FINITE-AMPLITUDE VIBRATION OF ORTHOTROPIC AXISYMMETRIC VARIABLE THICKNESS ANNULAR PLATE

by

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Dedicated to my parents.

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NOMENCLATURE

r, θ, z	cylindrical coordinates used to describe the undeformed configuration of the plate.
$h(r), a$	thickness function and radius of plate.
h_0	thickness at center.
t	time variable.
u, w	radial and transverse displacement of the middle plane
$\varepsilon_r, \varepsilon_\theta$	radial and circumferential strains.
σ_r, σ_θ	radial and circumferential stresses.
a_{11}, a_{12}, a_{22}	stress-strain relation coefficients.
N_r, N_θ	middle plane forces per unit length.
M_r, M_θ	bending moments per unit length.
$Q(\xi), Q^*(\xi)$	dimensionless loading distributions.
$q(r,t)$	loading intensity.
K	kinetic energy of the plate.
V_s, V_b	strain energy due to stretching of the middle plane and due to bending of the plate respectively.
W	work done on the plate by the external forces.
ν	Poisson's ratio = $-a_{12}/a_{22}$
c	ratio of elastic constants = a_{11}/a_{22}
$D(r)$	flextural rigidity of the plate = $a_{22}h^3/12(a_{11}a_{22}-a_{12}^2)$
ψ, ϕ	stress functions
ξ, τ	dimensionless space and time variables respectively.
χ	dimensionless transverse displacement
$g(\xi), f(\xi)$	shape functions of vibration.
A, α	amplitude parameters.

λ	nondimensional nonlinear eigenvalue.
$\omega = (\lambda)^{1/2}$	nondimensional angular frequency.
$\bar{Y}$, $\bar{Z}$, $\bar{H}$	(6x1) vector functions
M, N	coefficient matrices.
$\bar{0}$	(3x1) null matrix
’r, ’t	partial derivatives with respect to r & t
$\eta(\xi)$	variable thickness function.
τ	frequency parameter.
$\delta\chi$	first variation of $\chi = \delta G \sin\omega\tau$
$\bar{\eta}$	adjustable data in the related initial value problem.
{ }	indicates a column vector.
Δ	Del operator.

INTRODUCTION

Composite materials find large application in design of structural elements in the present age. These structures which are mainly in the form of plates, are subjected to severe operational conditions, and should thus be able to withstand large amplitudes of vibration. If the amplitude of vibration is of the same order of magnitude as the thickness of the plate, then the deformation of the mid-plane can no longer be neglected. In the development of a suitable thin plate theory, anisotropic properties and geometric non-linearities arising in the coupling of membrane and bending theories should be included. The resulting governing differential equations can be solved by approximate numerical methods due to the complexity of the problem.

In 1960, Kazimierz Borsuk, determined a method to solve in an accurate manner the problem of free vibration of circular cylindrically orthotropic plates. In 1969 A. P. Salzman and S. A. Patel used the method of separation of variables along with Frobenius' method to determine the frequencies of clamped or simply supported solid circular variable thickness orthotropic plates. In 1971, K. Vijayakumar and C. V. Joga Rao determined the axisymmetric vibration and buckling of polar orthotropic circular plates. In 1973 C. L. Huang and H. K. Woo used the Ritz-Kantorovich method to determine large oscillations of orthotropic annular plates. In 1974, G. K. Ramiah and K. Vijayakumar, determined the vibration of polar orthotropic annular plates.

The above and many other investigators (18-30) have worked on either solid, circular, variable thickness, anisotropic plates or annular orthotropic plates. This present investigation is thus concerned with harmonic, large amplitude, free vibrations of orthotropic, axisymmetric, annular plates of variable thickness.

The essence of the approximate method is to approximate the continuous system by a discrete one having a finite number of degrees of freedom. The discrete representation is achieved through an assumed space mode. Substitutuion of this in the differential equations along with the requirement that some measure of the error be minimized, the assumed space mode can be eliminated. The problem thus reduces to a nonlinear ordinary differential equation with time as an independent variable. This equation is similar to a one-degree of freedom Duffings equation (5).

This present work assumes the existence of harmonic vibrations. The time variable is eliminated by the application of a Ritz-Kantorovich averaging method. The basic governing equations thus reduce to a pair of ordinary differential equations, with a reformulated set of boundary conditions. A numerical study of these equations is proposed by introducing the related initial value problem.

The cases considered are, a parabolic variable thickness annular plate, with the variable thickness function of the form

$$\eta = 0.815 - 0.5\, X^2$$

and a convex variable thickness annular plate, with the variable thickness function of the form

$$\eta = 1.0 - 0.5\, X^{1/2} .$$

Both the above plates have the same volume and the same boundary conditions. The boundary conditions are free on the outside and fixed on the inside. The corresponding curves for the frequency responses, bending stresses, and membrane stresses are presented.

CHAPTER I

DERIVATION OF THE GOVERNING EQUATIONS

Consider a thin annular orthotropic plate, the elastic properties of which are different in the radial and circumferential directions. The fundamental assumptions made as regards to the flexural deformations of the plate are:

1. The loads and deflections are symmetric with respect to the z axis which passes through the center of the annulus.
2. The normals to the middle plane in the undeformed plate remain straight and normal to the middle plane in the deformed plate.
3. It follows the Hooke's Law.
4. Transverse shearing deformations are not included.
5. The maximum thickness of the plate is small in comparison to the radius of the plate.

Keeping in mind the above assumptions, the following strain-displacement relations are written:

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - z\,\frac{\partial^2 w}{\partial r^2}$$

or in indicial notation as:

$$\varepsilon_r = u_{,r} + \frac{1}{2} w_{,r}^2 - z w_{,rr} \qquad (1)$$

In the θ direction:

$$\varepsilon_\theta = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r}$$

or in indicial notation as:

$$\varepsilon_\theta = \frac{u}{r} - \frac{z}{r} w_{,r} \qquad (2)$$

In view of the orthotropy of the plate considered, the Hooke's Law can be written as:

$$\varepsilon_\theta = a_{11}\sigma_\theta + a_{12}\,\sigma_r \tag{3a}$$

$$\varepsilon_r = a_{12}\sigma_\theta + a_{22}\,\sigma_r \tag{3b}$$

From the above two equations, it follows

$$\sigma_r = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}\left(\varepsilon_r - \frac{a_{12}}{a_{11}}\,\varepsilon_\theta\right) \tag{4a}$$

$$\sigma_\theta = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}\left(\varepsilon_\theta - \frac{a_{12}}{a_{22}}\,\varepsilon_r\right) \tag{4b}$$

where a_{11}, a_{22}, a_{12} are the elastic constants and σ_r, σ_θ are the radial and circumferential stresses.

Resubstituting ε_θ and ε_r from (1) and (2), we have

$$\sigma_r = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}\left(u,_r + \frac{1}{2}\,w,_r^2 - \frac{a_{12}}{a_{11}}\left(\frac{u}{r} - \frac{z}{r}\,w,_r^2\right)\right) \tag{5a}$$

$$\sigma_\theta = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}\left(\frac{u}{r} - \frac{a_{12}}{a_{22}}\left(u,_r + \frac{1}{2}\,w,_r^2 - z\,\frac{\partial^2 w}{\partial r^2}\right) - \frac{z}{r}\,\frac{\partial w}{\partial r}\right) \tag{5b}$$

Expressions for the radial and circumferential forces per unit length, N_r and N_θ, are obtained by integrating the respective stresses across the thickness of the plate.

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta\,dz = \frac{a_{22}h(r)}{a_{11}a_{22} - a_{12}^2}\left(\frac{u}{r} - \frac{a_{12}}{a_{22}}\left(\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right)\right) \tag{6a}$$

$$N_r = \int_{-h/2}^{h/2} \sigma_r\,dz = \frac{a_{11}h(r)}{a_{11}a_{22} - a_{12}^2}\left(\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{a_{12}}{a_{11}}\left(\frac{u}{r}\right)\right) \tag{6b}$$

or,

$$N_r = \frac{h(r)}{a_{22}(c-\nu^2)} \left(c(u,_r + \frac{1}{2} w,_r^2) + \frac{\nu u}{r}\right) \tag{7a}$$

$$N_\theta = \frac{h(r)}{a_{22}(c-\nu^2)} \left(\frac{u}{r} + \nu u,_r + \frac{\nu}{2} (w,_r)^2\right) \tag{7b}$$

where,

$c = \frac{a_{11}}{a_{22}}$ = ratio of properties in radial and circumferential directions.

$\nu = -\frac{a_{12}}{a_{22}}$ = poisson's ratio.

The radial and circumferential moments per unit length, M_r and M_θ, are obtained by integrating the moments of the forces about the middle plane across the thickness of the plate.

$$M_r = \int_{-h/2}^{h/2} \sigma_r \, zdz = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \int_{-h/2}^{h/2} \left\{\varepsilon_r - \frac{a_{12}}{a_{11}} \varepsilon_\theta\right\} zdz \tag{8a}$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, zdz = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \int_{-h/2}^{h/2} \left\{\varepsilon_\theta - \frac{a_{12}}{a_{22}} \varepsilon_r\right\} zdz \tag{8b}$$

or,

$$M_r = -D\left(\frac{\nu}{r} w,_r + c\, w,_{rr}\right) \tag{9a}$$

$$M_\theta = -D\left(\frac{1}{r} w,_r + \nu\, w,_{rr}\right) \tag{9b}$$

where,

$$D = \frac{a_{22}h^3}{12(a_{11}a_{22} - a_{12}^2)} = \frac{E_\theta h^3}{12} = \frac{h^3}{12\beta}$$

$$\beta = \frac{a_{11}a_{22} - a_{12}^2}{a_{22}}$$

The Energy Method

The extended Hamilton's Principle, which states that, within the interval of time, t_1 and t_2, the first variation of the action integral is equal to zero, is made use of here; i.e.;

$$\delta \int_{t_1}^{t_2} I dt = 0 \qquad (10)$$

Here the Lagrangian is $L = K - V_s - V_b + W$ (11)

where

K = Kinetic Energy

V_s = strain energy due to stretching of the middle plane

V_b = strain energy due to the bending of the plate.

W = work done by the time dependent external forces.

1. Neglecting the radial part of inertia force, $\partial u/\partial t << \partial w/\partial t$, the kinetic energy is

$$K = \pi \int_c^a \rho h(r)\, w_t^2\, r dr \qquad (12)$$

2. The strain energy due to stretching of the middle plane is obtained as follows:

$$V_s = 2\pi \int_c^a \frac{N_r \bar{\varepsilon}_r}{2} + \frac{N_\theta \bar{\varepsilon}_\theta}{2}\; r dr \qquad (13)$$

Substituting values of N_r, N_θ, ε_r, ε_θ and rearranging,

$$V_s = \frac{\pi}{\beta} \int_c^a \{cu_r^2 + \frac{c}{4} w_r^4 + cu_r w_r^2 + 2\nu \frac{u}{r} u_r + \nu \frac{u}{r} w_r^2 + (\frac{u}{r})^2\} hrdr \qquad (14)$$

3. The strain energy due to bending of the plate:

$$V_b = -\pi \int_c^a \left\{ M_r \frac{\partial^2 w}{\partial r^2} + M_\theta \frac{1}{r} \frac{\partial w}{\partial r} \right\} rdr$$

Substituting M_r and M_θ and rearranging,

$$V_b = \pi \int_c^a D(r) \left\{ cw_{rr}^2 + 2 \frac{\nu}{r} w_r w_{rr} + \frac{1}{r^2} w_r^2 \right\} rdr \tag{15}$$

4. The work done by the exciting force function p(r,t),

$$W = -2\pi \int_c^a p(r,t) \; wrdr \tag{16}$$

Substituting equations (12), (14), (15) and (16) into (11) we obtain the Lagrangian as:

$$L = \pi \int_c^a \rho h(r) \; w_t^2 \; rdr$$

$$- \frac{\pi}{\beta} \int_c^a \{cu_r^2 + cu_r w_r^2 + \frac{c}{4} w_r^4 + 2 \frac{u}{r} u_r + \frac{u}{r} w_r^2 + (\frac{u}{r})^2\} \; hrdr$$

$$- \frac{\pi}{12\beta} \int_c^a \left\{ cw_{rr}^2 + 2 \frac{\nu}{r} w_r w_{rr} + \left(\frac{w_r}{r}\right)^2 \right\} h^3 rdr$$

$$+ 2\pi \int_c^a p(r,t) \; wrdr \tag{17}$$

Now the integral can be written symbolically as:

$$I = \int_{t_1}^{t_2} \pi \int_c^a f(t,r;w,u,w_r,u_r,w_t,w_{rr}) \; rdrdt$$

where,

$$f = \{\rho hrw_t^2 - \frac{h}{\beta} [cru_r^2 + \frac{c}{4} rw_r^4 + cu_r rw_r^2 + 2\nu uu_r + \nu uw_r^2 + \frac{u^2}{r}]$$

$$- \frac{h^3}{12\beta} [crw_{rr}^2 + 2\nu w_r w_{rr} + \frac{1}{r} w_r^2] + 2p(r,t)rw\}$$

The first variation of I vanishes

$$\delta I = \int_{t_1}^{t_2}\int_c^a \left\{\left(\frac{\partial f}{\partial w}\right)\delta w + \left(\frac{\partial f}{\partial u}\right)\delta u + \left(\frac{\partial f}{\partial w_r}\right)\delta w_r + \left(\frac{\partial f}{\partial u_r}\right)\delta u_r + \left(\frac{\partial f}{\partial w_t}\right)\delta w_t + \left(\frac{\partial f}{\partial w_{rr}}\right)\delta w_{rr}\right\} dr\, dt \quad (18)$$

Now,

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial w}\right)\delta w dr dt = \int_{t_1}^{t_2}\int_c^a (2pr)\,\delta w dr dt \quad (19)$$

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial u}\right)\delta u dr dt = \int_{t_1}^{t_2}\int_c^a -\frac{h}{\beta}\left(2\nu u_r + \nu w_r^2 + \frac{2u}{r}\right)\delta u dr dt \quad (20)$$

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial w_r}\right)\delta w_r dr dt = \int_{t_1}^{t_2}\frac{\partial f}{\partial w_r}\,\delta w\Big|_c^a dt - \int_{t_1}^{t_2}\int_c^a \frac{d}{dr}\left(\frac{\partial f}{\partial w_r}\right)\delta w dr dt \quad (21a)$$

by partial integration.

