VALUING AMERICAN PUT OPTIONS USING CHEBYSHEV POLYNOMIAL APPROXIMATION

Guglielmo Maria Caporale
Brunel University, London

Mario Cerrato
London Metropolitan University

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Abstract

This paper suggests a simple valuation method based on Chebyshev approximation at Chebyshev nodes to value American put options. It is similar to the approach taken in Sullivan (2000), where the option`s continuation region function is estimated by using a Chebyshev polynomial. However, in contrast to Sullivan (2000), the functional is fitted by using Chebyshev nodes. The suggested method is flexible, easy to program and efficient, and can be extended to price other types of derivative instruments. It is also applicable in other fields, providing efficient solutions to complex systems of partial differential equations. The paper also describes an alternative method based on dynamic programming and backward induction to approximate the option value in each time period.

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Corresponding author: *Professor Guglielmo Maria Caporale, Brunel Business School, Brunel University, Uxbridge, Middlesex UB8 3PH, UK. Tel.: +44 (0)1895 266713. Fax: +44 (0)1895 269770. Email: Guglielmo-Maria.Caporale@brunel.ac.uk*

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1. Introduction

American options are difficult to price because they can be exercised at any time during their lifetime, and therefore closed-form solutions for their values cannot be obtained. The binomial method introduced by Cox, Ross, and Rubinstein (1979) is still the most widely used valuation model, because it is easy to implement and produces reasonably accurate results. However, it has a major drawback, namely the fact that a high degree of accuracy can only be achieved with a high number of time steps, which reduces the computational speed and results in considerable efficiency costs. Furthermore, it cannot be used to price certain derivatives such as, for example, options on arithmetic average[1].

Various alternative methods have been suggested to enhance the computational efficiency of American option pricing in comparison to the binomial model and overcome its drawbacks. For instance, Breen (1991) proposes the accelerated binomial model, and shows that a satisfactory degree of accuracy can be achieved using only 150 steps. However, his model has the limitation that it is based on the extrapolation method of Geske and Johnson (1984), which may give rise to non-uniform convergence. Ju and Zhong (1999) derive an approximation formula to price American options along the lines of Barone-Adesi and Whaley (1987). They show that their approach produces accurate prices and is computationally efficient, though it has the drawback of not achieving convergence. More recently, Sullivan (2000) puts forward a method combining Gaussian quadrature and Chebyshev polynomial approximation to achieve the best combination of accuracy and computational efficiency in pricing options. He presents numerical evidence that this technique delivers a price that is as accurate as that generated by finite difference methods, and, on average, even faster. Finally, Longstaff and Schwartz (2001) suggest to combine Least Squares with Monte Carlo techniques to price American options. Their method is very easy to programme and can be extended to price American-Bermuda-Asian options and many others. However, the computational time it requires for achieving an accurate price is considerable. Moreover, very little is known (apart from

[1] Also, in a number of cases it is very difficult if not impossible to use the binomial model to price options on geometric average, or options written on more than three assets. The same criticism applies to finite difference methods.

their Proposition 1) about the asymptotic properties and convergence of the proposed Least Square estimator.

The method suggested in this paper is similar to Sullivan's (2000) in three respects, namely: our model is based on the risk-neutral pricing relation used by the binomial method; we consider a continuous (rather than a discrete) process generating stock prices; we keep the log-normal density instead of replacing it with the binomial one. However, in contrast to Sullivan (2000), we fit the functional at Chebyshev nodes[2]. The advantage of our approach is its flexibility, and the fact that it is easily implementable also for pricing options written on multiple assets or other kinds of options (i.e. geometric or arithmetic average options), similarly to the Longstaff and Schwartz (2001) method. In addition, it is applicable in other fields, providing efficient solutions to complex systems of partial differential equations. These features make our approach very attractive. One reason why polynomial approximations of this type are underutilised (in comparison to direct *ad hoc* approximation methods) by applied researchers might be lack of familiarity. Therefore, in Section 3, we provide some guidance on how to use them to solve systems of differential equations. We also describe an alternative method based on dynamic programming to approximate the value of the option in each time period.

The layout of the paper is the following. Section 2 describes the option pricing valuation model. Section 3 outlines the approximation method we advocate to obtain the solution to the option pricing problem. Section 4 evaluates its empirical performance by comparing it to the benchmark and other methods in terms of both accuracy and computational efficiency. Section 5 summarises the main findings of this study and offers some concluding remarks.

[2] Tzavalis and Wang (2003) use a similar approach based on Chebyshev approximation to approximate the optimal exercise boundary in the context of a stochastic volatility model. Their method also relies on extrapolation procedures.

2. The Valuation Model

American options have the feature of allowing the holder to exercise them before the expiration date, which gives rise to serious difficulties for valuation models. As already mentioned, at present the Cox, Ross, and Rubinstein (1979) binomial model for approximating the option value function is still seen as the benchmark technique against which alternative approaches should be evaluated. In this section we propose a simple and flexible model to evaluate American put options that is similar to the one used in the analysis of investment under uncertainty (see Dixit, 1992 and Pindyck, 1991).

