Therefore, the Chen and Klein[26] method has devised the index for ranking fuzzy numbers

$$I(Y_j, \tilde{R}) = \frac{\sum_{r=0}^{m}((y_j^R)_{\alpha_r} - \theta)}{\left[\sum_{r=0}^{m}((y_j^R)_{\alpha_r} - \theta) - \sum_{r=0}^{m}((y_j^L)_{\alpha_r} - \eta)\right]} \quad (17)$$

where $\theta = \min\ \{ (y_j^L)_{\alpha_r} | r = 1,2,\ldots,m;\ j = 1, 2, \ldots, n;\ 0 \le \alpha \le h\}$, $\eta = \max\ \{ (y_j^R)_{\alpha_r} \big| r = 1, 2, \ldots, m; j = 1, 2, \ldots, n; 0 \le \alpha \le h\}$ and $\tilde{R}$ is the referential rectangle, which is obtained by multiplying the maximum height of the membership functions h by the distance between the crisp maximizing and crisp minimizing barriers. Here, can be regarded as a fuzzy number. The numerator and denominator of Eqn (17) are, respectively, approximations of the positive area and area of the difference fuzzy number $Y_j - \tilde{R}$. Larger values of the index of difference are preferred. In this paper, m is set to 10.

By applying Eqn (17) the ranking indexes for five bridgeworks are calculated. Thus, onee can obtain $BW_1 = 0.4058$, $BW_2 = 0.3489$, $BW_3 = 0.3790$, $BW_4 = 0.5483$ and $BW_5 = 0.6977$. The higher the ranking score, the more preferred the prioritised consideration. Consequently, the five bridgeworks can be ranked as $BW_5 \succ$ (prioritized to) $BW_4 \succ BW_1 \succ BW_3 \succ BW_2$. That is, BW_5 is the priority for emergency repairs. Thus, the final fuzzy evaluation and results may provide the commander with informative references for decision making.

5. COMPUTING-BASED INTERFACE

In order to make computing and ranking the results much easier and to increase the recruiting productivity for the commander or engineer, an information system called the bridgework emergency repairing decision support system (BERDSS), shown in Fig. 3, has been developed. This prototype system was developed with Visual Basic 6 and ACCESS on a *N*-tier client server architecture. In BERDSS, the decision maker first needs to key in the numbers of bridgework and criteria as shown in Fig. 4. Then operators also need to input the scores of each criterion and weighted values on each criterion of bridgework, as illustrated in Fig. 5. The system can calculate the evaluated value for each bridgework. The result is shown in Fig. 6. The score of ranking is the largest. Thus, the bridgework is the prioritised selection choice on which to perform emergent repairing.

6. CONCLUSIONS

In combat circumstances, a key factor in winning battles is that commanders are able to prioritise, in a speedy manner, the work orders of their bridges to be repaired so that efficient combat support can be achieved. Since the measures from the criteria and relative importance may be vague and uncertain, they are treated as linguistic values. The evaluation of these prioritised emergency bridgeworks can be carried out by the fuzzy sets theory and fuzzy weighted average approach. In this paper, an enhanced fuzzy weighted average algorithm called MBMFWA is proposed, and an application of this algorithm for

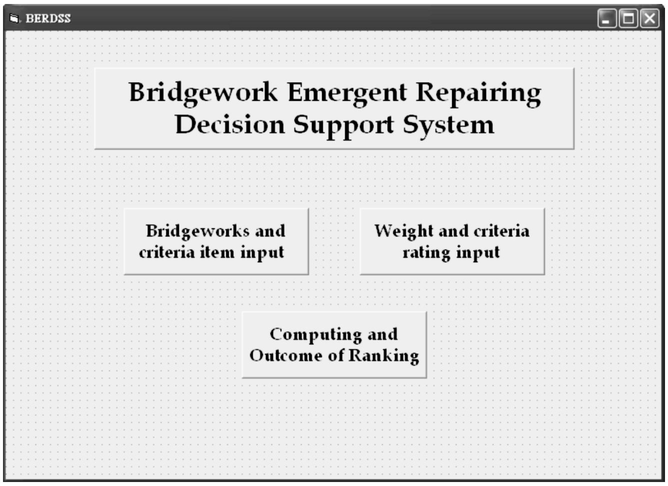

Figure 3. Functional interface of BERDSS.

prioritising emergency bridgeworks under military consideration is developed. The obtained results are also ranked through the ranking methods and provide appropriate references for the commanders. Furthermore, it has made computing and ranking the results much easier, and to increase the recruiting productivity, a computer-based BERDSS system has been developed to effectively aid commanders in dealing with fuzzy-set multi-criterion decision making problems. In future research, this approach will be extended to evaluate similar practical cases of multi-criterion decision problems in military contexts.