Substituting values of $\frac{d}{dr}\left(\frac{\partial f}{\partial w_r}\right)$ and $\frac{\partial f}{\partial w_r}$ into equation (21a) we obtain:

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial w_r}\right)\delta w_r dr dt = \int_{t_1}^{t_2} -\frac{h}{\beta}\left\{crw_r^3 + 2cu_r rw_r + 2\nu uw_r\right\}$$

$$- D(r)\left\{2\nu w_{rr} + \frac{2w_r}{r}\right\}\delta w\Big|_c^a dt$$

$$- \int_{t_1}^{t_2}\int_c^a \left\{-\frac{h}{\beta}\left[3crw_r^2 w_{rr} + cw_r^3 + 2cu_r w_r\right.\right.$$

$$+ 2\nu uw_{rr} + 2cu_r rw_{rr} + 2cu_{rr}\,\nu w_r +$$

$$+ 2\nu u_r w_r] - D_r[2\nu w_{rr} + \frac{2w_r}{r}]$$

$$- \frac{h_r}{\beta} [crw_r^3 + 2cu_r w_r + 2\nu u w_r]$$

$$- D(r)[2\nu w_{rrr} + \frac{2w_{rr}}{r} - \frac{2w_r}{r^2}\Big\} \delta w dr dt \qquad (21)$$

By similar method, substituting values of $\frac{d}{dr}\left(\frac{\partial f}{\partial u_r}\right)$ and $\frac{\partial f}{\partial u_r}$ in equation:

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial u_r}\right) \delta u_r dr dt = \int_{t_1}^{t_2} \frac{\partial f}{\partial u_r} \delta u\Big|_c^a dt - \int_{t_1}^{t_2}\int_c^a \frac{d}{dr}\left(\frac{\partial f}{\partial u_r}\right) \delta u dr dt$$

We get,

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial u_r}\right) \delta u_r dr dt = \int_{t_1}^{t_2} - \frac{h}{\beta} [2cru_r + crw_r^2 + 2\nu u]\delta u \Big|_c^a dt$$

$$- \int_{t_1}^{t_2}\int_c^a \Big\{- \frac{h}{\beta} [2cu_r + 2cru_{rr} + cw_r^2 + 2crw_r w_{rr}$$

$$+ 2\nu u_r] - \frac{h_r}{\beta} [2cru_r + crw_r^2 + 2\nu u\Big\} \delta u dr dt \qquad (22)$$

Substituting values of $\frac{\partial f}{\partial w_t}$ and $\frac{d}{dt}\left(\frac{\partial f}{\partial w_t}\right)$ in equation:

$$\int_{t_1}^{t_2}\int_c^a \frac{\partial f}{\partial w_t} \delta w_t dr dt = \int_c^a \frac{\partial f}{\partial w_t} \delta w \Big|_{t_1}^{t_2} dr - \int_{t_1}^{t_2}\int_c^a \left\{\frac{d}{dt}\left(\frac{\partial f}{\partial w_t}\right)\right\} \delta w dr dt$$

And as, $\delta w = 0$ at t_1, t_2, the first integral = 0, hence the above equation reduces to:

$$\int_{t_1}^{t_2}\int_c^a \frac{\partial f}{\partial w_t} \delta w_t dr dt = - \int_{t_1}^{t_2}\int_c^a (2\rho h\, r\, w_{tt})\, \delta w dr dt \qquad (23)$$

And finally we have,

$$\int_{t_1}^{t_2}\int_c^a \left(\frac{\partial f}{\partial w_{rr}}\right) \delta w_{rr}\, drdt = \int_{t_1}^{t_2} \frac{\partial f}{\partial w_{rr}} \delta w_r \Big|_c^a dt$$

$$- \int_{t_1}^{t_2} \frac{d}{dr}\left(\frac{\partial f}{\partial w_{rr}}\right) \delta w\Big|_c^a + \int_{t_1}^{t_2}\int_c^a \frac{d^2}{dr^2}\left(\frac{\partial f}{\partial w_{rr}}\right) \delta w drdt$$

after integrating by parts twice, and substituting the values of the various constituents of the equation we have,

$$\int_{t_1}^{t_2}\int_c^a \frac{\partial f}{\partial w_{rr}} \delta w_{rr}\, drdt = \int_{t_1}^{t_2} - D(r)[2cw_{rr} + 2\nu w_r]\ \delta w_r \Big|_c^a dt$$

$$- \int_{t_1}^{t_2} \Big\{-D(r)[2cw_{rr} + 2cr\, w_{rrr} + 2\nu w_{rr}$$

$$- D_r[2cr\, w_{rr} + 2\nu w_r]\Big\}\ \delta w \Big|_c^a dt$$

$$+ \int_{t_1}^{t_2}\int_c^a \Big\{-D(r)[4cw_{rrr} + 2crw_{rrrr} + 2\nu w_{rrr}]$$

$$- D_r[4crw_{rrr} + 4cw_{rr} + 4\nu w_{rr}]$$

$$- D_{rr}[2crw_{rr} + 2\nu w_r]\Big\}\ \delta w drdt \tag{24}$$

Substituting equations (19) to (24) into (18) we get:

$$\delta I = \int_{t_1}^{t_2}\int_c^a 2pr\ \delta w drdt + \int_{t_1}^{t_2}\int_c^a -\frac{h}{\beta}\left(2\nu u_r + \nu w_r^2 + \frac{2u}{r}\right) \delta u drdt$$

$$+ \int_{t_1}^{t_2} [-\frac{h}{\beta}(crw_r^3 + 2cu_r rw_r + 2\nu uw_r) - D(r)(2\nu w_{rr} + \frac{2w_r}{r})]\ \delta w \Big|_c^a dt$$

$$- \int_{t_1}^{t_2} \int_c^a \left\{ - \frac{h}{\beta} [3crw_r^2 w_{rr} + cw_r^3 + 2cu_{rr} w_r r + 2cu_r r w_{rr}\right.$$

$$+ 2cu_r w_r + 2\nu u w_{rr} + 2\nu u_r w_r] - \frac{h_r}{\beta} [crw_r^3 + 2cu_r r w_r$$

$$\left. + 2\nu u w_r] - D(r)[2\nu w_{rrr} + \frac{2w_{rr}}{r} - \frac{2}{r^2} w_r] - D_r(r)[2\nu w_{rr} + \frac{2}{r} w_r] \right\} \delta w dr dt$$

$$+ \int_{t_1}^{t_2} \left\{ \frac{h}{\beta} [2cru_r + crw_r^2 + 2\nu u] \right\} \delta u \Big|_c^a dt$$

$$- \int_{t_1}^{t_2} \int_c^a \left\{ - \frac{h}{\beta} [2cu_r + 2cru_{rr} + cw_r^2 + 2crw_r w_{rr} + 2\nu u_r] \right.$$

$$\left. - \frac{h_r}{\beta} [2cru_r + crw_r^2 + 2\nu u] \right\} \delta u dr dt$$

$$+ \int_{t_1}^{t_2} \left\{ -D(r) \; [2cw_{rr} + 2\nu w_r] \right\} \delta w_r \Big|_c^a dt$$

$$- \int_{t_1}^{t_2} \left\{ -D(r)[2cw_{rr} + 2crw_{rrr} + 2\nu w_{rr}] - D_r[wcrw_{rr} + 2\nu w_r] \right\}$$

$$- \delta w \Big|_c^a dt + \int_{t_1}^{t_2} \int_c^a \left\{ -D(r)[4cw_{rrr} + 2crw_{rrrr} + 2\nu w_{rrr}] \right.$$

$$\left. - D_r \; [4crw_{rr} + 4\nu w_{rr} + 4cw_{rr}] - D_{rr}[2crw_{rr} + 2\nu w_r] \right\} \delta w dr dt$$

$$- \int_{t_1}^{t_2} \int_c^a (2\rho h r w_{tt}) \; \delta w dr dt = 0 \qquad (25)$$

For equation (25) to hold true, the integrands in the double and single integrals should vanish separately.

The double integral yields the Euler-Lagrange equations:

$$D(r)\ [cw_{rrrr} + \frac{2c}{r} w_{rrr} - \frac{w_{rr}}{r^2} + \frac{w_r}{r^3}] + D_r[2cw_{rrr} + \frac{2c}{r} w_{rr} + \frac{\nu}{r} w_{rr} - \frac{w_r}{r^2}]$$

$$+ D_{rr}[cw_{rr} + \frac{\nu w_r}{r}] + \rho h(r) w_{tt} = p(r,t) + \frac{1}{\beta}\left\{h[c(u_r w_{rr} + u_{rr} w_r\right.$$

$$\left. + \frac{u_r w_r}{r} + \frac{3}{2} w_r^2 w_{rr} + \frac{w_r^3}{2r}) + \frac{\nu}{r}(uw_{rr} + u_r w_r)] + h_r(cu_r w_r + \frac{c}{2} w_r^3 + \frac{\nu}{r} uw_r)\right\}$$

(26)

and

$$h\left\{cu_{rr} + \frac{cu_r}{r} - \frac{u}{r^2} + (c-\nu)\frac{w_r^2}{2r} + cw_r w_{rr}\right\}$$

$$+ h_r\left\{cu_r + \frac{c}{2} w_r^2 + \frac{\nu u}{r}\right\} = 0 \qquad (27)$$

The single integrals yield, the boundary conditions,

$$w{=}0 \quad \text{or} \quad -\frac{h}{\beta} w_r \left(\frac{c}{2} w_r^2 + cu_r + \frac{\nu}{r}\right) + D(r)\left(-\frac{w_r}{r^2} + cw_{rrr} + \frac{cw_{rr}}{r}\right)$$

$$+ D_r(cw_{rr} + \frac{\nu}{r} w_r) = 0 \ \rightarrow\ \text{shear} = 0$$

deflection = 0

$w_r = 0$ or $D(r)(2crw_{rr} + 2\nu w_r) = 0 \rightarrow$ Moment = 0

slope = 0

and

$u = 0$ or $rh(r)\ (cu_r + \frac{cw_r^2}{2} + \frac{u}{r}) = 0 \rightarrow$ Force = 0

Equation (26) can be expressed as:

$$D(cw_{rrrr} + \frac{2c}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{r^3} w_r) + D_r(2cw_{rrr} + (2c+\nu) \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r)$$

$$+ D_{rr}(cw_{rr} + \frac{\nu}{r} w_r) + \rho h w_{tt} = p(r,t) + \frac{1}{\beta} \frac{1}{r} \frac{\partial}{\partial r} [hrw_r(cu_r + \frac{c}{2} w_r^2 + \frac{\nu}{r} u)] \tag{28}$$