Suppose that the price of a non-dividend-paying asset in period 0 is P_0, and denote with K the strike price of a put option written on that asset. Assume that from time 1 onwards the price of the asset is $P_1 \rightarrow P_0 u(t_1)$ with probability q, and $P_1 \rightarrow P_0 / u(t_1)$ with probability $1-q$, where $u = \exp^{(\sigma\sqrt{\Delta t})}$, $q = \frac{1}{2} + \frac{\sqrt{\Delta t}}{2\sigma}(r - \frac{1}{2}\sigma^2)$, r is the annualised interest rate and σ the annualised volatility of the asset price. The following is also assumed:

Assumption 1: The option can only be exercised today or never.

Under Assumption 1, and if we denote with V_0 the present value from exercising the option, then:

$$V_0 = P_0 + [P_0 uq + P_0 / u(1-q)][\frac{1}{1+r} + \frac{1}{(1+r)^2}...]$$

$$= P_0 + [P_0 uq + P_0 / u(1-q)][\frac{1}{r}]$$

Therefore if $V_0 < K$ the option is in the money, and if exercised it generates the following payoff:

$$\pi_0 = \max(K - V_0, 0). \quad (1)$$

Assumption 2: The holder does not exercise the option but instead waits until the next period when new information becomes available.

Assumption 3: Given the above probability structure of the model, the next period's asset price P_1 will stay the same in periods 2,3…

Under assumptions (2)-(3), discounting back the income stream we have:

$$V_1 = P_1(1+r)/r$$

For each movement in the price P, the holder will exercise the option if $V_1 < K$, with the following payoff:

$$F_1 = \max(K - V_1, 0) \tag{2}$$

Combining equations (1) and (2) the net present value, in period 0, from exercising the option is:

$$F_0 = \max(K - V_0, \frac{1}{1+r} E_0(F_1)) \tag{3}$$

where[3] $E_0(F_1) = q\max[p_0 u(1+r)/r - K, 0] + 1 - q\max[p_0 / u(1+r)/r - K, 0]$

Assumption 4: In any period t, the holder observes the state of the system s_t, and takes an action x_t, earning a reward $V_t(s_t, x_t)$.

Assumption 5: $s_t/(t+1)$ is a closed set containing all the information on P at $t+1$, such that $P_r(s_{t+1}/s_t) = P_r^{'}(s_t, x_t)$. That is, the probability of next period`s state, conditional on current information, depends only on the current state and the investor`s actions.

Assumption 6: In period $t+1$, the state is s_{t+1}, the holder takes an action and the outcome is defined by $V_{t+1}(s_{t+1}, x_{t+1})$.

[3] Note that we take the expectation of F since future payoffs are unknown.

Under assumptions (4)-(6), exercising the option in period t results in the following payoff[4]:

$$V_t(x_t, s_t) + \frac{1}{1+r} E_t[V_{t+1}(s_{t+1})] \qquad (4)$$

Therefore the holder will choose an x_t to maximise:

$$V_t(s_t) = \max_{x_t} \{V_t(x_t, s_t) + \frac{1}{1+r} E_t[V_{t+1}(s_{t+1})]\} \qquad (5)$$

As shown in Dixit and Pindyck (1994), the Bellman equation for this optimal stopping time has a solution in continuous time that takes the following form:

$$rV\ (s,t) = \max_x [V(x,s,t) + \frac{1}{dt} E(dV)] \qquad (6)$$

with $(\frac{1}{dt})E(dV) = \lim \Delta t_{\rightarrow\infty} \frac{E(\Delta V)}{\Delta t}$

Equation (6) represents a "fundamental arbitrage condition" that is interpreted as saying that, in order for the holder to keep the option, he must earn at least the risk-free return[5].

The solution of the Bellman equation given by (6) is the value of the option. However, as it stands, equation (6) is of little practical use, unless $E(dV)\frac{1}{dt}$ can be evaluated. We propose a method based on approximating this function in order to obtain an approximate solution[6].

Suppose the process for s is described by the following geometric Brownian motion:

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[4] The first part of equation (4) is the value of the option if immediately exercised, whilst the second describes its continuation value.
[5] The right-hand side of (6) represents the total return to the holder from holding the option.
[6] Abadir and Rockinger (2003) advocate a useful technique to fit options data to extrapolate risk-neutral densities.

$$ds = \mu s dt + \delta s dz$$

where dz is a standard increment of a Wiener process, μ is a drift parameter and δ the variance parameter.

