APPLICATION OF QUASILINEARIZATION TO INDUSTRIAL MANAGEMENT SYSTEMS

by 45

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CHAPTER 1

INTRODUCTION

1.1 IDEA OF DECISION MAKING IN MANAGEMENT

The administration of a modern business enterprise has become an enormously complex undertaking. During the past few years there has been an increasing tendency to turn to quantitative techniques and models as a potential means for solving the problems that arise in such an enterprise.

Engineering has been defined as concerned with the design, improvement and installation of integrated systems of men, machines and materials for the service of society. Every working day, the typical executive of a modern industrial organization makes a number of complex decisions in order to optimize his company's performance. This emphasizes the importance of quantitative techniques as a useful means for decision making in industrial management systems.

In any problem solving situation, there are variables or factors which influence the outcome of whatever decision is made. These variables can be classified as those which the decision maker controls, called the control variables, and a class of those which he cannot control, called the state variables. After identifying the control and state variables, they should be combined in some logical manner so that they form a model of the problem. The object of the decision maker is not to construct a model as close as possible to the reality of the problem, but rather a simplest model that predicts the outcomes reasonably well. Next step is to develop a measure of effectiveness, called the objective function, to predict the behavior of the model. A model is then solved for different values of the control variables. These will be called feasible solutions.

In general, decision making can be described as a process whereby management when confronted with a problem, selects a specific course of action, called the optimal policy, from a set of feasible solutions.

Many of the mathematical models in engineering, physical sciences and other disciplines involve non-linear differential equations of the two point boundary value type. Unfortunately, no general analytical method exists for solving them. Several kinds of non-linear differential equations have been solved analytically, but the solution of each has required a method unique to that type. Various methods have been used to solve non-linear differential equations numerically. Among them are graphical methods, methods based upon successive approximations and methods based upon iterative procedures.

1.2 PURPOSE OF THIS STUDY

Industrial Engineers work with a wide variety of optimization problems. For this reason they should be familiar with the most efficient techniques for solving the decision-making problems. Because of the relatively recent origin of operations research, more efficient techniques are not needed in most cases.

The purpose of this research is to study the effectiveness of a recently developed method, quasilinearization, in solving industrial management problems which involve non-linear differential equations.

More specifically, the object of this work is to investigate the computational features of this technique with respect to different problems. The second object is to provide the systems analysts a new tool for optimization.

Other computational techniques, such as the gradient technique, the second variation method, and invariant imbedding can also be used for solving the problems with non-linear differential equations, but considering the object of this study, they will not be discussed here.

CHAPTER 2

SOLUTION OF TWO POINT BOUNDARY VALUE PROBLEMS

2.1 INTRODUCTION

The mathematical formulation of many problems in science and engineering leads to differential equations. Problems in which the conditions to be satisfied by the solution of a differential equation of order two or greater may be specified at both ends of an interval are known as two point boundary value problems. If the conditions are specified at more than two points in the interval, the problems are known as multi-point boundary value problems. The latter type of problems do not appear very often in engineering models.

Initial value problems are those in which all conditions are imposed at one point. This may be the initial or final point of that interval.

Consider a system of differential equations

$$\frac{dx_1}{dt} = f(x,y)$$

$$t_i \leq t \leq t_f \qquad (1)$$

$$\frac{dx_2}{dt} = g(x,y)$$

If the conditions for both x and y are given at the same point,

$$x_1(t_i) = x_1^0 \qquad x_2(t_i) = x_2^0$$

or

$$x_1(t_f) = x_1^1 \qquad x_2(t_f) = x_2^1 \tag{2}$$

the problem is called the initial value problem. However, if the conditions for both x and y are not at the same point,

$$x_1(t_i) = x_1^0 \qquad x_2(t_f) = x_2^1$$

or

$$x_1(t_f) = x_1^1 \qquad x_2(t_i) = x_2^0 \tag{3}$$

the problem is called two point boundary value problem.

A higher order differential equation can always be replaced by a set of first order differential equations by introducing auxiliary variables [9]. For this reason, only first order differential equations will be discussed throughout this work.

2.2 NUMERICAL SOLUTION OF INITIAL VALUE PROBLEMS

Since all of this work will be based on numerical methods of obtaining solutions of ordinary differential equations, a best known and most frequently used scheme for solving initial value problems, Runge-Kutta method, is discussed here.

In this method, the increments of the functions are calculated once for all by means of a definite set of formulas, and the calculations for the first increment are exactly same as for any other increment. These processes are self-starting. Advantage of this method is the independent choice of the size of the step, which may be increased to speed up the progression or decreased to lower truncation errors without recalculation of previous data.

The fourth-order formulas for the Runge-Kutta method, [9], for Equations (1) are

$$x_1(t_{k+1}) = x_1(t_k) + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$x_2(t_{k+1}) = x_2(t_k) + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4) \tag{4}$$

where

$$m_1 = f(x_1(t_k), x_2(t_k), t_k)\,\Delta t$$

$$m_2 = f(x_1(t_k) + \frac{m_1}{2}, x_2(t_k) + \frac{n_1}{2}, t_k + \frac{\Delta t}{2})\,\Delta t$$

$$m_3 = f(x_1(t_k) + \frac{m_2}{2}, x_2(t_k) + \frac{n_2}{2}, t_k + \frac{\Delta t}{2})\,\Delta t$$

$$m_4 = f(x_1(t_k) + m_3, x_2(t_k) + n_3, t_k + \Delta t)\,\Delta t$$

$$n_1 = g(x_1(t_k), x_2(t_k), t_k)\,\Delta t$$

$$n_2 = g(x_1(t_k) + \frac{m_1}{2}, x_2(t_k) + \frac{n_1}{2}, t_k + \frac{\Delta t}{2})\,\Delta t$$

$$n_3 = g(x_1(t_k) + \frac{m_2}{2}, x_2(t_k) + \frac{n_2}{2}, t_k + \frac{\Delta t}{2})\,\Delta t$$

$$n_4 = g(x_1(t_k) + m_3, x_2(t_k) + n_3, t_k + \Delta t)\,\Delta t. \tag{5}$$

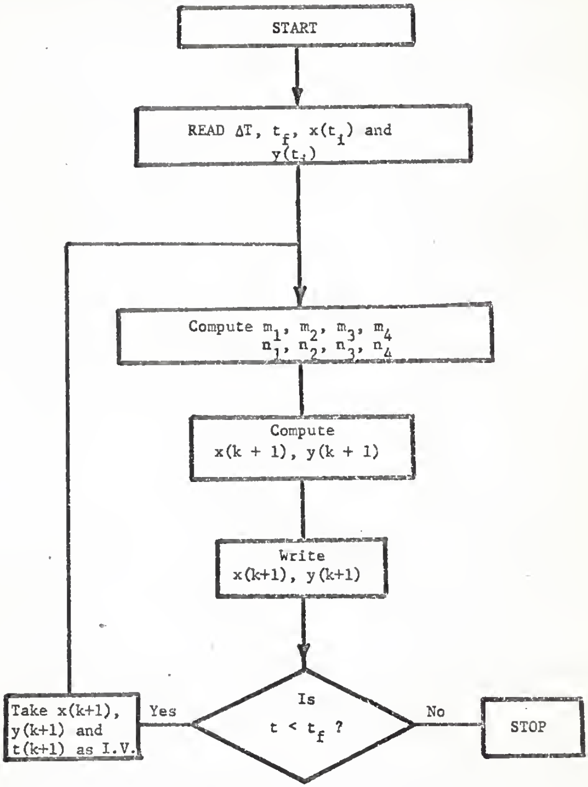

Fig. 1. Computer Logic Diagram for Runge-Kutta Method

Knowing the initial values of $x_1(t_i)$, $x_2(t_i)$, and step size Δt, values of $x_1(t_i + \Delta t)$ and $x_2(t_i + \Delta t)$ can be calculated using the above formulas. Similarly $x_1(t_i + 2\Delta t)$ and $x_2(t_i + 2\Delta t)$ can be calculated using $x_1(t_i + \Delta t)$ and $x_2(t_i + \Delta t)$. Hence incrementing t everytime by Δt, the final values, $x_1(t_f)$ and $x_2(t_f)$ can be calculated.

The truncation error in this method is $O(\Delta t^5)$. A simplified computational scheme is shown in Figure 1.

2.3 DIFFICULTIES IN TWO POINT BOUNDARY VALUE PROBLEMS

The numerical solution of any ordinary differential equation requires the knowledge of initial values of all the variables. Starting with the initial values, the solutions are constructed step by step in small intervals of the variables. Because of this nature, they are also called the marching techniques.

In an initial value problem, all the initial (or final) values are known. Hence the solution is relatively easy. In a two point boundary value problem, some of the initial (or final) values are unknown. Hence, the numerical techniques, like Runge-Kutta method, cannot be applied directly. For this reason, this type of problems are very difficult to solve.

In general, the procedure for solving this type of problems is to assume the missing initial (or final) conditions and solve for all the grid points and then compare the values of the calculated and given final (or initial) conditions. If they are uot the same within allowable error, a new set of missing initial (or final) values is assumed and the same procedure is repeated. By this trial and error procedure, a suitable set of initial (or final) conditions can be determined.

This procedure becomes very tedious if the problem has many differential equations and is very complex in nature. The relatively slow convergence during the process of numerical solution can make the generally used trial and error procedure impractical.

Unfortunately, most of the mathematical models in quantitative analysis are very complex having many variables. Problems of this type are most subtle and difficult and are not well suited for modern digital computers. There is no general proof of existence and uniqueness of solutions to problems of this type.

2.4 SUPERPOSITION PRINCIPLE

A two point boundary value problem is not too difficult if the performance equations are linear. This is because of the fact that superposition principle is applicable to linear differential equations.

Consider the following two simultaneous first order linear differential equations

$$\frac{dx_1}{dt} = a_1(t) + b_1(t)\ x_1 + c_1(t)\ x_2 \tag{6}$$

$$\frac{dx_2}{dt} = a_2(t) + b_2(t)\ x_1 + c_2(t)\ x_2 \tag{7}$$

$$x_1(t_I) = x_1^0 \quad \text{and} \quad x_2(t_f) = x_2^1 \tag{8}$$

where a_1, b_1, c_1, a_2, b_2, and c_2 are functions of the independent variable, t. x_1^0, x_2^1 are known constants at the initial and final values of t respectively.

Using any arbitrarily assumed initial conditions for x_1 and x_2, say, $x_{1p}(t_i) = 1$ and $x_{2p}(t_i) = 0$, Equations (6) and (7) can be solved numerically to obtain a set of particular solutions, $x_{1p}(t)$ and $x_{2p}(t)$, $t_i \leq t \leq t_f$.

Two sets of non-trivial homogeneous solutions, $x_{1,1h}(t)$, $x_{2,1h}(t)$ and $x_{1,2h}(t)$, $x_{2,2h}(t)$, can be obtained with any two different sets of arbitrarily assumed initial conditions, say $x_{1,1h}(t_i) = 1$, $x_{2,1h}(t_i) = 0$ and $x_{1,2h}(t_i) = 0$, $x_{2,2h}(t_i) = 1$, from the homogeneous equations of Equations (6) and (7).

The homogeneous equations are obtained by setting the constant terms equal to zero.

$$\frac{dx_1}{dt} = b_1(t)\, x_1 + c_1(t)\, x_2 \tag{9}$$

$$\frac{dx_2}{dt} = b_2(t)\, x_1 + c_2(t)\, x_2 \tag{10}$$

It is important to note that the particular and homogeneous solutions can be obtained numerically using a step by step integration method, like the Runge-Kutta method. The reader is referred to Ince [7] and Lee [9] for detailed discussion.