Proceeding further with the stress formulation, and substituting the following into equations (27) & (28)

$$\Psi = rN_r \quad \text{and} \quad \frac{\partial \Psi}{\partial r} = N_\theta$$

$$\frac{\partial u}{\partial r} + \frac{1}{r} w_r^2 = \frac{a_{22}}{h} (N_r - \nu N_\theta) = \frac{a_{22}}{h} (\frac{\Psi}{r} - \nu \Psi_r)$$

$$\frac{u}{r} = \frac{a_{22}}{h} (c\Psi_r - \frac{\nu}{r} \Psi)$$

We have the following equation:

$$D(cw_{rrrr} + \frac{2c}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{r^3} w_r) + D_r(2cw_{rrr} + (2c+\nu) \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r$$

$$+ D_{rr}(cw_{rr} + \frac{\nu}{r} w_r) + \rho h w_{tt} = p(r,t) + \frac{1}{r} [w_r \Psi]_r \tag{29}$$

$$[c\Psi_{rr} + \frac{c\Psi_r}{r} - \frac{\Psi}{r^2}] + \frac{h_r}{h} [-c\Psi_r + \frac{\nu\Psi}{r}] + \frac{h}{2a_{22}r} w_r^2 = 0 \tag{30}$$

and the boundary conditions become:

$$w = 0 \quad \text{or} \quad D[cw_{rrr} + \frac{c}{r} w_{rr} - \frac{w_r}{r^2}] + D_r[cw_{rr} + \frac{\nu}{r} w_r] - \frac{1}{r} w_r \Psi = 0$$

$$w_r = 0 \quad \text{or} \quad cw_{rr} + \frac{\nu}{r} w_r = 0$$

$$\Psi = 0 \quad \text{or} \quad c\Psi_r - \nu \frac{\Psi}{r} = 0$$

Using the substitutions,

$$\chi = \frac{w}{a}$$

$$\xi = \frac{r}{a}$$

$$\tau = t\left[\frac{D_0}{ha^4(1-\nu^2)}\right]^{1/2}$$

$$\phi = \frac{a_{22}}{h_0 a}\,\Psi$$

$$D = \frac{h^3}{12\beta} \quad ; \quad \beta = a_{22}(c-\nu^2) = \frac{h_o^3\,\eta^3}{12(c-\nu^2)a_{22}}$$

$$D_0 = \frac{h_0^3}{12a_{22}}$$

we get the following non-dimensional form:

$$\eta^3[c\chi'''' + \frac{2c}{\xi}\chi''' - \frac{1}{\xi^2}\chi'' + \frac{1}{\xi^3}\chi'] + (\eta^3)_\xi\,[2c\chi''' + (2c+\nu)\,\frac{1}{\xi}\chi''$$

$$- \frac{1}{\xi^2}\chi'] + (\eta^3)_{\xi\xi}\,[c\chi'' + \frac{\nu}{\xi}\chi'] + \eta\left(\frac{c-\nu^2}{1-\nu^2}\right)\chi_{\tau\tau}$$

$$= 12(c-\nu^2)a_{22}(\frac{a}{h_o})^3\,p + 12(c-\nu^2)\,(\frac{a}{h_o})^2\,\frac{1}{\xi}\,[\chi'\phi]' \tag{31}$$

and,
$$[c\phi'' + \frac{c\phi'}{\xi} - \frac{\phi}{\xi^2}] + \frac{\eta_r}{\eta}\,[-c\phi' + \nu\,\frac{\phi}{\xi}] + \frac{\eta}{2\xi}\,[\chi_\xi]^2 \tag{32}$$

The boundary conditions become,

$$\chi' = 0 \quad \text{or} \quad c\chi'' + \frac{\nu}{\xi}\chi' = 0 \tag{33a}$$

$$\chi = 0 \quad \text{or} \quad [c\chi''' + \frac{c}{\xi}\chi'' - \frac{1}{\xi^2}\chi'] + \frac{3\eta'}{\eta}\,[c\chi'' + \frac{\nu}{\xi}\chi']$$

$$-12(c-\nu^2)\,(\frac{a}{h_o})^2\,\frac{1}{\eta^3\xi}\,(\chi'\phi) = 0 \tag{33b}$$

$$\phi = 0 \quad \text{or} \quad c\phi' - \frac{\nu}{\xi}\,\phi = 0 \tag{33c}$$

CHAPTER II

APPROXIMATE ANALYSIS

There is, at present, no exact method known, for the solution of the differential equations (31) and (32), also the standard fourier analysis used in linear vibration problems is not applicable, because the nonlinear character of the differential equation, causes coupling of vibration modes.

Consequently, this nonlinear coupled problem, can only be solved by some approximate numerical method. Approximate solutions of large amplitude vibrations can be achieved by separation of variables method, or implementing function space methods to eliminate the space coordinate with an assumed mode shape function. The problem is thus reduced to a non-linear ordinary differential equations with time t, as an independent variable. The resulting one degree of freedom Duffings equation is solved and the solutions are in terms of elliptical functions. This is called the assumed - space - mode solution. The Kantorovich Averaging method is proposed to find an assumed time-mode solution of the equations (31) and (32) and the boundary conditions equations (33).

Kantorovich Averaging Method:

A sinusoidal form of the loading intensity is assumed here:

$$P(\xi,\tau) = Q(\xi)\ \mathrm{Sin}\ \omega\tau \tag{34}$$

also, the steady state response can be closely approximated by the expressions,

$$X(\xi,\tau) = G(\xi)\ \mathrm{Sin}\ \omega\tau \tag{35a}$$

$$\phi(\xi,\tau) = F(\xi)\ \mathrm{Sin}^2\omega\tau \tag{35b}$$

where $G(\xi)$ and $F(\xi)$ are the undetermined shape functions of vibrations.

Substituting equations (35) into equation (32), we have

$$c\,\frac{d^2F}{d\xi^2} + \frac{c}{\xi}\frac{dF}{d\xi} - \frac{F}{\xi^2} + \frac{h}{2\xi}\,(G')^2 + \frac{\eta_r}{\eta}\,[-c\,\frac{dF}{d\xi} + \nu\,\frac{F}{\xi}] = 0 \tag{36}$$

As the expressions (34) and (35) cannot satisfy equations (31) for all τ, the integral,

$$\begin{aligned} I_A = \int_R^1 \Big\{ & \eta^3[c\chi'''' + \frac{2c}{\xi}\chi''' - \frac{1}{\xi^2}\chi'' + \frac{1}{\xi^3}\chi'] \\ & + (\eta^3)_\xi\,[2c\chi''' + \frac{(2c+\nu)}{\xi}\chi'' - \frac{1}{\xi^2}\chi'] \\ & + (\eta^3)_{\xi\xi}\,[c\chi'' + \frac{\nu}{\xi}\chi'] + \eta\left(\frac{c-\nu^2}{1-\nu^2}\right)\chi_{\tau\tau} \\ & - (c-\nu^2)P - 12(c-\nu^2)\,(\frac{a}{h_0})^2\,\frac{1}{\xi}\,(\chi'\phi)'\Big\}\,\delta X\xi d\xi \end{aligned} \tag{37}$$

where $\quad P = 12a_{22}\,(\frac{a}{h})^3\,p,$

is used to obtain a governing equation which closely resembles equation (31), within the limits of assumed form of motion and loading as given in equations (34) and (35).

Substituting expressions (34) and (35) into (37) and equating the average virtual work over a period of oscillation to zero, or explicitly:

$$I' = \int_0^{2\pi/w} I_A d\tau = 0$$

yields:

$$\begin{aligned} & \eta^3[cG'''' + \frac{2c}{\xi}G''' - \frac{1}{\xi^2}G'' + \frac{1}{\xi^3}G'] + (\eta^3)_\xi\,[2cG''' + \frac{(2c+\nu)}{\xi}G'' \\ & \quad - \frac{1}{\xi^2}G'] + (\eta^3)_{\xi\xi}[cG'' + \frac{\nu}{\xi}G'] - \omega^2\eta\,\frac{(c-\nu^2)}{(1-\nu^2)}\,G \\ & \quad - 9\,(c-\nu^2)\,(\frac{a}{h_o})^2\,\frac{1}{\xi}\,[G'\,F]' = (c-\nu^2)\,Q \end{aligned} \tag{38}$$

The problem therefore becomes governed by a pair of nonlinear coupled ordinary differential equations (36) and (38). For convenience in conducting a parametric study, let

$$G(\xi) = A\, g(\xi) \qquad (39a)$$

$$F(\xi) = A^2 f(\xi) \qquad (39b)$$

where A is amplitude parameter, and $g(\xi)$ and $f(\xi)$ are shape functions. Substituting these into equations (35) and (38) we get,

$$c \frac{d^2 f}{d\xi^2} + c(1 - \frac{\eta'}{\eta} \xi) \frac{f'}{\xi} - (1 \frac{\nu'}{\eta} \nu\xi) \frac{f}{\xi^2} + \frac{\eta}{2\xi} (g')^2 = 0 \qquad (40a)$$

$$c\eta^3 g'''' + (\frac{2c}{\xi} \eta^3 + 6c\eta'\eta^2) g''' + (- \frac{\eta^3}{\xi^2} + \frac{3(2c+\nu)\eta'\eta^2}{\xi}$$

$$+ 3c\eta''\eta^2 + 6c\eta(\eta')^2)\, g''$$

$$+ (\frac{1}{\xi^3} \eta^3 - 3\eta'\eta^2 \frac{1}{\xi^2} + (3\eta''\eta^2 + 6\eta(\eta')^2)\frac{\nu}{\xi})\, g' - \eta\lambda \frac{(c-\nu^2)}{(1-\nu^2)} g\ .$$

$$- 9(c-\nu^2)\alpha \frac{1}{\xi} (g'f') = (c-\nu^2) \frac{Q^*}{\sqrt{\alpha}} \qquad (40b)$$

This can be expressed as:

$$A_1 g'''' + A_2 g''' + A_3 g'' + A_4 g' - \eta\lambda \frac{(c-\nu^2)}{(1-\nu^2)} g - \frac{9(c-\nu^2)}{\xi} \alpha (g'\ f)'$$

$$= \frac{(c-\nu^2)}{\sqrt{\alpha}} Q^* \qquad (40b)$$

where

$$A_1 = c\eta^3$$

$$A_2 = \frac{2c}{\xi} \eta^3 + 6c\eta'\eta^2$$

$$A_3 = - \frac{1}{\xi^2} \eta^3 + 3(2c+\nu) \frac{\eta'\eta^2}{\xi} + 3c\eta''\eta^2 + 6c(\eta')^2\, \eta$$

$$A_4 = \frac{1}{\xi^3} \eta^3 - \frac{3\eta'\eta^2}{\xi^2} + (3\eta''\eta^2 + 6\eta(\eta')^2) \frac{\nu}{\xi}$$

$$\alpha = A(\frac{a}{h_o})^2$$

$$\lambda = \omega^2$$

$$Q^* = \frac{Aa}{h_o} Q$$

The above equations together with the boundary conditions selected from Table I, constitute a two-point boundary problem which is solved through the solution of the related initial value problem.

Table-I

Type of Edge		Boundary Condition at Edge $\xi_1 = R$ or 1
Clamped Immovable	$g = 0$ $g' = 0$	$cf' - \frac{\nu}{\xi} f = 0$
Clamped Movable	$g = 0$ $g' = 0$	$\frac{f}{\xi} = 0$
Hinged Immovable	$g = 0$ $cg'' + \frac{\nu}{\xi} g' = 0$	$cf' - \frac{\nu}{\xi} f = 0$
Hinged Movable	$g = 0$ $cg'' + \frac{\nu}{\xi} g' = 0$	$\frac{f}{\xi} = 0$
Free	$cg'' + \frac{\nu}{\xi} g' = 0$ $cg''' + c[\frac{1}{\xi} + \frac{3\eta'}{\eta}]g'' - [\frac{1}{\xi^2} - \frac{3\eta'}{\eta} \frac{\nu}{\xi}]g' = 0$	$\frac{f}{\xi} = 0$

CHAPTER III

NUMERICAL ANALYSIS

The solutions of nonlinear boundary values and nonlinear eigenvalue problems, are very complicated and hence these are solved by converting them to initial value problems.

Initial Value Method

Due to the extremely nonlinear form of these equations, after conversion to an initial value problem the shooting technique is used. An associated variational problem is developed and used in Newton-Raphson iteration scheme.

The governing equations (40a) and (40b) can be written as a system of six first order differential equations,

$$\frac{dY}{d\xi} = \bar{H}(\xi, \bar{Y}, \alpha, \lambda, Q^*) \quad , \quad R < \xi < 1 \tag{41}$$

where

$$\bar{Y}(\xi) = \begin{Bmatrix} g \\ g' \\ g'' \\ g''' \\ f \\ f' \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} \quad , \quad (\ )' = \frac{d}{d\xi} \quad , \text{ and}$$

$\bar{H}$ is the appropriately defined (6x1) vector function:

$$\bar{H} = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ \\ y_5' \\ y_6' \end{Bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ -\frac{A_2}{A_1} y_4 - \frac{A_3}{A_1} y_3 - \frac{A_2}{A_1} y_2 + \frac{\eta\lambda}{A_1} y_1 \frac{(c-\nu^2)}{(1-\nu^2)} \\ + \frac{9(c-\nu^2)}{\xi A_1} \alpha (y_3 y_5 + y_2 y_6) + \frac{(c-\nu^2)}{\sqrt{\alpha}\, A_1} Q^* \\ y_6 \\ (1 - \frac{\eta'}{\eta} \nu\xi) \frac{y_5}{\xi} - (1 - \frac{\eta'}{\eta} \xi) \frac{y_6}{\xi} - \frac{\eta}{2\xi} (y_2)^2 \end{Bmatrix}$$

The parameters α and λ are at present not known and hence two additional restraints are imposed to evaluate these. One component of $\bar{Y}(1)$ is normalized to fulfill the requirement.