We can expand $E(dV)\frac{1}{dt}$, using Ito`s Lemma and the stochastic process above to obtain:

$$rV(s,t) = \max_x \, [V(x,s,t) + rsV_s + V_t + \frac{1}{2}\delta^2 s^2 V_{ss})] \qquad (7)$$

If we set $V(s,t) \approx \phi(s)c(t)$, where ϕ is a suitable basis for an n-dimensional family of approximating functions and $c(t)$ is an n-vector of time-varying coefficients, equation (7) can be re-written as follows[7]:

$$\phi(s)c'(t) \approx [rs\phi'(s) + \frac{1}{2}\delta^2 s^2 \phi''(s) - r\phi(s)]c(t) \approx \psi(s)c(t) \qquad (8)$$

To determine $c(t)$, we select n-values of s, s_i, and solve (8) for that particular set of values**.** Given the n-dimensional family of basis functions chosen, (8) can now be re-written in the form of a system as follows:

$$\Phi c'(t) = \Psi c(t) \qquad (9)$$

where Φ and Ψ are two $n \times n$ matrices.

Therefore the solution to our option pricing problem is given by:

$$V(s,t) \approx \Phi^{-1} \max[V(s,t), \Phi(s)c'(t)] \qquad (10)$$

[7] See Appendix 1 for a proof.

3. Polynomial Approximation

In the previous section we suggested solving the Bellman equation, which describes the option pricing problem, by using approximations. In this section we describe in greater detail the approximation method adopted in this paper.

Let $V \in \Re^{n+1}$ be a function defined on the interval $[a,b]$ that is not tractable analytically, and assume that P is a polynomial that interpolates V at the distinct $n+1$ points $s_i \in [a,b]$, with $P(s)=\sum_{i=0}^{n} c_i \phi_i(s)$. In order to solve the option pricing problem by approximation we need to define: (a) the family of basis functions to approximate the function V, (b) the interpolation nodes, s_i. In this section we show that Chebyshev polynomials in conjunction with Chebyshev nodes offer the best solution to our problem.

Theorem 1: if $V \in \Re[a,b]$, then for all $\varepsilon > 0$ there exists a polynomial $P(s)$ such that $\forall s \in [a,b] \, | V(s) - P(s) | \leq \varepsilon$.

Remark 1. The above theorem is known as the Weierstrass theorem. It states that any continuous function can be approximated with a certain degree of accuracy by using a polynomial. Although very important theoretically, this theorem is of little practical use since it does not give any indication of what polynomial is the most appropriate to use, or even what order polynomial is needed to achieve a certain degree of accuracy.

The error made by using a polynomial of order n to approximate the function given in Theorem 1 can be easily calculated as:

$$V(s) - P(s) = \frac{1}{n+1} V^{(n+1)}(\varepsilon) \prod_{i=0}^{n} (s - s_i)$$

The objective of using such an *efficient* polynomial consists in choosing a set of nodes s_i so as to make the term $\prod_{i=0}^{n}(s-s_i)$ as small as possible (Kenneth, 1998). One possibility is to approximate the function V at the n-evenly spaced nodes. However, it is well known that in general, even for smooth functions, polynomials of this type do not produce very good approximations.[8] Therefore, we suggest approximating the function over the interval $[a,b]$, at the Chebyshev nodes defined as:

$$s_i = \cos(\frac{2i+1}{2n+2}\pi), i = 0,1,\ldots,n$$

Our approach can be justified by appealing to Rivlin's theorem, stating that Chebyshev node polynomial interpolants are nearly optimal polynomial approximants (Rivlin, 1990), and has been shown to perform well empirically (Rivlin, 1990). Chebyshev nodes are also known to possess a further convenient property, i.e. equi-oscillation [9](Kenneth, 1998).

As important as the choice of the nodes interpolants is that of a family of functions from which the approximant P will be drawn. We suggest using a Chebyshev polynomial. This is defined as[10]:

$$\Gamma_i(s) = \cos(ia\cos(s)) \quad i = 0,1,\ldots,n$$

with $\Gamma_0(s) = 1$, $\Gamma_1(s) = s$, and $\Gamma_{n+1}(s) = 2s\Gamma_n - \Gamma_{n-1}(s)$

Therefore:

$$V(s) = \sum_{i=0}^{n} c_i \Gamma_i(s) \qquad (11)$$

where $c_0 = \frac{1}{n+1}\sum_{i=0}^{n} V(s_k)$ and $c_i = \frac{2}{n+1}\sum_{i=0}^{n} V(s_k)\cos(ia\cos(s_k))$, $i = 1,\ldots,n$

[8] A classic example is Runge`s function (Rivlin, 1990).

[9] This property states that the maximum error of a cubic function, for example, shall be reached at least five times, and the sign of this error should alternate between the interpolation points.

[10] Note that in this application we use the general formula for the Chebyshev basis, however there exists also a recursive formula.