The superposition principle states that because of the additive property of the solution of a linear system, the general solution of Equations (6) and (7) is,

$$x_1(t) = x_{1p}(t) + A_1 x_{1,1h}(t) + A_2 x_{1,2h}(t) \tag{11}$$

$$x_2(t) = x_{2p}(t) + A_1 x_{2,1h}(t) + A_2 x_{2,2h}(t) \tag{12}$$

where A_1 and A_2 are integration constants.

A_1 and A_2 can be obtained by substituting the boundary values, Equation (8), into Equations (11) and (12) with the results of the particular and homogeneous solutions. Once the values of A_1 and A_2 are known, the right hand sides of Equations (11) and (12) are completely known. This gives the solution for $x_1(t)$ and $x_2(t)$ at all the grid points.

This approach can be generalized to a set of n simultaneous first order linear differential equations

$$\frac{dx_i}{dt} = g_i(x_1, x_2, \ldots, x_n, t) \qquad i = 1, 2, \ldots n \tag{13}$$

$$x_j(t_f) = x_j^1 \qquad j = 1, 2, \ldots m \tag{14}$$

$$x_k(t_i) = x_k^0 \qquad k = m + 1, m + 2, \ldots n \tag{15}$$

The general solution by the superposition principle is

$$x_i(t) = x_{ip}(t) + \sum_{k=1}^{n} A_k x_{i,kn}(t) \qquad i = 1, 2, \ldots n \tag{16}$$

In this general case, we have to assume n initial conditions, $x_{ip}(t_i) = x_{ip}^0$, for the particular solution, and n sets of initial conditions

$x_{i,kh}(t_i) = x^0_{i,kh}$, for n sets of homogeneous solutions. Integration constants A_k are determined from the n known boundary conditions and the assumed and computed boundary conditions for the particular and homogeneous solutions.

Usually n sets of homogeneous solutions are required to obtain the general solution. However, if the assumed initial values for the particular solution are properly selected, only m sets of homogeneous solutions need be obtained.

Consider the Equations (6) and (7), their general solution is given by Equations (11) and (12). Suppose the initial values for the particular solution are chosen as

$$x_{1p}(t_i) = x^0_1 \qquad\qquad x_{2p}(t_i) = 0$$

and the initial values for the homogeneous solutions are given as before, then Equations (11) and (12) at the initial time t_i, reduce to

$$\begin{aligned} x_1(t_i) &= x_{1p}(t_i) + A_1 x_{1,1h}(t_i) + A_2 x_{1,2h}(t_i) \\ x^0_1 &= x^0_1 \qquad + A_1 \quad 1 \qquad + A_2 \quad 0 \qquad (17) \\ A_1 &= x^0_1 - x^0_1 = 0. \end{aligned}$$

Since $A_1 = 0$, the first set of homogeneous solutions is not needed. This shows how to select the appropriate initial conditions so as to reduce the set of homogeneous solutions needed from n to m. For further discussion, the reader is referred to Lee [9].

The procedure can be divided into essentially two steps. First, the problem is converted into initial value problems and these problems are solved numerically. Then, the integration constants are obtained by solving a set of algebraic equations. Combination of these results is the general solution of the original problem.

CHAPTER 3

QUASILINEARIZATION

3.1 INTRODUCTION

The advantages of superposition principle in solving two point boundary value problem lead to the idea of linearizing the non-linear differential equations so that the superposition principle can be applied. This is the basic concept of quasilinearization.

Quasilinearization technique was developed by Bellman [1] and Kalaba [8] and applied extensively to chemical engineering problems by Lee [9 , 10, 11] in obtaining numerical solutions of certain classes of non-linear ordinary differential equations of the boundary value type encountered in chemical engineering, optimization, the boundary layer theory and in control problems.

This technique essentially linearizes the set of non-linear differential equations. Conceptually, this method is very close to Newton Raphson method of finding roots of an equation; however, since the unknowns to be determined in this method are functions and not fixed valued roots as in Newton Raphson method, both the computational and theoritical aspects are much more complicated.

In addition to linearizing the non-linear equations, the quasilinearization technique provides a sequence of functions which in general converges rather rapidly to the solution of the original non-linear equations. Usually, the latter is more important. A rough initial approximation for the unknown function can lead to the solution of the original equation through a sequence of functions. In general, for most practical problems, this rough initial approximation can be obtained from engineering experiences and intuitions.

3.2 COMPUTATIONAL PROCEDURE

In many operations research techniques, the verbal description of the algorithm is far more difficult than the algorithm itself. Hence the logic will be developed and explained with an illustration.

Consider a set of nonlinear differential equations

$$\frac{dx}{dt} = f(x,y)$$

$$t_i \leq t \leq t_f \qquad (18)$$

$$\frac{dy}{dt} = g(x,y)$$

with boundary values

$$x(t_i) = x^0 \qquad \text{and} \qquad y(t_f) = y^1 \qquad (19)$$

Using the Taylor series expansion f(x,y) and g(x,y) can be linearized around x = a and y = b as follows:

$$f(x,y) = f(a,b) + (x - a)\ f_a(a,t) + (y - b)\ f_b(b,t)$$

$$g(x,y) = g(a,b) + (x - a)\ g_a(a,t) + (y - b)\ g_b(b,t) \qquad (20)$$

which is the Taylor series with second and higher terms omitted. Symbol $f_a(a,t)$ represents the partial derivative of f with respect to x at x = a.

From Equations (18) and (20), we obtain

$$\frac{dx}{dt} = f(a,b) + (x - a)\ f_a(a,t) + (y - b)\ f_b(b,t)$$

$$\frac{dy}{dt} = g(a,b) + (x - a)\ g_a(a,t) + (y - b)\ g_b(b,t) \qquad (21)$$

Since a and b are known functions of t, Equations (21) are linear differential equations with variable coefficients. The boundary conditions for Equations (21) are given by Equations (19).

A recurrence relation can now be established. Choose an initial approximation for a and b, say $a = x_0$ and $b = y_0$. Substituting these approximations into Equations (21), it is possible to solve these first order linear differential equations for x and y using a step by step integration method and the boundary conditions given by Equation (19). Call this new solution of x and y as x_1 and y_1. Now using x_1 and y_1, it is possible to find improved values of x and y. Call these improved functions x_2 and y_2. Next using x_2 and y_2, x_3 and y_3 can be determined. This iterative procedure is continued until the desired accuracy is obtained.

The recurrence relation can be written as

$$\frac{dx_1}{dt} = f(x_o, y_o) + (x_1 - x_o)\ f_{x_o}(x_o, y_o) + (y_1 - y_o)\ f_{y_o}(x_o, y_o) \qquad (22)$$

$$\frac{dy_1}{dt} = g(x_o, y_o) + (x_1 - x_o)\ g_{x_o}(x_o, y_o) + (y_1 - y_o)\ g_{y_o}(x_o, y_o)$$

$$\frac{dx_2}{dt} = f(x_1, y_1) + (x_2 - x_1) f_{x_1}(x_1, y_1) + (y_2 - y_1) f_{y_1}(x_1, y_1) \qquad (23)$$

$$\frac{dy_2}{dt} = g(x_1, y_1) + (x_2 - x_1) g_{x_1}(x_1, y_1) + (y_2 - y_1) g_{y_1}(x_1, y_1)$$

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$$\frac{dx_{N+1}}{dt} = f(x_N, y_N) + (x_{N+1} - x_N) f_{x_N}(x_N, y_N) + (y_{N+1} - y_N) f_{y_N}(x_N, y_N) \qquad (24)$$

$$\frac{dy_{N+1}}{dt} = g(x_N, y_N) + (x_{N+1} - x_N) g_{x_N}(x_N, y_N) + (y_{N+1} - y_N) g_{y_N}(x_N, y_N)$$

The boundary conditions given in Equation (19) are used in solving Equations (22) through (24).

For a number of problems Equations (22) through (24) have been proved to converge monotomically to the solution of Equation (18). The convergence rate is quadratic in the sense that each iteration approximately doubles the number of digits of accuracy.

3.3 SUMMARY

The procedure can be summarized in the following steps.

1. The nth order non-linear ordinary differential equations are first converted into a system of simultaneous first order ordinary differential equations.

2. This set of equations is then linearized using Equation (21).
3. The recurrence relation for the set of linearized first order differential equations is constructed using Equations (22) through (24).
4. The appropriate initial approximation, $x_{i,o}(t)$, is assumed for each unknown dependent variable as a function of independent variable t.
5. The results of the first iteration, $x_{i,1}(t)$, can be obtained by substituting $x_{i,o}(t)$ into the recurrence relation and using the superposition principle.
6. A further improved solution is obtained by repeating Step 5. The procedure is continued until the solution converges to the desired accuracy.

3.4 DISCUSSION

The main advantage of this technique is that if the procedure converges, it converges quadratically to the solution of the original equation. Quadratic convergence means that the error in the (n+1)st iteration tends to be proportional to the square of the error in the nth iteration. All the computational features of Newton-Raphson technique are retained in this technique.

In spite of all the advantages, this technique also has its difficulties. There are two main difficulties. The first difficulty arises from the fact that in using the superposition principle, a set of algebraic equations must be solved. Thus the ill-conditioning phenomenon in solving a set of linear algebraic equations can make the superposition principle useless. Another difficulty is the convergence problem. If the initial approximation is not within the interval of convergence a solution cannot be obtained. For a detailed mathematical treatment of this topic, the reader is referred to Lee [9].

CHAPTER 4

APPLICATION TO AN ADVERTISEMENT PROBLEM

In this chapter, the computational aspects of this technique will be discussed with respect to its application to an inventory and advertisement model having two state variables and one control variable.

4.1 DEVELOPMENT OF THE MODEL

The diffusion model for advertisement was originally developed by Teichroew [14]. Consider a group of people in which only certain members possess a particular piece of information, say, about a manufacturing company's product. Suppose that the total number of persons in this group remain constant and that the diffusion of information occurs only through personal contact. The number of contacts made by an average informed person in an arbitrary unit of time is given by a contact coefficient. This coefficient is same for all members of the group. In a contact, the contactee receives information if he does not already have it; if he already has it, the contact is wasted in the sense that it did not increase the number of informed people.

Let $Q(0) = Q_o$ = number of informed people at time t_i.

N = total number of persons

c_c = contact coefficient; the number of contacts made by one informed person per unit time.

$Q(t)$ = number of informed persons at time t.

$Q(t)/N$ = proportion of informed persons at time t.

$1 - Q(t)/N$ = proportion of uninformed persons at time t.

$c_c\, Q(t)\, dt$ = contacts made during a time interval dt.