The boundary conditions can be expressed as:

$$M\bar{Y}(1) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (42a)$$

$$N\bar{Y}(R) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (42b)$$

where, M&N are (4x6) and (3x6) coefficient matrices of rank 4 and 3 respectively. The first row of M normalizes a component of $\bar{Y}(1)$, and the remaining rows of M&N are obtained by taking into consideration the boundary conditions at the two ends.

The corresponding initial value problem may be expressed as

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi, \bar{Z}; \alpha, \lambda, Q^*) \quad (43a)$$

$$\bar{Z}(1) = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix} = \bar{\gamma} \qquad (43b)$$

where $\bar{\gamma}$ is a (6x1) vector of initial values.

Substitution of these initial values $\bar{Z}(1) = \bar{\gamma}$ into the equation (42a) yields:

$$M\bar{Z}(1) = M\bar{\gamma} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (44)$$

As M is of rank four, two additional values are required, and by the implicit function theorem,

$$\bar{\gamma} = \bar{\gamma}^*(\eta_1, \eta_2)$$

is a solution of equation (44), and η_1, η_2 are arbitrary initial values.

The related initial value problem can thus be written as:

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi, \bar{Z}; \alpha, \lambda, Q^*) \qquad (45a)$$

$$\bar{Z}(1) = \bar{\gamma}^*(\eta_1; \eta_2) \qquad (45b)$$

The above contains the initial values, which satisfy the boundary conditions (42a).

Assuming a continuous function $Q^*(\xi)$, a solution of the initial value problem (45) is obtained over a closed interval [R,1], and is denoted by

$$\bar{Z} = \bar{Z}(1;\ \bar{\eta},\ \alpha) \quad , \quad \bar{\eta} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \lambda \end{Bmatrix}$$

From the boundary condition (42b),

$$N\bar{Z}(R;\ \bar{\eta},\ \alpha) = 0 \tag{46}$$

Stating a well-know matrix theorem, "For a system of equations $N\bar{Z}(\xi;\ \bar{\eta},\ \alpha) = \bar{0}$ a necessary and sufficient condition for a unique solution $\bar{\eta} = \bar{\eta}(\alpha)$, is that the determinant of the Jacobian matrix, $J = \frac{\partial}{\partial\bar{\eta}}\ [N\bar{Z}(R,\ \bar{\eta},\ \alpha)]$ is not equal to zero," assuming also that $\bar{Z}$ is continuously differentiable with respect to $\bar{\eta}$ and α.

Thus there exists a locally unique function at $\xi = R$, such that,

$$N\bar{Z}(R,\ \eta(\alpha),\ \alpha) = \bar{0}$$

Or, we can put it as

$$\bar{Y}(\xi,\alpha) = \bar{Z}(\xi,\ \bar{\eta}(\alpha),\ \alpha)\ .$$

This forms an α-dependent family of solutions to (42) each of which is a solution to the initial value problem.

For a fixed value of α, say α^o, equations (46) reduce to three trancedental equations,

$$N\bar{Z}(R,\ \bar{\eta},\ \alpha^o) = \bar{0} \tag{47}$$

A root η^c of (47) may be obtained by Newtons iteration method. Starting with an initial guess, $\bar{\eta} = \bar{\eta}_1$ the iterative sequence,

$$\bar{\eta}_{k+1} = \bar{\eta}_k + \Delta\bar{\eta}_k \quad , \quad k = 1,2,3,\ldots \tag{48a}$$

is generated.

This can be expanded in the Taylor's Series. Retaining only the linear terms, gives,

$$\Delta\bar{\eta}_k = -[N \frac{\partial}{\partial\bar{\eta}_k} Z(R,\eta_k, \alpha^o)]^{-1} N\bar{Z}(R; \bar{\eta}_k, \alpha^o) \tag{48b}$$

where, at the k^{th} step, the (6x3) matrix J_1 is defined as,

$$(J_1) = \left(\frac{\partial\bar{Z}}{\partial\bar{\eta}}\right)_{\xi=R} = \left(\frac{\partial Z_i}{\partial\eta_j}\right)_{\xi=R} \qquad \begin{matrix} i = 1,2,\ldots,6 \\ j = 1,2,3 \end{matrix} \tag{49}$$

Physically, this represents the change of final values with respect to $\bar{\eta}$.

The expression $N\bar{Z}(\xi, \bar{\eta}_k, \alpha^o)$ represents the k^{th} error vector.

If the initial guess $\bar{\eta}$, is in the neighborhood of η^o, then the convergence of the sequence $\bar{\eta}_k$ to the root $\bar{\eta}^o$ is feasible.

In order to generate the sequence $\bar{\eta}_k$, it is necessary to evaluate the matrix $(J_1)_k$ at each step, k, of the iteration process. To do this, an associated variational problem is introduced.

Formally differentiating (45) with respect to $\bar{\eta}$, gives

$$\frac{d}{d\xi}\left(\frac{\partial\bar{Z}}{\partial\eta}\right) = \frac{\partial\bar{H}}{\partial\eta} + \left(\frac{\partial\bar{H}}{\partial\bar{Z}}\right)\left(\frac{\partial\bar{Z}}{\partial\eta}\right) \tag{50a}$$

$$\left(\frac{\partial\bar{Z}}{\partial\eta}\right)_{\xi=1} = \frac{\partial\bar{\gamma}^*}{\partial\eta} \tag{50b}$$

which constitute eighteen first order equations, and a corresponding set of initial values.

$$\frac{dZ_1}{d\xi} = Z_2$$

$$\frac{dZ_2}{d\xi} = Z_3$$

$$\frac{dZ_3}{d\xi} = Z_4$$

$$\frac{dZ_4}{d\xi} = \eta^2\omega^2 \frac{(c-\nu^2)}{(1-\nu^2)} y_1 - A_4y_2 - A_3y_3 - A_2y_4 \tag{51}$$

$$+ \frac{9(c-\nu^2)\frac{\alpha}{\xi}(y_3y_5 + y_2y_6)}{A_1} + \frac{(c-\nu^2)\ Q^*}{\sqrt{\alpha}\ A_1}$$

$$\frac{dZ_5}{d\xi} = Z_6$$

$$\frac{dZ_6}{d\xi} = \frac{1}{c}\left[1 - \frac{\eta'}{\eta}\xi\nu\right]\frac{1}{\xi^2} y_5 - \left[1 - \frac{\eta'}{\eta}\xi\right]\frac{1}{\xi} y_6 - \frac{\eta}{2\xi c} y_2^2$$

Differentiating the above with respect to $(\eta_1, \eta_2, \lambda)$ we get the following variational equations:

$$\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial \eta_1}\right) = \frac{\partial Z_2}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial \eta_1}\right) = \frac{\partial Z_3}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial \eta_1}\right) = \frac{\partial Z_4}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial \eta_1}\right) = \left(\eta^2\omega^2 \frac{(c-\nu^2)}{(1-\nu^2)} \frac{\partial Z_1}{\partial \eta_1} - A_4 \frac{\partial Z_2}{\partial \eta_1} - A_3 \frac{\partial Z_3}{\partial \eta_1} - A_2 \frac{\partial Z_4}{\partial \eta_1}\right. \tag{52a}$$

$$+ 9(c-\nu^2)\frac{\alpha}{\xi}\left(\frac{\partial Z_3}{\partial \eta_1} Z_5 + \frac{\partial Z_5}{\partial \eta_1} Z_3 + \frac{\partial Z_6}{\partial \eta_1} Z_2\right.$$

$$\left.\left. + \frac{\partial Z_2}{\partial \eta_1} Z_6\right)\right) / A_1$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial \eta_1}\right) = \frac{\partial Z_6}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial \eta_1}\right) = [1 - \frac{\eta'}{\eta}\,\xi\nu]\,\frac{\partial Z_5}{\partial \eta_1} \,/\, c\xi^2 - [1 - \frac{\eta'}{\eta}\,\xi]\,\frac{\partial Z_6}{\partial \eta_1} \,/\, \xi$$

$$- \frac{\eta}{c\xi}\, Z_2\, \frac{\partial Z_2}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial \eta_2}\right) = \frac{\partial Z_2}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial \eta_2}\right) = \frac{\partial Z_3}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial \eta_2}\right) = \frac{\partial Z_4}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial \eta_2}\right) = \left(\eta^2\omega^2\,\frac{(c-\nu^2)}{(1-\nu^2)}\,\frac{\partial Z_1}{\partial \eta_2} - A_4\,\frac{\partial Z_2}{\partial \eta_2} - A_3\,\frac{\partial Z_3}{\partial \eta_2} - A_2\,\frac{\partial Z_4}{\partial \eta_2}\right.$$

$$+ 9(c-\nu^2)\,\frac{\alpha}{\xi}\left(\frac{\partial Z_3}{\partial \eta_2}\, Z_5 + \frac{\partial Z_5}{\partial \eta_2}\, Z_3 + \frac{\partial Z_2}{\partial \eta_2}\, Z_6\right.$$

$$\left.\left. + \frac{\partial Z_6}{\partial \eta_2}\, Z_2\right)\right) \,/\, A_1 \qquad (52b)$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial \eta_2}\right) = \frac{\partial Z_6}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial \eta_2}\right) = [1 - \frac{\eta'}{\eta}\,\xi\nu]\,\frac{\partial Z_5}{\partial \eta_2} \,/\, c\xi^2 - [1 - \frac{\eta'}{\eta}\,\xi]\,\frac{\partial Z_6}{\partial \eta_2} \,/\, \xi$$

$$- \frac{\eta}{\xi c}\, Z_2\, \frac{\partial Z_2}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial \lambda}\right) = \frac{\partial Z_2}{\partial \lambda}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial \lambda}\right) = \frac{\partial Z_3}{\partial \lambda}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial \lambda}\right) = \frac{\partial Z_4}{\partial \lambda}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial \lambda}\right) = \left(\eta^2\omega^2 \frac{(c-\nu^2)}{(1-\nu^2)} \frac{\partial Z_1}{\partial \lambda} - A_4 \frac{\partial Z_2}{\partial \lambda} - A_3 \frac{\partial Z_3}{\partial \lambda} - A_2 \frac{\partial Z_4}{\partial \lambda}\right.$$

$$+ \eta^2 Z_1 + 9(c-\nu^2) \frac{\alpha}{\xi} \left(\frac{\partial Z_3}{\partial \lambda} A_5 + \frac{\partial Z_5}{\partial \lambda} Z_3\right. \tag{52c}$$

$$\left.\left. + \frac{\partial Z_2}{\partial \lambda} Z_6 + \frac{\partial Z_6}{\partial \lambda} Z_2\right)\right) / A_1$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial \lambda}\right) = \frac{\partial Z_6}{\partial \lambda}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial \lambda}\right) = [1 - \frac{\eta'}{\eta} \xi\nu] \frac{\partial Z_5}{\partial \lambda} / c\xi^2 - [1 - \frac{\eta'}{\eta}] \frac{\partial Z_6}{\partial \lambda} / \xi$$

$$- \frac{\eta}{\xi c} Z_2 \frac{\partial Z_2}{\partial \lambda}$$

For a given vector $\bar{\eta}$ and $\alpha = \alpha^o$, this derived problem along with the initial value problem (45) may be integrated simultaneously on the interval [R,1]. Corresponding to a given value of $\bar{\eta}$ and $\alpha = \alpha^o$, the calculation of the resulting solution to the variational problem at $\xi = 1$ provides the Jacobian (J_1). By setting $\bar{\eta} = \bar{\eta}_1$ and integrating equations (45) & (50) from $\xi=1$ to $\xi=R$, gives the first correction vector $\bar{\eta}_2$. By repeating this procedure, the desired sequence $\bar{\eta}_k$ is obtained, which converges to $\bar{\eta}^o$ within a specified error bound to the accuracy of the system.

Having obtained $\bar{\eta}^o$, corresponding to α^o, the value of α can now be perturbed,

$$\alpha = \alpha^o + \Delta\alpha^o = \alpha^1$$

The problem is reinstated, for this value of α, starting from $\bar{\eta} = \bar{\eta}^o$. If $\Delta\alpha^o$ is small, then $\bar{\eta}^o$ is contained in the new contraction domain of Newton's method, the iterations converging to $\bar{\eta}^1$ corresponding to $\alpha = \alpha^1$. Successive completion of this operation j number of times, yields,

$$\bar{\eta}^i = \bar{\bar{\eta}}^i\,(\alpha^i) \quad , \quad i = 0,1,2,\ldots,j$$

By setting $\alpha = \alpha^i + \Delta\alpha^j = \alpha^{j+1}$ and starting integration from $\bar{\bar{\eta}}^j$, one obtains $\bar{\bar{\eta}}^{j+1}$ provided $\Delta\alpha^j$ results in convergence.

The range of α is limited as the elastic plate cannot withstand unbounded amplitudes.

CHAPTER IV

NUMERICAL COMPUTATIONS

The above theoretical analysis suggests the use of a numerical integration technique.

Use is made of a fourth order Runge-Kutta-Gill method to integrate the initial value problems (45) and (50) over the interval [R,1]. The following approach is suggested.