A Chebyshev basis polynomial, in conjunction with Chebyshev interpolation nodes, produces an efficient interpolation equation which is very accurate and stable over n. Furthermore, such a polynomial should be able to replicate, not just the function V at $s_1, s_2, ..., s_n$, but also its derivatives $s_1^{'}, s_2^{'}, ..., s_n^{'}$. Therefore the approximant that solves our problem can be defined as follows[11]:

$$\sum_{i=1}^{n} c_i \Gamma_i(s) = V(s_i), \forall_i = 1, ..., n_1$$
$$\sum_{i=1}^{n} c_i \Gamma'_i(s') = V'(s_i), \forall_i = 1, ..., n_2$$

with $n_1 + n_2 = n$.

Once the basis functions (approximants) have been chosen and the approximant nodes defined, the basis coefficients c_i can be obtained. If we define the following Chebyshev-Vandermode type matrix T:

$$\mathrm{T} = \begin{bmatrix} \Gamma_0(s_1) & \Gamma_1(s_1) & . & ...\Gamma_{n-1}(s_1) \\ \Gamma_0(s_2) & \Gamma_1(s_2) & . & \Gamma_{n-1}(s_2) \\ . & . & . & . \\ \Gamma_0(s_n) & \Gamma_1(s_n) & . & \Gamma_{n-1}(s_{n)} \end{bmatrix}$$

then the coefficients c_i; $c = (c_0, c_1, ..., c_{n-1})'$ of $V(s)$ solve $\mathrm{T}c = V$, with $\Gamma_{ij} = \Gamma_i(s_i)$ being the j basis function evaluated at the i-th interpolation node. When s is allowed to vary over some other interval, say $[t, T] \neq [-1,1]$, we rescale the value of s to s^* where $s^* = \frac{1}{2}((T - t)s + (T + t))$ [12].

[11] Note that, although one can also use Hermite polynomials to approximate the functional and the slopes, the latter are inefficient (Kenneth, 1998).

[12] An interesting issue here is the non-singularity of the Vandermode matrix over Chebyshev basis as above. In theory, there is no guarantee that the matrix is non-singular. However, in practice, in general applications such as ours, we can conjecture that as long as the number of indeterminates exceeds the sparsity with respect to T, non-singularity should hold. Alternatively, we suggest two ways to overcome

As an example of using different basis functions, consider equation (8) to price a European option. In this case, the ordinary differential equation has a known solution and one can use it to calculate the approximation error. We use three different basis functions (i.e. Chebyshev basis, spline basis, and linear spline basis). The approximation error is shown in Figures (1-3).

Insert Figures (1-3) here

As can be seen, when the approximation is calculated using Chebyshev basis functions the error is of the order of 1×10^{-7} for a polynomial of order 10. Spline and linear spline functions do not achieve a comparable degree of accuracy even increasing the order of the polynomial to 30.

4. Numerical Results and Comparisons

To evaluate the empirical performance of the proposed method we compare it in this section, in terms of both accuracy and efficiency, to the benchmark, i.e. the binomial tree model with 10,000 time steps, as well as the analytical approximation of Barone-Adesi and Whaley (1987 - BAW henceforth), and a method based on solving equation (3) by dynamic programming (DP), which is explained below. We use the root mean squared errors (RMSE) as a measure of accuracy. Computational efficiency is measured in CPU time (seconds) required to compute the price of the entire set of options.

Table 1 shows the results for the entire set of options considered. We specify the set of parameters as in Ju and Zhong (1999). We also report the results using the Black and Scholes method (1973 - B&S henceforth)[13].

Insert Table 1 here

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the problem: (a) simply use the singular value decomposition of T; (b) use the generalised Vandermode matrix over Chebyshev. In fact, for this type of matrix Werther (1993) proves that, as long as the indeterminates take a value $[1, \infty]$, the generalised Vandermode matrix over Chebyshev basis is non-singular.

As can be seen, the BAW (1987) method is quite accurate, but the method based on dynamic programming gives the best results. The latter requires approximating the Bellman equation in (3) using backward recursion and specifying a reward function f, and a transition probability function P. Once the terminal value function, v_{T+1}, has been set, one can easily solve equation (3) by recursively calculating at each step the optimal value function v_t and the state of the system x_t. v_1 will give the price of the option.

Table 1 also shows that the DP method produces very similar results to those generated by the binomial tree with 10,000 time steps. In other words, it is qualitatively equivalent to the accelerated binomial tree method proposed by Breen (1991), but it has the advantage that it does not rely on extrapolation techniques[14].

Figure (4) shows the solution to the option price problem using dynamic programming for different values of the underlying stock.

Insert Figure 4 here

Table 1, column 7 shows the option prices obtained using the least squares Monte Carlo approach suggested by Longstaff and Schwartz (2001 – LS henceforth). Clearly, at least in this simple case and for this particular parameter specification, their method does not outperform the alternative ones previous methods above [15] [16]. As in Stentoft (2003), we

[13] Although the Black and Scholes (1973) method does not apply to American-style options, we decided also to report option prices calculated in this way for the sake of completeness.