The increase in the total number of informed people during a short interval of time Δt is obtained by multiplying the number of contacts by the proportion of uninformed persons, because an increase in informed members is caused only by contacts with uninformed group. Hence,

$$dQ(t) = c_c \quad Q(t) \quad dt \quad (1 - Q(t)/N)$$

$$\frac{dQ(t)}{dt} = c_c \quad Q(t) \quad (1 - \frac{Q(t)}{N}) \tag{25}$$

Suppose now that the manufacturing company can influence the number of contacts by spending money for advertising. Specifically, it can increase the number of contacts made by the informed people by an additional number A per unit of time. Thus,

$$\frac{dQ(t)}{dt} = Q(t) \quad (c_c + A(t)) \quad [1 - \frac{Q(t)}{N}] \tag{26}$$

If each informed person buys c_q units of the company's product and if S(t) represents the sale at time t, then

$$S(t) = c_q \quad Q(t) \tag{27}$$

Let $c_q = 1$, and substitute for Q(t) in Equation (26),

$$\frac{dS(t)}{dt} = S(t) \quad (c_c + A(t)) \quad [1 - \frac{S(t)}{N}] \tag{28}$$

The rate of change of the company's inventory, I(t), is given by

$$\frac{dI(t)}{dt} = P(t) - S(t) \tag{29}$$

where P(t) = production rate at time t. The production rate is assumed to be a linear function given by

$$P(t) = a + bt$$

where a, b are constants and t is time.

This is a typical industrial management problem where the management wishes to maximize the profit given by Equation (30).

$$J = \int_{t_i}^{t_f} [c\ S(t) - c_I\ (I_m - I(t))^2 - c_A\ S(t)A^2(t)]dt \tag{30}$$

where J is the net total profit, c is the revenue from sale of one unit of the product, c_I is the inventory carrying cost, I_m can be considered as the capacity for the storage of inventory, and c_A is the cost of advertising.

The role of the management in this particular case is to select the optimal policy from among all feasible solutions which gives the maximum profit.

4.2 DEFINITION OF THE PROBLEM

Maximize

$$J = \int_{t_i}^{t_f} [c\ S(t) - c_I(I_m - I(t))^2 - c_A\ S(t)\ A^2(t)]dt \tag{30}$$

subject to

$$P(t) = a + bt \tag{31}$$

$$\frac{dI(t)}{dt} = P(t) - S(t) \tag{32}$$

$$\frac{dS(t)}{dt} = S(t)\ (c_c + A(t))\ [1 - \frac{S(t)}{N}] \tag{33}$$

with boundary conditions

$$I(t_1) = I^0 \qquad \text{and} \qquad S(t_1) = S^0 \tag{34}$$

4.3 FORMULATION OF THE PROBLEM

The above optimization problem can be solved by calculus of variations with the help of the quasilinearization technique. For detailed treatment of the calculus of variations, the reader is referred to Bliss [3] and Elsgolc [5].

Equations (31) through (34) can be rewritten as

$$\frac{dI(t)}{dt} - (a + bt) + s(t) = 0 \tag{35}$$

$$\frac{dS(t)}{dt} - S(t)\ (c_c + A(t))[1 - \frac{S(t)}{N}] = 0 \tag{36}$$

We have two state variables, I and S, and one control variable A. Introduce Lagrange multipliers λ_1, λ_2 and constant multipliers θ_1, θ_2 and define the following functions.

$$F = [\lambda_I(\dot{I} - (a + bt) + S) + \lambda_2(\dot{S} - S(c_c + A - \frac{Sc_c}{N} - \frac{SA}{N}))$$

$$+ cS - c_I(I_m - I)^2 - c_A\, S\, A^2] \tag{38}$$

and

$$G = [\theta_1(I(0) - I^0) + \theta_2(S(0) - S^0)] \tag{39}$$

where the notation $\dot{I}$ represents the first differential $\frac{dI}{dt}$. The Euler-Lagrange equations [11]

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right) - \frac{\partial F}{\partial y_i} = 0 \tag{40}$$

and

$$\frac{\partial F}{\partial A} = 0 \tag{41}$$

can now be applied to Equation (38) to obtain relations for the Lagrange multipliers λ_1, λ_2.

$$\frac{d\lambda_1}{dt} = 2(I(I_m - I) \tag{42}$$

$$\frac{d\lambda_2}{dt} = \lambda_1 + c - c_A\, A^2 - c_c\, \lambda_2 - \lambda_2\, A + \frac{2c_c S\lambda_2}{N} + \frac{2AS\lambda_2}{N} \tag{43}$$

We need boundary conditions for the Lagrange multipliers λ_1, λ_2. They were obtained by applying the transversality condition [11].

$$\left.\frac{\partial G}{\partial y_i}\right|_{t_i} - \left.\frac{\partial F}{\partial \dot{y}_i}\right|_{t_i} = 0 \qquad \text{or} \qquad \left.\frac{\partial G}{\partial y_i}\right|_{t_f} - \left.\frac{\partial F}{\partial \dot{y}_i}\right|_{t_f} = 0$$

Applying this condition to Equations (38) and (39), we obtain

$$\begin{aligned} 0 - \lambda_1(t_f) &= 0 \qquad & \lambda_1(t_f) &= 0 \\ 0 - \lambda_2(t_f) &= 0 \qquad & \lambda_2(t_f) &= 0 \end{aligned} \tag{44}$$

Now we have four differential equations with two initial and two final conditions, which make the problem, a two point boundary value type. Applying condition (41) to Equation (38), we obtain

$$A = \frac{\lambda_2}{2c_A}\left(\frac{S}{N} - 1\right) \tag{45}$$

Since it is possible to express the control variable explicity in terms of the state variables, let us eliminate A from all the performance equations.

$$\frac{dI}{dt} = a + bt - S \tag{46}$$

$$\frac{dS}{dt} = c_c S - \frac{c_c S^2}{N} + \frac{S^2\lambda_2}{c_A N} - \frac{S\lambda_2}{2c_A} - \frac{S^3\lambda_2}{2c_A N^2} \tag{47}$$

$$\frac{d\lambda_I}{dt} = 2c_I(I_m - I) \tag{48}$$

$$\frac{d\lambda_2}{dt} = \lambda_1 + c + \frac{3S^2\lambda_2^2}{4c_A N^2} + \frac{\lambda_2^2}{4c_A} - c_c\lambda_2 + \frac{2c_c S\lambda_2}{N} \tag{49}$$

The boundary conditions are given by Equations (34) and (44).

4.4 QUASILINEARIZATION

Observe that Equations (47) and (49) are non-linear. They should be linearized. The linearization procedure is thc same as described in Chapter 3. Referring to Equation (21), we need the expressions for f_a, f_b, g_a, g_b. In other words, we require

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial \lambda_2} \\ \frac{\partial g_2}{\partial S} & \frac{\partial g_2}{\partial \lambda_2} \end{bmatrix}$$

This matrix can be obtained from Equations (47) and (49).

$$J = \begin{bmatrix} c_c - \frac{2c_c S}{N} + \frac{2S\lambda_2}{c_A N} - \frac{\lambda_2}{2c_A} - \frac{3S^2\lambda_2}{2c_A N^2} \; ; & \frac{S^2}{c_A N} - \frac{S}{2c_A} - \frac{S^3}{2c_A N^2} \\ \frac{3S\lambda_2^2}{2c_A N^2} - \frac{\lambda_2^2}{c_A N} + \frac{2c_c\lambda_2}{N} & ; \; - c_c + \frac{3S^2\lambda_2}{2c_A N^2} + \frac{\lambda_2}{2c_A} - \frac{2S\lambda_2}{c_A N} + \frac{2c_c S}{N} \end{bmatrix}$$

The linearized equations and the recurrance relations can be developed in accordance with Equations (21) through (24).

$$\frac{dI_{n+1}}{dt} = a + bt - s_{n+1} \tag{50}$$

$$\frac{ds_{n+1}}{dt} = [c_c s_n - \frac{c_c s_n^2}{N} + \frac{s_n^2 \lambda_{2,n}}{c_A N} - \frac{s_n \lambda_{2,n}}{2c_A} - \frac{s_n^3 \lambda_{2,n}}{2c_A N^2}]$$

$$+ (s_{n+1} - s_n)[c_c - \frac{2c_c s_n}{N} + \frac{2s_n \lambda_{2,n}}{c_A N} - \frac{\lambda_{2,n}}{2c_A} - \frac{3s_n^2 \lambda_{2,n}}{2c_A N^2}]$$

$$+ (\lambda_{2,n+1} - \lambda_{2,n})[\frac{s_n^2}{c_A N} - \frac{s_n}{2c_A} - \frac{s_n^3}{2c_A N^2}] \tag{51}$$

$$\frac{d\lambda_{1,n+1}}{dt} = 2c_I I_m - 2C_I I_{n+1} \tag{52}$$

$$\frac{d\lambda_{2,n+1}}{dt} = [\lambda_{1,n+1} + c - c_c \lambda_{2,n} + \frac{3s_n^2 \lambda_{2,n}^2}{4c_A N^2} + \frac{\lambda_{2,n}^2}{4c_A} - \frac{s_n \lambda_{2,n}^2}{c_A N} + \frac{2c_c s_n \lambda_{2,n}}{N}]$$

$$+ (s_{n+1} - s_n)[\frac{3s_n \lambda_{2,n}^2}{2c_A N^2} - \frac{\lambda_{2,n}^2}{c_A N} + \frac{2c_c \lambda_{2,n}}{N}]$$

$$+ (\lambda_{2,n+1} - \lambda_{2,n})[\frac{\lambda_{2,n}}{2c_A} - c_c + \frac{3s_n^2 \lambda_{2,n}}{2c_A N^2} - \frac{2s_n \lambda_{2,n}}{c_A N} + \frac{2c_c s_n}{N}] \tag{53}$$

The boundary conditions are given by equations (34) and (44).

Equations (50) through (53) are ordinary linear differential equations and with the boundary conditions given by equations (34) and (44), they form a two point boundary value problem. This problem can now be solved by superposition principle. Selecting the initial conditions, for the particular and homogeneous solutions such that they satisfy the given initial conditions, the general solution can be written as

$$I(t) = I_p(t) + A_1 I_{1h}(t) + A_2 I_{2h}(t)$$

$$S(t) = S_p(t) + A_1 S_{1h}(t) + A_2 S_{2h}(t)$$

$$\lambda_1(t) = \lambda_{1,p}(t) + A_1 \lambda_{1,1h}(t) + A_2 \lambda_{1,2h}(t)$$

$$\lambda_2(t) = \lambda_{2,p}(t) + A_1 \lambda_{2,1h}(t) + A_2 \lambda_{2,2h}(t) \qquad (54)$$

These equations are derived in accordance with Equations (11) and (12).

After obtaining the final solution with the superposition principle, Equations (30) and (45) can he solved for the profit and advertisement, respectively. This completes one iteration. Further iteration was allowed until desired accuracy was obtained.

4.5 NUMERICAL ASPECTS

In order to solve this problem, the constants were assumed to have the following values.

$a = 70$	$c_A = 1.5$	$I(0) = I^0 = 20$
$b = 100$	$C_I = 0.15$	$S(0) = S^0 = 20$
$c_c = 2$	$N = 150$	$t_i = 0 \quad t_f = 1.0$
$c = 10$	$I_m = 50$	$\Delta t = 0.01$

As discussed before, we need initial approximations to start the solution. Since only two Equations, (51) and (53), were non-linear, we needed the initial approximations for S and λ_2 only. These values were obtained from intuition and knowledge about the system. Various aets of initial approximations used in this problem are listed in Table 1.

Solution of this problem by the superposition principle requires a aet of particular solutions and four sets of homogeneous solutions. However, as discussed previously, if the initial values for the particular and homogeneous solutions are chosen such that they aatisfy the given initial conditions, only two sets of homogeneous solutions are needed. The following set of values at the initial time satisfy this condition and hence they were used as the initial values for the particular solution.

$$I_p(0) = 20 \qquad \lambda_{1,p}(0) = 0.0$$
$$S_p(0) = 20 \qquad \lambda_{2,p}(0) = 0.0 \tag{56}$$

The initial values for the two sets of homogeneous solutions were assumed as

	$I_{ih}(0)$	$S_{ih}(0)$	$\lambda_{1,ih}(0)$	$\lambda_{2,ih}(0)$	
Set 1	0	0	1	0	
Set 2	0	0	0	1	(57)

4.6 COMPUTATIONAL ASPECTS

Using the initial values given by Equation (56), the set of linear differential Equations (50) through (53) were solved using the Runge-Kutta method to obtain the particular solution.

