A STUDY OF PROBABILITY DENSITY FUNCTIONS RELATED TO THE RADAR BACKSCATTERING

by

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TABLE OF CONTENTS

PAGE

LIST OF FIGURES iv

LIST OF TABLES v

CHAPTER

I. INTRODUCTION 1

1.1 Motivation and General Purpose 1

1.2 The Problem 2

1.3 Summary of Chapter Development 3

II. THE ANALYTICAL METHODS OF FINDING THE DENSITY FUNCTION OF A FUNCTION OF TWO OR MORE RANDOM VARIABLES 5

2.1 Introduction 5

2.2 The Density Function of a Function of Two or More Random Variables 6

2.3 The Density Functions of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ 8

III. THE MONTE CARLO METHOD AND CURVE FITTING TECHNIQUE 14

3.1 Introduction 14

3.2 Generation of Random Numbers 14

3.3 The Accuracy of the Monte Carlo Method . . 17

3.4 Curve Fitting Technique 20

IV. THE RESULTS OF DENSITY FUNCTIONS BY THE MONTE CARLO METHOD 26

4.1 Introduction 26

4.2 The Result of the Generated Random Numbers 27

4.3 Comparisons of the Results by the Monte Carlo Method and the Analytical Method . 28

CHAPTER PAGE

4.4 The Density Function of cos θ_1 by the Monte Carlo Method 30

V. CONCLUSIONS AND RECOMMENDATIONS 57

5.1 Conclusions 57

5.2 Recommendations 58

REFERENCES 60

ACKNOWLEDGMENTS 62

APPENDIX A 64

A.1 The Derivation of the Probability Density of cos θ_1 64

APPENDIX B 67

B.1 The Derivation of the First Kind of Curve Fitting Algorithm 67

B.2 The Algorithm for Finding an Optimum p . . 69

APPENDIX C 71

C.1 Program for Generation Uniform Random Numbers 71

C.2 Program for Generating Independent Normal Random Numbers 73

C.3 Program for Generating New Random Numbers (Function of Normal Random Pairs) . . . 75

C.4 Program for the First Kind of Curve Fitting 77

C.5 Program for the Second Kind of Curve Fitting 82

LIST OF FIGURES

FIGURE		PAGE
1.	Density Function of S with $\sigma = 1$ by the Analytical Method	12
2.	Density Function of V with $\sigma = 1$ by the Analytical Method	13
3.	Flow Chart of the First Kind of Curve Fitting . .	23
4.	Flow Chart of the Second Kind of Curve Fitting .	24
5.	Histogram of the Generated Normal Distribution of Zero Mean and Standard Deviation One by the Monte Carlo Method	29
6.	Density Functions of S, with $\sigma = 1.0$, $\rho = 0.4$.	32
7.	Density Functions of S, with $\sigma = 1.0$, $\rho = 0.9$.	33
8.	Density Functions of V, with $\sigma = 1.0$, $\rho = 0.4$.	34
9.	Density Functions of V, with $\sigma = 1.0$, $\rho = 0.9$.	35
10.	Density Functions of $\cos \theta_1$, with $\sigma = 0.1$, $\theta = 30^\circ$	37
11.	Density Functions of $\cos \theta_1$, with $\sigma = 0.1$, $\theta = 45^\circ$	38
12.	Density Functions of $\cos \theta_1$, with $\sigma = 0.1$, $\theta = 60^\circ$	39
13.	Density Functions of $\cos \theta_1$, with $\sigma = 3$, $\theta = 30^\circ$	40
14.	Density Functions of $\cos \theta_1$, with $\sigma = 3$, $\theta = 45^\circ$	41
15.	Density Functions of $\cos \theta_1$, with $\sigma = 3$, $\theta = 60^\circ$	42
16.	Density Functions of $\cos \theta_1$, with $\sigma = 100$, $\theta = 30^\circ$	43
17.	Density Functions of $\cos \theta_1$, with $\sigma = 100$, $\theta = 45^\circ$	44
18.	Density Functions of $\cos \theta_1$, with $\sigma = 100$, $\theta = 60^\circ$	45
19.	A Physical Interpretation of the Densities of $\cos \theta_1$ when σ is Large	47

CHAPTER I

INTRODUCTION

1.1 Motivation and General Purpose

In the determination of the amount of reflected electromagnetic energy from a surface, called the radar backscattering,[2] one of the major methods of modeling the reflecting surface is to assume the surface height above some mean surface can be described by a random process.[9] By the tangent plane approximation which assumes the reflected body is isotropic and homogeneous, the radar backscatter can be obtained in an integral form. Essentially the integrand of this integral is a new random process which is a function of the surface height random process and its derivatives. The form of this function changes as the shape of the average surface changes, i.e., a sphere or a plane. It is desired to find the probability density of the integrand and the integral for each case.

Upon the realization of the magnitude of this problem, a sub-problem is decided upon which could be utilized as an interim step regardless of the form of the average surface. The integrands in most cases can be expressed in terms of a power series of $\cos \theta_1$ times an exponential term whose exponent is a function of only the surface height, where θ_1 is the angle between the surface normal and the direction to the receiver. Therefore as a first step, it is decided to determine the probability density of $\cos \theta_1$ and then this density can later be applied to a particular surface.

1.2 The Problem

For the problem of radar backscattering from a rough spherical surface,[9] normal spherical coordinates ($\vec{a}_{r_0}$, $\vec{a}_\theta$, $\vec{a}_\phi$) are defined. The variation in the radius of the rough sphere from the average at a point (θ, ϕ) is denoted by $H(\theta, \phi)$. The random surface is assumed continuous in the mean and differentiable over a finite region. The radius vector from the origin to the surface point is $\vec{r}_0$. ($\vec{r}_0 = r_0 \vec{a}_{r_0}$). Then the equation of the surface can be written as

$$\Psi = r_0 - \left[(a + H(\theta,\phi)\right] \equiv 0 \qquad (1.1.1)$$

where a is an average radius of the rough surface. By normalizing the gradient of the equation (1.1.1), the unit outward surface normal vector $\vec{a}_n$ is found.

$$\vec{a}_n = \frac{\nabla \Psi}{|\nabla \Psi|} = \left[\frac{\partial \Psi}{\partial r_0} a_{r_0} + \frac{1}{r_0}\frac{\partial \Psi}{\partial \theta}\vec{a}_\theta + \frac{1}{r_0 \sin\theta}\frac{\partial \Psi}{\partial \phi}\vec{a}_\phi\right]\frac{1}{|\nabla \Psi|} \qquad (1.1.2)$$

Performing the indicated operations of (1.1.2) on Ψ as defined in (1.1.1), the above equation yields

$$\vec{a}_n = \left[a_{r_0} - \frac{1}{r_0}\frac{\partial H(\theta,\phi)}{\partial \theta}\vec{a}_0 - \frac{1}{r \sin\theta}\frac{\partial H(\theta,\phi)}{\partial \phi}\vec{a}_\phi\right] J^{-1} \qquad (1.1.3)$$

where

$$J = \sqrt{1 + \left[\frac{1}{r_0}\frac{\partial H(\theta,\phi)}{\partial \theta}\right]^2 + \left[\frac{1}{r_0 \sin\theta}\frac{\partial H(\theta,\phi)}{\partial \phi}\right]^2} \qquad (1.1.4)$$

Under the assumption that $H(\theta, \phi)$ is a normal variable, its partial derivatives with respect to its variables are also

normal.[11,13] Therefore two normal variables are defined as

$$X = \frac{1}{r_0} \frac{\partial}{\partial \theta} H(\theta, \phi)$$

$$Y = \frac{1}{r_0 \sin \theta} \cdot \frac{\partial}{\partial \phi} H(\theta, \phi)$$

$\vec{a}_R$ denotes the direction of the receiver from the point on the rough surface given by r_0, θ, ϕ. The angle between $\vec{a}_R$ and $\vec{a}_n$ is defined as θ_1. The expression of cos θ_1 then is found.

$$\cos \theta_1 = \frac{\cos \theta + Y \sin \theta}{\sqrt{1 + X^2 + Y^2}} \qquad (1.1.5)$$

Since cos θ_1 involves the function $\sqrt{1 + X^2 + Y^2}$, the problem of finding the density function of cos θ_1 starts from finding the densities of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ with a desire to find the former density by the information contained in the latter two.

Hopefully, the probability density of cos θ_1 can be determined analytically and the variations of this density with each different standard deviation σ, correlation coefficient ρ, and angle θ can be studied. If it fails, a valid approximation is desired. The Monte Carlo method is thus introduced to solve the aforementioned problem numerically due to the failure of finding the density function of cos θ_1 analytically.

1.3 Summary of Chapter Development

Chapter I contains the statement of the problem. Chapter II

presents two general analytical methods for finding the probability density function. The explicit densities of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ are obtained by using these two methods. Appendix A presents the derivation of the density function of $\cos \theta_1$ by analytical methods. However, the problem is not solved by these methods.

Chapter III states the Monte Carlo method used to obtain the desired probability density. First, the method of generating random numbers is discussed. Chi-square test is mentioned here to check the goodness of fit of the generated random numbers. Two curve fitting techniques are developed for finding the explicit expression by the given data from the Monte Carlo method. More details of the curve fitting algorithm are given in Appendix B.

Chapter IV presents the results of densities by using the methods of Chapter III. A comparison is made to the density function of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ with the results by the analytical methods. The densities of $\cos \theta_1$ for certain parameters are also obtained. Appendix C is the computer programs used in this chapter. The various densities of $\cos \theta_1$ corresponding to different parameters are discussed and further researches are recommended in Chapter V.

CHAPTER II

THE ANALYTICAL METHODS OF FINDING THE DENSITY FUNCTION OF A FUNCTION OF TWO OR MORE RANDOM VARIABLES

2.1 Introduction

The difficulty of finding the probability density of a function of random variables depends on the number of random variables and the complexity of the functional relationships involved. If these random variables are independent, it makes the problem easier than if they are dependent. The densities can be quickly calculated for linear functions of random variables. In most physical problems, either the random variables are not independent or the function is nonlinear. When either or both of these cases occur the algebraic solutions require very complicated integrations which are not always solvable in closed form or expressed as a well-known series.

In this chapter two general analytical methods of finding the density function will be discussed. The algebraic solutions for the density functions of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ are obtained by these methods. These explicit densities will be compared with the densities obtained by the Monte Carlo method in order to estimate the accuracy and validity of the Monte Carlo method.

2.2 The Density Function of a Function of Two or More Random Variables

Under certain conditions,[13,8] the function of random variables can be considered as another random variable. There are many methods of finding the density function of this new random variable. Only two different methods are discussed here.

The first method consists of finding the distribution function $F_Z(z)$ first, then by taking derivative with respect to z the probability density function is obtained.

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) \tag{2.2.1}$$

In order to determine $F_Z(z)$ for a given z, the probability of the event $\{Z \leq z\}$ must be found. For the function of two random variables case, D_Z is denoted by the region of XY plane such that $g(x,y) = z$. Then $\{Z \leq z\} = \{(X,Y) \in D_Z\}$. Hence it suffices to find the probability mass in the region D_Z. This mass is given by the integration

$$F_Z(z) = P(Z \leq z) = p(X, Y) \in D_Z$$

$$= \int_{D_Z} \int f_{XY}(x, y)\,dx\,dy \tag{2.2.2}$$

where $f_{XY}(x, y)$ is the original joint density function of random variables X and Y.

Similarly for a function of n random variables, that is $z = g(x_1, x_2, \ldots, x_n)$, D_Z is denoted by the region of n-dimensional space such that $g(x_1, x_2, \ldots, x_n) = z$ and

$\{Z \leq z\} = \{(X_1, X_2, \ldots, X_n) \in D_Z\}$. Then

$$F_Z(z) = P(Z \leq z) = P\{(X_1, X_2, \ldots, X_n) \in D_Z\}$$

$$= \int_{D_Z} \cdots \int f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n \qquad (2.2.3)$$

where $f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$ is the original joint density function of random variables $X_1, X_2, \ldots, X_n$.

The second method consists of introducing an auxiliary variable and finding the joint density function of the desired random variable and the auxiliary variable. Then by integrating the joint density function over the entire range of the auxiliary variable, the marginal density function of the random variable is obtained.

For the case of the function of two random variables, the auxiliary variable W is assumed to be $W = g_2(X, Y)$. With $Z = g_1(X, Y)$ the joint density is

$$f_{ZW}(z,w) = \frac{f_{XY}(x_1,y_1)}{|J(x_1,y_1)|} + \ldots + \frac{f_{XY}(x_n,y_n)}{|J(x_n,y_n)|} \qquad (2.2.4)$$

where (x_1,y_1), (x_2,y_2) . . . (x_n,y_n) are all the real solutions of the equation, $g_1(x,y) = z$ with $g_2(x,y) = w$, and $J(x,y)$ is the Jacobian of the transformation. The unknown density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w) dw \qquad (2.2.5)$$

The second method can be extended to a function of n random variables. Given $Z_1 = g_1(X_1, X_2, \ldots, X_n)$, then n - 1 auxiliary

variables as

$$Z_2 = g_2(X_1,X_2,\ldots,X_n),\ Z_3 = g_3(X_1,X_2,\ldots,X_n),\ \ldots,$$

$$Z_n = g_n(X_1,X_2,\ldots,X_n)$$

are assumed. The choice of the auxiliary variables depends on which assumption can simplify the problem of finding the joint density. By solving these n-system equations, k sets of real solution $(x_{1i}, x_{2i}, x_{3i}, \ldots, x_{ni})$ are obtained, where $i = 1, 2, 3, \ldots, k$. Therefore the joint density function is

$$f_{Z_1Z_2\cdots Z_n}(z_1,z_2,\ldots,z_n) = \sum_{i=1}^{k} \frac{f_{X_1X_2\ldots X_n}(x_{1i},x_{2i},\ldots,x_{ni})}{|J(x_{1i},x_{2i},\ldots,\ \ldots,\ldots,x_{ni})|} \tag{2.2.6}$$

where $J(x_1, x_2, \ldots, x_n)$ is the Jacobian of the transformation of these n-system equations. Then

$$f_{Z1}(z_1) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{Z_1Z_2\ldots Z_n}(z_1,z_2,\ldots,z_n)\,dz_2dz_3\ldots dz_n \tag{2.2.7}$$

The above two methods are commonly used to find the density function of a function of random variables. They will be used in the next section.

2.3 The Density Functions of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$

For finding the density functions of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$, the polar coordinates are used here due to their simplicity in the operation. Assuming $Z = X^2 + Y^2$, the density function of Z is obtained by the first method of the last section. After that, a new random variable W is defined as Z + 1.

Its density is found merely by shifting the variable of the density function of Z to W - 1. The square root of W and one over this square root are other random variables. These two densities are obtained by the second method of the last section based on the density of W.

X and Y are two random variables with zero means and equal standard deviation σ and correlation coefficient ρ. The joint probability density function of X and Y is

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2+y^2 - 2\rho_{xy}}{2\sigma^2(1-\rho^2)}\right\} \qquad (2.3.1)$$

By starting with $Z = X^2 + Y^2$, then D_Z is a circle with radius of $\sqrt{z}$. In polar form, $r = \sqrt{z}$, $x = r\cos\theta$, $y = r\sin\theta$, then

$$F_Z(z) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \int_{D_Z}\int \exp\left\{-\frac{1}{2\sigma^2(1-\rho^2)} \cdot (x^2+y^2-2\rho_{xy})\right\} dxdy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \int_0^{\sqrt{z}} r \exp\left\{\frac{-r^2}{2\sigma^2(1-\rho^2)}\right\} \int_0^{2\pi} \exp\left\{\frac{\rho(\sin 2\theta)r^2}{2\sigma^2(1-\rho^2)}\right\} d\theta dr$$

The second part of the above integration is the modified Bessel function of first kind of order zero[1] denoted as I_o; then

$$F_Z(z) = \frac{1}{\sigma^2\sqrt{1-\rho^2}} \int_0^{\sqrt{z}} r \exp\left\{\frac{-r^2}{2\sigma^2(1-\rho^2)}\right\} I_o\left\{\frac{\rho r^2}{2\sigma^2(1-\rho^2)}\right\} dr$$

By taking the derivative of $F_Z(z)$ with respect to z and using the generalized Leibnitz formula,[4] the density function of Z is obtained.

$$f_Z(z) = \frac{1}{2\sigma^2\sqrt{1-\rho^2}} \exp\left\{\frac{-z}{2\sigma^2(1-\rho^2)}\right\} I_o\left\{\frac{\rho z}{2\sigma^2(1-\rho^2)}\right\} u(z) \tag{2.3.2}$$

where u(z) is the unit step function.