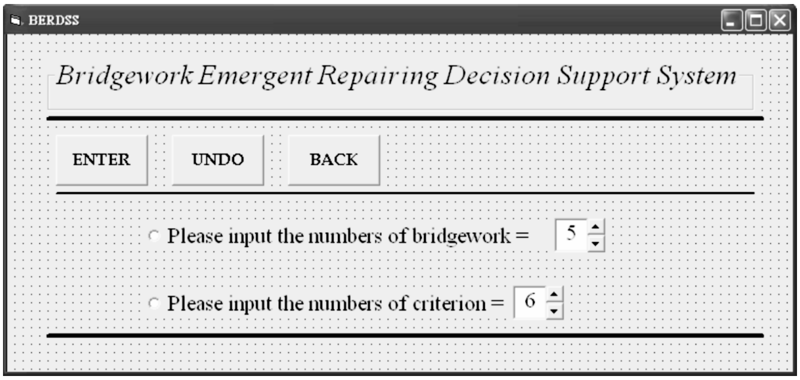

Figure 4. Input the numbers of bridgework and criterion.

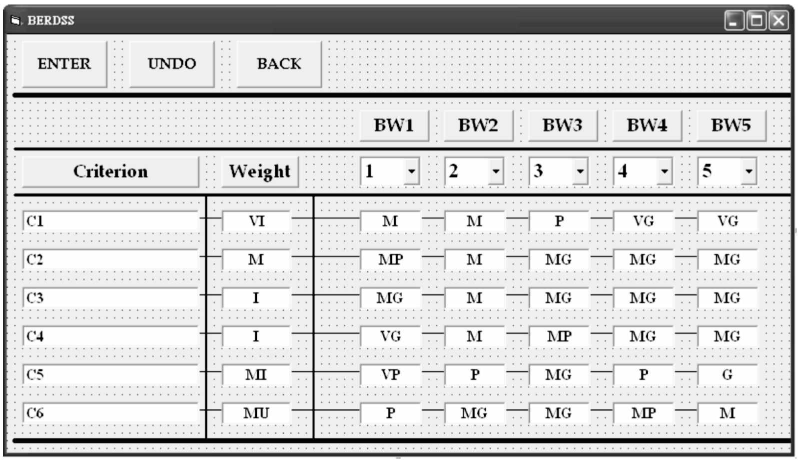

Figure 5. Input weight and input evaluation value of each bridgework on each criterion.

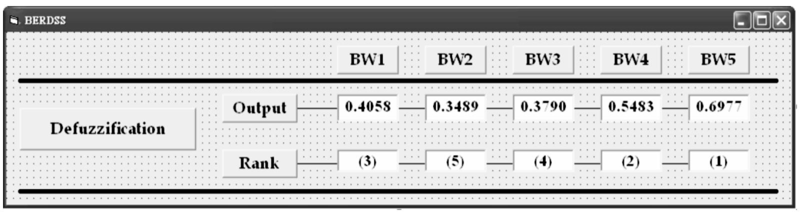

Figure 6. The outcomes of ranking by Chen and Klein's[26] ranking methods.

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Appendix 1

The proof for the complexity using the MBMFWA algorithm:

Theoretically the worst case for MBMFWA may be figured out as following:

First, for $\min\{f_L\}$ in the first phase, because of Chang[21], *et al.*'s algorithm, the worst case happens when the initial evaluation is larger than the largest-but-one term but less than the largest term, but $\min\{f_L\}$ is larger than the smallest term but less than the smallest-but-one term of $(c_{ji}^L)_\alpha$'s. In phase 2, therefore by Guu's algorithm, the worst case happens when the final evaluation from phase 1 and $\min\{f_L\}$ are both larger than the largest-but-one term but less than the largest term of the remaining $(c_{ji}^L)_\alpha$'s. For $\max\{f_R\}$, the worst case may be figured out analogously. Based on this theoretical worst case, the complexity of the MBMFWA may be proved as follows.