For homogeneous solutions, the known terms in Equations (50) through (53) were set to zero and the modified equations were solved by Runge-

Table 1. List of initial approximations.

Set No.	$S_0(t)$	$\lambda_{2,0}(t)$
1	450.0	20.0
2	350.0	15.0
3	300.0	12.5
4	300.0	10.0
5	275.0	10.0
6	250.0	10.0
7	200.0	7.0
8	150.0	2.0
9	100.0	1.5
10	50.0	- 1.0
11	28.0	- 0.25
12	25.0	- 0.50
13	22.0	0.125
14	20.0	0.0
15	5.0	- 3.0

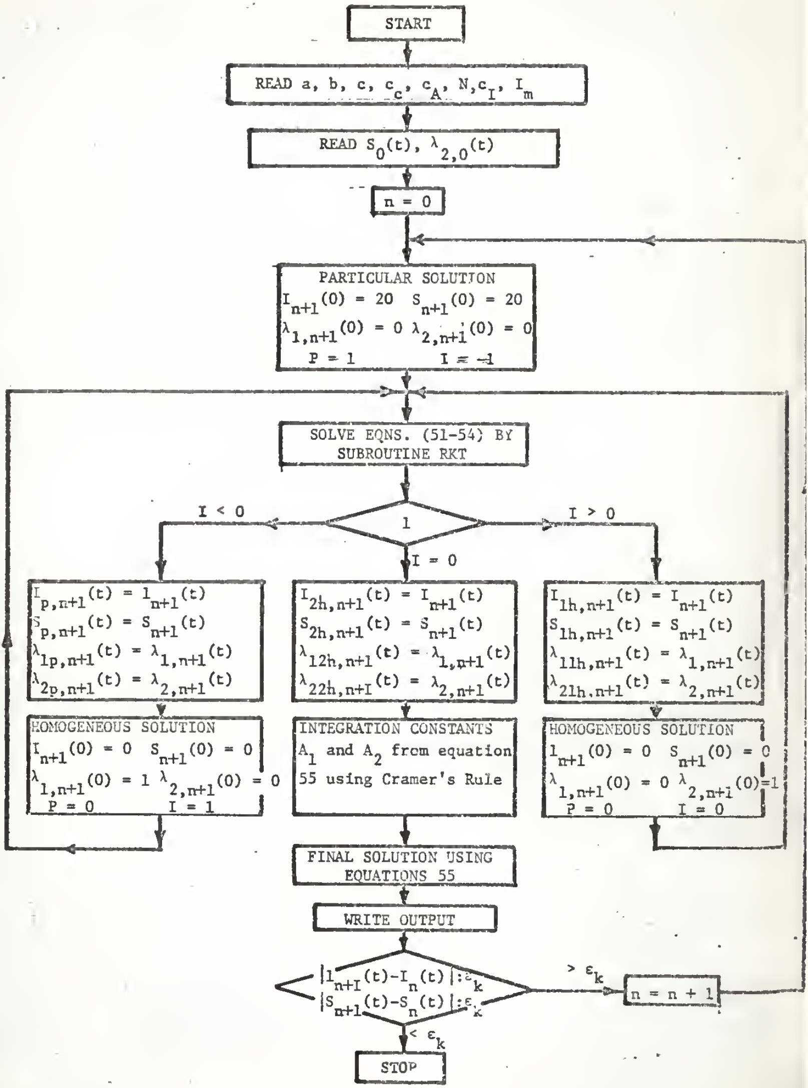

Fig. 2. Computer Logic Diagram for an Advertisement Problem.

Kutta method using the initial conditions given by Equation (57). These were the first and the second set of homogeneous solutions. Last two Equations in (54) at final time t = 1 were used to solve for the two integration constants, A_1 and A_2. The solution was obtained by Cramer's rule. Next, the general solutions for the two state variables and two Lagrange multipliers were obtained by using the superposition principle, Equation (54).

The control variable A and objective function, J, were obtained next using Equations (45) and (30) respectively. For simplicity the following approximation was used to calculate the total profit.

$$J = \sum_{t_i}^{t_f} [cS(t) - c_I(I_m - I(t))^2 - c_A S(t) A^2(t)]\Delta t$$

In general, 9 iterations were allowed.

The IBM 360/50 computer system was used for all these computations. Computer logic diagram is shown in Fig. 2. The computer program is given in Appendix 2.

4.7 RESULTS

The optimal profit in this program was J = 587.80 and the optimal initial and final values are

I(0) = 20	s(0) = 20	A(0) = 3.98
I(1) = 66.15	s(1) = 115.69	A(1) = 0

Out of the 15 different sets of initial approximations listed in Table 1, the first five sets did not produce convergence.

In set 1, the particular and homogeneous solutions of all the four variables at the final time, t_f, involve terms of the order 10^{40}. As a result, the calculation of integration constants by Cramer's rule involves terms of the order 10^{80}, which cannot be handled by IBM 360 computer and exponential overflow was resulted.

Set 2 encountered a similar problem. The lowest term in the particular and homogeneous solutions at final time t_f, was of the order 10^{13}. This did not cause any difficulty in the calculation of integration constants, but resulted in exponential overflow in the computation of final solution of first iteration.

Sets 3, 4, and 5 encountered basically the same problem. For explanation, results of set 4 are used here. In the final solution of first iteration, the following results were obtained.

$$A(1) = -0.187 \times 10^{14} \qquad I(1) = 0.706 \times 10^{7}$$

$$J = 0.118 \times 10^{34} \qquad s(1) = -0.141 \times 10^{9}$$

as a result of such large numbers, the computer experienced exponential overflow and stopped computation while calculating the particular solution of the second iteration.

Sets 6 through 15 converged to the same optimal solution in about 4-5 iterations. The convergence rates of sales, inventory, and advertisement are shown in Figs. 3 through 11. The initial approximations used are sets

10, 12, and 14 in Table 1. Fig. 12 shows the convergence rate of the profit function with set 14 of Table 1 as the initial approximation. The convergence rates of the initial and final values of the variables are tabularized in Tables 2 through 5. The IBM 360/50 computer took about 3.72 minutes to complete 9 iterations for this problem with WATFOR compiler.

4.8 DISCUSSION

The results show that this problem converged with ten different and far from optimal initial approximations. The optimal curves show that the profiles of the state and control variables were either monotonically decreasing or monotonically increasing. This made quasilinearization method more effective.

It was observed from Tables 2-5 that convergence was obtained in 4-5 iterations for all sets of initial approximations that converged. It was concluded that

1. The quasilinearization method converges quadratically, whenever it converges.

and 2. The convergence rate is almost independent of the choice of initial approximation, if the latter values are within the convergence interval or range.

A note on the choice of initial approximation is in order. In this problem, the optimal solution of sales is between 20.0 and 115.69. But any initial approximation of sales between 5.0 and 250.0 would converge to the optimal solution. Hence, choosing this value should not be a problem. The author feels that, in general, the basic knowledge of physical

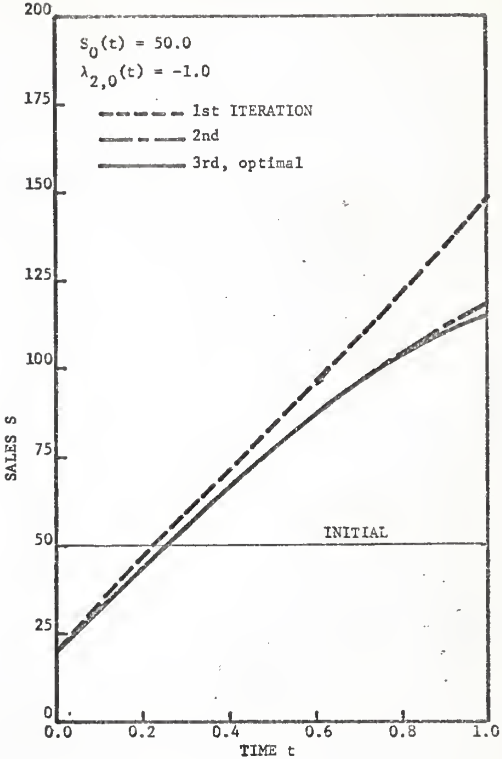

Fig. 3. Convergence Rate of Sales in an Advertisement Problem

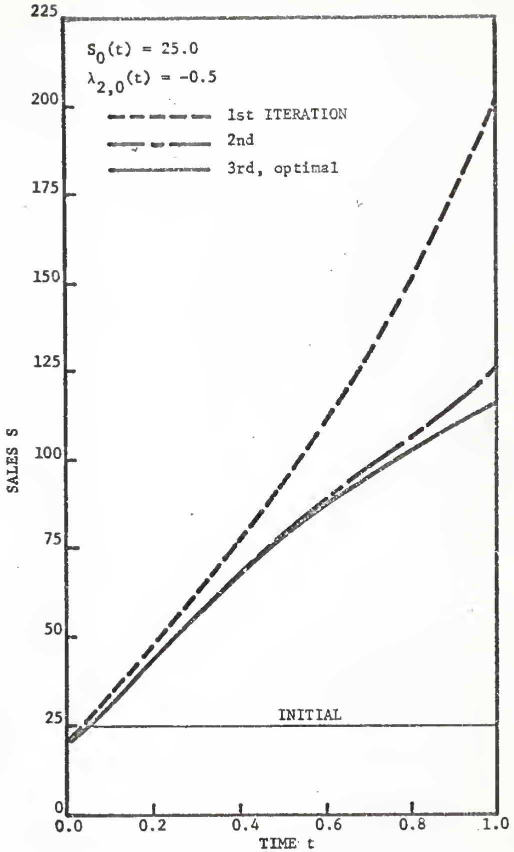

Fig. 4. Convergence Rate of Sales in an Advertisement Problem.

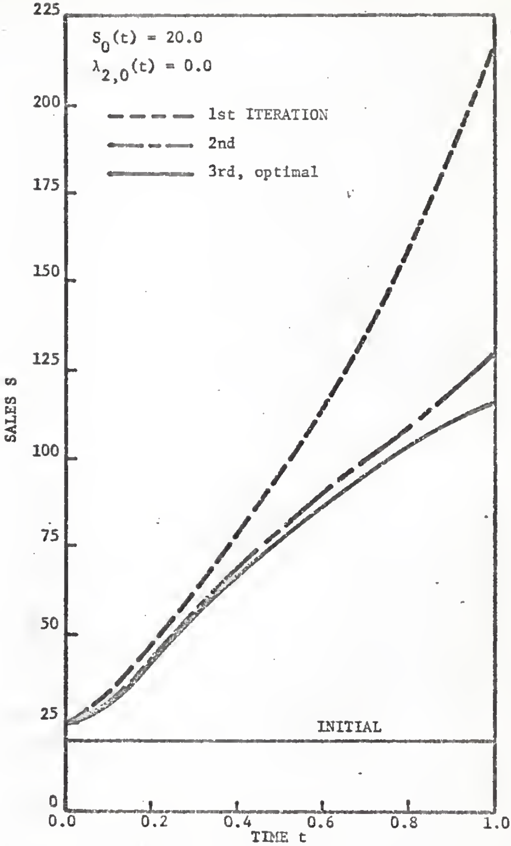

Fig. 5. Convergence Rate of Sales in an Advertisement Problem.