Defining W = Z + 1, with $f_W(w) = f_Z(w - 1)$, then

$$f_W(w) = \frac{1}{2\sigma^2\sqrt{1-\rho^2}} \exp\left\{\frac{-(w-1)}{2\sigma^2(1-\rho^2)}\right\} I_o\left\{\frac{\rho(w-1)}{2\sigma^2(1-\rho^2)}\right\} u(w-1) \tag{2.3.3}$$

For $S = \sqrt{W} = \sqrt{1 + X^2 + Y^2}$, with $f_S(s) = 2s\, f_W(s^2)$, then

$$f_S(s) = \frac{2s}{2\sigma^2\sqrt{1-\rho^2}} \exp\left\{\frac{-(s^2-1)}{2\sigma^2(1-\rho^2)}\right\} I_o\left\{\frac{\rho(s^2-1)}{2\sigma^2(1-\rho^2)}\right\} u(s-1) \tag{2.3.4}$$

For $V = \frac{1}{S} = \frac{1}{\sqrt{1+X^2+Y^2}}$ with $f_V(v) = \frac{1}{v^2} f_S(\frac{1}{v})$, then

$$f_V(v) = \frac{2}{2\sigma^2\sqrt{1-\rho^2}\, v^3} \exp\left\{\frac{-(1-v^2)}{2\sigma^2(1-\rho^2)v^2}\right\} I_o\left\{\frac{(1-v^2)\rho}{2\sigma^2(1-\rho^2)v^2}\right\} \left[u(v) - u(v-1)\right] \tag{2.3.5}$$

Equations (2.3.4) and (2.3.5) are the solutions needed to compare with the results of the Monte Carlo method in Chapter IV. They are also plotted for $\sigma = 1$ and $\rho = 0.4$, 0.9 as Fig. 6 through Fig. 9 in Chapter IV for a comparison.

For the limiting case, when $\rho = 1$, equations (2.3.4) and (2.3.5) yield

$$f_S(s) = \frac{s}{\sigma\sqrt{\pi(s^2-1)}} \exp\left\{\frac{-(s^2-1)}{4\sigma^2}\right\} u(s-1) \tag{2.3.6}$$

$$f_V(v) = \frac{1}{\sigma v^2 \sqrt{\pi(1-v^2)}} \exp\left\{-\frac{(1-v^2)}{4\sigma^2 v^2}\right\}\left[u(v) - u(v-1)\right] \qquad (2.3.7)$$

When $\rho = 0$, they yield

$$f_S(s) = \frac{s}{\sigma^2} \exp\left\{-\frac{(s^2 - 1)}{2\sigma^2}\right\} u(s - 1) \qquad (2.3.8)$$

$$f_V(v) = \frac{1}{\sigma^2 v^3} \exp\left\{-\frac{(1-v^2)}{2\sigma^2 v^2}\right\}\left[u(v) - u(v - 1)\right] \qquad (2.3.9)$$

The graphs of equations (2.3.6) - (2.3.9) are shown in Fig. 1 and Fig. 2 for $\sigma = 1$ case.

All the results of the above density functions are verified by integrating over the entire range of the variable and obtaining the number one.

The derivation of the density function of $\cos \theta_1$ is shown in Appendix A. With the methods used here and the method by Fourier transformation, all involve a difficult integration. It cannot end with a closed form or a well-known series expression. Therefore the Monte Carlo method is used in Chapter IV which gives an approximate result with certain fixed parameters.

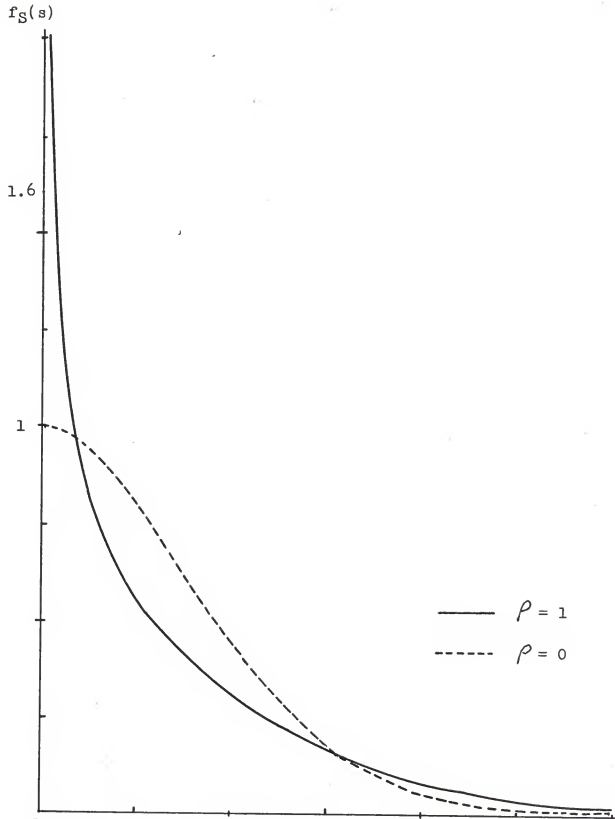

Fig. 1. Density function of S with $\sigma = 1$ by the analytical method ($S = \sqrt{1 + X^2 + Y^2}$).

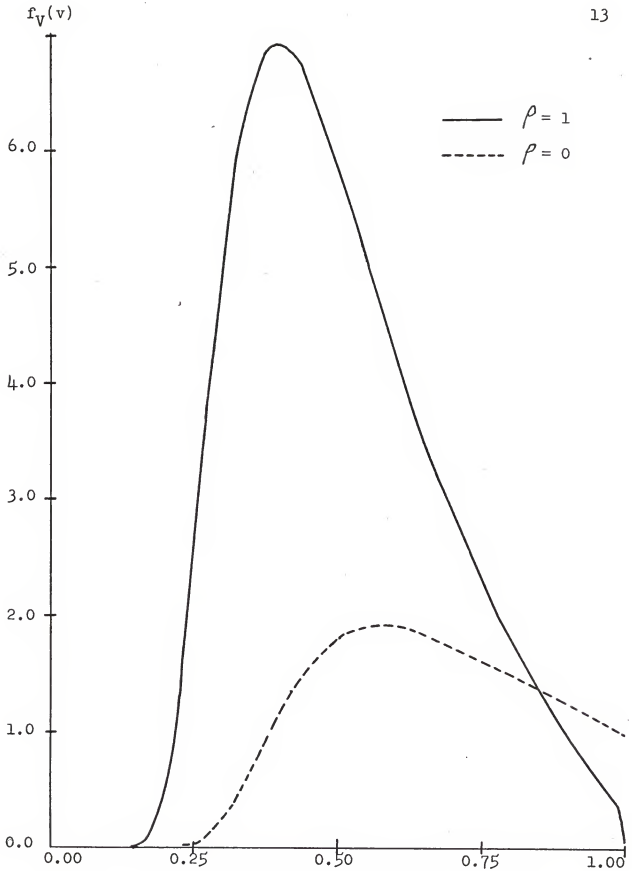

Fig. 2. Density function of V with $\sigma = 1$ by the analytical method ($V = 1/\sqrt{1 + X^2 + Y^2}$).

CHAPTER III

THE MONTE CARLO METHOD AND CURVE FITTING TECHNIQUE

3.1 Introduction

As previously indicated, analytical methods of obtaining probability densities for functions of random variables do not always yield closed form solutions. In this chapter, the Monte Carlo method is introduced. When it is combined with curve fitting techniques, an approximate closed form solution can be obtained.

The important job of the Monte Carlo method is to generate a set of random numbers with a certain distribution. How to generate random numbers of uniform distribution and how to change them to another distribution are discussed in section 3.2. This job will be done by the computer, IBM 360/50. With a limited sampled size and computer round-off errors, the accuracy of the Monte Carlo method has to be considered. It is discussed in section 3.3.

Finally, two curve fitting techniques which will be used to find the density function explicitly with the data from the Monte Carlo method are developed in section 3.4.

3.2 Generation of Random Numbers

The Monte Carlo method[10,12,15] is a kind of simulation technique. The generation of random numbers plays an important role in this method. First, the way to generate random numbers

with uniform distribution is discussed. Then by a simple computation, they are changed to random numbers with a normal distribution. This set of numbers will be used to solve the problem of finding the probability density function of a function of the normal random variables.

There are several ways to generate random numbers with a uniform distribution.[6,12] Most of the available schemes use the multiplicative congruential method. This method is concerned with generating sequences of nonnegative integers by means of a congruence relation, then they are divided by their mode to get the numbers between 0 and 1.

The congruence relation is

$$U_{n+1} = a\, U_n \pmod{p^b} \tag{3.2.1}$$

where p denotes the number of numerals in the number system utilized by the computer and b denotes the number of digits in a word. a is a constant multiplier of the form $a = 8t \pm 3$ (t is a positive integer), and U_0, the starting value, is an odd integer to assure maximal periods for the sequences generated by this method.

The principal value of the uniform distribution for simulation techniques lies in its simplicity and in the fact that it can be used to simulate random variables from almost any kind of probability distribution.[12] Therefore when the random variable of uniform distribution has been generated, the random variable of the desired distribution is obtained by means of a simple computation.

In general, there are three ways to generate normal random numbers from the uniform ones. These are the central limit approach, the direct approach, and the fast procedure.[12] The direct approach is discussed here due to its faster calculation and an exact result compared with the others.

Given two uniform random variables R_1 and R_2 that are independent and defined on the (0, 1) interval, then

$$Z_1 = (-2 \log_e R_1)^{1/2} \cos 2\pi R_2 \qquad (3.2.2)$$

$$Z_2 = (-2 \log_e R_1)^{1/2} \sin 2\pi R_2 \qquad (3.2.3)$$

are two independent normal variables.

It is also very easy to change Z_1 and Z_2 into a pair of correlated normal variables with mean vector $\vec{\mu}$ and covariance matrix $\vec{V}$. There exists a unique lower triangular matrix C such that

$$\vec{X} = \vec{C}\,\vec{Z}$$

where Z is a standard uncorrelated vector of random variables as generated by equation (3.2.2) and (3.2.3). X is the desired correlated normal vector, and $\vec{V} = \vec{C} \cdot \vec{C}^{\,T}$. (T means transpose.)

When two random normal variables X_1 and X_2 are given, their means, variance and correlation coefficient are then defined as $E_{X_1}(x_1) = \mu_1$, $E_{X_2}(x_2) = \mu_2$, $Var_{X_1}(x_1) = \sigma_1^2$, $Cor_{X_1X_2}(x_1x_2) = \rho\sigma_1\sigma_2$.

Consequently,

$$\vec{V} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \qquad (3.2.4)$$

By equation (3.2.4), then

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{C} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \vec{\mu} = \begin{bmatrix} \sigma_1 Z_1 \\ \sigma_2(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2) \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad (3.2.5)$$

Equation (3.2.5) is the equation to find two correlated normal random variables. The test of the accuracy of these random variables will be discussed in the next section.

3.3 The Accuracy of the Monte Carlo Method

The accuracy of the Monte Carlo method mainly depends on the sample size of the generating numbers and the validity of randomness. By the law of large numbers,[13] it is shown that

$$P\left\{|\bar{X}_j - p_j| < \epsilon\right\} \geq 1 - \frac{p_j q_j}{n\epsilon^2}$$

$$\bar{X}_j = \frac{\bar{X}_{j1} + \bar{X}_{j2} + \cdots + \bar{X}_{jn}}{n} \qquad (3.3.1)$$

where ϵ is any infinitesimal positive number,

p_j is the actual probability of occurrence in the j^{th} interval,

and $q_j = 1 - p_j$.

n is the sample size, and

X_{ji} = 1 when the event occurs in the j^{th} interval,
otherwise
X_{ji} = 0.

If the k is the total number of occurrences out of n trials in that interval, then $X_j = k/n$. The value is obtained by the Monte Carlo method. Thus as $n \longrightarrow \infty$, equation (3.3.1) yields

$$P\left\{ \left| \bar{X}_j - p_j \right| \leq \epsilon \right\} \longrightarrow 1 \qquad (3.3.2)$$

The above equation illustrates the probability of the difference between the value by the Monte Carlo method and the actual value which is less than a small quantity. As n approaches infinity, this probability becomes one. This means the difference is less than a small quantity ϵ when n is sufficiently large. No matter how small the ϵ is, the equation (3.3.2) holds. The larger n the smaller ϵ can be expected.

However, it is impossible to increase the sample size n to infinity. A compromise is made considering the economic consideration of computer running time, the size of memory of computer and how much accuracy is needed.

Next question is the validity of randomness. The statistical properties of random numbers generated by the methods outlined in the previous section should coincide with the statistical properties of numbers generated by an idealized chance device that selects numbers from a certain interval independently where each number has a certain probability. Clearly the random numbers produced by computer programs are not random in this

sense, since they are completely determined by the starting data and have limited precision. But so long as the generated random numbers can pass the set of statistical tests implied by the aforementioned idealized chance device, these numbers can be treated as truly random. There are several tests used to test the validity of randomness.[3,5,12] Here only chi-square test for the goodness of fit is discussed.

With the assumption that in each of n independent trials precisely one of r events, $A_1, A_2, \ldots, A_r$ must happen, $v_1, v_2, \ldots, v_n$ are the numbers of successes out of n in each event by the Monte Carlo method. With the actual probabilities $p_{1o}, p_{2o}, \ldots, p_{ro}$ are numbers with $\sum_{j=1}^{r} p_{jo} = 1$. In order to test the hypothesis $p_1 = p_{1o}, p_2 = p_{2o}, \ldots, p_r = p_{ro}$, the following statistic is considered:

$$D^2 = \sum_{j=1}^{r} \frac{(v_j - np_{jo})^2}{np_{jo}} = \sum_{j=1}^{r} \frac{n(p_j - p_{jo})^2}{p_{jo}} \qquad (3.3.3)$$

It can be shown that D^2 has asymptotically a chi-square distribution with r = 1 degree of freedom.

The test of the hypothesis at the 100 α per cent significance level is obtained by choosing a number b such that $P\{\chi^2 > b\} = \alpha$. Where χ^2 has chi-square distribution with r - 1 degrees of freedom and rejecting the hypothesis if a value of D^2 greater than b is actually observed. The chi-square value for a different degree of freedom can be found in a good mathematical table.[1]

3.4 Curve Fitting Technique

Two kinds of fitting curves are used in this paper. The first kind corresponds to a polynomial multiplied by an exponential term. This curve can fit smooth data with a few terms and match its peak value by the exponential term very easily. The second kind is a finite Fourier series of sine terms only. It can fit rough data faster than the first kind. The fitting technique is based on minimizing the least square error between the assumed curve and fitting data. The iterative method is used to find the coefficients of the fitting curve, with the assumption that the first kind of fitting curve is

$$f(x) = \sum_{r=o}^{n} a_r x^r \, e^{-px^2} \tag{3.4.1}$$

Since the integration of the density function over its entire range of the variable is unity, the above equation is modified in order to satisfy this constraint. Introducing a Lagrangian multiplier λ, the modified equation is

$$\overline{f}(x) = f(x) + \lambda\left(\int_{-\infty}^{\infty} \overline{f}(x)dx - 1\right) \tag{3.4.2}$$

with the assumption that $g(x_i)$ is the i^{th} given datum of the fitting set, and $f(x_i)$ is the i^{th} datum of the fitted set. Then the error vector is $E = \left\{e_i\right\} = \left\{g(x_i) - f(x_i)\right\}$.