[14] Breen (1991) uses a Richardson extrapolation procedure. Such procedures (e.g., Richardson`s or Geske and Johnson`s) are known to have a problem of non-uniform convergence which, although it might not matter in practice (see Breen, 1991), remains a serious difficulty on a theoretical level. To solve this problem, Chang et al (2002) suggest a modified Richardson extrapolation method.

[15] We have used Laguerre basis as well as exponential basis. The number of replications was set equal to 100,000 and the number of time steps to 50. Laguerre basis seems to produce a more accurate price. Following LS (2001), we report options prices obtained by using Laguerre basis. There have also been other applications of the LS (2001) method, as, for example in Moreno and Novas (2001); however, these authors only apply it to obtain the option price for one set of parameters. In our opinion, option pricing methods should be tested by applying them for various parameter specifications, as in this paper.

[16] However, we would stress again that the LS (2001) method has the advantage of being applicable also in the case of more complex derivative instruments. In practice, in order to achieve high accuracy, one could apply it for each option n times (say n = 100), use different seeds in the Monte Carlo simulations, and

find non-monotonicity in the convergence of the option price given by this model towards the true option price. This limits the usefulness of Proposition 1 in LS (2001).

Finally, in the last column of Table 1, we report option prices obtained by using our suggested methodology. We use the first three (plus an intercept) Chebyshev basis to estimate the parameters c_i in (11)[17]. We assume 50 time steps and 100,000 Monte Carlo replications to generate stock prices. The basis number has been chosen using Theorem 6.4.2 in Kenneth (1998). We note that, although our proposed method produces less accurate prices than the Binomial, the BAW (1997) and the DP methods, its RMSE appears to be in the bid-ask spread range for traded stock options. Further, as already pointed out, in common with the LS (2001) method, and in contrast to the other methods presented above, it is easily implementable to price more complex derivatives. Finally, it appears to outperform the LS (2001) method.

We also verify that the function we have approximated is smooth across the early exercise boundary, that is $\frac{dP}{ds}(s_p(t-T,t)=-1$, where $s_p(t)$ is the critical value for which the value of the option is equal to its exercise price. This can be done by plotting $\frac{dP}{ds}$ against s near the exercise boundary. Figure (5) shows the plot for the case $t-T=0.5$. We estimated that, for this value of $t-T$, $\frac{dP}{ds}=-0.9985$.

Next, we assess the computational efficiency of the algorithm we use in comparison with other methods. The routines were written in MatLab 6.0 and run on a Pentium 4 1.6GHz-M, 256MB. As already pointed out by several researchers (see for example Breen, 1991), efficiency depends highly on the particular hardware/software used. To partially tackle the problem we decided to calculate the CPU time over the entire set of options listed in the Table 1. The results are reported in Table 2.

Insert Table 2 here

finally use the average as an estimate for the option price. This is likely to result in an efficiency loss, though. The computer routines used for this method are available from the authors on request.

It appears that the BAW method outperforms the DP one. By contrast, the latter has a very similar performance to the accelerated binomial method of Ju and Zhong (1999) applied to the same set of options. In fact, these authors report a CPU time of 1.177 compared to our 1.951. This result, once again, suggests that, qualitatively, the DP method is equivalent to the accelerated binomial method and slightly more accurate.

Computational time appears to be a problem for the LS (2001) method, much less so for the method proposed in this paper.[18]

5. Conclusions

This study contributes to the literature on American option pricing by suggesting a valuation method based on Chebyshev approximation at Chebyshev nodes to estimate the log-normal density. This method is employed to price a large set of American put options, and is shown to produce reasonably accurate prices for the options considered. We also investigate its efficiency, and find that it outperforms alternatives methods such as the LS (2001) one. In our view, though, the main advantage of our approach consists in providing a simple and reliable framework which can be applied to price more complex derivative instruments. Evaluating the hedging performance of our method would also be of considerable interest. These issues will be investigated in future papers.

We also describe an alternative method based on dynamic programming to approximate the option value in each time period. This approach appears to be qualitatively equivalent to the Accelerated Binomial tree proposed by Breen (1991), but has the advantage of not being affected by problems of non-uniform convergence.

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[17] Note that we estimate these coefficients using Chebyshev basis and Chebyshev nodes.

[18] We also used our method with only ten time steps, and in most cases the prices we obtained were quite similar. It should be possible to improve accuracy (and computational speed) in the case of ten time steps by using variance reduction techniques. However, given the lack of theory on the effects of these techniques on the estimates of the American option prices, we have not applied them here. We leave this issue for future research.