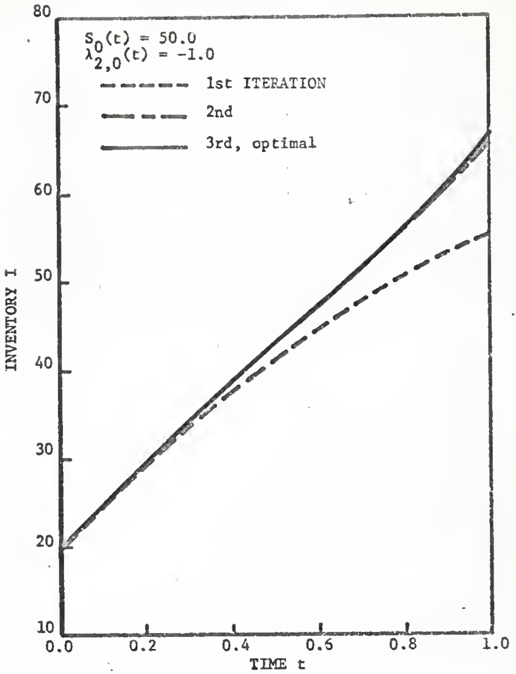

Fig. 6. Convergence Rate of Inventory in an Advertisement Problem.

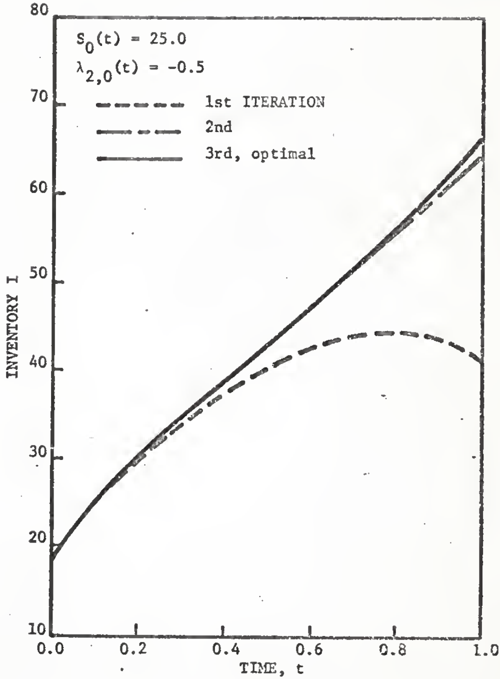

Fig. 7. Convergence Rate of Inventory in an Advertisement Problem.

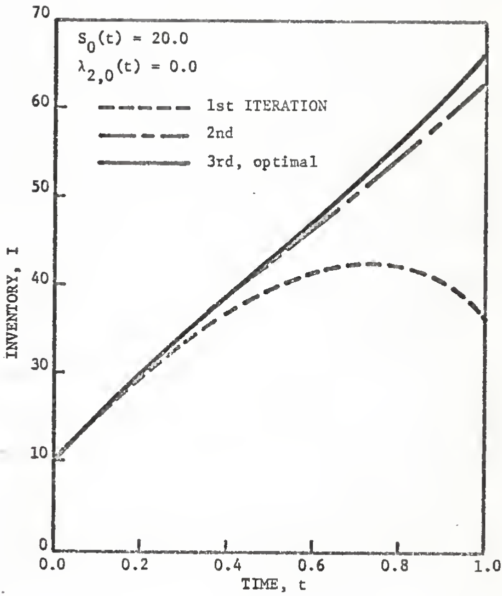

Fig. 8. Convergence Rate of Inventory in an Advertisement Problem.

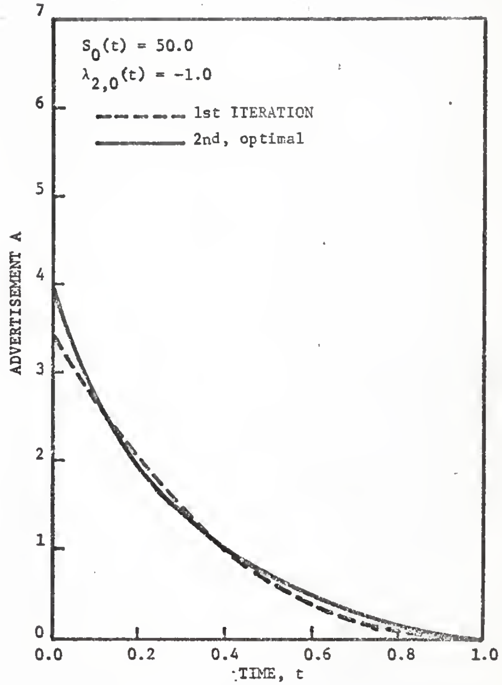

Fig. 9. Convergence Rate of Advertisement in an Advertisement Problem.

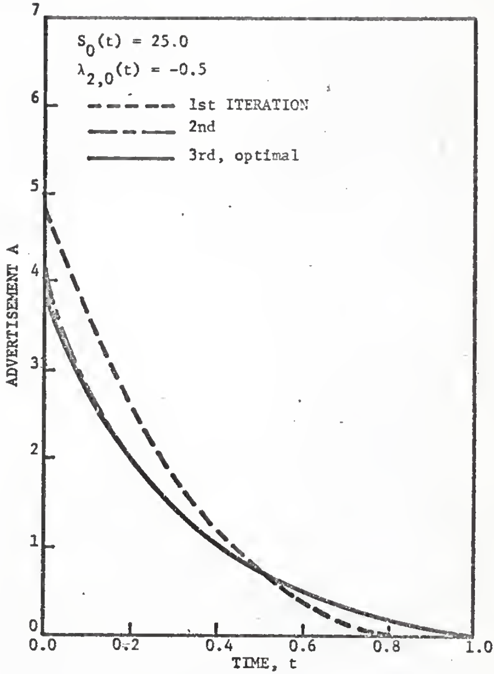

Fig. 10. Convergence Rate of Advertisement in an Advertisement Problem.

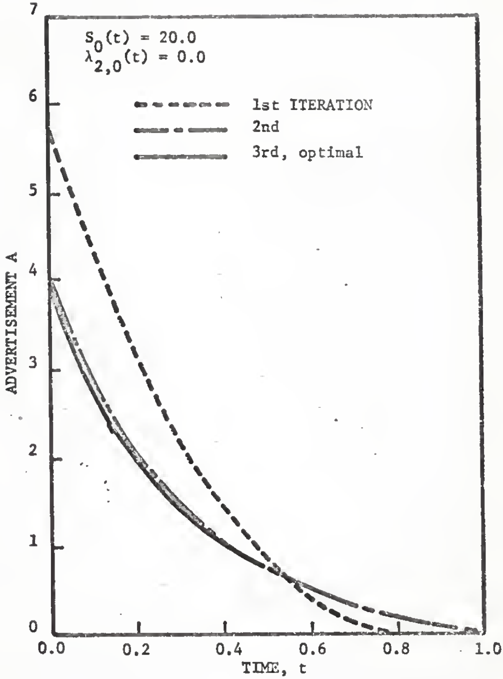

Fig. 11. Convergence Rate of Advertisement in an Advertisement Problem.

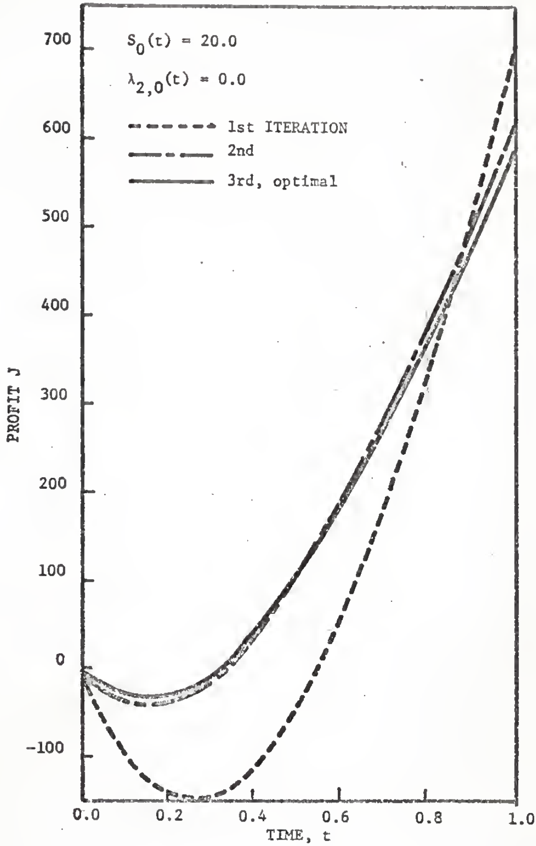

Fig. 12. Convergence Rate of Profit Function in an Advertisement Problem

Table 2. Convergence rate of $S(t_f)$

$S_0(t)$	250.0	200.0	150.0	100.0	50.0	28.0	25.0	22.0	20.0	5.0
$\lambda_{2,0}(t)$	10.0	7.0	2.0	1.5	-1.0	-0.25	-0.5	0.125	0.0	-3.0
Iteration										
0	250.00	200.00	150.00	100.00	50.0	28.00	25.00	22.00	20.00	5.00
1	231.00	174.49	132.40	115.54	148.14	199.76	203.01	214.09	216.39	209.27
2	166.20	135.32	116.46	115.65	117.40	125.93	125.88	128.97	129.04	120.58
3	130.27	116.18	115.70	115.69	115.69	115.80	115.78	115.91	115.88	115.68
4	117.39	115.69	115.69	115.69	115.69	115.69	115.69	115.69	115.69	115.69
5	115.69	115.69	115.69			115.69	115.69	115.69	115.69	115.69
6	115.69									

Table 3. Convergence rate of $I(t_f)$

$S_0(t)$	250.0	200.0	150.0	100.0	50.0	28.0	25.0	22.0	20.0	5.0
$\lambda_{2,0}(t)$	10.0	7.0	2.0	1.5	-1.0	-0.25	-0.5	0.125	0.0	-3.0
Iteration										
1	-54.56	-49.72	46.20	66.04	55.61	40.34	41.29	36.26	36.86	65.68
2	12.54	60.68	66.67	66.21	65.85	63.85	64.32	62.77	63.05	70.04
3	67.84	66.63	66.14	66.15	66.15	66.14	66.15	66.12	66.13	66.29
4	64.83	66.15	66.15	66.15	66.15	66.15	66.15	66.15	66.15	66.15
5	66.15	66.15	66.15			66.15		66.15	66.15	66.15
6	66.15									

Table 4. Convergence rate of $A(t_1)$

$s_0(t)$	250.0	200.0	150.0	100.0	50.0	28.0	25.0	22.0	20.0	5.0
$\lambda_{2,0}(t)$	10.0	7.0	2.0	1.5	-1.0	-0.25	-0.5	-0.125	0.0	-3.0
Iteration										
1	-3.97	-2.64	3.39	2.01	3.43	4.86	4.85	5.69	5.79	-4.65
2	-3.21	3.12	3.16	3.90	3.97	4.07	4.01	4.13	4.09	2.06
3	1.13	3.89	3.98	3.98	3.98	3.98	3.98	3.98	3.98	3.88
4	4.14	3.97	3.98	3.98	3.98	3.98	3.98	3.98	3.98	3.98
5	3.97	3.98								3.98
6	3.98	3.98								
7	3.98									

Table 5. Convergence rate of J.