The squared error is

$$||E||^2 = \left\{e_i\right\}^T \left\{e_i\right\} \tag{3.4.3}$$

where $i = 1, 2, \ldots n$.

To minimize the squared error, a set of equations are obtained by setting $\frac{\partial \|E\|^2}{\partial a_{r_o}} = 0$ for a particular a_{r_o} and $\frac{\partial \|E\|^2}{\partial \lambda} = 0$. Based on these two derivatives, a matrix equation is written as

$$\left[A\right]^* \left\{ \frac{\{a_i\}}{\lambda} \right\} = \{C\} + \lambda \left\{ Y(a_i) \right\} \tag{3.4.4}$$

where the notation $[\ ]$ denotes a square or rectangular matrix and $\{\ \}$ denotes a column vector. The details of the above equation are presented in Appendix B.

For a given p and setting $\lambda = o$, the coefficient vector $\{a_i\}$ is obtained with a value $[A]^{-1}\{C\}$. Then by assuming a $\lambda \neq o$, and the previous coefficient vector $\{a_i\}$ substituted into $\{Y(a_i)\}$, a new coefficient and vector can be obtained.

$$\left\{ \frac{\{a_i\}}{\lambda} \right\} = [A]^{-1} \left(\{C\} + \lambda \left\{ Y(a_i) \right\} \right) \tag{3.4.5}$$

Every time a new coefficient and λ vector is obtained by putting the old one of a step before into equation (3.4.5). After several steps it is expected $\{a_i\}$ and λ will converge to some value. That is a vector satisfying the least square error fitting requirement under a preassigned p. The error $\|E\|^2$ can be computed by (3.4.3) immediately. The variation of this error

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*Note: When $[A]$ is not a square matrix, both sides of equation of (3.4.4) are then multiplied by transpose of $[A]$, i.e., $[A]^T$. $[A]^T[A] = B$ is a square matrix. Thus it makes the inversion possible. So that the least square solution can be obtained. Later on, all the notation of $[\ ]$ is assumed as a square matrix to make the discussion of the problem simple.

function by changing p can be used as a criterion finding optimum p; i.e., when $||E_i||^2 - ||E_{i-1}||^2 < o$ and $||E_{i+1}||^2 - ||E_i||^2 > o$, there is a local optimum Po such that $Pi < Po < Pi+1$, $||E|| = f(p)$ in the small neighborhood about Po can be found by the technique mentioned above. The optimum Po is achieved by minimizing f(p).

A Lagrangian multiplier λ is used to adjust the integrating area of the density function to one. It can be assumed as a small value less than 0.5, because the result of the density functions by the Monte Carlo method does not cause a large error.

Figure 3 is the flow chart of the algorithm used in the first kind of curve fitting. Appendix B has a detail deviation of the first kind curve fitting algorithm.

The second kind of the curve fitting is of the form

$$f(x) = \sum_{k=o}^{n} a_k \sin kx \qquad (3.4.6)$$

Given $f(j \Delta x)$ is the j^{th} fitting data, the x is the increment of the fitting interval. The equation (3.4.6) is written as a matrix form

$$\left\{f(j\Delta x)\right\} = \left[\sin (jk\Delta x)\right] \left\{a_k\right\} \qquad (3.4.7)$$

with

$$\left[\sin(jk\Delta x)\right]^{-1} = \left(\frac{2}{n+1}\right) \left[\sin(jk\Delta x)\right],$$

therefore

$$\left\{a_k\right\} = \left(\frac{2}{n+1}\right) \left[\sin(jk\Delta x)\right] \left\{f(j\Delta x)\right\} \qquad (3.4.8)$$

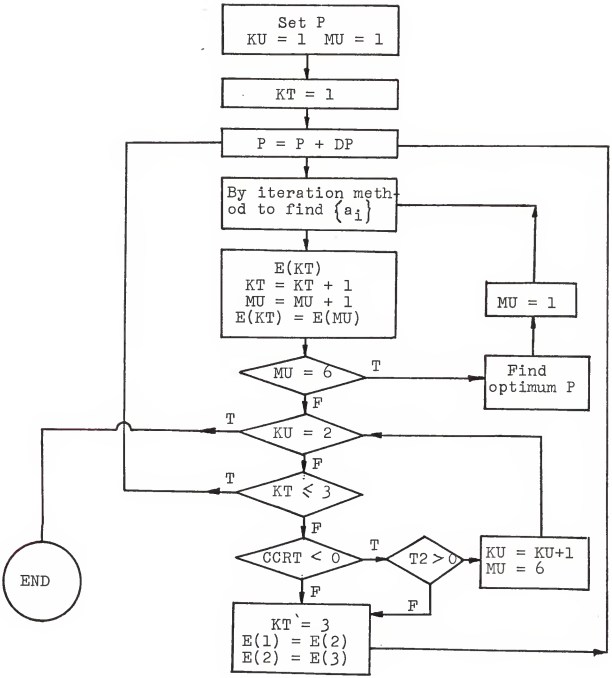

Fig. 3. Flow chart of the first kind curve fitting

$$f(x) = \sum_{r=0}^{n} a_r x^r e^{-px^2} .$$

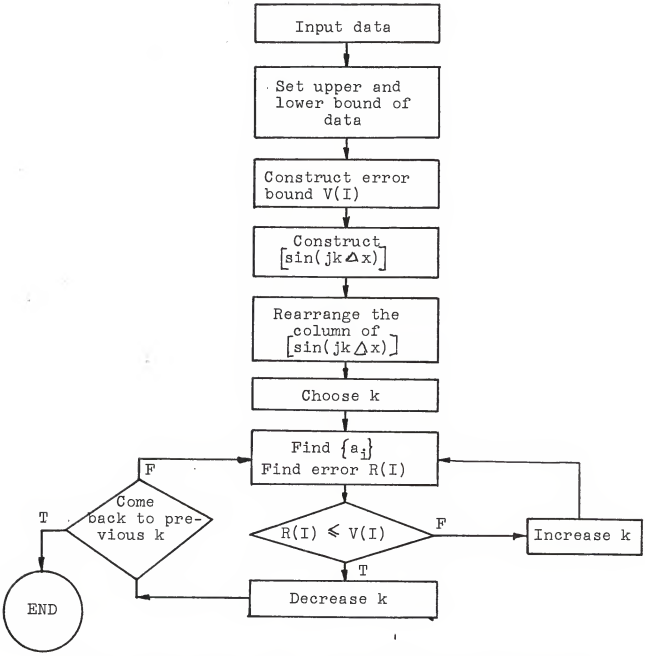

Fig. 4. Flow chart of the second kind of fitting curve

$$f(x) = \sum_{k=1}^{n} a_k \sin kx.$$

By considering the above equation, the inner products of the columns of the sine matrix with respect to the data vector $f(j\Delta x)$ denote how much is the contribution of the corresponding column to the fitting. A simple algorithm called the constrained least mean square solution is therefore suggested. It is a method to arrange the columns of the sine matrix according to their inner products. Then, with a given error bound, it can also adjust the mean squared error of the fitting to within this bound. Since this method is designed to fit the data occupied between the interval 0 to π and all harmonic terms of sine function at 0 and π are zero, therefore if both or either terminals of the fitting data are nonzero, it is necessary to set them to zero by adjusting the length of the interval due to the convergent consideration. Also for any given interval T, the scale factor T/π is used to normalize this interval to the designed one. Equation (3.4.8) is thus rewritten as

$$f(t) = \sum_{k=1}^{n} a_k \sin \frac{k\pi}{T} t \qquad (3.4.9)$$

Figure 4 is the flow chart of the algorithm used in the second kind of curve fitting.

The methods discussed in this chapter will be used to find density function of $\cos \theta_1$ in the next chapter.

CHAPTER IV

THE RESULTS OF DENSITY FUNCTIONS BY THE MONTE CARLO METHOD

4.1 Introduction

This chapter is going to use the methods of Chapter III to find the desired probability function.

The subroutine RANDU of IBM 360 is used to generate two sets of independent uniform random variables. These uniform random variables are changed to dependent normal pairs. Based on these pairs, a new random variable, that is, a function of normal random pairs, is obtained. The chi-square test is used to check the randomness of uniform and normal random variables. A simple sorting program is run on these random numbers to find the frequency ratios of the distribution. After having the frequency ratio, the curve fitting technique is used to find an explicit expression for the desired density function. In order to decrease the round-off errors due to the computation, all of the computer programs of this chapter use the double precision. In section 4.3, a comparison is made to the densities of S and V by using the Monte Carlo method and the analytical method for the $\sigma = 1$ and $\rho = 0.4, 0.9$ cases. The densities of $\cos \theta_1$ are obtained with explicit expressions for certain parameters in the last section.

4.2 The Result of the Generated Random Numbers

First of all, the IBM scientific subroutine RANDU is used here to generate a pair of uniform random variables. By equations (3.2.2), (3.2.3), and (3.2.5), a transformation is made on them to pairs of dependent variables.

This subroutine RANDU is specific to the system 360 and will produce 2^{29} uniform random numbers before repeating. The seed a in equation (3.2.1) is suggested to be 65,539. 214,358,881 and 776,179,721 are the two odd numbers used to generate a pair of uniform random variables. The sample size is chosen to be 10^4 throughout this paper.

After the uniform random variables are obtained, they are sorted according to the magnitude of their values and put into fifty equal distance cells from zero to one. Then the frequency ratio of each cell is the number of random numbers on that cell over the length of cell multiplied by the total sample size. Since the actual frequency ratio is unity everywhere for a uniform distribution, chi-square values of these pairs of uniform random numbers are calculated by the definition of section 3.3. They are 65.49 and 43.46 respectively. For the case of 49 degrees of freedom, $P\left\{\chi^2 > 77.7334\right\} = 0.005$ is the value. Because the above two values are less than 77.7334, these two sets of uniform random numbers are not rejected for 0.5 per cent significance level.

By equations (3.22) and (3.23) two independent normal variables with zero means and standard deviations of one are obtained. Although the range of these random numbers is between

positive and negative infinities, it is cut down to a value from +3.5 to -3.5 (i.e., = 3.5 σ). That is reasonable since the probability that the normal random numbers are outside the aforementioned range is less than 0.047. For finding the chi-square of the generated normal random variables, the same procedure is done as uniform random variable case. The only difference is its range of the variable. In this case, the chi-square values are 76.20 and 64.20, respectively. They are also not rejected for the 0.5 per cent significance level. Figure 5 shows a histogram of one of the generated normal density function. The dot points express the value in the exact normal density function. By using a curve fitting method to fit the histogram, those irregular deviations caused by errors are smoothed out.

4.3 Comparisons of the Results by the Monte Carlo Method and the Analytical Method

The density functions of S and V are obtained in this section by the Monte Carlo method. They are compared with the results obtained by analytical methods in section 2.3.

The two sets of independent normal variables which are found in the last section are changed to dependent ones with a given standard deviation vector and correlation coefficient. The mean vector is assumed to be zero and the two random variables X and Y have the same standard deviation throughout this paper. The random variables, S and V, are calculated by their functional relationship with the generated random variables. The range of V is from zero to one and S is cut down from one

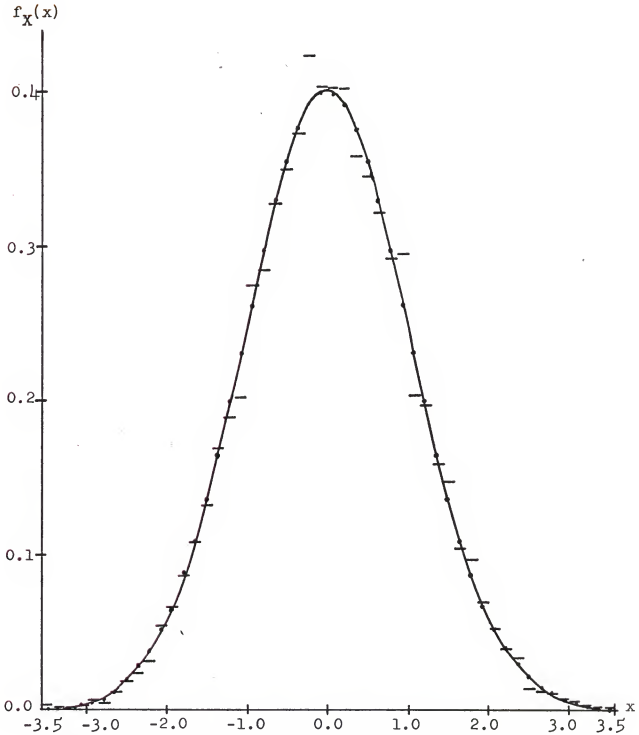

Fig. 5. Histogram of the generated normal distribution of zero mean and standard deviation one by the Monte Carlo method.

to 5.0. By the same method as used in the last section, they are sorted according to the magnitude of their values into fifty equal distance cells in the range of the variable. The frequency ratios of each cell is calculated thereafter. In other words, the histograms of S and V are found. The first kind of fitting curve is then used to fit these histograms. The coefficients of the fitted equations S and V for $\sigma = 1$ and $\rho = 0.4, 0.9$ are listed in Table 1. Actually, the fitted equations express the density functions of V and S with the given parameters. Their curves are plotted in Fig. 6 to Fig. 9. They are compared with the corresponding curves obtained by the Monte Carlo method. These comparisons show that fairly close results are obtained by both methods. Because the inverse quantity might cause larger round-off error, the densities of V produce more error than the densities of S by the Monte Carlo method. However, the errors are not over ±3 per cent. These results verify that the Monte Carlo method is a good valid approximation. By using the Monte Carlo method the probability density of $\cos \theta_1$ will be obtained in the next section.

4.4 The Density Function of $\cos \theta_1$ by the Monte Carlo Method

Although by using analytical methods to find an algebraic solution of the density of the random variable $\cos \theta_1$ is very hard, the Monte Carlo method can do this job numerically without too many difficulties.

Table 1. Coefficients of fitting curves of S and V with $\sigma = 1$

$$f(x) = \sum_{r=0}^{4} a_r x^r e^{-p(x-p_1)^2}.$$

	S		V	
	$\rho = 0.4$	$\rho = 0.9$	$\rho = 0.4$	$\rho = 0.9$
p_1	0.705	0.0	0.0	0.0
p	11.7784	1.5137	0.779	0.552
a_0	-18.00380	1.37614	-7.37264	42.52669
a_1	132.42396	-22.27794	17.24265	-86.40062
a_2	-292.89944	104.01710	-9.15744	69.11961
a_3	272.25791	-157.98395	0.50329	-24.80488
a_4	-86.77027	85.05811	1.01621	3.45576

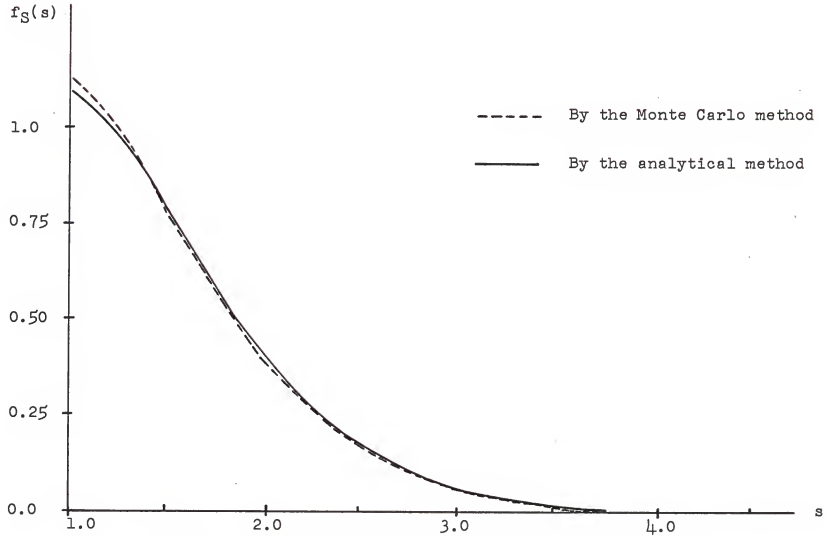

Fig. 6. Density functions of S with $\sigma = 1.0$, $\rho = 0.4$.