The problem is first reduced to that of a free vibration by setting $Q^* = 0$ and $\alpha^o = 0$. By subjecting this equation to a particular set of boundary conditions the linear eigenvalues and mode shape functions are determined.

This information leads to a basis for making a reasonable starting guess, η_1, required by the initial value method.

For $\bar{\eta} = \bar{\eta}_1$, the initial value problems (45) and (50) are integrated numerically over [R,1] with a step size $\Delta\mu = 1/40$. Successive correction is carried out till all equations in (47) satisfy the range of prescribed error; this being consistent with the order $0\ (|\Delta\mu|^4)$, of Runge Kutta Gill method.

By gradually incrementing the value of α and restarting the correction and integration procedure from the values of $(\eta_1, \eta_2, \lambda)$, obtained from the solution corresponding to the previous α, the resonance curve and other solutions are evaluated. This procedure is terminated at a particular value of α^m, because of reasons mentioned earlier.

Cases Considered:

The cases considered are:

(1) Annular circular plate with convex variable thickness and free on the outside, fixed on the inside.

(2) Annular circular plate with parabolic variable thickness and free on the outside, fixed on the inside.

The figures pertaining to the above two cases are as shown in Appendix I.

The governing equations and boundary conditions are written as:

$$\frac{d\bar{Y}}{d\xi} = \bar{H}(\xi, \bar{Y}; \alpha, \lambda, Q^*) \quad ; \quad R < \xi < 1 \tag{53a}$$

$$M\,\bar{Y}(1) = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{53b}$$

$$N\,\bar{Y}(R) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{53c}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & c & 0 & 0 & 0 \\ 0 & -(1 - \frac{3\eta'}{\eta}\nu) & c(-1 + \frac{3\eta'}{\eta}) & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\nu}{R} & c \end{bmatrix} .$$

The related initial value problem, is defined as,

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi,\ \bar{Z};\ \alpha,\ \lambda,\ Q^*) \tag{54a}$$

$$\bar{Z}(1) = \bar{\gamma}^*(\eta_1,\eta_2) = \begin{Bmatrix} 1 \\ \eta_1 \\ -\frac{\nu}{c}\,\eta_1 \\ \frac{(1+\nu)}{c}\,\eta_1 \\ 0 \\ \eta_2 \end{Bmatrix} \tag{54b}$$

and the variational problem is

$$\frac{d}{d\xi}\left\{\frac{\partial\bar{Z}}{\partial\eta_1}\right\} = \left\{\frac{\partial\bar{H}}{\partial\bar{Z}}\right\}\left\{\frac{\partial\bar{Z}}{\partial\eta_1}\right\} \tag{53a}$$

$$\frac{d}{d\xi}\left\{\frac{\partial\bar{Z}}{\partial\eta_1}\right\} = \begin{Bmatrix} 0 \\ 1 \\ -\frac{\nu}{c} \\ \frac{1+\nu}{c} \\ 0 \\ 0 \end{Bmatrix} \tag{55b}$$

$$\frac{d}{d\xi}\left\{\frac{\partial\bar{Z}}{\partial\eta_2}\right\} = \left\{\frac{\partial\bar{H}}{\partial\bar{Z}}\right\}\left\{\frac{\partial\bar{Z}}{\partial\eta_2}\right\} \tag{55c}$$

$$\left\{\frac{\partial\bar{Z}}{\partial\eta_2}\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{55d}$$

$$\frac{d}{d\xi}\left\{\frac{\partial Z}{\partial \lambda}\right\} = \left[\frac{\partial \bar{H}}{\partial \bar{Z}}\right]\left\{\frac{\partial \bar{Z}}{\partial \lambda}\right\} + \left\{\frac{\partial \bar{H}}{\partial \lambda}\right\} \tag{55e}$$

$$\left\{\frac{\partial \bar{Z}}{\partial \lambda}\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{55f}$$

The unification of the above is as symbolized in equation (50) with $\bar{\eta} = (\eta_1, \eta_2, \lambda)$ while the value of α is held constant.

$$\begin{Bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\eta_3 \end{Bmatrix} = - \begin{bmatrix} \frac{\partial Z_1}{\partial \eta_1} & \frac{\partial Z_1}{\partial \eta_2} & \frac{\partial Z_1}{\partial \lambda} \\ \frac{\partial Z_2}{\partial \eta_1} & \frac{\partial Z_2}{\partial \eta_2} & \frac{\partial Z_2}{\partial \lambda} \\ c\,\frac{\partial Z_6}{\partial \eta_1} - \frac{\nu}{\xi}\frac{\partial Z_5}{\partial \eta_1} & c\,\frac{\partial Z_6}{\partial \eta_2} - \frac{\nu}{\xi}\frac{\partial Z_5}{\partial \eta_2} & c\,\frac{\partial Z_6}{\partial \lambda} - \frac{\nu}{\xi}\frac{\partial Z_5}{\partial \lambda} \end{bmatrix}^{-1}_{\xi=R} \begin{Bmatrix} Z_1 \\ Z_2 \\ cZ_6 - \frac{\nu}{\xi} Z_5 \end{Bmatrix}_{\xi=R}$$

provides the linear correction of the estimated values $(\eta_1, \eta_2, \lambda)$, where Z_i are components of $\bar{Z}$.

For each value of α^i, a sequence which defines discrete values of α, successive corrections of $(\eta_1, \eta_2, \lambda)$ were performed, till the final values of $\bar{Z}(R)$ are satisfied,

$$\max_{1 \le i \le 3} \left| \sum_{j=1}^{6} n_{ij}\; Z_j(R) \right| \le 0.1 \times 10^{-5} \tag{57}$$

where $n_{ij} = N$.

Perturbing the amplitude α, the process is started using the values of $\bar{\eta}$ obtained after the first cycle is completed. At least five to six iterations were required for most values of α.

Stresses:

From all the discussion carried out so far, it is obvious that the amplitude influences the distribution of bending stress to a greater extent, as these are related to the derivatives of the transverse shape function $g(\xi)$.

The expressions for the bending and membrane stresses are:

$$\sigma_r^b = -\frac{6M_r}{h^2} = \frac{h_o}{a}\frac{\eta(\xi)}{2a_{22}(c-\nu^2)}[c\chi'' + \frac{\nu}{\xi}\chi']$$

$$\sigma_\theta^b = -\frac{6M_\theta}{h^2} = \frac{h_o}{a}\frac{\eta(\xi)}{2a_{22}(c-\nu^2)}[\frac{1}{\xi}\chi' + \chi'']$$

$$\sigma_r^m = \frac{N_r}{h} = \frac{1}{a_{22}\eta(\xi)}\frac{\phi}{\xi}$$

$$\sigma_\theta^m = \frac{N_\theta}{h} = \frac{1}{a_{22}\eta(\xi)}\phi'$$

These are the radial bending stress, circumferential bending stress, radial membrane stress and circumferential membrane stress respectively, in terms of the dimensionless deflection, χ, and stress function ϕ, respectively. Taking the previous assumption into consideration, i.e.,

$$\chi(\xi,\tau) = Ag(\xi)\ \mathrm{Sin}\ \omega\tau$$

$$\phi(\xi,\tau) = A^2 f(\xi)\ \mathrm{Sin}^2\omega\tau$$

and also taking into consideration the fact that when time, τ, is equal to the odd multiple of $\pi/2w$, we have the maximum stresses,

$$\frac{\sigma_r^b a^2 a_{22}}{h_o^2} = \pm\frac{\sqrt{\alpha}\ \eta(\xi)}{2(c-\nu^2)}(cg_{\xi\xi} + \frac{\nu}{\xi}g_\xi) \quad (58a)$$

$$\frac{\sigma_\theta^b a^2 a_{22}}{h_o^2} = \pm\frac{\sqrt{\alpha}\ \eta(\xi)}{2(c-\nu^2)}(\nu g_{\xi\xi} + \frac{1}{\xi}g_\xi) \quad (58b)$$

$$\frac{\sigma_r^m a^2 a_{22}}{h_o^2} = \frac{\alpha}{\eta(\xi)} \left(\frac{f}{\xi}\right) \tag{58c}$$

$$\frac{\sigma_\theta^m a^2 a_{22}}{h_o^2} = \frac{\alpha}{\eta(\xi)} (f') \tag{58d}$$

The above equations were used in the computer program.

CHAPTER 5: CONCLUSIONS

This study is based on the supposition of harmonic oscillations. The assumed solutions (35) contradict the inseparability of modes in Von Karman's dynamic equations. Nevertheless, for moderate amplitude of vibrations, physical arguments may be made to justify such assumptions.

The time coordinate function is assumed and eliminated by a time averaging method. By elimination of the time variable, an infinite number of degrees of freedom in the space coordinate function is achieved. By the numerical integration technique used the solution of the continuous system is obtained at a number of discrete points. This reduces the number of degrees of freedom to the number of points considered.

The two cases studied are an annular plate with parabolic variable thickness and of convex variable thickness. Both are of the same volume and have the same boundary conditions of free on the outside and fixed on the inside.

The responses of the plates exhibit a behavior similar to that of a hard spring.

The parabolic variable thickness plate is stiffer than the convex variable thickness one, as is evident from the frequency responses obtained.

The bending stresses of the first plate are slightly higher than those of the second plate and the membrane stresses are just the reverse.

The membrane stresses have significant magnitudes even at relatively low amplitudes. This is due to a stress concentration factor at the edge of the hole, and is called the boundary layer.

The results obtained were compatible with those obtained by Sandman [2]. In this study the higher modes and stability of vibration have not been considered. So also, various other boundary conditions are possible, these are thus left open for future investigation.

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Appendix A

Figures

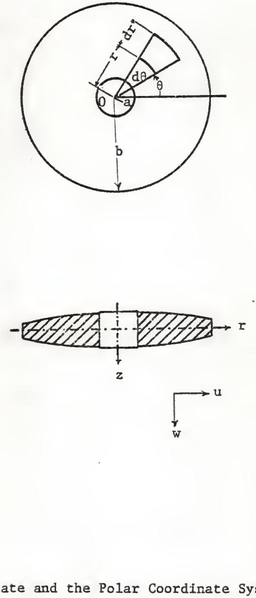

Fig. 1. Circular Plate and the Polar Coordinate System.

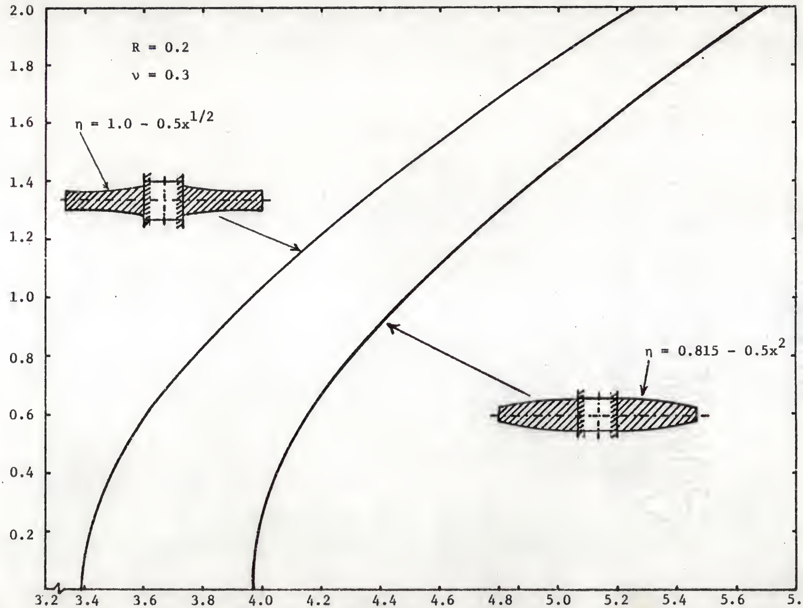

Fig. 2. Frequency responses of the two plates.

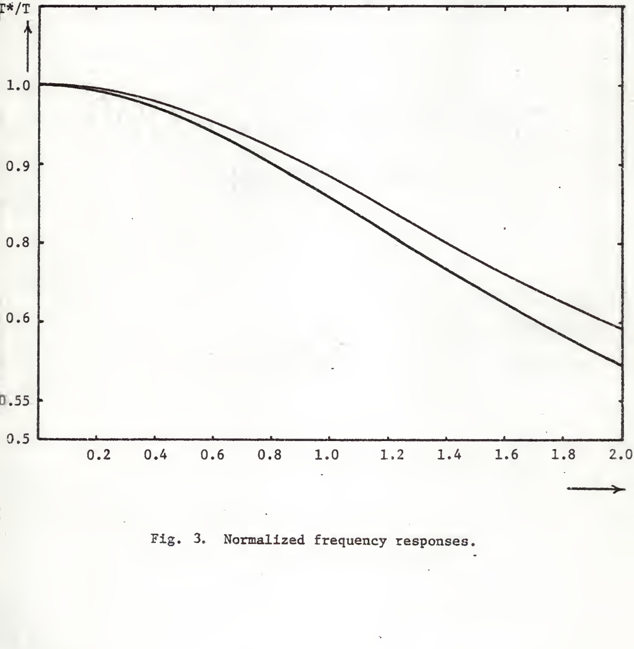

Fig. 3. Normalized frequency responses.