$S_0(t)$	250.0	200.0	150.0	100.0	50.0	28.0	25.0	22.0	20.0	5.0
$\lambda_{2,0}(t)$	10.0	7.0	2.0	1.5	-1.0	-0.25	-0.5	0.125	0.0	-3.0
Iteration										
1	998.33	1154.12	890.67	638.38	694.25	752.86	752.86	711.31	703.49	538.95
2	1106.76	769.32	611.64	588.37	591.53	608.15	606.02	613.53	612.26	551.07
3	674.99	578.17	587.59	587.80	587.80	587.93	587.87	588.14	588.06	590.66
4	489.78	588.00	587.80	587.80	587.80	587.80	587.80	587.80	587.80	587.80
5	587.92	587.80	587.80			587.80	587.80	587.80	587.80	587.80
6	587.80	587.80								
7	587.80									

behavior of the system is enough to make correct choice. Furthermore, many numerical schemes have been devised to overcome the convergence problem. One such scheme is the data perturbation technique [4].

In order to further investigate the convergence and other computational aspects of this problem, the following constants were used.

$$\begin{array}{lll} a = 0.7 & c_A = 1.0 & I(0) = I^0 = 0.2 \\ b = 1.0 & c_I = 0.15 & S(0) = S^0 = 0.2 \\ c_c = 2.0 & N = 1.5 & t_i = 0.0 \quad t_f = 1.0 \\ c = 10.0 & I_m = 1.0 & \Delta t = 0.01 \end{array} \tag{58}$$

The initial approximations used were

$$S_0(t) = 0.2 \qquad \lambda_{2,0}(t) = 0.0$$

The initial values for the one particular and two homogeneous solutions were

	$I(0)$	$S(0)$	$\lambda_1(0)$	$\lambda_2(0)$
Particular soln.	0.2	0.2	0.0	0.0
Homo. soln. set 1	0.0	0.0	1.0	0.0
Homo. soln. set 2	0.0	0.0	0.0	1.0

The convergence rates are shown in Table 6. The problem converged in 4 iterations. The optimal profiles of I, S, and A are shown in Fig. 13.

Table 6. Convergence rates for the modified problem, Equation (58)

Iteration Number	Time t	I(t)	S(t)	$\lambda_1(t)$	$\lambda_2(t)$	A(t)	J
0	0.0		0.20		0.0		
	0.25		0.20		0.0		
	0.50		0.20		0.0		
	0.75		0.20		0.0		
	1.00		0.20		0.0		
1	0.0	0.200	0.200	-0.225	-0.225	9.763	
	0.25	0.292	0.714	-0.169	-13.563	3.553	
	0.50	0.313	1.284	-0.118	-7.337	0.529	
	0.75	0.240	1.985	-0.064	-3.010	-0.487	
	1.00	0.027	2.915	-0.0	0.0	0.0	8.631
2	0.0	0.200	0.200	-0.187	-13.903	6.025	
	0.25	0.308	0.592	-0.131	-6.851	2.072	
	0.50	0.383	0.945	-0.082	-3.526	0.653	
	0.75	0.448	1.146	-0.038	-1.985	0.234	
	1.00	0.532	1.433	0.0	0.0	0.0	7.070
3	0.0	0.200	0.200	-0.180	-12.369	5.360	
	0.25	0.313	0.552	-0.124	-6.308	1.994	
	0.50	0.403	0.861	-0.076	-3.994	0.851	
	0.75	0.490	1.078	-0.034	-2.173	0.306	
	1.00	0.595	1.223	0.0	0.0	0.0	6.682
4	0.0	0.200	0.200	-0.180	-12.411	5.378	
	0.25	0.313	0.552	-0.124	-6.333	2.001	
	0.50	0.403	0.860	-0.076	-4.015	0.856	
	0.75	0.490	1.077	-0.034	-2.175	0.307	
	1.00	0.596	1.219	0.0	0.0	0.0	6.669
5	0.0	0.200	0.200	-0.180	-12.410	5.378	
	0.25	0.313	0.552	-0.124	-6.332	2.001	
	0.50	0.403	0.860	-0.076	-4.015	0.856	
	0.75	0.490	1.077	-0.034	-2.175	0.307	
	1.00	0.596	1.219	0.0	0.0	0.0	6.669

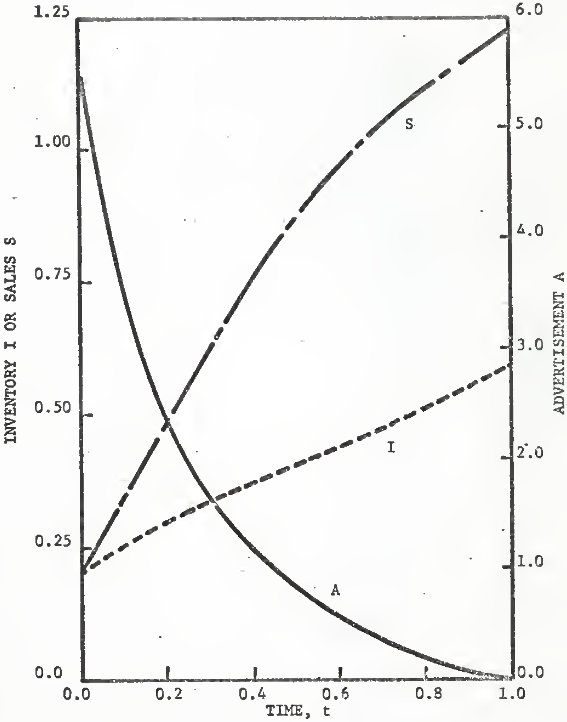

Fig. 13. Optimal Profiles of I(t), S(t) and A(t) for modified Problem, Eqn. (59).

Most important advantage of this technique is that the control variable can be eliminated from the performance equations. Hence, initial guess for the control variable is not required in solving for the state variables. For this reason, this technique is superior to the gradient technique because any error in guessing the initial control can result in failure to obtain the solution.

It may be interesting to note that the results of this problem, using the initial values listed in Equation (55), compare favorably with the results obtained by the first variational gradient technique [15]. The modified problem with numerical values given by Equation (58) has also been solved by the second variation technique [13]. The present results again compare favorably with this result.

It will be observed in the next chapter, that even if the control variable cannot be canceled from the performance equations, quasilinearization method still works well.

CHAPTER 5

APPLICATION TO AN ADVERTISEMENT AND PRODUCTION PROBLEM

We now wish to apply the quasilinearization technique to a more complex problem, namely an advertisement and production problem. This problem has six state variables and three control variables. In addition, the profiles are fairly unstable due to the rapid change of the variables with time.

5.1 DEVELOPMENT OF THE MODEL

Consider the manufacturing process shown in Fig. 14. There are two chemical reactors in which the following consecutive reactions take place

$$A \longrightarrow B \longrightarrow C$$

Both these reactions are first order. The component B is the desired product and C is the waste product. Suppose B is a new product which needs advertisement to boost the sales. Furthermore, to protect against fluctuations in demand, an inventory will be assumed for B. A and C are assumed to have unlimited market at fixed price and they are sold as soon as manufactured.

Let x_i and y_i, $i = 1, 2$, represent the concentration of A and B respectively. Under steady state conditions, from material balance, we have

$$\begin{matrix}\text{amount of} \\ \text{A in}\end{matrix} = \begin{matrix}\text{amount of} \\ \text{A out}\end{matrix} + \begin{matrix}\text{amount of A} \\ \text{transformed to B}\end{matrix} \tag{59}$$

$$\begin{matrix}\text{amount of} \\ \text{B in}\end{matrix} = \begin{matrix}\text{amount of} \\ \text{B out}\end{matrix} + \begin{matrix}\text{amount of B} \\ \text{transformed to C}\end{matrix} - \begin{matrix}\text{amount of B} \\ \text{produced from A}\end{matrix} \tag{60}$$

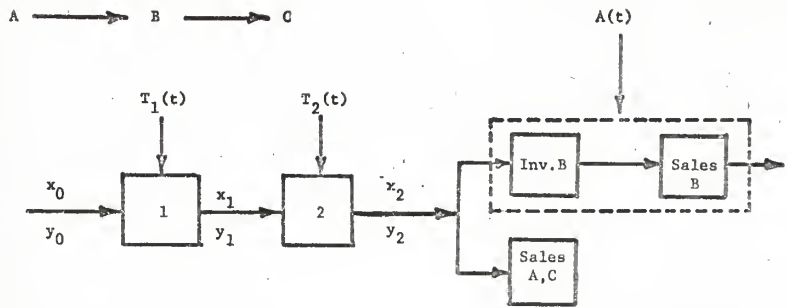

Fig. 14. Advertisement and Production Model

Let

v_i = volume of chemical reactor i, i = 1, 2.

q = flow rate

k_{ai} = reaction rate constant of the first reaction in reactor i

k_{bi} = reaction rate constant of the second reaction in reactor i

G_a, G_b = frequency constants of the first and second reactions, respectively

E_a, E_b = activation energies of the first and second reactions, respectively

R = gas constant

T_i = temperature in reactor i

The kinetics of the reactions can now be written as

$$qx_0 = qx_1 + v_1 k_{a1} x_1$$

or

$$q(x_0 - x_1) - v_1 k_{a1} x_1 = 0$$

at steady state. Under unsteady state conditions we have

$$v_1 \frac{dx_1}{dt} = q(x_0 - x_1) - v_1 k_{a1} x_1 \tag{61}$$

similarly, eqn. (60) can be rewritten as

$$qy_0 = qy_1 + v_1 k_{b1} y_1 - v_1 k_{a1} x_1$$

or

$$q(y_0 - y_1) - v_1 k_{b1} y_1 + v_1 k_{a1} x_1 = 0$$

at steady state. Under steady state conditions, we have

$$v_1 \frac{dy_1}{dt} = q(y_0 - y_1) - v_1 k_{b1} y_1 + v_1 k_{a1} x_1 \tag{62}$$

With similar arguments, the kinetics of the reactions in the second reactor can be written as

$$v_2 \frac{dx_2}{dt} = q(x_1 - x_2) - v_2 k_{a2} x_2 \tag{63}$$

and

$$v_2 \frac{dy_2}{dt} = q(y_1 - y_2) - v_2 k_{b2} y_2 + v_2 k_{a2} x_2 \tag{64}$$

The reaction rate constants are defined as

$$k_{a1} = G_a \exp(-\frac{E_a}{RT_1}), \qquad k_{b1} = G_b \exp(-\frac{E_b}{RT_1})$$

$$k_{a2} = G_a \exp(-\frac{E_a}{RT_2}), \qquad k_{b2} = G_b \exp(-\frac{E_b}{RT_2}) \tag{65}$$

As indicated before B is the desired product and it needs inventory and advertisement. The performance equation for the inventory is

$$\begin{matrix}\text{rate of change} \\ \text{of inventory}\end{matrix} = \begin{matrix}\text{production} \\ \text{rate}\end{matrix} - \begin{matrix}\text{sales} \\ \text{rate}\end{matrix}$$

$$\frac{dI}{dt} = qy_2 - S \tag{66}$$

where I is the inventory and S is sales.