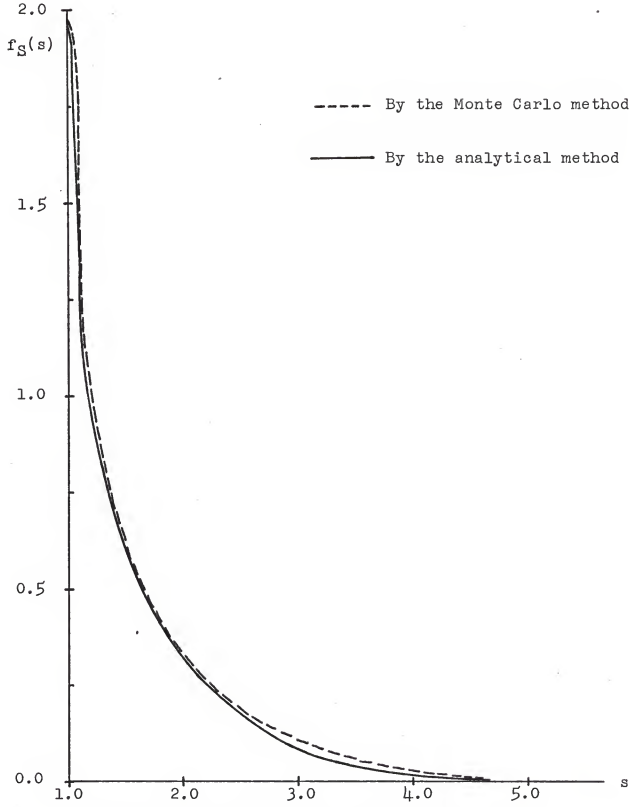

Fig. 7. Density functions of S with $\sigma = 1.0$, $\rho = 0.9$.

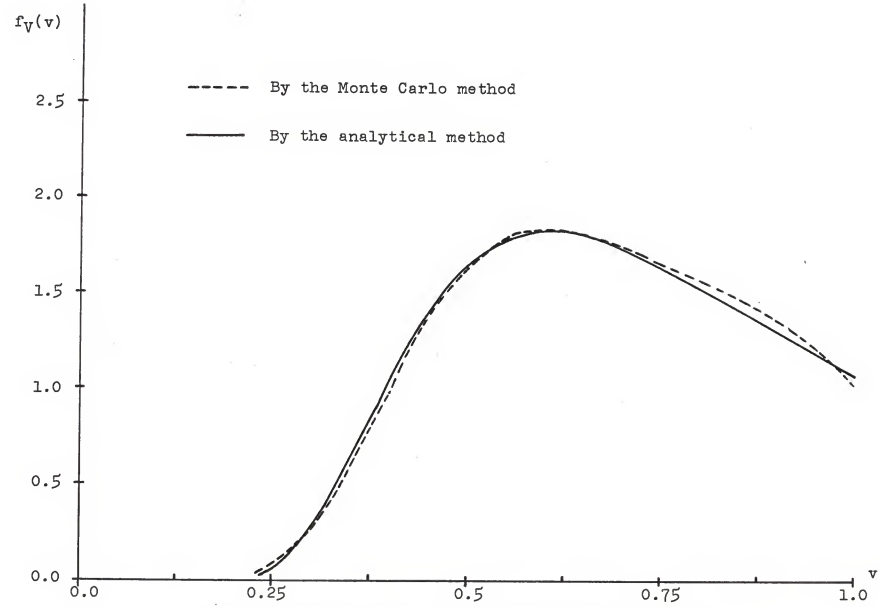

Fig. 8. Density functions of V with $\sigma = 1$, $\rho = 0.4$.

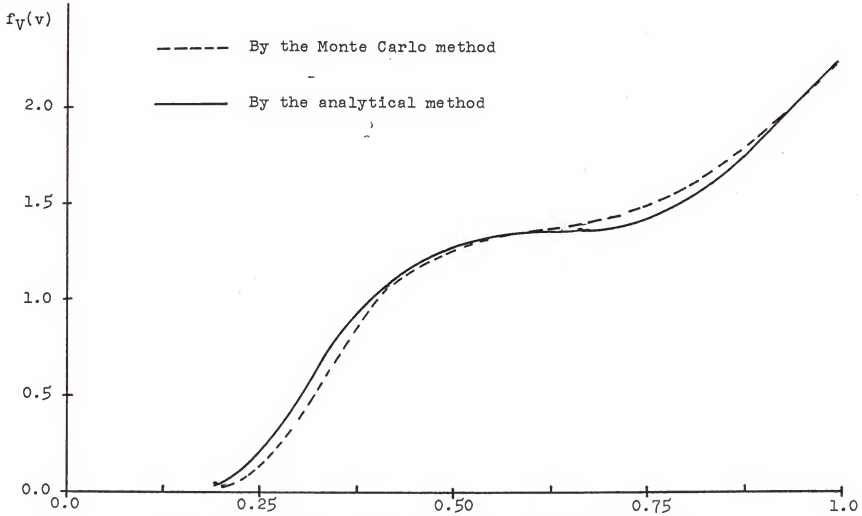

Fig. 9. Density functions of V with $\sigma = 0.1$, $\rho = 0.9$.

By the same method as section 4.2, the new random variable $\cos \theta_1$ is obtained by its functional relationship to X and Y in the cases σ = 0.1, 3, 100, ρ = 0.4, 0.9, and $\theta = 30^\circ, 45^\circ, 60^\circ$. Frequency ratios of each $\cos \theta_1$ with different parameters are calculated. The first kind of curve fitting technique is then used for σ = 0.1 case. However, for $\sigma > 1$, the second kind of curve fitting is used due to the histograms changing so rapidly. In Table 2 to Table 7 are listed all the coefficients of the fitted curves. Pairs of density curves of $\cos \theta_1$ with equal σ and θ but different ρ are plotted from Fig. 10 to Fig. 18. Since the total areas under the densities are unity, the areas in Fig. 15 are calculated by the trapezoidal method as a valid check. They are 0.951 for ρ = 0.9, and 1.0098 for ρ = 0.4, respectively.

From these curves and tables, observations can be made with respect to changing of parameters σ, ρ, and θ. The range of the random variable $\cos \theta_1$ is between +1 and -1. For σ = 0.1 and the different ρ cases, the curve occupies only the positive axis with one peak value (Figs. 10-12). When σ increases, it extends to the negative axis with two major peak values in the positive and negative sides respectively (Figs. 13-18). The amplitudes of these densities are also affected by σ. If the standard deviation is smaller, then the amplitude of the peak is larger.

The correlation coefficient does not cause a significant difference of the densities for the σ = 0.1 case (Figs. 10-12). It does not affect the density greatly when σ increases. The

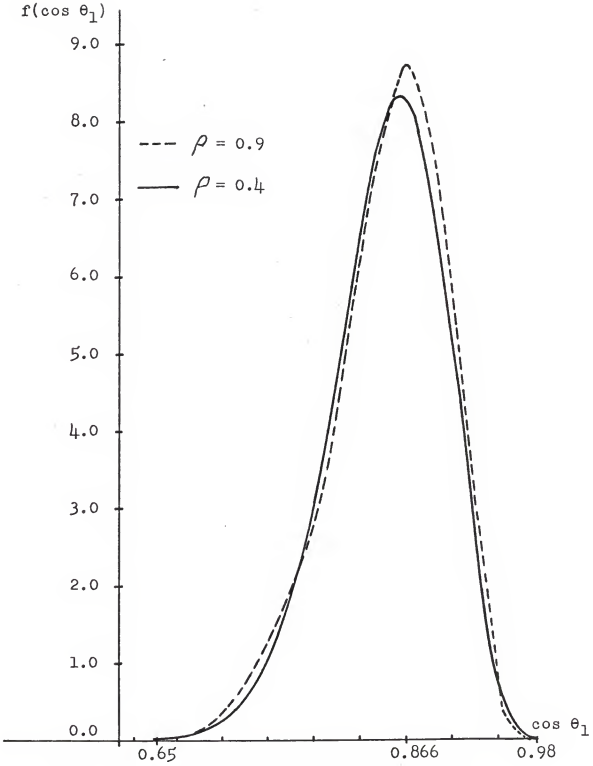

Fig. 10. Density functions of $\cos \theta_1$ with $\sigma = 0.1$, $\theta = 30^{\circ}$.

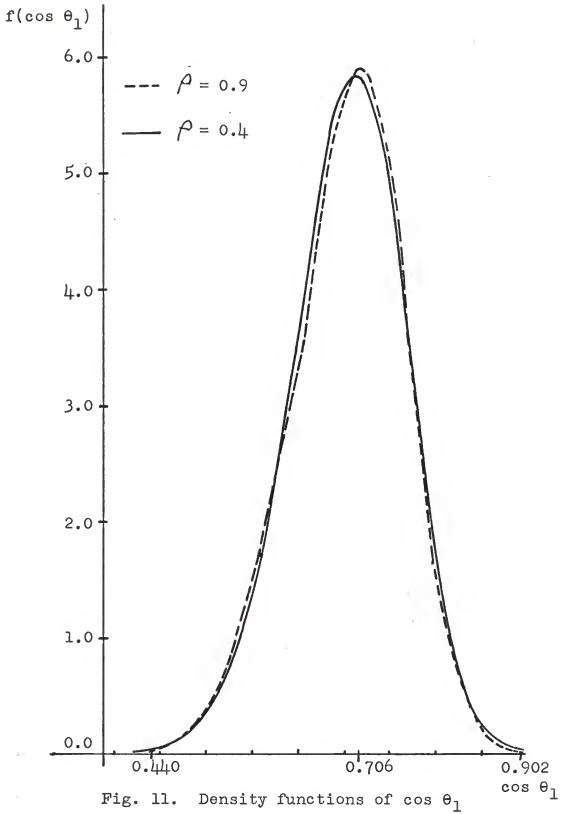

Fig. 11. Density functions of $\cos \theta_1$ with $\sigma = 0.1$, $\theta = 45^{\circ}$.

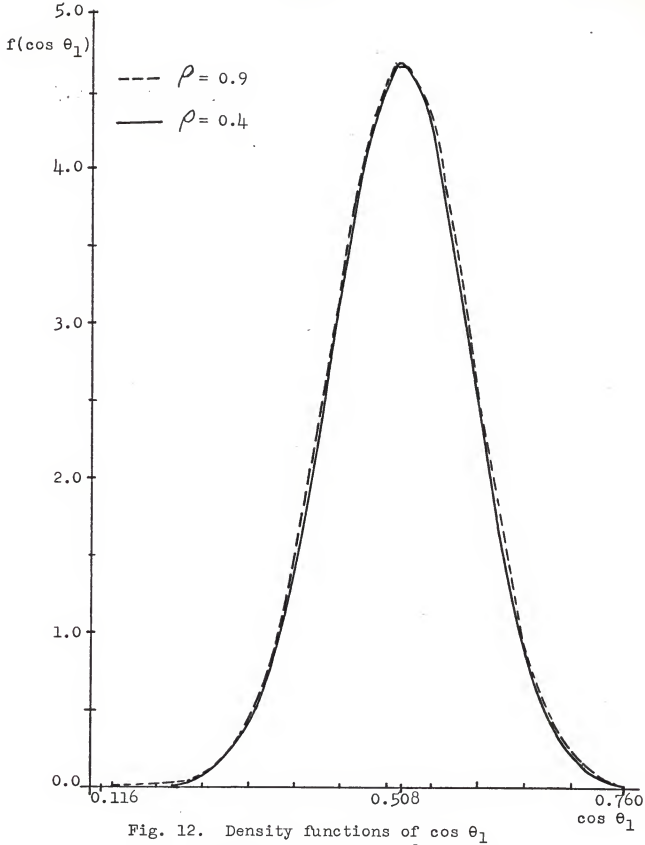

Fig. 12. Density functions of $\cos \theta_1$ with $\sigma = 0.1$, $\theta = 60^{\circ}$.

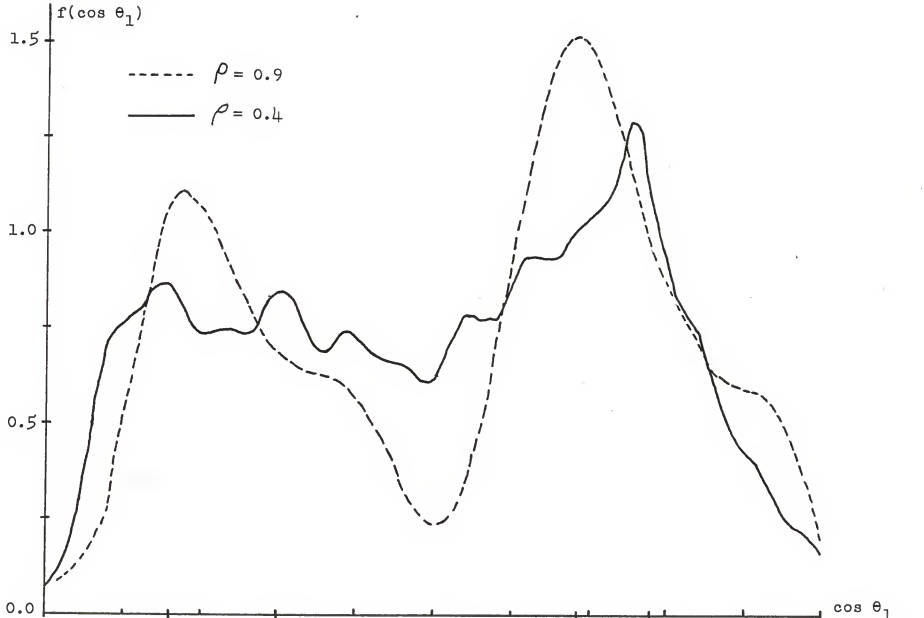

Fig. 13. Density functions of $\cos\theta_1$ with $\sigma = 3$, $\theta = 30^\circ$.

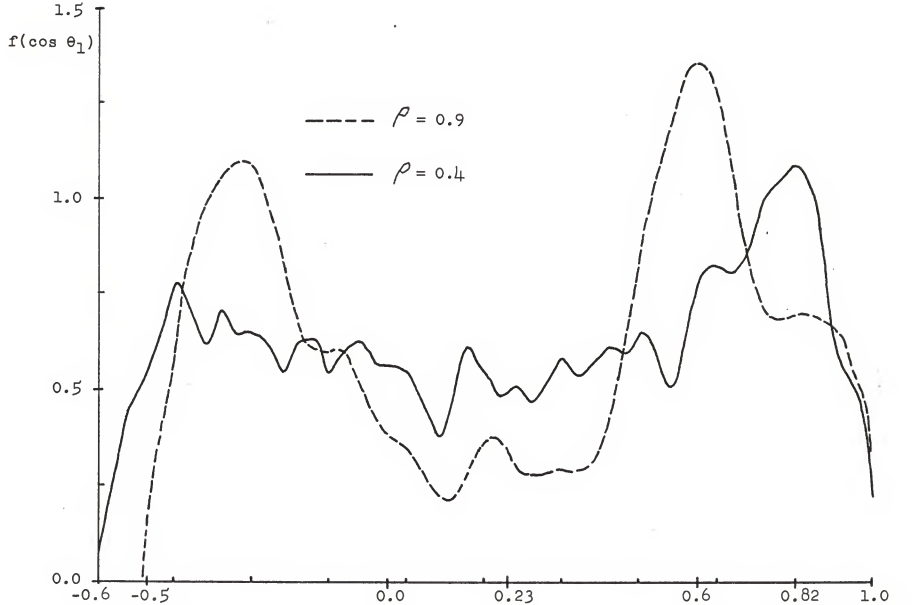

Fig. 14. Density functions of cos θ_1 with $\sigma = 3$, $\theta = 45^\circ$.