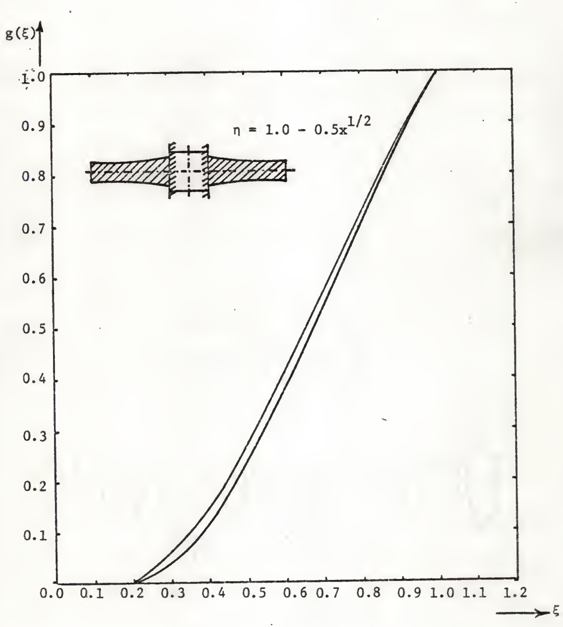

Fig. 4. Shape function for annular, convex-variable thickness plate.

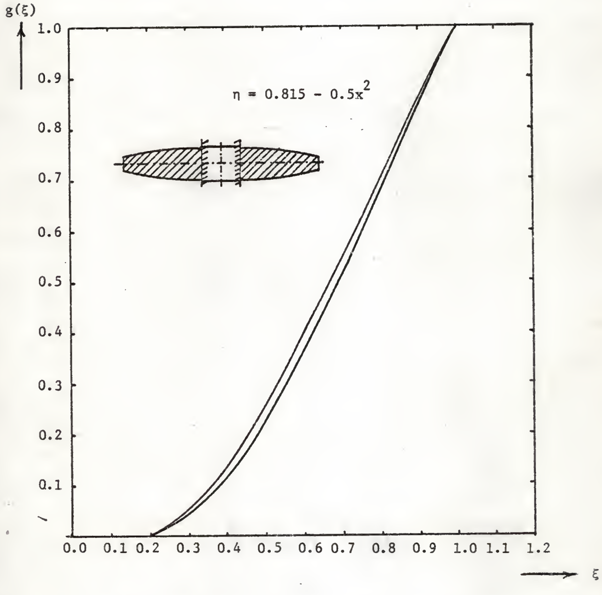

Fig. 5. Shape function for annular, parabolic, variable thickness plate.

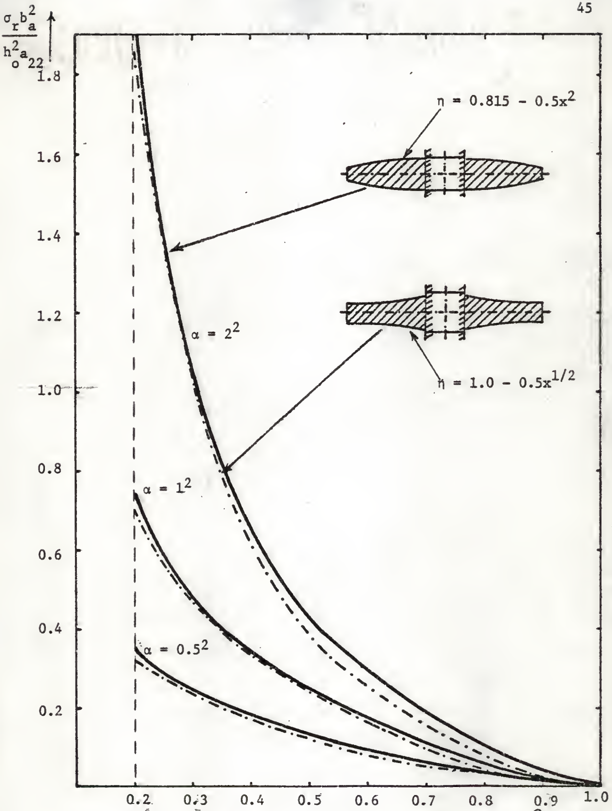

Fig. 6. Radial bending stresses

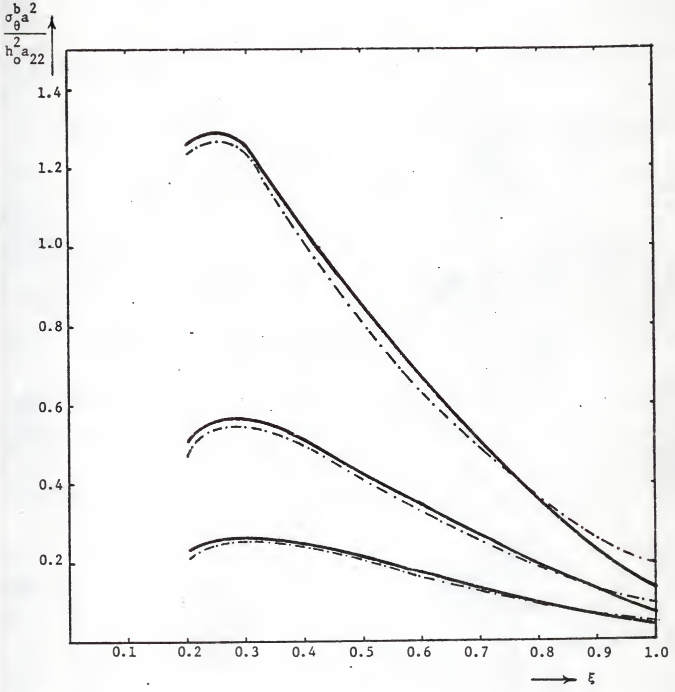

Fig. 7. Circumferential bending stresses.

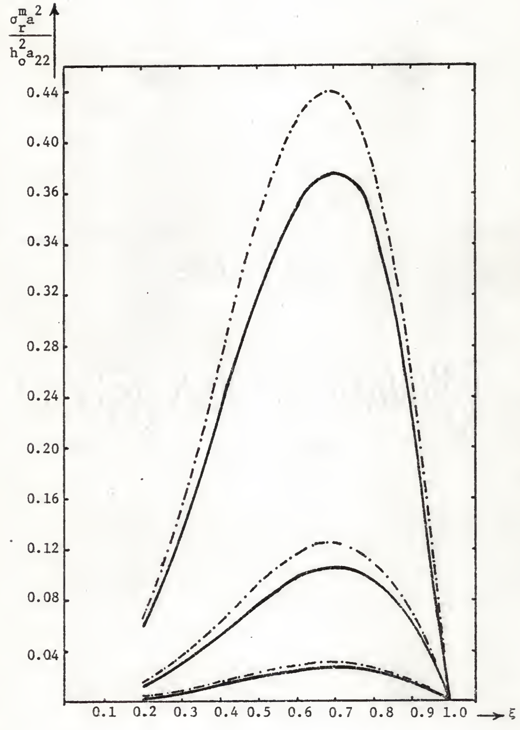

Fig. 8. Radial membrane stresses.

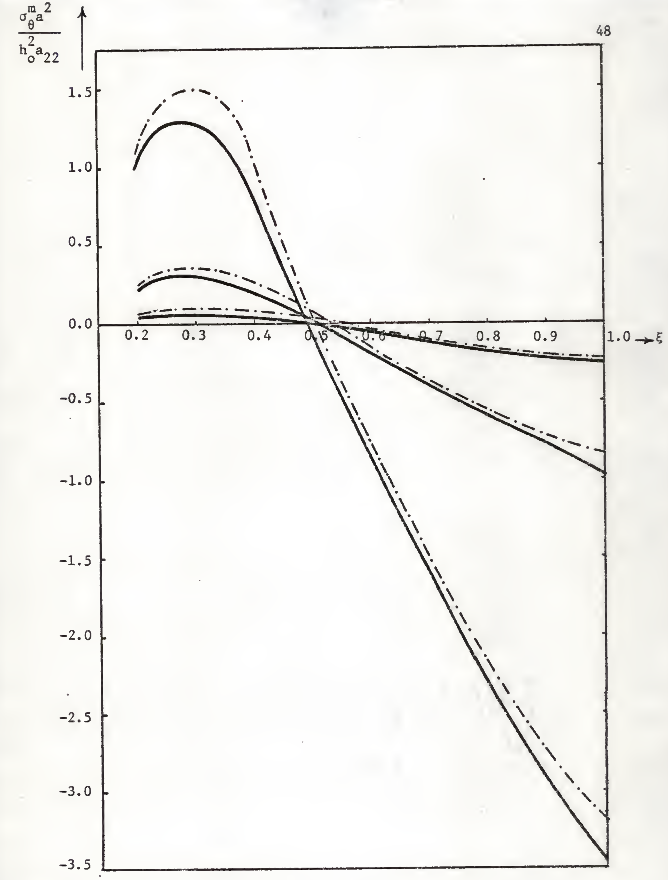

Fig. 9. Circumferential membrane stresses.