The performance equation for advertisement is the same as Equation (26) in the last chapter. Again, let $c_q = 1$. According to Equation (27), we have

$$S(t) = c_q\, Q(t) = Q(t)$$

Thus,

$$\frac{dS}{dt} = S(c_c + A)[1 - \frac{S}{N}] \qquad (67)$$

where c_c is the contact coefficient. A is the advertisement and N represents the total number of people in the group.

Equations (61), (62), (63), (64), (66), and (67) describe the system completely. We have six state variables, x_1, y_1, x_2, y_2, I, and S, and three control variables, T_1, T_2, and A.

The management in this particular industrial system is confronted with the problem of selecting three control variables such that the following profit function, J, is maximized.

$$\text{profit} = \int_{t_i}^{t_f} [\text{revenue of B} + \text{revenue of A} + \text{revenue of C} - \begin{matrix}\text{inventory}\\ \text{cost}\end{matrix} - \begin{matrix}\text{advertisement}\\ \text{cost}\end{matrix} - \begin{matrix}\text{manufacturing}\\ \text{cost}\end{matrix}]\, dt$$

Mathematically,

$$J = \int_{t_i}^{t_f} [c_1 c_q S + c_2 q x_2 + c_3 q(1 - x_2 - y_2) - c_I (I_m - I)^2$$

$$- c_A A^2 S^2 - c_T\{(T_{1m} - T_1)^2 + (T_1 - T_2)^2\}]dt \tag{68}$$

5.2 DEFINITION OF THE PROBLEM

Maximize the functional

$$J = \int_{t_i}^{t_f} [c_1 c_q S + c_2 q x_2 + c_3 q(1 - x_2 - y_2) - c_I (I_m - I)^2$$

$$- c_A A^2 S^2 - c_T\{(T_{1m} - T_1)^2 + (T_1 - T_2)^2\}]dt \tag{69}$$

subject to the constraints of

$$v_1 \frac{dx_1}{dt} = q(x_0 - x_1) - v_1 k_{a1} x_1 \tag{70}$$

$$v_1 \frac{dy_1}{dt} = q(y_0 - y_1) - v_1 k_{b1} y_1 + v_1 k_{a1} x_1 \tag{71}$$

$$v_2 \frac{dx_2}{dt} = q(x_1 - x_2) - v_2 k_{a2} x_2 \tag{72}$$

$$v_2 \frac{dy_2}{dt} = q(y_1 - y_2) - v_2 k_{b2} y_2 + v_2 k_{a2} x_2 \tag{73}$$

$$\frac{dI}{dt} = q y_2 - S \tag{74}$$

$$\frac{dS}{dt} = S(c_c + A)[1 - \frac{S}{N}] \tag{75}$$

with boundary conditions

$$x_1(t_i) = x_1^0 \qquad y_2(t_i) = y_2^0$$

$$y_1(t_i) = y_1^0 \qquad I(t_i) = I^0 \qquad I(t_f) = I^1$$

$$x_2(t_i) = x_2^0 \qquad S(t_i) = S^0 \tag{76}$$

5.3 FORMULATION OF THE PROBLEM

It was required to find the optimal value of the state variables and control variables so that the objective function is maximized. This problem can be solved by calculus of variations. The procedure for obtaining the solution remains essentially the same.

Equations (70) through (76) can be rewritten as

$$\dot{x}_1 - \frac{q}{v_1}(x_0 - x_1) + G_a e^{-\frac{E_a}{RT_1}} x_1 = 0 \tag{77}$$

$$\dot{y}_1 - \frac{q}{v_1}(y_0 - y_1) + G_b e^{-\frac{E_b}{RT_1}} y_1 - G_a e^{-\frac{E_a}{RT_1}} x_1 = 0 \tag{78}$$

$$\dot{x}_2 - \frac{q}{v_2}(x_1 - x_2) + G_a e^{-\frac{E_a}{RT_2}} x_2 = 0 \tag{79}$$

$$\dot{y}_2 - \frac{q}{v_2}(y_1 - y_2) + G_b e^{-\frac{E_b}{RT_2}} y_2 - G_a e^{-\frac{E_a}{RT_2}} x_2 = 0 \tag{80}$$

$$\dot{I} - qy_2 + S = 0 \tag{81}$$

$$\dot{S} - (c_c S + AS)[1 - \frac{S}{N}] = 0 \tag{82}$$

The symbol $\dot{x}_1$ represents $\frac{dx_1}{dt}$.

Introduce lagrange multipliers, λ_i, i = 1, . . ., 6, and constant multipliers θ_j, j = 1, . . ., 7, and define the following functions.

$$F = [\lambda_1(\dot{x}_1 - \frac{q}{v_1}(x_0 - x_1) + G_a e^{-\frac{E_a}{RT_1}} x_1)$$

$$+ \lambda_2(\dot{y}_1 - \frac{q}{v_1}(y_0 - y_1) + G_b e^{-\frac{E_a}{RT_1}} y_1 - G_a e^{-\frac{E_a}{RT_1}} x_1)$$

$$+ \lambda_3(\dot{x}_2 - \frac{q}{v_2}(x_1 - x_2) + G_a e^{-\frac{E_a}{RT_2}} x_2$$

$$+ \lambda_4(\dot{y}_2 - \frac{q}{v_2}(y_1 - y_2) + G_b e^{-\frac{E_b}{RT_2}} y_2 - G_a e^{-\frac{E_a}{RT_2}} x_2)$$

$$+ \lambda_5(\dot{I} - qy_2 + S)$$

$$+ \lambda_6(\dot{S} - cS - AS + \frac{cS^2}{N} + \frac{AS^2}{N})$$

$$+ c_1 c_q S + c_2 q x_2 + c_3 q(1 - x_2 - y_2) - c_I(I_m - I)^2$$

$$- c_A A^2 S^2 - c_T\{(T_{1m} - T_1)^2 + (T_1 - T_2)^2\}] \qquad (83)$$

and

$$G = [\theta_1(x_1(t_i) - x_1^0) + \theta_2(y_1(t_i) - y_1^0) + \theta_3(x_2(t_i) - x_2^0)$$

$$+ \theta_4(y_2(t_i) - y_2^0) + \theta_5(I(t_i) - I^0)$$

$$+ \theta_6(I(t_f) - I^1) + \theta_7(S(t_i) - S^0)] \qquad (84)$$

The Euler - Lagrange equations [11],

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right) - \frac{\partial F}{\partial y_i} = 0 \tag{85}$$

and

$$\frac{\partial F}{\partial Z} = 0 \tag{86}$$

can now be applied to Equations (83) to obtain the relationships for the six lagrange multiplier equations.

$$\frac{d\lambda_1}{dt} = q\left(\frac{\lambda_1}{v_1} - \frac{\lambda_3}{v_2}\right) + (\lambda_1 - \lambda_2)G_a\, e^{-\frac{E_a}{RT_1}} \tag{87}$$

$$\frac{d\lambda_2}{dt} = q\left(\frac{\lambda_2}{v_1} - \frac{\lambda_4}{v_2}\right) + \lambda_2 G_b\, e^{-\frac{E_b}{RT_1}} \tag{88}$$

$$\frac{d\lambda_3}{dt} = \frac{\lambda_3 q}{v_2} + (\lambda_3 - \lambda_4)G_a\, e^{-\frac{E_a}{RT_2}} + q(c_2 - c_3) \tag{89}$$

$$\frac{d\lambda_4}{dt} = \frac{\lambda_4 q}{v_2} + \lambda_4 G_b\, e^{-\frac{E_b}{RT_2}} - q(c_3 + \lambda_5) \tag{90}$$

$$\frac{d\lambda_5}{dt} = 2c_I I_m - 2c_I I \tag{91}$$

$$\frac{d\lambda_6}{dt} = c_1 + \lambda_5 - c\lambda_6 - A\lambda_6 + \frac{2cS\lambda_6}{N} + \frac{2AS\lambda_6}{N} - 2c_A A^2 S \tag{92}$$

Application of Equations (86) to (83) yields

$$A = \frac{\lambda_6}{2c_A} \left(\frac{1}{N} - \frac{1}{S} \right) \tag{93}$$

$$\frac{(\lambda_1 - \lambda_2)G_a E_a x_1}{RT_1^2} e^{-\frac{E_a}{RT_1}} + \frac{\lambda_2 G_b E_b y_1}{RT_1^2} e^{-\frac{E_b}{RT_1}} - 2c_T(2T_1 - T_2 - T_{1m}) = 0 \tag{94}$$

$$\frac{(\lambda_3 - \lambda_4)G_a E_a x_2}{RT_2^2} e^{-\frac{E_a}{RT_2}} + \frac{\lambda_4 G_b E_b y_2}{RT_2^2} e^{-\frac{E_b}{RT_2}} + 2c_T(T_1 - T_2) = 0 \tag{95}$$

Equation (93) gives explicit expression of the control variable A. Hence A can be eliminated in all the performance equations. However, the control variables, T_1 and T_2, appear implicitly in the above equations and cannot be eliminated.

Substituting the expressions for A into Equations (82) and (92), we obtain

$$\frac{dS}{dt} = cS - \frac{cS^2}{N} + \frac{S\lambda_6}{c_A N} - \frac{\lambda_6}{2c_A} - \frac{S^2\lambda_6}{2c_A N^2} \tag{96}$$

$$\frac{d\lambda_6}{dt} = c_1 + \lambda_6 - c\lambda_6 - \frac{\lambda_6^2}{2c_A N} + \frac{2cS\lambda_6}{N} + \frac{S\lambda_6^2}{2c_A N^2} \tag{97}$$

5

Notice that these two equations are non-linear.

Equations (77) through (81), (87) through (91), (96) and (97) represent the system. For these 12 differential equations we have only 7 boundary conditions given by Equation (76). The additional 5 boundary conditions can be obtained by applying the transversality condition [11].

$$\frac{\partial G}{\partial y_i}\bigg|_{t_i} - \frac{\partial F}{\partial \dot{y}_i}\bigg|_{t_i} = 0 \quad \text{or} \quad \frac{\partial G}{\partial y_i}\bigg|_{t_f} - \frac{\partial F}{\partial \dot{y}_i}\bigg|_{t_f} = 0$$

to Equations (83) and (84)

$$\begin{aligned} 0 - \lambda_1(t_f) &= 0 & \lambda_1(t_f) &= 0 \\ 0 - \lambda_2(t_f) &= 0 & \lambda_2(t_f) &= 0 \\ 0 - \lambda_3(t_f) &= 0 & \lambda_3(t_f) &= 0 \\ 0 - \lambda_4(t_f) &= 0 & \lambda_4(t_f) &= 0 \\ 0 - \lambda_6(t_f) &= 0 & \lambda_6(t_f) &= 0 \end{aligned} \tag{98}$$

The boundary conditions given by Equations (76) and (98) make this system, a two point boundary value problem.

Let us consider the case that the final condition on the inventory was not given. Equations (69) through (75) remain unchanged. Equation (76) can be modified as

$$\begin{aligned} x_1(t_i) &= x_1^0 & y_2(t_i) &= y_2^0 \\ y_1(t_i) &= y_1^0 & I(t_i) &= I^0 \\ x_2(t_i) &= x_2^0 & S(t_i) &= S^0 \end{aligned} \tag{99}$$

Definition of the function F, Equation (83), remains the same, but the function G is modified as

$$G = [\theta_1(x_1(t_i) - x_1^0) + \theta_2(y_1(t_i) - y_1^0) + \theta_3(x_2(t_i) - x_2^0)$$
$$+ \theta_4(y_2(t_i) - y_2^0) + \theta_5(I(t_i) - I^0)$$
$$+ \theta_6(S(t_i) - S^0)] \tag{100}$$

Since F remains unchanged, Equations (87) through (97) are valid in this case.