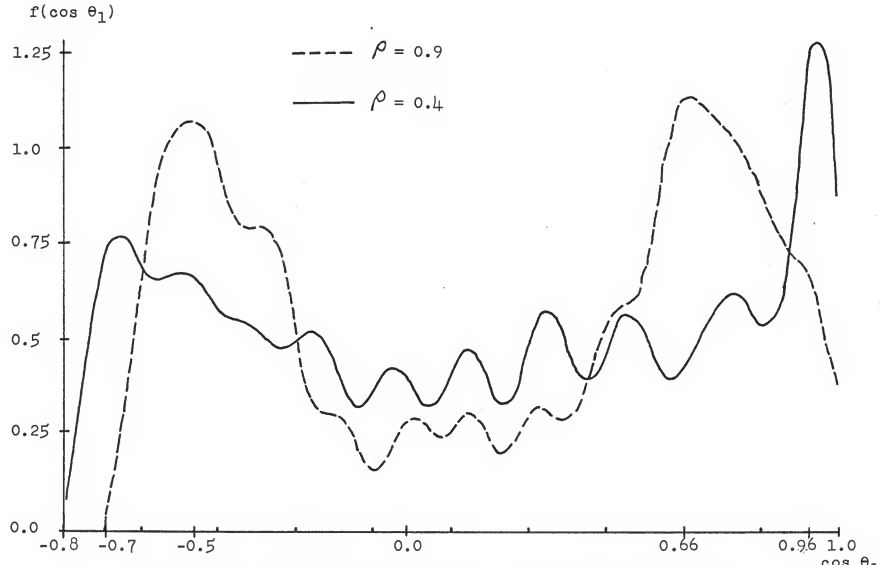

Fig. 15. Density functions of cos θ_1 with $\sigma = 3$, $\theta = 60°$.

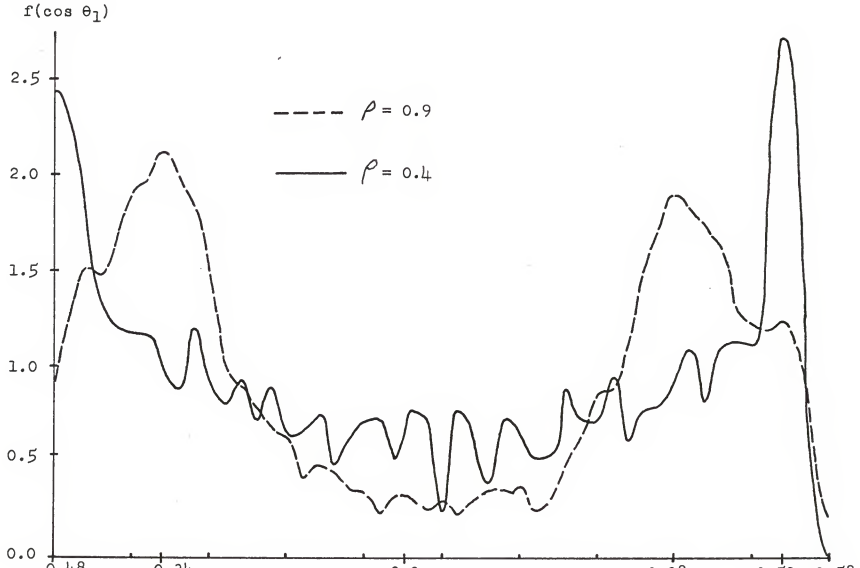

Fig. 16. Density functions of cos θ_1 with σ = 100, θ = 30°.

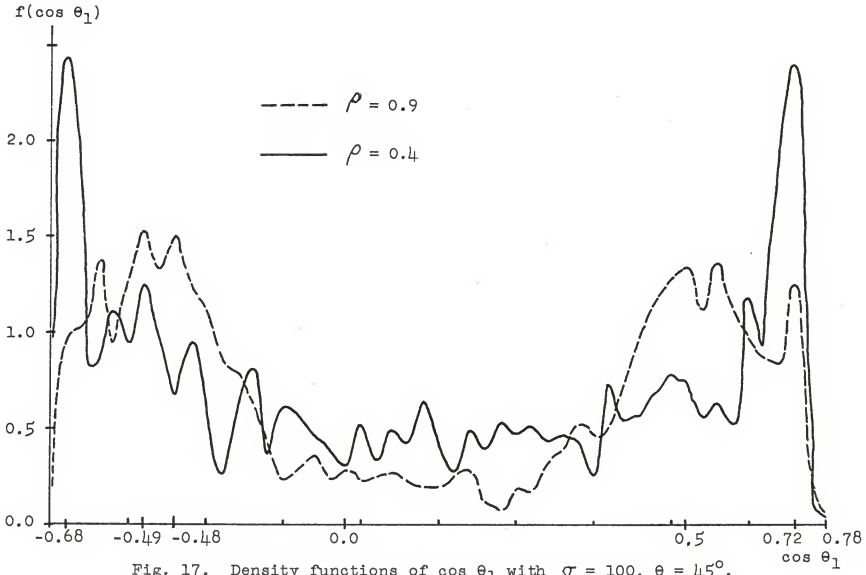

Fig. 17. Density functions of cos θ_1 with $\sigma = 100$, $\theta = 45^{\circ}$.

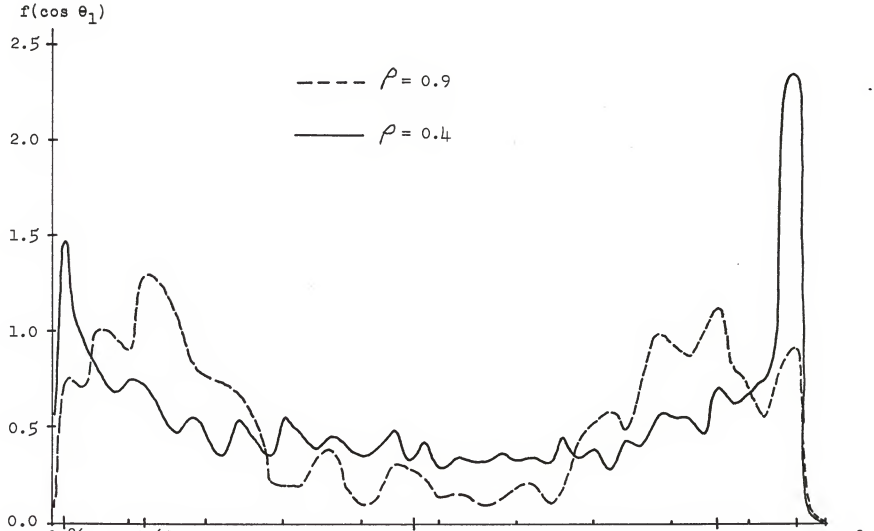

Fig. 18. Density functions of cos θ_1 with σ = 100, θ = 60°.

larger ρ produces a higher peak. The range of cos θ_1 also increases when ρ decreases (Figs. 13-18).

When σ is equal to 0.1, the angle θ exactly locates the point where the peak value of the densities occurs. That is the value of cos θ (Figs. 10-12). However, when σ is equal to 3 and 100, it is hard to see where the peak value does occur. A conservative estimate is around the value of cos θ for the major peak of the positive axis. The major peak of the negative axis is closer to the origin than the peak of the positive axis. This estimate is better for the $\sigma = 3$ case than the $\sigma = 100$ case (Figs. 13-18).

As a physical interpretation, when σ is large sharp slopes in the reflecting body are expected. As example $\sigma = 100$, over 90 per cent of the surface slopes are between 45° and 87°. Especially when θ is very large, the most values of cos θ_1 are either in the same direction as $\vec{a}_R$ or in the opposite direction (i.e., cos $\theta \longrightarrow \pm 1$). When θ decreases, a considerable number of cos θ_1 values approach $\theta_1 = 90^\circ$ (i.e., cos $\theta_1 = 0$). These phenomena are clearly shown in Fig. 19. They are actually observed in Figs. 13 to 18.

From the tables, when $\sigma = 0.1$, the densities of cos θ_1 are approximately normal distributions with mean 0.5 for $\theta = 60^\circ$ as shown (Table 3). The densities are centered at 0.707 and 0.866 for $\theta = 45^\circ$ and $\theta = 60^\circ$, respectively, with the same standard deviation 0.1 (Table 2, Table 4). They are somewhat like Rice distribution for the latter two cases. The existence of higher order harmonic terms in the fitting curve of cos θ_1

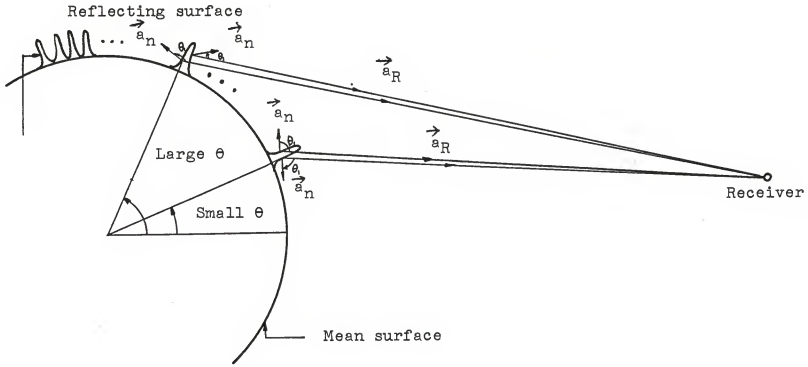

Fig. 19. A physical interpretation of the densities of cos θ_1 when σ is large.

for the cases $\sigma = 3$ and $\sigma = 100$ means more roughness in the data as given by the Monte Carlo method. Theoretically, such roughness cannot occur. This is an accumulated error due to the curve fitting and the computer simulation. The error of curve fitting is not so great because it uses the minimum least square sense. The computer simulation error is actually the origin of the error. By increasing the sample size, a better result is expected. However, the computer program used in this paper has to use a double precision and pairs of generated random numbers have to be stored. Therefore by using IBM 360/50 with the double precision and the sample size 10,000 (for pairs, i.e., 20,000), the size cannot be increased significantly. But to avoid the problem of the limited memory of the computer, one way is to store the generated random numbers on tapes. This is a recommended suggestion for the future research.

Table 2. Coefficients of fitting curves of $\cos \theta_1$, with $\sigma = 0.1$, $\theta = 30^\circ$,

$$f(x) = \sum_{r=0}^{3} a_r x^r e^{-p(x-p_1)^2}.$$

$\sigma = 0.1$, $\theta = 30^\circ$		
	$\rho = 0.4$	$\rho = 0.9$
p_1	0.866	0.866
p	192.247	204.370
a_0	3363.18032	7859.27189
a_1	-11844.72088	-27538.79014
a_2	13917.84435	32136.8961
a_3	-5443.13619	-12477.08512

Table 3. Coefficients of fitting curves of $\cos \theta_1$ with $\sigma = 0.1$, $\theta = 60$, $f(x) = Ae^{-p(x-p_1)^2}$

$\sigma = 0.1$, $\theta = 60$		
	$\rho = 0.4$	$\rho = 0.9$
p_1	0.5	0.5
p	69.928	67.889
A	4.72168	4.65298

Table 4. Coefficients of fitting curves of $\cos\theta_1$ with $\sigma = 0.1$, $\theta = 45^\circ$,

$$f(x) = \sum_{r=0}^{3} a_r x^r e^{-p(x-p_1)^2}.$$

$\sigma = 0.1$, $\theta = 45^\circ$		
	$\rho = 0.4$	$\rho = 0.9$
p_1	0.707	0.707
p	101.859	120.00
a_0	265.329	692.26728
a_1	-1130.96678	-2877.58864
a_2	1635.07361	3998.56342
a_3	-784.38536	-1841.23884

Table 5. Coefficients of fitting curves of $\cos \theta_1$ with $\sigma = 3$, $\theta = 30^\circ$.

$\rho = 0.4$		$\rho = 0.9$	
Order No.	Coefficient	Order No.	Coefficient
1	0.99702	1	0.98888
3	0.29310	3	0.41367
4	0.10095	4	0.36384
6	0.07049	7	0.22471
2	-0.06940	2	-0.15871
10	-0.06015		
12	-0.03525		
18	-0.03384		
27	0.03349		
5	-0.03190		
13	-0.02420		
Range	-0.4 ∽ 1.0	Range	-0.4 ∽ 1.0
Terms of fitting	11	Terms of fitting	5

Table 6. Coefficients of fitting curves of $\cos \theta_1$ with $\sigma = 3$, $\theta = 45^\circ$.

$\rho = 0.4$		$\rho = 0.9$	
Order No.	Coefficient	Order No.	Coefficient
1	0.77254	1	0.76162
3	0.39098	3	0.60205
5	0.18252	6	0.14228
7	0.08083	2	-0.13047
4	-0.079098	9	0.10959
2	-0.07457	10	-0.10710
6	-0.04739	4	0.05369
21	0.02976	5	0.08159
15	-0.03256	8	-0.05959
11	-0.03114	12	-0.04141
20	0.03166	20	0.03795
38	0.08162		
33	-0.02759		
28	-0.02775		
Range	-0.6 ∽ 1.0	Range	-0.5 ∽ 1.0
Terms of fitting	14	Terms of fitting	11

Table 7. Coefficient of fitting curve of $\cos \theta_1$ with $\sigma = 3$, $\theta = 60^\circ$.

$\rho = 0.4$		$\rho = 0.9$	
Order No.	Coefficient	Order No.	Coefficient
1	0.61517	1	0.62455
3	0.34107	3	0.60531
5	0.20143	5	0.26235
7	0.16137	14	-0.05642
9	0.11458	12	-0.04905
16	-0.08781	22	-0.04840
12	-0.08355	2	-0.03867
11	0.07622	16	-0.03540
10	0.07668		
13	0.07392		
8	-0.07460		
18	-0.07502		
6	-0.06877		
14	-0.06860		
15	0.06057		
17	0.04955		
22	-0.05024		
Range	-0.8 ∽ 1.0	Range	-0.7 ∽ 1.0
Terms of fitting	17	Terms of fitting	8

Table 8. Coefficients of fitting curves of $\cos \theta_1$ with $\sigma = 100$, $\theta = 30^\circ$.

$\rho = 0.4$				$\rho = 0.9$			
Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient
1	1.00944	14	0.14006	3	1.11162	8	0.05313
3	0.68269	26	0.09201	1	1.01148	22	0.05175
5	0.48581	12	0.11103	5	0.41252	24	0.05059
7	0.37063	15	0.10281	6	0.19378	41	0.04973
9	0.29316	28	0.05442	4	0.13130		
11	0.21800	10	0.08699	13	0.09265		
18	0.20250	31	0.11149	11	0.07872		
16	0.19150	35	0.11179	16	0.07023		
20	0.18180	32	0.02277	15	0.06885		
22	0.16734	43	0.09621	18	0.06152		
24	0.13745			10	-0.05963		
13	0.14415			14	0.05315		
Range	-0.48 ∽ 0.58			Range	-0.48 ∽ 0.58		
Terms of fitting	22			Terms of fitting	16		

Table 9. Coefficients of fitting curves of $\cos \theta_1$ with $\sigma = 100$, $\theta = 45^\circ$.