In this algorithm, Step 1) computes the initial benchmarks and requires one time evaluation of ℓ_0 and ρ_0. In the worst case, each step of 2.1) and 4.1) in phase 1 requires 4 times of evaluation of ℓ_p (ρ_q) because p, $q < 5$, and also in the worst case set ΔI_p (ΔJ_q) has only one element. In phase 2, each of the Steps 2.4) and 4.4) requires $\log_2(n-6)$ times of evaluation of ℓ_p (ρ_q) according to the median finding technology (Blum[24], *et al.*; Gurwitz[25]) since theoretically, in the worst case, $\log_2(n - 6)$ times of evaluation for $n - 6$ element searches may be required. Overall, the number of evaluations is $2 + 2 \times 4 + 2(\log_2(n-6))$. Furthermore, the arithmetical operations of these steps may be figured as: in phase 1 when $p = 0$ and $q = 0$, ℓ_0 (ρ_0) each requires $2(n - 1)$ additions, n multiplications and one division. For p and $q = 1$, ℓ_1 (ρ_1) each requires at most $2(n - 2)$ additions, $2(n - 2)$ subtractions, $(n - 2)$ multiplications, and one division. For p and $q = 2, 3, 4$, ℓ_p (ρ_q) each requires at most 4 subtractions, one multiplication and one division due to the worst case where ΔI_p (ΔJ_q) has only one element. Also, each of the Steps 2.4) and 4.4) in phase 2 when p $(q) = 5, 6, \ldots, \log_2(n - 6)$ requires at most $2(n-6)/2^p$ additions, $2(n-6)/2^p$ subtractions, $(n-6)/2^p$ multiplications and one division for ℓ_p (ρ_q) due to the worst case that $I_p^{(1)}$ $(I_q^{(1)})$ in the algorithm has at most $(n - 1)/2^p$ elements according to (Blum[24], *et al.*; Gurwitz[25]). Thus, the total number of arithmetical operations in the worst case is

$$2[2(n-1)+n+1]+2[2(n-2)+2(n-2)+(n-2)+1]$$
$$+\,2[3\times(4+1+1)]+2\sum_{p=5}^{\log_2(n-6)}\left(\frac{5(n-6)}{2^p}+1\right)$$

$$=2\left[\begin{array}{l}8n+8+\displaystyle\sum_{p=1}^{\log_2(n-6)}\left(\frac{5(n-6)}{2^p}\right)-\\ \displaystyle\sum_{p=1}^{4}\left(\frac{5(n-6)}{2^p}\right)+\log_2(n-6)-4\end{array}\right]$$

$$=\frac{133n+34}{8}-\frac{10(n-6)}{2^{\log_2(n-6)}}+2\log_2(n-6),$$

for $n \geq 6$, which is obviously less than $\frac{133n+34}{8}+2\log_2(n-6)$. For $6 > n \geq 2$, it is $2(8n + 8)$ for phase 1. Therefore, the complexity is $O(n)$.

MBMFWA- algorithm

Step 1: Compute the initial benchmarks ℓ_0 and ρ_0 (Eqns 14(a) and 14(b)).

Let $I = \{1, 2, \ldots, n\}$, $I_0 = \left\{i \in I \mid (c_{ji}^L)_\alpha < \ell_0 \text{ and } i \neq z\right\}$

$J_0 = \left\{i \in I \mid (c_{ji}^R)_\alpha > \rho_0 \text{ and } i \neq u\right\}$, and $p = q = 1$.

If $I_0 = \varnothing$ then $\ell_0 = (y_j^L)_\alpha = \min\{f_L\}$. *If* $J_0 = \varnothing$ then $\rho_0 = (y_j^R)_\alpha = \max\{f_R\}$.