APPENDIX B

Computer program for annular orthotropic
convex variable thickness plate --
Backward shooting

```
$JOB
C**********************************************************************
C       INITIAL VALUE METHOD - FREE VIBRATION OF AN ANNULAR
C       ORTHOTROPIC ,CONVEX VARIABLE THICKNESS PLATE, WITH
C       BOUNDARY CONDITIONS AS FIXED ON THE INSIDE AND
C       FREE ON THE OUTSIDE
C**********************************************************************
C**********************************************************************
C       S=RATIO OF ELASTIC CONSTANTS
C       E=POISSONS RATIO
C       QL=UNIFORM LOADING INTENSITY
C       A=AMPLITUDE
C       R=RATIO OF INNER TO OUTER RADIUS
C       P=EIGENVALUE
C       H=STEP SIZE
C       ETA=THICKNESS FUNCTION
C       DETA=FIRST DERIVATIVE OF ETA
C       DDET=SECOND DERIVATIVE OF ETA
C**********************************************************************
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION ETA(41),XX(41),Y(24),C(24),TP(3,3),D(6,41)
      DIMENSION O(3),LW(3),MW(3),ER(3)
      DIMENSION RBS(45),CBS(45),RMS(45),CMS(45)
  112 FORMAT(5X,'AMF=',D22.14,3X,'FREQ=',D22.14,3X,'FRER=',
     1D22.14)
  113 FORMAT(9X,'W',19X,'DW',18X,'DDW',17X,'DDDW')
  114 FORMAT(4D22.14)
  115 FORMAT(//9X,'F',15X,'DF')
  117 FORMAT(1H )
  120 FORMAT(5X,'S=',F10.3,5X,'E=',F10.3,5X,'QL=',F10.3)
  121 FORMAT(//9X,'STA',19X,'PROF')
  122 FORMAT(6X,'RBS',18X,'CBS',18X,'RMS',18X,'CMS')
  123 FORMAT(5X,'ITER=',I2)
      S=0.5
      H=1./40.
      LL=41
      KK=9
      JK=LL+1-KK
      IK=1
      QL=0.0
      A=0.0
      E=1./3.
      R=0.2D-0
      DA=0.25
      P=8.3**2
      ETA1=1.4482
      ETA2=-0.2426
      BE=0.5
C**********************************************************************
C       CONSTRUCT INITIAL VALUES
C**********************************************************************
  500 IT=1
      DO 9 I=1,24
    9 Y(I)=0.0D-0
      Y(1)=1.0D-0
      Y(2)=ETA1
      Y(3)=-(E*Y(2))/S
      Y(4)=((1.+E)*Y(2))/S
      Y(6)=ETA2
      Y(8)=1.0D-0
```

```
      Y(9)=-E/S
      Y(10)=(1.0+E)/S
      Y(18)=1.
C***********************************************************
C      X=INDEPENDENT VARIABLE
C     INTEGRATION BY BACKWARD SHOOTING
C***********************************************************
  600 X=1.0D-0
      DO 623 I=1,24
  623 Q(I)=0.0D-0
      DO 620 I=1,6
  620 D(I,LL)=Y(I)
      HR=-H
      DO 624 J=2,JK
      M=LL+1-J
      CALL RKG(X,HR,Y,C,P,A,S,E,QL,BE)
      DO 615 L=1,6
  615 D(L,M)=Y(L)
  624 CONTINUE
C***********************************************************
C     ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS
C***********************************************************
      ER(1)=D(1,KK)
      ER(2)=D(2,KK)
      ER(3)=S*D(6,KK)-(E*D(5,KK))/R
C***********************************************************
C     CONSTRUCT ERROR NORM
C***********************************************************
       DO 26 I=1,4
      DER=DABS(ER(I))
      IF(DER.GT.0.1D-05) GO TO 28
   26 CONTINUE
       GO TO 900
   28 CONTINUE
C***********************************************************
C     NEWTONS METHOD (ERROR NORM=(8)Y=0
C     TP(I,J)= THE JACOBIAN OF THE INITIAL VALUES
C     C(I)=CORRECTION VECTOR
C***********************************************************
      TP(1,1)=Y(7)
      TP(2,1)=Y(8)
      TP(3,1)=S*Y(12)-(E*Y(11))/R
      TP(1,2)=Y(13)
      TP(2,2)=Y(14)
      TP(3,2)=S*Y(18)-(E*Y(17))/R
      TP(1,3)=Y(19)
      TP(2,3)=Y(20)
      TP(3,3)=S*Y(24)-(E*Y(23))/R
      DET=0.0D-0
      CALL DMINV(TP,3,DET,LW,MW)
      DO 75 I=1,3
      C(I)=0.0
      DO 75 J=1,3
   75 C(I)=C(I)-TP(I,J)*ER(J)
C***********************************************************
C     CORRECTED VALUES
C***********************************************************
      DO 76 I=1,6
   76 Y(I)=D(I,LL)
      Y(2)=Y(2)+C(1)
```

```
      Y(6)=Y(6)+C(2)
      P=P+C(3)
      Y(3)=-(E*Y(2))/S
      Y(4)=((1.+E)*Y(2))/S
      DO 77 I=7,24
   77 Y(I)=0.0D-0
      Y(8)=1.0D-0
      Y(9)=-E/S
      Y(10)=(1.0+E)/S
      Y(18)=1.0
      IT=IT+1
       IF(IT.GT.10) GO TO 550
       GO TO 600
C*****************************************************
C     FINAL RESULTS
C*****************************************************
  900 SRA=DSQRT(A)
      SP=DSQRT(P)
      DO 795 J=KK,LL,4
      DJ=J-1
      XX(J)=DJ*H
      ETA(J)=1.-BE*(XX(J)**(0.5))
      IF(XX(J).GT.0.0) GO TO 905
      RBS(J)=SRA*C(3,J)/2.*(1.-E)
      CBS(J)=RBS(J)
      RMS(J)=A*D(6,J)
      CMS(J)=RMS(J)
      GO TO 795
  905 RBS(J)=SRA*ETA(J)*(S*D(3,J)+E*D(2,J)/XX(J))/2.*(S-E**2)
      CBS(J)=SRA*ETA(J)*(D(2,J)/XX(J)+E*D(3,J))/2.*(S-E**2)
      RMS(J)=A*D(5,J)/ETA(J)*XX(J)
      CMS(J)=A*D(6,J)/ETA(J)
  795 CONTINUE
C*****************************************************
C     FOR FREQUENCY RATIO
C*****************************************************
      IF(A.GT.0.0) GO TO 906
      SPO=SP
  906 SPR=SP/SPO
      WRITE(6,117)
      WRITE(6,120) S,E,CL
      WRITE(6,117)
      WRITE(6,112) SRA,SP,SPR
      WRITE(6,117)
      WRITE(6,113)
      DO 901 J=KK,LL,4
  901 WRITE(6,114) (D(I,J),I=1,4)
      WRITE(6,115)
      DO 902 J=KK,LL,4
  902 WRITE(6,114) (D(L,J),L=5,6)
      WRITE(6,117)
      WRITE(6,122)
      DO 903 J=KK,LL,4
  903 WRITE(6,114) RBS(J),CBS(J),RMS(J),CMS(J)
      WRITE(6,117)
      WRITE(6,123) IT
      WRITE(6,121)
      DO 924 J=KK,LL,4
  924 WRITE(6,114) XX(J),ETA(J)
      WRITE(6,117)
```

```
C************************************************************
C     PERTURBATION OF AMPLITUDE
C************************************************************
      A=A+DA
      IK=IK+1
      IF(IK.GT.26) GO TO 550
      ETA1=C(2,LL)
      ETA2=C(6,LL)
      P=(SP-0.3)**2
      GO TO 500
  550 STOP
      END
      SUBROUTINE RKG(X,H,Y,Q,P,AP,S,E,QL,BE)
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),Q(24),DY(24),A(2)
      A(1)=0.2928932188135
      A(2)=1.7071067811865
      H2=0.5*H
      CALL DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)
      DO 13 I=1,24
      R=H2*DY(I)-Q(I)
      Y(I)=Y(I)+R
   13 Q(I)=Q(I)+3.0*R-H2*DY(I)
      X=X+H2
      DO 60 J=1,2
      CALL DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)
      DO 20 I=1,24
      R=A(J)*(H*DY(I)-Q(I))
      Y(I)=Y(I)+R
   20 Q(I)=Q(I)+3.0*R-A(J)*H*DY(I)
   60 CONTINUE
      X=X+H2
      CALL DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)
      DO 26 I=1,24
      R=(H*DY(I)-2.0*Q(I))/6.0
      Y(I)=Y(I)+R
   26 Q(I)=Q(I)+3.0*R-H2*DY(I)
      RETURN
      END
      SUBROUTINE DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),DY(24)
      ETA=1.-BE*(X**(0.5))
      DETA=-0.5*BE*(X**(-0.5))
      DDET=0.25*BE*(X**(-1.5))
      DO 10 I=1,3
   10 DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 12 I=7,9
   12 DY(I)=Y(I+1)
      DY(11)=Y(12)
      DO 15 I=13,15
   15 DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 16 I=19,21
   16 DY(I)=Y(I+1)
      DY(23)=Y(24)
      IF(X.GE.0.1D-02) GO TO 17
```

```
      DY(4)=3.*P*Y(1)/8.*(ETA**2)+27.*(1.-E**2)*AP*Y(3)*Y(6)
     1/4.*(ETA**3)
      DY(4)=DY(4)-9.*(1.+E)*DDET*Y(3)/8.*ETA
      IF(AP) 18,18,19
   19 DY(4)=DY(4)+3.*(1.-E**2)*QL/8.*(ETA**3)*DSQRT(AP)
   18 DY(4)=DY(4)
      DY(10)=3.*P*Y(7)/8.*(ETA**2)+27.*(1.-E**2)*AP*(Y(9)*
     1Y(6)+Y(3)*Y(12))/4.*(ETA**3)
      DY(10)=DY(10)-9.*(1.+E)*DDET*Y(9)/8.*ETA
      DY(16)=3.*P*Y(13)/8.*(ETA**2)+27.*(1.-E**2)*AP*(Y(15)*
     1Y(6)+Y(18)*Y(3))/4.*(ETA**3)
      DY(16)=DY(16)-9.*(1.+E)*DDET*Y(15)/8.*ETA
      DY(22)=3.*P*Y(19)/8.*(ETA**2)+3.*Y(1)/8.*(ETA**3)+27.
     1*(1.-E**2)*AP*(Y(21)*Y(6)+Y(3)*Y(24))/4.*(ETA**3)
      DY(22)=DY(22)-9.*(1.+E)*DDET*Y(21)/8.*ETA
      DY(6)=0.0
      DY(12)=0.0
      DY(18)=0.0
      DY(24)=0.0
      GO TO 70
   17 B0=(9.*(S-(E**2)))/(S*X*(ETA**3))
      B1=(S-E**2)/((1.-E**2)*(S*(ETA**2)))
      B2=((6.*DETA)/ETA)+(2./X)
      B3=(-1./(S*(X**2)))+(3.*DETA*(2.*S+E))/(S*X*ETA)+(3.*
     1DDET)/ETA+(6.*(DETA**2))/(ETA**2)
      B4=1./(S*(X**3))-(3.*DETA)/(S*(X**2)*ETA)+(3.*E*DDET)/
     1(ETA*S*X)+6.*E*(DETA**2)/(S*X*(ETA**2))
      DY(4)=B0*AP*(Y(3)*Y(5)+Y(2)*Y(6))+B1*P*Y(1)-B4*Y(2)
     1-B3*Y(3)-B2*Y(4)
      IF(AP) 100,100,101
  101 DY(4)=DY(4)+(S-(E**2))*QL/(S*(ETA**3)*DSQRT(AP))
  100 DY(4)=DY(4)
      DY(6)=(1.-E*X*DETA/ETA)*Y(5)/(S*(X**2))-(1.-X*DETA/ETA
     1)*Y(6)/X-(ETA*(Y(2)**2))/(2.*X*S)
      DY(10)=B0*AP*(Y(6)*Y(8)+Y(5)*Y(9)+Y(2)*Y(12)+Y(3)*Y(11
     1))+B1*P*Y(7)-B2*Y(10)-B3*Y(9)-B4*Y(8)
      DY(12)=(1.-E*X*DETA/ETA)*Y(11)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(12)/X-ETA*Y(2)*Y(8)/(S*X)
      DY(16)=B0*AP*(Y(5)*Y(15)+Y(3)*Y(17)+Y(14)*Y(6)+Y(2)*
     1Y(18))+B1*P*Y(13)-B2*Y(16)-B3*Y(15)-B4*Y(14)
      DY(18)=(1.-E*X*DETA/ETA)*Y(17)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(18)/X-ETA*Y(2)*Y(14)/(S*X)
      DY(22)=B0*AP*(Y(21)*Y(5)+Y(3)*Y(23)+Y(20)*Y(6)+Y(2)*
     1Y(24))+B1*P*Y(19)+B1*Y(1)-B2*Y(22)-B3*Y(21)-B4*Y(20)
      DY(24)=(1.-E*X*DETA/ETA)*Y(23)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(24)/X-ETA*Y(2)*Y(20)/(S*X)
   70 RETURN
      END

      SUBROUTINE DMINV(A,N,D,L,M)
      DIMENSION A(9),L(3),M(3)
      DOUBLE PRECISION A,D,BIGA,HOLD,DABS
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
```

```
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI)=HOLD
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI)=HOLD
   45 IF(BIGA) 48,46,48
   46 D=0.0
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
   65 CONTINUE
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
      D=D*BIGA
      A(KK)=1.0/BIGA
   80 CONTINUE
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
```