$\rho = 0.4$				$\rho = 0.9$			
Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient
1	0.73180	44	-0.13236	3	0.80616	27	-0.06488
3	0.49717	16	0.13527	1	0.71468	42	-0.05182
5	0.37588	42	-0.11895	5	0.32176	40	-0.05171
7	0.29024	24	0.12506	6	0.09813	16	0.04123
9	0.23121	36	-0.12445	13	0.07023	25	-0.05101
11	0.16809	14	0.12319	44	-0.07021	24	0.03620
22	0.15313	27	-0.09990	4	0.06139		
18	0.15035	15	0.05217	11	0.05147		
38	-0.14206	29	-0.09413	21	-0.06185		
13	0.09973	26	0.11080	29	-0.05998		
20	0.14376			23	-0.05352		
40	-0.12922			46	-0.05910		
Range	-0.68 ~ 0.78			Range	-0.68 ~ 0.78		
Terms of fitting	22			Terms of fitting	18		

Table 10. Coefficients of fitting curves of cos θ_1 with $\sigma = 100$, $\theta = 60^\circ$.

$\rho = 0.4$				$\rho = 0.9$			
Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient	Order No.	Coefficient
1	0.57913	27	-0.07782	3	0.61967	36	-0.03946
3	0.38075	38	-0.07357	1	0.56811	40	-0.04306
5	0.29065	26	0.07253	5	0.24517		
7	0.22918	16	0.06938	6	0.09974		
9	0.16796	46	-0.06857	13	0.02534		
11	0.12681	29	-0.06596	25	-0.06943		
22	0.10099	36	-0.05705	23	-0.07201		
24	0.09534	43	0.05604	4	0.05738		
20	0.09171	48	-0.05523	21	-0.07097		
13	0.09045	41	0.05337	44	-0.04892		
44	-0.08523	14	0.05113	11	0.01302		
18	0.08226	28	0.04784	38	-0.04401		
40	-0.08090	15	0.04715	24	0.05995		
25	-0.08044	21	-0.04657	18	0.06820		
23	-0.07956	31	-0.04382	42	-0.04563		
42	-0.07765	47	0.04348	10	-0.01168		
Range :	-0.86 ~ 0.98			Range :	-0.86 ~ 0.98		
Terms of fitting	32			Terms of fitting	18		

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The goal of finding the probability density of $\cos \theta_1$ which is a function of two correlated random variables X and Y does not succeed by using the analytical methods. Therefore the Monte Carlo method is used. When this method is combined with the curve fitting techniques, approximate solutions are obtained in a closed form for different parameters σ, ρ, θ. Three conclusions are made to these results.

1. When the standard deviations of the two random variables X and Y are less than unity, the densities of $\cos \theta_1$ are slightly affected by the changing of the correlation coefficient. They are approximately Rice distributions centered at the value $\cos \theta$.
2. When the standard deviations of X and Y are larger than unity, the effect on the densities due to increasing the correlation coefficient is significant. Two major peak values of the densities occur in positive and negative sides, respectively.
3. The larger the standard deviation of X and Y, the larger the occupied range of the density of $\cos \theta_1$ is. However, the range is limited between -1 to 1.

5.2 Recommendations

1. One way to find the density of $\cos \theta_1$ by analytical method might be very interesting. Because the density of $\cos \theta_1$ can be obtained by the densities of $Y/\sqrt{1 + X^2 + Y^2}$ which is the derivative of $\sqrt{1 + X^2 + Y^2}$ with respect to Y. In other words, with given probability densities of f(X, Y), the study of finding the densities of $\frac{\partial}{\partial X} f(X, Y)$ or $\frac{\partial}{\partial Y} f(X, Y)$ is suggested. Hopefully, this approach will not be tied up by hard integrations.

2. In order to solve the problem of the limited memory of the computer, storing the generated random numbers on tapes is suggested. Thus the sample size of the Monte Carlo method can be increased greatly. A better accuracy is then expected by this method in finding the densities of $\cos \theta_1$.

3. With a larger sample size, the use of the Monte Carlo method to find the probability density functions of the reflection coefficients $V_{pp}(\theta_1)$ and $V_{nn}(\theta_1)$ is recommended, where

$$V_{pp}(\theta_1) = \frac{n_1^2 \cos \theta_1 - \sqrt{n_2^2 + \cos^2 \theta_1}}{n_1^2 \cos \theta_1 + \sqrt{n_2^2 + \cos^2 \theta_1}}$$

$$V_{nn}(\theta_1) = \frac{n_3 \cos \theta_1 - \sqrt{n_2^2 + \cos^2 \theta_1}}{n_3 \cos \theta_1 + \sqrt{n_2^2 + \cos^2 \theta_1}}$$

n_1, n_2, n_3 are deterministic constants.

4. The densities of $\cos \theta_1$ change from Rice distribution to two-peak distributions when σ are altered from 0.1 to 3. An interesting problem is to find the value of σ when this changing exactly occurs.

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APPENDICES

APPENDIX A

A.1 The Derivation of the Probability Density of $\cos \theta_1$

$$\left\langle a + by \right\rangle = a + b \langle y \rangle = a \tag{A.1.1}$$

$$\left\langle \frac{a + by}{\sqrt{1 + x^2 + y^2}} \right\rangle = a \left\langle \frac{1}{\sqrt{1 + x^2 + y^2}} \right\rangle + b \left\langle \frac{y}{\sqrt{1 + x^2 + y^2}} \right\rangle \tag{A.1.2}$$

By using polar coordinates, with $c = 2\sigma^2(1 - \rho^2)$, then

$$\left\langle \frac{y}{\sqrt{1+x^2+y^2}} \right\rangle = \int_0^\infty \int_0^{2\pi} \frac{\sqrt{1-\rho^2}\, r^2 \sin\theta}{\pi c \sqrt{1 + r^2}} \exp\left\{ -\frac{r^2}{c} (1 - \rho \sin 2\theta) \right\} d\theta dr$$

$$= \int_0^\infty \frac{\sqrt{1-\rho^2}\, r^2}{\pi c \sqrt{1+r^2}} \exp\left(-\frac{r^2}{c}\right) \int_0^{2\pi} \sin\theta \exp\left\{ \frac{r^2 \rho}{c} \sin 2\theta \right\} d\theta dr$$

where the second integration of the above equation is

$$= \int_0^{2\pi} \sin\theta \left\{ I_o\left(\frac{\rho r^2}{c}\right) + 2 \left(\sum_{k=0}^{\infty} (-1)^k I_{2k+1}\left(\frac{\rho r^2}{c}\right) \sin(2(2k+1)\theta) \right.\right.$$

$$\left.\left. + \sum_{k=1}^{\infty} (-1)^k I_{2k}\left(\frac{\rho r^2}{c}\right) \cos 4k\theta \right) \right\} d\theta = 0 \tag{A.1.3}$$

The mean value of $\cos \theta_1$ is

$$\left\langle \frac{a + by}{\sqrt{1 + x^2 + y^2}} \right\rangle = a \left\langle \frac{1}{\sqrt{1 + x^2 + y^2}} \right\rangle$$

$$= 2a \int_0^\infty \frac{r\sqrt{1-\rho^2}}{c\sqrt{1+r^2}} \exp\left(-\frac{r^2}{c}\right) I_o\left(\frac{r^2\rho}{c}\right) dr \tag{A.1.4}$$

Assume $z = \dfrac{y}{\sqrt{1 + x^2 + y^2}}$

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z\sqrt{1+x^2}/\sqrt{1-z^2}} \frac{1}{2\sigma^2 \sqrt{1 - \rho^2}\, \pi}$$

$$\exp\left\{ - \frac{(x^2 + y^2 - 2\rho xy)}{2\sigma^2(1 - \rho^2)} \right\} dxdy$$

By taking the derivative with respect to z, it yields

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\sqrt{1-\rho^2}\sqrt{1+x^2}}{c\pi(1-z^2)^{3/2}} \exp\left\{ - \frac{(x^2+z^2)}{c(1-z^2)} \right\} \exp\left\{ \frac{2\rho xz\sqrt{1+x^2}}{c\sqrt{1 - z^2}} \right\} dx \qquad \text{(A.1.5)}$$

For $\rho = 0$ case, $c = 2\sigma^2$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\sqrt{1 + x^2}}{2\pi\sigma^2(1-z^2)^{3/2}} \exp\left\{ - \frac{(x^2+z^2)}{2\sigma^2(1-z^2)} \right\} dx \qquad \text{(A.1.6)}$$

Equations (A.1.4), (A.1.5), (A.1.6) have not been solved to obtain the solutions which can be expressed in algebraic form or with well-known series expressions. The moment theorem states that

$$\underline{\Phi}(w) = \sum_{n=0}^{\infty} \frac{(jw)^n}{n} m_n$$

where m_n is the n^{th} moments. If they are known, then $\underline{\Phi}(w)$ can be obtained. Then,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\Phi}(w)\, e^{-jwx}\, dx$$

$$\left\langle \cos^n\theta_1 \right\rangle = \left\langle \left(\frac{a + by}{\sqrt{1+x^2+y^2}} \right)^n \right\rangle = \left\langle \frac{a^n + nba^{n-1}y + \frac{n(n-1)}{2!} b^2a^{n-2}y^2 + \ldots + b^n y^n}{(1 + x^2 + y^2)^{n/2}} \right\rangle$$

$$= \left\langle \frac{a^n}{(1+x^2+y^2)^{n/2}} \right\rangle + \left\langle \frac{nba^{n-1}y}{(1+x^2+y^2)^{n/2}} \right\rangle + \left\langle \frac{\frac{n(n-1)}{2!} b^2a^{n-2}y^2}{(1 + x^2 + y^2)^{n/2}} \right\rangle + \ldots$$

$$+ \left\langle \frac{b^n y^n}{(1 + x^2 + y^2)^{n/2}} \right\rangle$$

The average value of the terms with odd power of y vanish in the above equation. The terms with even powers of y, by using polar coordinates and assuming n = 2m, then

$$\left\langle \frac{y^{2m}}{(1+x^2+y^2)^{n/2}} \right\rangle = \int_0^\infty \int_0^{2\pi} \frac{(r \sin\theta)^{2m} \sqrt{1-\rho^2}}{(1+r^2)^{n/2} \pi c} \exp\left\{ -\left[\frac{r^2}{c} (1 - \rho \sin 2\theta) \right] \right\} d\theta dr$$

where c is defined as above, it yields

$$\left\langle \frac{y^{2m}}{(1+x^2+y^2)^{n/2}} \right\rangle = \int_0^\infty \frac{r^{2m} \sqrt{1-\rho^2}}{2^m \pi c (1+r^2)^{n/2}} e^{-r^2/c} I_o\left(\frac{\rho r^2}{c}\right) dr \quad \text{(A.1.7)}$$

The above equation has not been solved to find a solution which can be expressed algebraically or with a well-known series expression.

APPENDIX B

B.1 The Derivation of the First Kind of Curve Fitting Algorithm

By substituting equation (3.4.1) into (3.4.2), then

$$\overline{f}(x) = \Big(\sum_{r=0}^{m} a_r x^r\Big)\, e^{-px^2} + \lambda\Big(\int_{-\infty}^{\infty} \sum_{r=0}^{m} a_r x^r\, e^{-px^2}\, dz - 1\Big)$$

the square error is

$$\|E\|^2 = \sum_{i=1}^{n} \Big[g(x_i) - \sum_{r=0}^{m} x_i^r\, e^{-px_i^2} + \lambda\Big(\sum_{r=0}^{m} a_r \int_{-\infty}^{\infty} x^r e^{-px^2}\, dx - 1\Big)\Big]^2$$

$$\frac{\partial \|E\|^2}{\partial \lambda} = 0, \text{ then}$$

$$\sum_{r=0}^{m} a_r \int_{-\infty}^{\infty} x^r\, e^{-px^2}\, dx = 1 \qquad \text{(B.1.1)}$$

$$\frac{\partial \|E\|^2}{\partial a_{r_0}} = 0, \text{ with equation (B.1), then}$$

$$\sum_{i=1}^{n} \sum_{r=0}^{m} a_r x_i^r x_i^{r_0}\, e^{-2px_i^2} + \lambda \sum_{i=1}^{n} g(x_i) \int_{-\infty}^{\infty} x^{r_0}\, e^{-px^2}\, dx$$

$$= \lambda \sum_{i=1}^{n} \sum_{r=0}^{m} a_r x_i^r\, e^{-px_i^2} \int_{-\infty}^{\infty} x^{r_0}\, e^{-px^2}\, dx + \sum_{i=1}^{n} x_i^{r_0} e^{-px_i^2}\, g(x_i) \qquad \text{(B.1.2)}$$

where $r_0 = 0, 1, 2, \ldots, m$.

Equations (B.1) and (B.2) are combined and written in a matrix form.

$$A \begin{Bmatrix} \{a_i\} \\ \cdots \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \{c\} + \lambda \{Y(a_i)\} \end{Bmatrix} \quad * \qquad (B.1.3)$$

where

$$\{a_i\} = \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{Bmatrix}$$

$$\{c\} = \begin{Bmatrix} \sum x_i^0 \, e^{-px_i^2} \, g(x_i) \\ \sum x_i^1 \, e^{-px_i^2} \, g(x_i) \\ \cdot \\ \cdot \\ \cdot \\ \sum x_i^m \, e^{-px_i^2} \, g(x_i) \end{Bmatrix}$$

$$Y(a_i) = \begin{Bmatrix} \int x^0 \, e^{-px^2} \, dx \\ \int x \, e^{-px^2} \, dx \\ \cdot \\ \cdot \\ \cdot \\ \int x^m \, e^{-px^2} \, dx \\ 0 \end{Bmatrix} \quad \left(\sum_{i=1}^{n} \sum_{r=0}^{m} a_r x_i^r \, e^{-px_1^2} \right)$$

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

A is a (m+2) x (m+2) square matrix

A_{11} is (m+1) x (m+1), A_{12} is (m+1) x 1, A_{21} is 1 x (m+1), A_{22} is 0.

---

*See the matrix notation in the footnote of section 3.4.

$$A_{11} = \begin{bmatrix} \sum_{i=1}^{n} x_i^0 x_i^0 e^{-2px_i^2}, & \sum_{i=1}^{n} x_i^1 x_i^0 e^{-2px_i^2}, \ldots, & \sum_{i=1}^{n} x_i^m x_i^0 e^{-2px_i^2} \\ \sum_{i=1}^{n} x_i^0 x_i^1 e^{-2px_i^2}, & \sum_{i=1}^{n} x_i^1 x_i^1 e^{-2px_i^2}, \ldots, & \sum_{i=1}^{n} x_i^m x_i^1 e^{-2px_i^2} \\ \vdots & \vdots & \\ \sum_{i=1}^{n} x_i^0 x_i^m e^{-2px_i^2}, & \sum_{i=1}^{n} x_i^1 x_i^m e^{-2px_i^2}, \ldots & \sum_{i=1}^{n} x_i^m x_i^m e^{-2px_i^2} \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} \sum_{i=1}^{n} g(x_i) \int x^0 e^{-px^2} dx \\ \sum_{i=1}^{n} g(x_i) \int x^1 e^{-px^2} dx \\ \vdots \\ \sum_{i=1}^{n} g(x_i) \int x^m e^{-px^2} dx \end{bmatrix}$$

$$A_{21} = \left[\int x^0 e^{-px^2} dx, \int x^1 e^{-px^2} dx, \ldots, \int x^m e^{-px^2} dx \right]$$

B.2 The Algorithm for Finding an Optimum P

With the assumption that the exact error curve is

$$E_1(p) = a_0 + a_1 p + a_2 p^2$$

The calculated error value is $E_d(p_i)$, then the square error between these two values is

$$\| e_i \|^2 = \sum_{i=0}^{n} \left[E_1(p_1) - E_d(p_i) \right]^2$$

$$\frac{\partial \|e_i\|^2}{\partial a_r} = 0 \text{ , where } r = 0, 1, 2, \text{ then}$$

$$\sum_{i=0}^{n} \left[2 E_1(p_i) - 2 E_d(p_i) \right] = 0$$

$$\sum_{i=0}^{n} \left[2 p_i E_1(p_i) - 2 p_i E_d(p_i) \right] = 0$$

$$\sum_{i=0}^{n} \left[2 p_i^2 E_1(p_i) - 2 p_i^2 E_d(p_i) \right] = 0$$

The above three equations are combined and written in a matrix form.

$$i=1 \quad \begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} p_i & \sum_{i=1}^{n} p_i^2 \\ \sum_{i=1}^{n} p_i & \sum_{i=1}^{n} p_i^2 & \sum_{i=1}^{n} p_i^3 \\ \sum_{i=1}^{n} p_i^2 & \sum_{i=1}^{n} p_i^2 & \sum_{i=1}^{n} p_i \end{bmatrix} \begin{Bmatrix} a_0 \\ a_i \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} d_i \\ \sum_{i=1}^{n} p_i d_i \\ \sum_{i=1}^{n} p_i^2 d_i \end{Bmatrix} \qquad \text{(B.2.1)}$$

The coefficient vector $\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}$ can be obtained by the equation (B.2.1). The optimum p is then $p = \frac{-a_1}{2a_2}$.