If $I_0 = \varnothing$ and $J_0 = \varnothing$, stop
else if $I_0 \neq \varnothing$ and $J_0 = \varnothing$ then go to *Step 2*
else if $I_0 = \varnothing$ and $J_0 \neq \varnothing$ then go to *Step 3*
else go to *Step 2*
end

Step 2: For $\min\{f_L\} = (y_j^L)_\alpha$ *:*

2.1 If $p = 1$ then $\ell_{p=1} := \beta_{L,1}/\gamma_{L,1} =$

$$\left.\left(\begin{array}{l}\beta_{L,0}+\displaystyle\sum_{i\in I_0}\left((w_i^R)_\alpha-(w_i^{Avg})_\alpha\right)\cdot(c_{ji}^L)_\alpha-\\ \displaystyle\sum_{i\notin I_0 \text{ and } i\neq g}\left((w_i^{Avg})_\alpha-(w_i^L)_\alpha\right)\cdot(c_{ji}^L)_\alpha\end{array}\right)\right/$$
$$\left(\gamma_{L,0}+\sum_{i\in I_0}\begin{array}{l}\left((w_i^R)_\alpha-(w_i^{Avg})_\alpha\right)-\\ \displaystyle\sum_{i\notin I_0 \text{ and } i\neq g}\left((w_i^{Avg})_\alpha-(w_i^L)_\alpha\right)\end{array}\right)$$

else

$\ell_p := \beta_{L,p}/\gamma_{L,p} =$

$$\left.\left(\beta_{L,p-1}-\sum\nolimits_{i\in\Delta I_{p-1}}\left((w_i^R)_\alpha-(w_i^L)_\alpha\right)\cdot(c_{ji}^L)_\alpha\right)\right/$$
$$\left(\gamma_{L,p-1}-\sum\nolimits_{i\in\Delta I_{p-1}}\left((w_i^R)_\alpha-(w_i^L)_\alpha\right)\right)$$

end

2.2 Compute $I_p = \left\{ i \in I_{p-1} \,|\, (c_{ji}^L)_\alpha < \ell_p \right\}$ and $\Delta I_p = I_{p-1} \setminus I_p$.
If $\Delta I_p = \varnothing$ then $\ell_p = (y_j^L)_\alpha = min\{f_L\}$ and stop *Step(2)*
else
let $p = p + 1$
if $p < \psi$ then return to *Step (2.1)*
else
go to *Step (2.3)*
end
end

2.3 Find $(c_{jk}^L)_\alpha = MEDIAN\{ (c_{ji}^L)_\alpha \mid i \in I_{p-1}\}$

and let $I_p^{(1)} = \left\{ i \in I_{p-1} \,|\, (c_{ji}^L)_\alpha < (c_{jk}^L)_\alpha \right\}$,

$I_p^{(2)} = \left\{ i \in I_{p-1} \,|\, (c_{ji}^L)_\alpha \geq (c_{jk}^L)_\alpha \right\}$ and

$I_p^{(3)} = \left\{ i \in I_{p-1} \,|\, (c_{ji}^L)_\alpha > (c_{jk}^L)_\alpha \right\}$.

2.4 Compute $\ell_p = \beta_{L,p}/\gamma_{L,p} =$

$$\left(\beta_{L,p-1} - \sum_{i \in I_p^{(2)}} \left((w_i^R)_\alpha - (w_i^L)_\alpha\right) \cdot (c_{ji}^L)_\alpha \right) \Big/ \left(\gamma_{L,p-1} - \sum_{i \in I_p^{(2)}} \left((w_i^R)_\alpha - (w_i^L)_\alpha\right)\right)$$

If $\ell_p < (c_{jk}^L)_\alpha$ then
$I_p = I_{p-1} \setminus I_p^{(2)}$ and let $p = p + 1$, then return to *Step (2.3)*
else
if $I_p^{(3)} \neq \varnothing$ then let $(c_{jt}^L)_\alpha = min\left\{ (c_{ji}^L)_\alpha \,\middle|\, i \in I_p^{(3)} \right\}$
if $\ell_p > (c_{jt}^L)_\alpha$ then
$I_p = I_{p-1} \setminus I_p^{(1)}$, $\beta_{L,p} = \beta_{L,p-1}$, $\gamma_{L,p} = \gamma_{L,p-1}$
and let $p = p + 1$, then return to *Step (2.3)*
else
$\ell_p = (y_j^L)_\alpha = min\{f_L\}$ and stop *Step (2)*
end
else
$\ell_p = (y_j^L)_\alpha = min\{f_L\}$ and stop *Step (2)*
end
end

Step 3: If $J_0 \neq \varnothing$ then go to *Step (4)*
else
stop all steps
end

Step 4: Analogous to *Step (2),* for solving the $max\{f_R\} = (y_j^R)_\alpha$ by iterative operations.