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Appendix 1

Recall equation (6):

$$rV\ (s,t) = \max_x\ [V(x,s,t) + \frac{1}{dt}E(dV)]$$

The investment in the option does not generate any cashflow up to the time the option is exercised. However, we can still express the return on this investment for the holder of the option in terms of its capital appreciation. Therefore we can reasonable assume that the Bellman equation above takes the following form in the continuation region:

$$rV = \frac{1}{dt}E(dV)$$

$$dtrV = E(dV)$$

Using Ito`s-lemma

$$dtrV = V_s ds + V_t dt + \frac{1}{2}V_{ss} ds^2$$

where $V_{(.)}$ represents the derivative with respect to the argument in the subscript.

By substituting the geometric Brownian motion process of section 1, we obtain[19]:

$$dtrV = V_s(rsdt + \delta sdz) + V_t dt + \frac{1}{2}V_{ss}[rsdt + \delta sdz]^2$$

After some algebra manipulation, this can be written as:

[19] Note: to ensure the existence of an optimum, we set $\mu = r$.

$$rV = rsV_s + V_t + \frac{1}{2}s^2\delta^2 V_{ss}$$

Set $V \approx \phi(s)c(t)$, then

$$r\phi(s)c(t) \approx rs\phi'(s)c(t) + \phi(s)c'(t) + \frac{1}{2}\delta^2 s^2 \phi''(s)c(t)$$

where $\phi'(s)$ and $c'(t)$ are derivatives with respect to the argument in parentheses. Therefore:

$$\phi(s)c'(t) \approx rs\phi'(s)c(t) + \frac{1}{2}\delta^2 s^2 \phi''(s)c(t) - r\phi(s)c(t)$$

and finally equation (8) can be written as:

$$\phi(s)c'(t) \approx [rs\phi'(s) + \frac{1}{2}\delta^2 s^2 \phi''(s) - r\phi(s)]c(t)$$

Tables and Figures

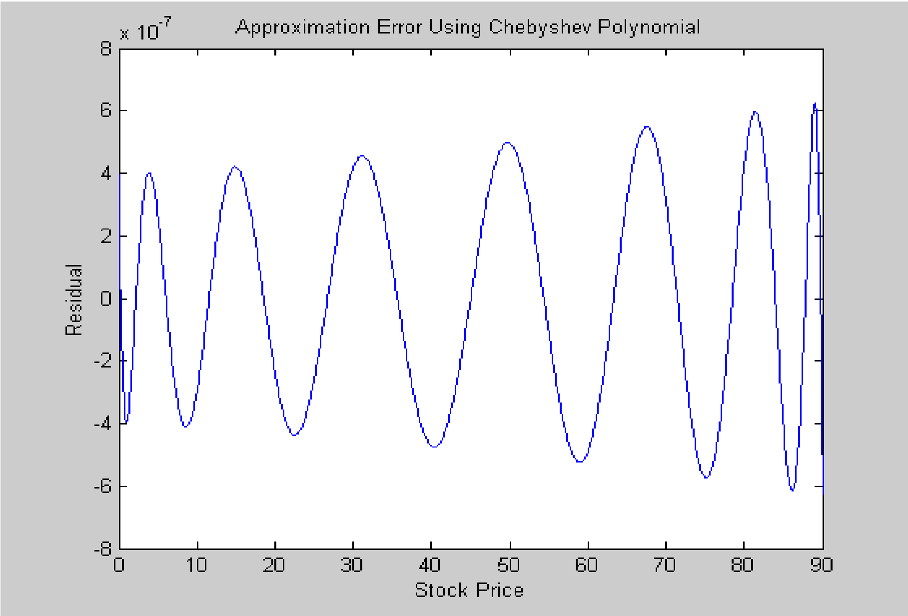

Figure 1

Approximation error using (8) and Chebyshev polynomial when volatility is equal to 0.2 and the interest rate is 0.048.

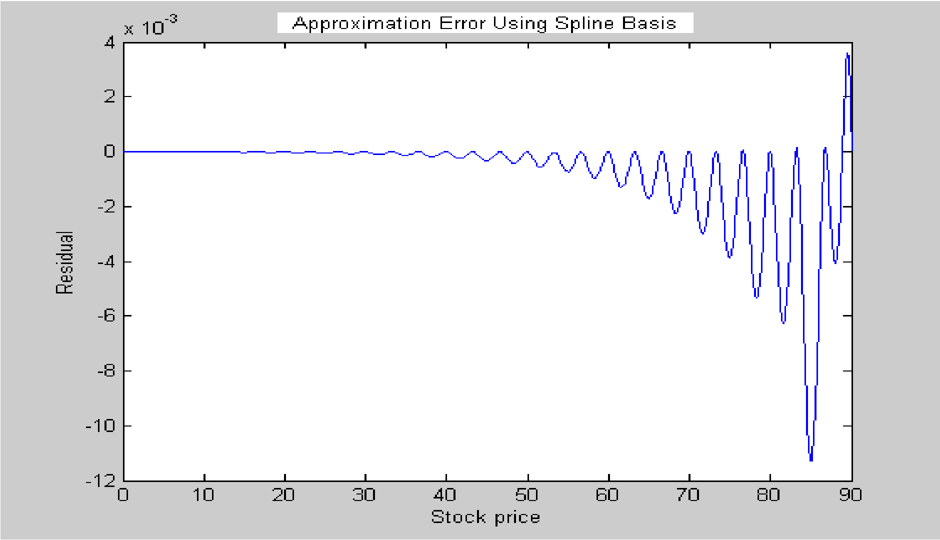

Figure 2

Approximation error using (8) and splines basis when volatility is equal to 0.2 and the interest rate is 0.048.