APPENDIX C

C.1 PROGRAM FOR GENERATION UNIFORM RANDOM NUMBERS

```
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION R(2,10000),C(2,50),B(50),FR(2,50)
      DIMENSION AM(2),DM(2),SM(2),DD(2)
    2 FORMAT (2I10)
    6 FORMAT (14X,'DX',6X,'UR',8X,'FR1',7X,'FR2')
    8 FORMAT (/12X,F5.2,3X,F7.4,3X,F7.4,3X,F7.4)
   40 FORMAT (/12X,'MEAN=',F9.7,3X,'  VARIANCE=       ',F9.7)
   42 FORMAT (////12X,'UNIFORM DISTRIBUTION ',I1)
   78 FORMAT (/12X,'CHI SQUARE VALUE=',F9.4)
      READ (1,2) NN,M
      DO 3 K=1,2
      READ (1,2) IX
      DO 3 I=1,M
      IY=IX*65539
      IF (IY) 85,86,86
   85 IY=IY+2147483647+1
   86 YFL=IY
      YFL=YFL*.4656613E-9
      IX=IY
    3 R(K,I)=YFL
      CM=M
      BN=NN
      B(1)=0.0
      DO 10 I=2,NN
   10 B(I)=B(I-1)+0.02
      DO 9 K=1,2
      DO 9 I=1,NN
    9 C(K,I)=0.0
      DO 11 K=1,2
      DO 11 J=1,M
      H=0.02
      DO 15 I=1,NN
      IF (R(K,J).LE.H) GO TO 11
   15 H=H+0.02
   11 C(K,I)=C(K,I)+1.0
      DO 18 K=1,2
      DD(K)=0.0
      AM(K)=0.0
      SM(K)=0.0
      DO 18 I=1,NN
      FR(K,I)=C(K,I)*BN/CM
      DD(K)=DD(K)+((FR(K,I)-1.0)**2)*200
      AM(K)=(C(K,I)*(B(I)+.01)/CM)+AM(K)
   18 SM(K)=(C(K,I)*((B(I)+.01)**2)/CM)+SM(K)
      DO 30 K=1,2
      IF (AM(K).EQ.0.) GO TO 31
      DM(K)=SM(K)-(AM(K)**2)
```

```
      GO TO 30
   31 DM(K)=SM(K)
   30 CONTINUE
      U=1.
      WRITE (3,6)
      WRITE (3,8) (B(I),U,FR(1,I),FR(2,I),I=1,NN)
      DO 60 I=1,2
      WRITE (3,42) I
      WRITE (3,40) AM(I),DM(I)
   60 WRITE(3,78) CD(I)
      STOP
      END
```

C.2 PROGRAM FOR GENERATING INDEPENDENT NORMAL RANDOM NUMBERS

```
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Z(2,10000),C(2,50),B(51),FR(2,50),D(51)
      DIMENSION AM(2),DM(2),SM(2),DD(2),G(50)
    2 FORMAT (2I10)
    6 FORMAT (14X,'DX',6X,'NR',8X,'FR1',7X,'FR2')
    8 FORMAT (/12X,F5.2,3X,F7.4,3X,F7.4,3X,F7.4)
   40 FORMAT (/12X,'MEAN=',F9.7,3X,'  VARIANCE      =',F9.7)
   42 FORMAT (////12X,'NORMAL DISTRIBUTION ',I1)
   78 FORMAT (/12X,'CHI SQUARE VALUE=',F9.4)
      READ (1,2) NN,M
      N1=NN+1
      READ (1,2) IX1,IX2
      DO 33 I=1,M
      IY1=IX1*65539
      IF (IY1) 51,61,61
   51 IY1=IY1+2147483647+1
   61 YFL1=IY1
      YFL1=YFL1*.4656613E-9
      IX1=IY1
      IY2=IX2*65539
      IF (IY2) 52,62,62
   52 IY2=IY2+2147483647+1
   62 YFL2=IY2
      YFL2=YFL2*.4656613E-9
      IX2=IY2
      R1=YFL1
      R2=YFL2
      Z(1,I)=((-2.*DLOG(R1))**.5)*DCOS(R2*6.2832)
   33 Z(2,I)=((-2.*DLOG(R1))**.5)*DSIN(R2*6.2832)
      CM=M
      BN=NN
      B(1)=-3.5
      DO 10 I=2,N1
   10 B(I)=B(I-1)+0.14
      DO 9 K=1,2
      DO 9 I=1,NN
    9 C(K,I)=0.0
      DO 11 K=1,2
      DO 11 J=1,M
      H=-3.36
      DO 15 I=1,NN
      IF (Z(K,J).LE.H) GO TO 11
   15 H=H+0.14
   11 C(K,I)=C(K,I)+1.0
      DO 14 I=1,N1
   14 D(I)=B(I)/1.414
      DO 18 K=1,2
      DD(K)=0.0
      AM(K)=0.0
```

```
      SM(K)=C.0
      DO 18 I=1,NN
      G(I)=C.5*(DERF(D(I+1))-DERF(D(I)))/C.14
      FR(K,I)=C(K,I)/(CM*.14)
      DD(K)=DD(K)+200*((FR(K,I)-G(I))**2)/G(I)
      AM(K)=C(K,I)*(B(I)+0.07)/CM+AM(K)
   18 SM(K)=SM(K)+(C(K,I)*((B(I)+0.07)**2)/CM)
      DO 30 K=1,2
      IF (AM(K).EQ.0.0) GO TO 31
      DM(K)=SM(K)-(AM(K)**2)
      GO TO 30
   31 DM(K)=SM(K)
   30 CONTINUE
      WRITE (3,6)
      WRITE (3,8) (B(I),G(I),FR(1,I),FR(2,I),I=1,NN)
      DO 60 I=1,2
      WRITE (3,42) I
      WRITE (3,40) AM(I),DM(I)
   60 WRITE (3,78) DD(I)
      STOP
      END
```

```
*C.3 PROGRAM FOR GENERATING NEW RANDOM NUMBERS (FUNCTION OF NORMAL
     RANDOM PAIRS)

     IMPLICIT INTEGER*4(I-N),REAL*8(A-H,C-Z)
     DIMENSION C(101),B(101),FR(101),Z1(10000),Z2(10000)
   2 FORMAT (2I10)
   3 FORMAT (2F7.2,3F5.2,F7.4)
   5 FORMAT (/5(2X,'DX',5X,'FR',4X))
   6 FORMAT (1H1,10X,'V1=',F7.2,1X,'V2=',F7.2,1X,'RHO=',F5.2,1X,'W1=',
    1F5.2,1X,'W2=',F5.2,1X,'ZTA=',F7.4)
   7 FORMAT (/5(1X,F6.3,1X,F7.4,1X))
   8 FORMAT (10F8.5)
  40 FORMAT(/12X,'MEAN=',F9.6,3X,'VARIANCE =',F9.6)
     READ (1,2) NN,M
     READ (1,2) IX1,IX2
     DO 33 I=1,M
     IY1=IX1*65539
     IF(IY1) 51,61,61
  51 IY1=IY1+2147483647+1
  61 YFL1=IY1
     YFL1=YFL1*.4656613E-9
     IX1=IY1
     IY2=IX2*65539
     IF (IY2) 52,62,62
  52 IY2=IY2+2147483647+1
  62 YFL2=IY2
     YFL2=YFL2*.4656613E-9
     IX2=IY2
     R1=YFL1
     R2=YFL2
     Z1(I)=((-2.*DLOG(R1))**.5)*DCOS(R2*6.2832)
  33 Z2(I)=((-2.*DLOG(R1))**.5)*DSIN(R2*6.2832)
     CM=M
     BN=NN
     B(1)=-1.0
     DO 9 I=2,NN
   9 B(I)=B(I-1)+0.02
     LS=1
  74 READ (1,3) V1,V2,RHO,W1,W2,ZTA
     CF=DCOS(ZTA)
     SF=DSIN(ZTA)
     C(1)=C.0
     DO 10 I=2,NN
  10 C(I)=C.0
     SM=0.C
     AM=0.C
     DO 11 J=1,M
     X1=V1*Z1(J)+W1
     X2=V2*(RHO*Z1(J)+Z2(J)*DSQRT(1.-RHO*RHO))+W2
     TT=DSQRT(1.+X1*X1+X2*X2)
     TR=(CF+X2*SF)/TT
```

```
      H=-1.C
      DO 15 I=1,NN
      IF (TR.LE.H) GO TO 11
      H=H+0.02
   15 CONTINUE
   11 C(I)=C(I)+1.0
      DO 30 I=1,NN
      FR(I)=C(I)/(CM*.02)
      AM=(C(I)*B(I)/CM)+AM
   30 SM=(C(I)*(B(I)**2)/CM)+SM
      IF (AM.EQ.0.) GO TO 31
      DM=SM-(AM**2)
      GO TO 32
   31 DM=SM
   32 WRITE (3,6) V1,V2,RHO,W1,W2,ZTA
      WRITE (3,5)
      WRITE (3,7) (B(J),FR(J),J=1,NN)
      WRITE (3,40) AM,DM
      WRITE (2,8) (FR(J),J=1,NN)
      LS=LS+1
      IF (LS.LE.6) GO TO 74
      STOP
      END
```

*Note: 1. This program used to find the random numbers $\cos\theta_1$.

2. When it is used to generate random numbers of $\sqrt{1 + X^2 + Y^2}$, then TR = TT, B(1) = 1.0, B(I) = B(I-1) + 0.04, H = 1.0, H = H + 0.04, FR(I) = C(I)/(CM * .04).

3. When it is used to generate random numbers of $1/\sqrt{1 + X^2 + Y^2}$, then TR = 1/TT, B(1) = 0.0, B(I) = B(I-1) + 0.01, H = 0.0, H = H + 0.01, FR(I) = C(I)/(CM * .01).

C.4 PROGRAM FOR THE FIRST KIND OF CURVE FITTING

```
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION Y(51),X(51),T(51,5),PE(51),A(6,6),G(51),TP(5),C(6),
     1B(5,5),TC(6),CRV(6),SV(5),D(2,6),CC(6),F(51),E(3),H(3,4),CA(3)
      DIMENSION ED(10)
    6 FORMAT (4F8.4)
    8 FORMAT (10X,2F9.5)
    9 FORMAT (10X,'P=',F7.3,6X,'E=',F10.4,6X,'KU=',I1)
   12 FORMAT (10X,'COEFFICIENTS')
   13 FORMAT (10X,8F12.5)
   14 FORMAT (10X,'RTA=',F12.5)
      KSU=1
   15 READ,(Y(I),I=1,34)
      READ (1,6) XO,XL,P,DP
      NU=1
      MU=1
      P1=0.707
      N=4
      DX=(XL-XO)/33.
      X(1)=XO+DX/2.
      DO 10 I=2,34
   10 X(I)=X(I-1)+DX
      DO 11 I=1,34
      T(I,1)=1.0
      DO 11 J=2,N
   11 T(I,J)=T(I,J-1)*X(I)
      KT=1
      KU=1
  100 DO 30 I=1,34
   30 PE(I)=DEXP(-P*(X(I)-P1)**2)
      DO 40 I=1,N
      DO 40 J=1,N
      A(I,J)=0.0
      DO 40 K=1,34
   40 A(I,J)=A(I,J)+T(K,I)*T(K,J)*PE(K)*PE(K)
      MN=N+1
      DO 41 J=1,N
      DO 45 I=1,34
   45 G(I)=PE(I)*T(I,J)
      DO 42 I=1,34
      TP(J)=0.0
   42 TP(J)=TP(J)+G(I)
      CALL INTG(G,34,DX,ARA)
   41 A(MN,J)=ARA
      SY=0.0
      DO 46 I=1,34
   46 SY=SY+Y(I)
      DO 47 I=1,N
   47 A(I,MN)=SY*A(MN,I)
      A(MN,MN)=0.0
```

```
      DO 50 I=1,N
      C(I)=C.0
      DO 50 J=1,34
   50 C(I)=C(I)+Y(J)*T(J,I)*PE(J)
      C(MN)=1.0
      DO 55 I=1,N
      DO 55 J=1,N
   55 B(I,J)=A(I,J)
      IF (N.EQ.1) GO TO 53
      CALL INVERS (B,N)
      GO TO 54
   53 B(I,J)=1./B(1,1)
   54 DO 60 I=1,N
      TC(I)=C.0
      DO 60 J=1,N
   60 TC(I)=TC(I)+B(I,J)*C(I)
      CRF=0.0
      DO 65 I=1,N
   65 CRF=CRF+TC(I)*TP(I)
      DO 66 I=1,N
   66 CRV(I)=CRF*A(MN,I)*0.3
      CRV(MN)=0.0
      DO 68 I=1,N
   68 SV(I)=A(MN,I)
      DO 67 I=1,MN
      TC(I)=C(I)
   67 C(I)=C(I)+CRV(I)
      CALL INVERS (A,MN)
   85 KK=1
   80 DO 70 I=1,MN
      D(KK,I)=0.0
      DO 70 J=1,MN
   70 D(KK,I)=D(KK,I)+A(I,J)*C(J)
      CALL CONST (MN,N,SV,TP,D,KK,TC,C)
      KK=KK+1
      IF(KK.LT.3) GO TO 80
      DO 88 I=1,MN
   88 DD(I)=CABS((D(KK-1,I)/D(KK-2,I))-1.0)
      CRT=DD(1)
      DO 89 I=2,MN
      IF (CRT.GE.DD(I)) GO TO 89
      CRT =DD(I)
   89 CONTINUE
      IF (CRT.LT.0.0001) GO TO 300
      DO 91 I=1,MN
   91 D(1,I)=D(2,I)
      KK=2
      GO TO 80
  300 DO 81 I=1,34
      F(I)=C.0
      DO 81 J=1,N
   81 F(I)=F(I)+T(I,J)*D(2,J)*PE(I)
      E(KT)=C.0
```

```
      DO 97 I=1,34
 97   E(KT)=E(KT)+DABS(Y(I)-F(I))
      WRITE (3,9) P,E(KT),KU
      WRITE (3,12)
      WRITE (3,13) (D(2,I),I=1,N)
      WRITE (3,14) D(2,MN)
      WRITE (3,8) (Y(I),F(I),I=1,34)
      ED(MU)=E(KT)
      MU=MU+1
      IF (MU.LT. 6) GO TO 899
      IF (NU.EQ. 3) GO TO 111
      CALL LSQP (ED,P,DP)
      MU=1
      KT=1
      NU=NU+1
      GO TO 100
899   IF (KU.EQ.2) GO TO 111
      KT=KT+1
      IF (KT.GE.4) GO TO 400
105   P=P+DP
      GO TO 100
400   CCRT=(E(KT-2)-E(KT-3))*(E(KT-1)-E(KT-2))
      IF (CCRT.LT.0.0) GO TO 600
650   E(1)=E(2)
      E(2)=E(3)
      KT=3
      GO TO 105
600   T2=E(1)-2.*E(2)+E(3)
      IF(T2.LT.0.0) GO TO 650
750   KU=KU+1
      KT=3
      DO 605 I=1,3
      H(I,1)=1.0
      AI=I
      H(I,2)=P-(3.0-AI)*DP
605   H(I,3)=H(I,2)*H(I,2)
      CALL INVERS (H,3)
      DO 620 I=1,3
      CA(I)=0.0
      DO 620 J=1,3
620   CA(I)=H(I,J)*E(J)+CA(I)
      P=-CA(2)/(2.*CA(3))
      GO TO 100
111   KSU=KSU+1
      IF (KSU.EQ.1) GO TO 15
      STOP
      END
```

```
      SUBROUTINE INTG(G,N,H,ARA)
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION G(34)
      ARA=G(1)-G(34)
      N1=N-1
      DO 10 I=2,N1,2
   10 ARA=ARA+4.0*G(I)+2.0*G(I+1)
      ARA=H*ARA/3.0
      RETURN
      END
      SUBROUTINE CONST(MN,N,SV,TP,D,KK,TC,C)
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION SV(5),TP(5),D(2,6),TC(6),CRV(6),C(6)
      CRF=0.0
      DO 10 I=1,N
   10 CRF=CRF+D(KK,I)*TP(I)
      CRF=D(KK,MN)*CRF
      DO 20 I=1,N
   20 CRV(I)=CRF*SV(I)
      CRV(MN)=0.0
      DO 30 I=1,MN
   30 C(I)=TC(I)+CRV(I)
      RETURN
      END
      SUBROUTINE LSQP (ED,P,DP)
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION ED(10),PT(10),C(5),B(3,3),A(3),D(3)
      PT(5)=P
      DO 10 I=1,4
      AI=I
   10 PT(I)=P-(5.-AI)*DP
      DO 20 I=1,5
      J=I-1
      C(I)=0.0
      DO 20 K=1,5
   20 C(I)=C(I)+PT(K)**J
      DO 30 I=1,3
      J=I-1
      D(I)=0.0
      DO 30 K=1,5
   30 D(I)=D(I)+ED(K)*(PT(K)**J)
      DO 40 I=1,3
      DO 40 J=1,3
      K=J+I-1
   40 B(I,J)=C(K)
      CALL INVERS (B,3)
      DO 50 I=1,3
      A(I)=0.0
      DO 50 J=1,3
   50 A(I)=A(I)+B(I,J)*D(J)
      P=-A(2)/(2.*A(3))
      RETURN
      END
```

```
      SUBROUTINE INVERS(A,N)
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION A(9,9),G(9),F(9)
      NN=N-1
      A(1,1)=1.0/A(1,1)
      DO 210 M=1,NN
      K=M+1
  250 DO 260 I=1,M
      G(I)=0.0
      DO 260 J=1,M
  260 G(I)=G(I)+A(I,J)*A(J,K)
      Q=0.0
      DO 270 I=1,M
  270 Q=Q+A(K,I)*G(I)
      Q=-Q+A(K,K)
      A(K,K)=1.0/Q
      DO 280 I=1,M
  280 A(I,K)=-G(I)*A(K,K)
      DO 290 J=1,M
      F(J)=0.0
      DO 290 I=1,M
  290 F(J)=F(J)+A(K,I)*A(I,J)
      DO 200 J=1,M
  200 A(K,J)=-F(J)*A(K,K)
      DO 210 I=1,M
      DO 210 J=1,M
  210 A(I,J)=A(I,J)-G(I)*A(K,J)
      RETURN
      END
```