Computer program for annular orthotropic
convex variable thickness plate --
Forward shooting.

```
$JOB
C*************************************************************
C      INITIAL VALUE METHOD - FREE VIBRATICN OF AN ANNULAR
C      ORTHCTRCPIC ,CCNVEX VARIABLE THICKNESS PLATE, WITH
C      BOUNDARY CCNDITICNS AS FIXED CN THE INSIDE AND
C      FREE ON THE OUTSIDE
C*************************************************************
C*************************************************************
C      S=RATIO CF ELASTIC CCNSTANTS
C      E=PCISSCNS RATIO
C      QL=UNIFORM LOADING INTENSITY
C      A=AMPLITUDE
C      R=RATIO CF INNER TO OUTER RADIUS
C      P=EIGENVALUE
C      H=STEP SIZE
C      ETA=THICKNESS FUNCTICN
C      DETA=FIRST CERIVATIVE OF ETA
C      DDET=SECCND DERIVATIVE CF ETA
C*************************************************************
        IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSICN ETA(41),XX(41),Y(30),C(30),TP(4,4),C(8,41)
      DIMENSICN C(4),MW(4),LW(4),ER(4)
      DIMENSICN RBS(45),CBS(45),RMS(45),CMS(45)
  112 FORMAT(5X,'AMP=',D22.14,3X,'FREQ=',D22.14,3X,'FRER=',
     1D22.14)
  113 FORMAT(9X,'W',19X,'DW',18X,'DDW',17X,'DDDW')
  114 FORMAT(4D22.14)
  115 FORMAT(//9X,'F',19X,'DF')
  117 FORMAT(1H )
  120 FORMAT(5X,'S=',F10.3,5X,'E=',F10.3,5X,'QL=',F10.3)
  121 FCRMAT(//9X,'STA',19X,'PROF')
  122 FORMAT(6X,'RBS',18X,'CBS',18X,'RMS',18X,'CMS')
  123 FORMAT(5X,'ITER=',I2)
       S=0.5
      H=1./40.
       LL=41
      KK=9
       JK=LL+1-KK
       IK=1
       QL=0.0
      A=0.0
      E=1./3.
      R=0.2D-0
      DA=0.25
        P=4.0**2
      ETA1=8.00
      ETA2=-58.0
      ETA3=0.06
      BE=0.5
C*************************************************************
C       CCNSTRUCT INITIAL VALUES
C*************************************************************
  500 IT=1
      DO 9 I=1,30
    9 Y(I)=0.0D-0
      Y(1)=1.0D-0
      Y(3)=ETA1
      Y(4)=ETA2
      Y(5)=ETA3
      Y(6)=E*ETA3/(R*S)
```

```
      Y(9)=1.0
      Y(16)=1.
      Y(23)=1.0
      Y(24)=E/(R*S)
C*****************************************************************
C       X=INDEPENDENT VARIABLE
C      INTEGRATION BY FORWARD SHOOTING
C*****************************************************************
  600 X=0.2D-0
      DO 623 I=1,30
  623 O(I)=0.0D-0
      DO 620 I=1,6
  620 D(I,KK)=Y(I)
      KJ=KK+1
      DO 624 J=KJ,LL
      CALL RKG(X,H,Y,O,P,A,S,E,QL,BE)
      DO 615 L=1,6
  615 D(L,J)=Y(L)
  624 CONTINUE
C*****************************************************************
C      ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS
C*****************************************************************
       ER(1)=D(1,LL)-1.0
      ER(2)=E*D(2,LL)+S*D(3,LL)
       ER(3)=S*D(4,LL)-S*0.5*D(3,LL)-(1.+1.5*E)*D(2,LL)
      ER(4)=D(5,LL)
C*****************************************************************
C      CONSTRUCT ERROR NORM
C*****************************************************************
       DO 26 I=1,4
      DER=DABS(ER(I))
      IF(DER.GT.0.1D-05) GO TO 28
   26 CONTINUE
       GO TO 900
   28 CONTINUE
C*****************************************************************
C      NEWTONS METHOD (ERROR NORM=(B)Y=0
C      TP(I,J)= THE JACOBIAN OF THE INITIAL VALUES
C      C(I)=CORRECTION VECTOR
C*****************************************************************
      TP(1,1)=Y(7)
      TP(2,1)=E*Y(8)+S*Y(9)
       TP(3,1)=-(1.+1.5*E)*Y(8)-S*0.5*Y(9)+S*Y(10)
      TP(4,1)=Y(11)
      TP(1,2)=Y(13)
      TP(2,2)=E*Y(14)+S*Y(15)
      TP(3,2)=-(1.+1.5*E)*Y(14)-S*0.5*Y(15)+S*Y(16)
       TP(4,2)=Y(17)
      TP(1,3)=Y(19)
      TP(2,3)=E*Y(20)+S*Y(21)
       TP(3,3)=-(1.+1.5*E)*Y(20)-S*0.5*Y(21)+S*Y(22)
      TP(4,3)=Y(23)
      TP(1,4)=Y(25)
      TP(2,4)=E*Y(26)+S*Y(27)
        TP(3,4)=-(1.+1.5*E)*Y(26)-S*0.5*Y(27)+S*Y(28)
      TP(4,4)=Y(29)
      DET=0.0D-0
      CALL DMINV(TP,4,DET,LW,MW)
      DO 75 I=1,4
       C(I)=0.0
```

```
      DO 75 J=1,4
   75 C(I)=C(I)-TP(I,J)*ER(J)
C*************************************************************
C     CORRECTED VALUES
C*************************************************************
      DO 76 I=1,6
   76 Y(I)=D(I,KK)
      Y(3)=Y(3)+C(1)
      Y(4)=Y(4)+C(2)
      Y(5)=Y(5)+C(3)
      P=P+C(4)
      Y(6)=E*Y(5)/(R*S)
      DO 77 I=7,30
   77 Y(I)=0.0D-0
      Y(9)=1.0
      Y(16)=1.0
      Y(23)=1.0
      Y(24)=E/(R*S)
      IT=IT+1
       IF(IT.GT.10) GO TO 550
       GO TO 600
C*************************************************************
C     FINAL RESULTS
C*************************************************************
  900 SRA=DSQRT(A)
      SP=DSQRT(P)
      DO 795 J=KK,LL,4
      DJ=J-1
      XX(J)=DJ*H
      ETA(J)=1.-BE*(XX(J)**(0.5))
      IF(XX(J).GT.0.0) GO TO 905
      RBS(J)=SRA*D(3,J)/2.*(1.-E)
      CBS(J)=RBS(J)
      RMS(J)=A*D(6,J)
      CMS(J)=RMS(J)
      GO TO 795
  905 RBS(J)=SRA*ETA(J)*(S*D(3,J)+E*D(2,J)/XX(J))/2.*(S-E**2)
      CBS(J)=SRA*ETA(J)*(D(2,J)/XX(J)+E*D(3,J))/2.*(S-E**2)
      RMS(J)=A*D(5,J)/ETA(J)*XX(J)
      CMS(J)=A*D(6,J)/ETA(J)
  795 CONTINUE
C*************************************************************
C     FOR FREQUENCY RATIO
C*************************************************************
      IF(A.GT.0.0) GO TO 906
      SPO=SP
  906 SPR=SP/SPO
      WRITE(6,117)
      WRITE(6,120) S,E,CL
      WRITE(6,117)
      WRITE(6,112) SRA,SP,SPR
      WRITE(6,117)
      WRITE(6,113)
      DO 901 J=KK,LL,4
  901 WRITE(6,114) (D(I,J),I=1,4)
      WRITE(6,115)
      DO 902 J=KK,LL,4
  902 WRITE(6,114) (D(L,J),L=5,6)
      WRITE(6,117)
      WRITE(6,122)
```

```
      DO 903 J=KK,LL,4
  903 WRITE(6,114) RBS(J),CBS(J),RMS(J),CMS(J)
      WRITE(6,117)
      WRITE(6,123) IT
      WRITE(6,121)
      DO 924 J=KK,LL,4
  924 WRITE(6,114) XX(J),ETA(J)
      WRITE(6,117)
C*********************************************************************
C     PERTURBATION OF AMPLITUDE
C*********************************************************************
      A=A+DA
      IK=IK+1
      IF(IK.GT.26) GO TO 550
      ETA1=C(3,KK)
      ETA2=C(4,KK)
      ETA3=C(5,KK)
       P=(SP-0.3)**2
      GO TO 500
  550 STOP
      END

      SUBROUTINE RKG(X,H,Y,Q,P,AP,S,E,QL,BE)
        IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(30),C(30),DY(30),A(2)
      A(1)=0.2928932188135
      A(2)=1.7071067811865
      H2=0.5*H
      CALL DERIV(X,H,Y,DY,P,AP,S,E,QL,BE)
      DO 13 I=1,30
      R=H2*DY(I)-Q(I)
      Y(I)=Y(I)+R
   13 Q(I)=Q(I)+3.0*R-H2*DY(I)
      X=X+H2
      DO 60 J=1,2
      CALL DERIV(X,H,Y,DY,P,AP,S,E,CL,BE)
      DO 20 I=1,30
      R=A(J)*(H*DY(I)-C(I))
      Y(I)=Y(I)+R
   20 O(I)=C(I)+3.0*R-A(J)*H*DY(I)
   60 CONTINUE
      X=X+H2
      CALL DERIV(X,H,Y,DY,P,AP,S,E,CL,BE)
      DO 26 I=1,30
      R=(H*DY(I)-2.0*Q(I))/6.0
      Y(I)=Y(I)+R
   26 Q(I)=C(I)+3.0*R-H2*DY(I)
      RETURN
      END

      SUBROUTINE DERIV(X,H,Y,DY,P,AP,S,E,CL,BE)
       IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(30),DY(30)
       ETA=1.-BE*(X**(0.5))
      DETA=-0.5*BE*(X**(-0.5))
      DDET=0.25*BE*(X**(-1.5))
      DO 10 I=1,3
   10 DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 12 I=7,9
```

```
   12 DY(I)=Y(I+1)
      DY(11)=Y(12)
      DO 15 I=13,15
   15 DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 16 I=19,21
   16 DY(I)=Y(I+1)
      DY(23)=Y(24)
      DO 20 I=25,27
   20 DY(I)=Y(I+1)
      DY(29)=Y(30)
      IF(X.GE.0.1D-02) GO TO 17
      DY(4)=3.*P*Y(1)/8.*(ETA**2)+27.*(1.-E**2)*AP*Y(3)*Y(6)
     1/4.*(ETA**3)
      DY(4)=DY(4)-9.*(1.+E)*DDET*Y(3)/8.*ETA
      IF(AP) 18,18,19
   19 DY(4)=DY(4)+3.*(1.-E**2)*QL/8.*(ETA**3)*DSQRT(AP)
   18 DY(4)=DY(4)
      DY(10)=3.*P*Y(7)/8.*(ETA**2)+27.*(1.-E**2)*AP*(Y(9)*
     1Y(6)+Y(3)*Y(12))/4.*(ETA**3)
      DY(10)=DY(10)-9.*(1.+E)*DDET*Y(9)/8.*ETA
      DY(16)=3.*P*Y(13)/8.*(ETA**2)+27.*(1.-E**2)*AP*(Y(15)*
     1Y(6)+Y(18)*Y(3))/4.*(ETA**3)
      DY(16)=DY(16)-9.*(1.+E)*DDET*Y(15)/8.*ETA
      DY(22)=3.*P*Y(19)/8.*(ETA**2)+3.*Y(1)/8.*(ETA**3)+27.
     1*(1.-E**2)*AP*(Y(21)*Y(6)+Y(3)*Y(24))/4.*(ETA**3)
      DY(22)=DY(22)-9.*(1.+E)*DDET*Y(21)/8.*ETA
      DY(6)=0.0
      DY(12)=0.0
      DY(18)=0.0
      DY(24)=0.0
      DY(30)=0.0
      GO TO 70
   17 B0=(9.*(S-(E**2)))/(S*X*(ETA**3))
      B1=(S-E**2)/((1.-E**2)*(S*(ETA**2)))
      B2=((6.*DETA)/ETA)+(2./X)
      B3=(-1./(S*(X**2)))+(3.*DETA*(2.*S+E))/(S*X*ETA)+(3.*
     1DDET)/ETA+(6.*(DETA**2))/(ETA**2)
      B4=1./(S*(X**3))-(3.*DETA)/(S*(X**2)*ETA)+(3.*E*DDET)/
     1(ETA*S*X)+6.*E*(DETA**2)/(S*X*(ETA**2))
      DY(4)=B0*AP*(Y(3)*Y(5)+Y(2)*Y(6))+B1*P*Y(1)-B4*Y(2)
     1-B3*Y(3)-B2*Y(4)
      IF(AP) 100,100,101
  101 DY(4)=DY(4)+(S-(E**2))*QL/(S*(ETA**3)*DSQRT(AP))
  100 DY(4)=DY(4)
      DY(6)=(1.-E*X*DETA/ETA)*Y(5)/(S*(X**2))-(1.-X*DETA/ETA
     1)*Y(6)/X-(ETA*(Y(2)**2))/(2.*X*S)
      DY(10)=B0*AP*(Y(6)*Y(8)+Y(5)*Y(9)+Y(2)*Y(12)+Y(3)*Y(11
     1))+B1*P*Y(7)-B2*Y(10)-B3*Y(9)-B4*Y(8)
      DY(12)=(1.-E*X*DETA/ETA)*Y(11)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(12)/X-ETA*Y(2)*Y(8)/(S*X)
      DY(16)=B0*AP*(Y(5)*Y(15)+Y(3)*Y(17)+Y(14)*Y(6)+Y(2)*
     1Y(18))+B1*P*Y(13)-B2*Y(16)-B3*Y(15)-B4*Y(14)
      DY(18)=(1.-E*X*DETA/ETA)*Y(17)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(18)/X-ETA*Y(2)*Y(14)/(S*X)
      DY(22)=B0*AP*(Y(21)*Y(5)+Y(3)*Y(23)+Y(20)*Y(6)+Y(2)*
     1Y(24))+B1*P*Y(19)+B1*Y(1)-B2*Y(22)-B3*Y(21)-B4*Y(20)
      DY(24)=(1.-E*X*DETA/ETA)*Y(23)/(S*(X**2))-(1.-X*DETA/
     1ETA)*Y(24)/X-ETA*Y(2)*Y(20)/(S*X)
      DY(28)=B0*AP*(Y(27)*Y(5)+Y(3)*Y(29)+Y(6)*Y(26)+Y(2)*
```

```
     1Y(30))+B1+P*Y(25)-B2*Y(28)-B3*Y(27)-B4*Y(26)+B1*Y(1)
      DY(30)=(1.-E*X*DETA/ETA)*Y(29)/(S*(X**2))
     1-(1.-X*DETA/ETA)*Y(30)/X-ETA*Y(26)*Y(2)/(S*X)
   70 RETURN
      END
      SUBROUTINE DMINV(A,N,D,L,M)
      DIMENSION A(16),L(4),M(4)
      DOUBLE PRECISION A,D,BIGA,HOLD,DABS
      D=1.0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI)=HOLD
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI)=HOLD
   45 IF(BIGA) 43,46,48
   46 D=0.0
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
      DC 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
```

```
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
   65 CONTINUE
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
      D=D*BIGA
      A(KK)=1.0/BIGA
   80 CONTINUE
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
  108 JO=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JO+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI)=HOLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI)=HOLD
      GO TO 100
  150 RETURN
      END
```

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FINITE-AMPLITUDE VIBRATION OF ORTHOTROPIC AXISYMMETRIC VARIABLE THICKNESS ANNULAR PLATE

by

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ABSTRACT

The problem of finite amplitude, axisymmetric free vibration of variable thickness orthotropic annular plates is formulated in terms of the Von Karman's dynamic equations. A kantorovich averaging technique is applied to convert the nonlinear boundary value problem into a corresponding eigenvalue problem by elimination of the time variable. A numerical study is proposed by introducing the related initial value problem. By making successive corrections and perturbations of the parameters in a numerical solution to the initial value problem, approximate solutions to the boundary value problem are obtained. The cases investigated are free outside and fixed inside, parabolic and convex variable thickness orthotropic annular plates.

The hard spring behavior is evident, and it is found that the mode shape, bending stresses and membrane stresses are nonlinear functions of the amplitude of vibration.