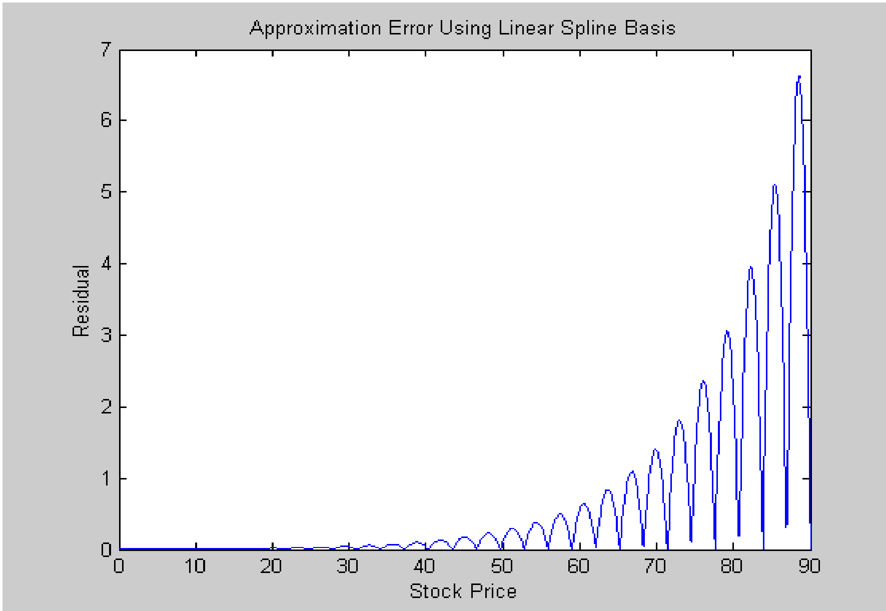

Figure 3

Approximation error using (8) and linear splines basis when volatility is equal to 0.2 and the interest rate is 0.048.

K	Sigma	t (yrs)	B&S	Binomial	BAW (1987)	LS (2001)	DP	CC (2005)
35	0.2	0.0833	0.0062	0.0061	0.0066	0.006	0.0066	0.0062
35	0.2	0.3333	0.196	0.2004	0.2044	0.1975	0.199	0.1998
35	0.2	0.5833	0.417	0.4328	0.4415	0.4276	0.432	0.429
40	0.2	0.0833	0.8403	0.8528	0.8502	0.8414	0.851	0.85
40	0.2	0.3333	1.5221	1.5797	1.577	1.5489	1.574	1.559
40	0.2	0.5833	1.8812	1.9906	1.9888	1.923	1.9887	1.957
45	0.2	0.0833	4.8399	5	5	4.863	5	5
45	0.2	0.3333	4.7804	5.087	5.0661	4.901	5.087	4.901
45	0.2	0.5833	4.84	5.265	5.2364	4.989	5.265	5.221
35	0.3	0.0833	0.196	0.0773	0.0781	0.0771	0.0755	0.078
35	0.3	0.3333	0.6866	0.6976	0.7014	0.693	0.6922	0.693
35	0.3	0.5833	1.1888	1.2199	1.228	1.2048	1.2105	1.209
40	0.3	0.0833	1.2988	1.31	1.3075	1.308	1.3083	1.3102
40	0.3	0.3333	2.4275	2.4822	2.4781	2.4533	2.4799	2.439
40	0.3	0.5833	3.0634	3.1694	3.1666	3.1089	3.1665	3.159
45	0.3	0.0833	4.9796	5.0597	5.047	4.9993	5.0584	5.051
45	0.3	0.3333	5.5288	5.7059	5.6793	5.598	5.7027	5.698
45	0.3	0.5833	5.9723	6.2438	6.215	6.1	6.2399	6.239
35	0.4	0.0833	0.2456	0.2462	0.247	0.246	0.2447	0.247
35	0.4	0.3333	1.3297	1.346	1.3489	1.3353	1.3347	1.347
35	0.4	0.5833	2.1127	2.1549	2.1618	2.153	2.1448	2.153
40	0.4	0.0833	1.7575	1.7667	1.7655	1.7597	1.7658	1.765
40	0.4	0.3333	3.3336	3.3869	3.3824	3.3499	3.3836	3.358
40	0.4	0.5833	4.2473	4.3523	4.3492	4.3047	4.348	4.331
45	0.4	0.0833	5.236	5.2863	5.2733	5.2557	5.2843	5.269
45	0.4	0.3333	6.3767	6.5096	6.4873	6.4389	6.5055	6.493
45	0.4	0.5833	7.1654	7.3829	7.3596	7.294	7.3715	7.329
RMSE					**0.0129**	**0.084**	**0.00479**	**0.0408**