C.5 PROGRAM FOR THE SECOND KIND OF CURVE FITTING

```
      DIMENSION A(55,55),X(55),Y(2,55),U(55),V(55),W(55),X1(55,55),R(55)
     1,T(55),DT(55),DM(55),DS(5),TY(55),C(55),B(55,55),DATA(2,98)
      DIMENSION F(98),DF(98)
860 FORMAT (1HO,5X,'COEFFICIENT',/)
865 FORMAT(1HO,5X,'APPROXIMATE VALUES',/)
866 FORMAT (1HO,5X,'U',/)
867 FORMAT (1HO,5X,'V',/)
868 FORMAT (1HO,5X,'N=',I4)
869 FORMAT(1H ,5X,'R')
870 FORMAT (5E18.8)
871 FORMAT (1H1)
879 FORMAT (4E18.8)
  5 FORMAT (10F8.5)
  7 FORMAT (3F8.5)
      KP=1
600 READ,MX,MK
      READ (1,7) (F(I),I=1,3)
      READ (1,5) (F(I),I=4,MX)
      DO 9 I=1,MX
      IF (F(I).LE. 0.0) GO TO 11
      GO TO 9
 11 F(I)=C.00001
  9 CONTINUE
      DO 10 I=1,MX
      DF(I)=0.4*SQRT(F(I))
      DATA(1,I)=F(I)+DF(I)
 10 DATA(2,I)=F(I)-DF(I)
      DO 666 KKA=1,MK
      READ,M
      CALL SET(M,B)
      CALL VECTOR (DATA,MX,M,Y)
      M1=M+1
      AM=M1
      S=0.0
      I=1
 15 U(I)=(Y(1,I)+Y(2,I))/2.0
      V(I)=(Y(1,I)-Y(2,I))/2.0
      S=S+V(I)**2
      I=I+1
      IF(I.LE.M) GO TO 15
      S=SQRT(S)
      CALL ARANGE(B,U,M,A)
      WRITE (3,871)
      N=M
      ND=N/2
      KL=-1
      NT=ND+2
100 N=N+KL*ND
      IF(N.EQ.M)GO TO 800
```

```
      WRITE (3,868) N
      I=1
   25 TY(I)=C.0
      I=I+1
      IF(I.LE.M) GO TO 25
      I=1
   33 J=1
   30 X1(I,J)=A(I,J)
      J=J+1
      IF(J.LE.N) GO TO 30
      I=I+1
      IF(I.LE.M) GO TO 33
      KT=1
      KK=1
      KS=1
  300 I=1
   38 W(I)=L(I)+TY(I)
      I=I+1
      IF(I.LE.M) GO TO 38
      I=1
   36 C(I)=C.0
      J=1
   37 C(I)=C(I)+X1(J,I)*W(J)
      J=J+1
      IF(J.LE.M) GO TO 37
      C(I)=C(I)*2.0/AM
      I=I+1
      IF(I.LE.N) GO TO 36
  200 I=1
   40 R(I)=-U(I)
      J=1
   41 R(I)=R(I)+X1(I,J)*C(J)
      J=J+1
      IF(J.LE.N) GO TO 41
      I=I+1
      IF(I.LE.M) GO TO 40
      WRITE (3,869)
      SR=0.C
      I=1
   45 SR=SR+R(I)**2
      I=I+1
      IF(I.LE.M) GO TO 45
      SR=SQRT(SR)
      IF(SR.LE.S) GO TO 70
      NT=N
      KL=1
      ND=ND/2
      IF(ND.GE.1) GO TO 100
      IF(N.LT.M) GO TO 80
      GO TO 500
   80 ND=1
      GO TO 100
   70 I=1
```

```
 72 IF(ABS(R(I)).LE.V(I)) GO TO 71
    GO TO 333
 71 I=I+1
    IF(I.LE.M) GO TO 72
555 KL=-1
    ND=ND/2
    IF(ND.GE.1) GO TO 100
    IF(N.GT.NT) GO TO 500
    ND=1
    IF(N.GE.2) GO TO 100
    GO TO 500
333 I=1
 85 T(I)=R(I)/V(I)
    IF(ABS(T(I)).LE.1.0) GO TO 95
    DT(I)=(ABS(T(I))-1.0)*R(I)
    GO TO 84
 95 DT(I)=0.0
    TY(I)=R(I)
    GO TO 86
 84 TY(I)=R(I)-DT(I)*1.10
 86 I=I+1
    IF(I.LE.M) GO TO 85
    DS(KT)=0.0
    I=1
105 DS(KT)=DS(KT)+ABS(DT(I))
    I=I+1
    IF(I.LE.M) GO TO 105
    WRITE (3,870) DS(KT)
    IF(DS(KT).LE.0.1E-20) GO TO 555
    KT=KT+1
    IF(KT.GE.5) GO TO 115
    GO TO 300
115 DM(KS)=DS(1)+DS(2)+DS(3)+DS(4)
    DM(KS)=DM(KS)/4.0
    KS=KS+1
    KT=1
    IF(KS.GE.4) GO TO 700
    GO TO 300
700 R2=DM(3)/DM(2)
    R1=DM(2)/DM(1)
    R3=ABS(R1/R2-1.0)
    IF(R3.LT.0.01) GO TO 710
    DM(1)=DM(2)
    DM(2)=DM(3)
    KS=3
    IF(KK.GT.M/4) GO TO 710
    KK=KK+1
    GO TO 300
710 KL=1
    NT=N
    ND=ND/2
    IF(ND.GE.1) GO TO 100
    IF(N.LT.M) GO TO 720
```

```
      GO TO 500
  720 ND=1
      GO TO 100
  800 I=1
  810 C(I)=C.0
      J=1
  820 C(I)=C(I)+A(I,J)*U(J)
      J=J+1
      IF(J.LE.M) GO TO 820
      C(I)=C(I)*2.0/AM
      I=I+1
      IF(I.LE.M) GO TO 810
      I=1
  830 TY(I)=C.0
      J=1
  840 TY(I)=TY(I)+A(I,J)*C(J)
      J=J+1
      IF(J.LE.M) GO TO 840
      I=I+1
      IF(I.LE.M) GO TO 830
      GO TO 765
  500 WRITE(3,868) N
      I=1
  760 TY(I)=C.0
      J=1
  761 TY(I)=TY(I)+X1(I,J)*C(J)
      J=J+1
      IF(J.LE.N) GO TO 761
      I=I+1
      IF(I.LE.M) GO TO 760
  765 WRITE(3,860)
      WRITE(3,870) (C(I),I=1,N)
      WRITE(3,865)
      WRITE(3,870) (TY(I),I=1,M)
      WRITE (3,866)
      WRITE(3,870) (U(I),I=1,M)
      WRITE (3,867)
      WRITE (3,870) (V(I),I=1,M)
  666 CONTINUE
      KP=KP+1
      IF (KP.LE.2) GO TO 600
      STOP
      END
```

```
      SUBROUTINE VECTOR(W,M1,M,Y)
      DIMENSION W(2,98),Y(2,98),X(98),Z(98)
      IF(M.EQ.M1) GO TO 60
      A1=M1+1
      A2=M+1
      DO 10 I=1,M1
      AI=I
   10 X(I)=AI/A1
      DO 15 I=1,M
      AI=I
   15 Z(I)=AI/A2
      JM=1
      MM=M1-1
      DO 30 I=1,M
      DO 35 J=JM,MM
      IF(Z(I).LT.X(J)) GO TO 35
      IF(Z(I).EQ.X(J)) GO TO 45
      CR=(Z(I)-X(J))*(Z(I)-X(J+1))
      IF(CR.GT.0.0) GO TO 35
      Y(1,I)=W(1,J)
      Y(2,I)=W(2,J)
      GO TO 65
   45 Y(1,I)=W(1,J)
      Y(2,I)=W(2,J)
      GO TO 75
   35 CONTINUE
   65 JM=J
      GO TO 30
   75 JM=J+1
   30 CONTINUE
      GO TO 70
   60 DO 85 I=1,M
      DO 85 J=1,2
   85 Y(J,I)=W(J,I)
   70 RETURN
      END
```

```
      SUBROUTINE ARANGE(A,U,M,C)
      DIMENSION A(66,66),U(66),B(66),KO(66),TP(66),C(66,66)
  200 FORMAT(1H1,5X,'INNER PRODUCTS'/)
  210 FORMAT(10E12.5)
  220 FORMAT (1HO,5X,'ORDER NO.'/)
  230 FORMAT(20I5)
  231 FORMAT (14I4)
      I=1
   10 B(I)=C.0
      J=1
   16 B(I)=B(I)+A(J,I)*U(J)
      J=J+1
      IF(J.LE.M) GO TO 16
      B(I)=ABS(B(I))
      I=I+1
      IF(I.LE.M) GO TO 10
      G=-1.C
      WRITE (3,200)
      L=1
      I=1
   20 J=1
   30 IF(B(J).LT.G) GO TO 31
      IF(B(J).EQ.G) GO TO 33
      G=B(J)
      L=J
      GO TO 31
   33 IF(L.EQ.1) GO TO 31
      IF(L.NE.KO(I-1)) GO TO 31
      L=J
   31 J=J+1
      IF(J.LE.M) GO TO 30
      G=-1.
      KO(I)=L
      WRITE (3,210) B(L)
      B(L)=-2.0
      I=I+1
      IF(I.LE.M) GO TO 20
      WRITE(3,220)
      WRITE (2,231) (KO(I),I=1,M)
      WRITE(3,230) (KO(I),I=1,M)
      I=1
   35 K=KO(I)
      J=1
   36 C(J,I)=A(J,K)
      J=J+1
      IF(J.LE.M) GO TO 36
      I=I+1
      IF(I.LE.M) GO TO 35
      RETURN
      END
```

```
      SUBROUTINE SET(M,C)
      IMPLICIT REAL*8(A,D),REAL*4(B,C,E-H,O-Z),INTEGER(I-N)
      DIMENSION C(100,100)
      M1=M+1
      AM=M1
      AP=3.14159265358979/AM+(.323846D-14)/AM
      A1=DSIN(AP)
      A2=DCOS(AP)
      AD1=A1
      AD2=A2
      C(1,1)=SNGL(A1)
      I=2
   10 AT=A2*AD1+A1*AD2
      A2=-A1*AD1+A2*AD2
      A1=AT
      C(I,1)=SNGL(A1)
      AD3=AT
      AD4=A2
      J=2
   20 AT=A2*AD3+A1*AD4
      A2=-A1*AD3+A2*AD4
      A1=AT
      C(I,J)=SNGL(A1)
      J=J+1
      IF(J.LE.I) GO TO 20
      A1=AD3
      A2=AD4
      I=I+1
      IF(I.LE.M) GO TO 10
      I=1
   40 II=I+1
      J=II
   41 C(I,J)=C(J,I)
      J=J+1
      IF(J.LE.M) GO TO 41
      I=I+1
      IF(I.LT.M) GO TO 40
      RETURN
      END
```

A STUDY OF PROBABILITY DENSITY FUNCTIONS RELATED TO THE RADAR BACKSCATTERING

by

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A study of probability density functions of the angle, θ_1, between the surface normal and the direction to the receiver of a rough surface in a radar backscattering problem is discussed. Both the analytical and the Monte Carlo methods for finding a probability density of a function of random variables are introduced. In order to get the algebraic expressions of the density from the data by the Monte Carlo method, two different curve fitting techniques are developed.

Because $\cos \theta_1$ involves the function $\sqrt{1 + X^2 + Y^2}$, where X and Y are two correlated normal random variables, the problem of finding the density function of $\cos \theta_1$ starts from finding the densities of $\sqrt{1 + X^2 + Y^2}$ and $1/\sqrt{1 + X^2 + Y^2}$ with a desire to find the former density by the information method contained in the latter two. The Monte Carlo method is used to find the densities of $\cos \theta_1$ with each different parameter σ, ρ, and θ due to the failure of finding the density function of $\cos \theta_1$ analytically. The results show that the densities of $\cos \theta_1$ are approximately Rice distributions centered at the value of $\cos \theta_1$ when σ is less than unity. The densities of $\cos \theta_1$ are also discussed by changing σ, ρ, and θ.