Table 1

Column 4 shows the results using the Black and Scholes (1973) method. The Binomial, in column 5, is based on 10,000 time steps.
Column 6 shows the results using the Baroni-Adesi and Whaley (1987) analytical approximation.
Column 7 shows the option prices using the Longstaff and Schwartz (2001) approach with Laguerre basis.
Column 8-9 show the results using dynamic programming and our suggested method (Caporale and Cerrato – CC). RMSE at the bottom of the table is the root mean square error.

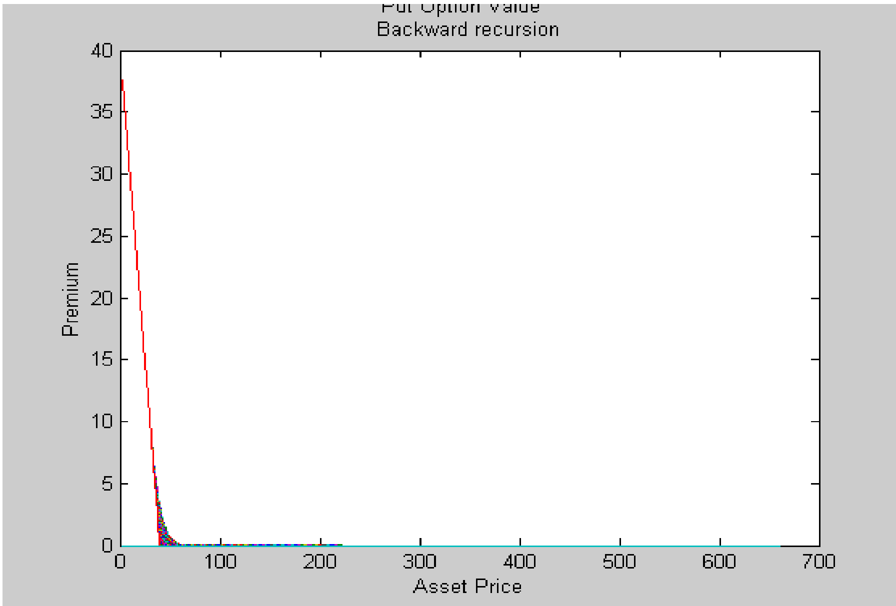

Figure 4

Put option value using dynamic programming, with strike price equal to 40, volatility to 0.3 and expiration time to 0.5833. The number of time steps was set equal to 150.

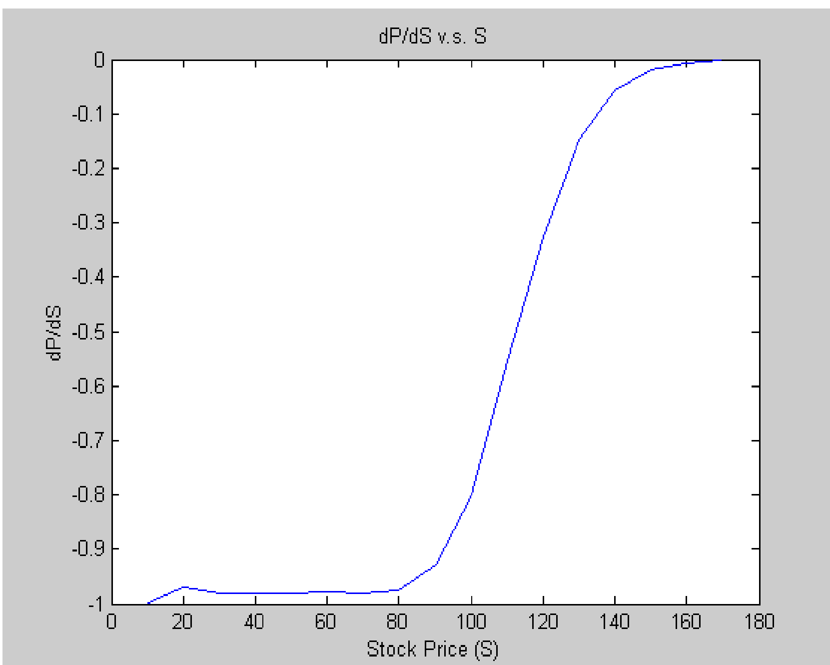

Figure 5

DP/ds v.s. s for s near to $s_p(t)$ with $t-T=0.5$.

B&S	BAW (1987)	LS (2001)	DP	CC (2005)
0.038	0.042	16.65	1.951	6.3958

Table 2

Average CPU time (in seconds) over 27 options.