Applying the transversality condition [11] to Equation (100), we have

$$\begin{aligned}
0 - \lambda_1(t_f) &= 0 \qquad & \lambda_1(t_f) &= 0 \\
0 - \lambda_2(t_f) &= 0 \qquad & \lambda_2(t_f) &= 0 \\
0 - \lambda_3(t_f) &= 0 \qquad & \lambda_3(t_f) &= 0 \\
0 - \lambda_4(t_f) &= 0 \qquad & \lambda_4(t_f) &= 0 \\
0 - \lambda_5(t_f) &= 0 \qquad & \lambda_5(t_f) &= 0 \\
0 - \lambda_6(t_f) &= 0 \qquad & \lambda_6(t_f) &= 0
\end{aligned} \tag{101}$$

5.4 QUASILINEARIZATION

Only Equations (96) and (97) are non-linear. The linearization procedure is the same as described in Chapter 3. Referring to Equation (21), we need the expressions for f_a, f_b, g_a, g_b. In other words, we need

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial \lambda_6} \\ \frac{\partial g_2}{\partial S} & \frac{\partial g_2}{\partial \lambda_6} \end{bmatrix}$$

This matrix can be obtained from Equations (96) and (97).

$$J = \begin{bmatrix} c - \frac{2cS}{N} + \frac{\lambda_6}{c_A N} - \frac{S\lambda_6}{c_A N^2} ; & \frac{S}{c_A N} - \frac{1}{2c_A} - \frac{S^2}{2c_A N^2} \\ \frac{2c\lambda_6}{N} + \frac{\lambda_6^2}{2c_A N^2} ; & \frac{2cS}{N} - c - \frac{\lambda_6}{c_A N} + \frac{\lambda_6}{c_A N^2} \end{bmatrix} \tag{102}$$

Linearization and recurrence relations were developed in accordance with equations (21) through (24).

$$\frac{dx_{1,n+1}}{dt} = \frac{q}{v_1}(x_0 - x_{1,n+1}) - G_a e^{-\frac{E_a}{RT_1}} x_{1,n+1} \tag{103}$$

$$\frac{dy_{1,n+1}}{dt} = \frac{q}{v_1}(y_0 - y_{1,n+1}) - G_b e^{-\frac{E_b}{RT_1}} y_{1,n+1} + G_a e^{-\frac{E_a}{RT_1}} x_{1,n+1} \tag{104}$$

$$\frac{dx_{2,n+1}}{dt} = \frac{q}{v_2}(x_{1,n+1} - x_{2,n+1}) - G_a e^{-\frac{E_a}{RT_2}} x_{2,n+1} \tag{105}$$

$$\frac{dy_{2,n+1}}{dt} = \frac{q}{v_2}(y_{1,n+1} - y_{2,n+1}) - G_b e^{-\frac{E_b}{RT_2}} y_{2,n+1} + G_a e^{-\frac{E_a}{RT_2}} x_{2,n+1} \tag{106}$$

$$\frac{dI_{n+1}}{dt} = qy_{2,n+1} - S_{n+1} \tag{107}$$

$$\frac{dS_{n+1}}{dt} = [cS_n - \frac{cS_n^2}{N} + \frac{S_n\lambda_{6,n}}{c_A N} - \frac{\lambda_{6,n}}{2c_A} - \frac{S_n^2\lambda_{6,n}}{2c_A N^2}]$$
$$+ (S_{n+1} - S_n)[c - \frac{2cS_n}{N} + \frac{\lambda_{6,n}}{c_A N} - \frac{S_n\lambda_{6,n}}{c_A N^2}$$

$$+ (\lambda_{6,n+1} - \lambda_{6,n})[\frac{S_n}{c_A N} - \frac{1}{2c_A} - \frac{S_n^2}{2c_A N^2}] \tag{108}$$

$$\frac{d\lambda_{1,n+1}}{dt} = q(\frac{\lambda_{1,n+1}}{v_1} - \frac{\lambda_{3,n+1}}{v_2}) + (\lambda_{1,n+1} - \lambda_{2,n+1})\ G_a\ e^{-\frac{E_a}{RT_1}} \tag{109}$$

$$\frac{d\lambda_{2,n+1}}{dt} = q(\frac{\lambda_{2,n+1}}{v_1} - \frac{\lambda_{4,n+1}}{v_2}) + \lambda_{2,n+1} G_b\ e^{-\frac{E_b}{RT_1}} \tag{110}$$

$$\frac{d\lambda_{3,n+1}}{dt} = \frac{q\lambda_{3,n+1}}{v_2} + (\lambda_{3,n+1} - \lambda_{4,n+1})G_a\ e^{-\frac{E_a}{RT_2}} + q(c_2 - c_3) \tag{111}$$

$$\frac{d\lambda_{4,n+1}}{dt} = \frac{q\lambda_{4,n+1}}{v_2} + \lambda_{4,n+1} G_b\ e^{-\frac{E_b}{RT_2}} - c_3 q - q\lambda_{5,n+1} \tag{112}$$

$$\frac{d\lambda_{5,n+1}}{dt} = 2c_I I_m - 2c_I I_{n+1}$$

$$\frac{d\lambda_{6,n+1}}{dt} = [c_1 + \lambda_{5,n+1} - c\lambda_{6,n} - \frac{\lambda_{6,n}^2}{2c_A N} + \frac{2cS_n\lambda_{6,n}}{N} + \frac{S_n\lambda_{6,n}^2}{2c_A N^2}]$$

$$+ (S_{n+1} - S_n)[\frac{2c\lambda_{6,n}}{N} + \frac{\lambda_{6,n}^2}{2c_A N^2}]$$

$$+ (\lambda_{6,n+1} - \lambda_{6,n})[\frac{2cS_n}{N} - c - \frac{\lambda_{6,n}}{c_A N} + \frac{S_n\lambda_{6,n}}{c_A N^2}] \tag{113}$$

The boundary conditions are given by Equations (76) and (98) or (76) and (101).

Equations (103) through (113) are ordinary linear differential equations and with the boundary conditions given by equations (76) and (98) or (76) and (101), they form a two point boundary value problem. This problem can now be solved by the superposition principle. If the initial values for the particular and homogeneous solutions are selected such that they satisfy the initial conditions, the general solution can be given as

$$x_1(t) = x_{1p}(t) + \sum_{k=1}^{6} A_k x_{1,kh}(t) \tag{114}$$

$$y_1(t) = y_{1p}(t) + \sum_{k=1}^{6} A_k y_{1,kh}(t) \tag{115}$$

$$x_2(t) = x_{2p}(t) + \sum_{k=1}^{6} A_k x_{2,kh}(t) \tag{116}$$

$$y_2(t) = y_{2p}(t) + \sum_{k=1}^{6} A_k y_{2,kh}(t) \tag{117}$$

$$I(t) = I_p(t) + \sum_{k=1}^{6} A_k I_{kh}(t) \tag{118}$$

$$S(t) = S_p(t) + \sum_{k=1}^{6} A_k S_{kh}(t) \tag{119}$$

$$\lambda_i(t) = \lambda_{ip}(t) + \sum_{k=1}^{6} A_k \lambda_{i,kh}(t) \qquad i = 1, \ldots, 6. \tag{120)-(125}$$

These equations are derived in accordance with Equations (11) and (12).

After obtaining the solution for the 6 state variables and 6 Lagrange multipliers by the superposition principle, Equations (69), (93), (94), and (95) can be solved for the profit and the three control variables, A, T_1, T_2, respectively.

$A_1 X_{1h1}(t) + A_2 X_{1h2}(t) + A_3 X_{1h3}(t)$
$X_{2h1}(t)$ X_{2h2}

All these calculations complete one iteration. Further iterations were allowed until desired accuracy was achieved.

5.5 NUMERICAL ASPECTS

Depending upon the value of the constants and the boundary conditions, this problem was divided into classes A, B, C, D, and E. The object was to investigate the convergence and other computational aspects of this technique from different angles.

Problem A

The following values were assumed for the various parameters

$$
\begin{array}{ll}
G_a = 0.535 \times 10^{11} \text{ per minute} & N = 100 \\
G_b = 0.461 \times 10^{18} \text{ per minute} & c = 1 \\
E_a = 18000 \text{ cal/mole} & c_T = 0.001 \text{ \$/°K} \\
E_b = 30000 \text{ cal/mole} & c_A = 0.01 \text{ \$} \\
R = 2 \text{ cal/mole °K} & c_1 = 5.0 \text{ \$} \\
q = 60 \text{ gal/min} & c_2 = c_3 = 0.0 \text{ \$} \\
v_1 = v_2 = 12 \text{ gallons} & c_q = 1.0 \quad c_I = 1.0 \text{ \$/gal.} \\
I_m = 10 \text{ gallons} & x_0(t) = 0.53 \quad t_i = 0.0 \\
T_{1m} = 340\text{°K} \qquad \Delta t = 0.01 & y_0(t) = 0.43 \quad t_f = 1.0
\end{array}
\qquad (126)
$$

The boundary conditions were

$$x_1(0) = 0.53 \quad y_1(0) = 0.43 \quad x_2(0) = 0.53 \quad y_2(0) = 0.43 \quad I(0) = 1.0$$

$$S(0) = 0.1 \qquad I(1) = 10.0$$

It should be emphasized that class A was the only problem which had final condition on the inventory.

There were only two non-linear differential equations, hence only two initial approximations, $S_0(t)$ and $\lambda_{6,0}(t)$, were required. The equations for the control variables, T_1 and T_2, are implicit and cannot be solved directly. Hence, the initial approximations for T_1 and T_2 were also required. The various sets of initial approximations used for problem A are listed in Table 7.

Problem B

The same parameters used in problem A were used here, except that the final condition on the inventory was removed. As a result of this change, according to Equation (101), $\lambda_5(1) = 0$. The boundary conditions are

$$x_1(0) = 0.53 \quad y_1(0) = 0.43 \quad x_2(0) = 0.53 \quad y_2(0) = 0.43 \quad I(0) = 1.0 \quad S(0) = 0.1$$

As in the last case, only $S_0(t)$, $\lambda_{6,0}(t)$, $T_{1,0}(t)$, and $T_{2,0}(t)$ were required as the initial approximations. They are listed in Table 8.

Problem C

Some of the parameters were changed. For clear understanding all of them are rewritten in the following

$G_a = 0.535 \times 10^{11}$ per minute $\qquad N = 100$

$G_b = 0.461 \times 10^{18}$ per minute $\qquad c = 1$

$E_a = 18000$ cal/mole $\qquad c_T = 0.0005$ \$/°k

$E_b = 30000$ cal/mole $\qquad c_A = 0.0002$ \$

$R = 2$ cal/mole °k $c_1 = 5.0$ \$

$q = 60$ gal./min. $c_2 = c_3 = 0.0$ \$

$v_1 = v_2 = 12$ gallons $c_q = 1.0$ $c_I = 1.0$ \$/gal.

$I_m = 20$ gallons $x_0(t) = 0.53$ $t_i = 0.0$

$T_{1m} = 340°k$ $\Delta t = 0.01$ $y_0(t) = 0.43$ $t_f = 1.0$

The boundary conditions were

$x_1(0) = 0.53$ $y_1(0) = 0.43$ $x_2(0) = 0.53$ $y_2(0) = 0.43$ $I(0) = 8.0$ $S(0) = 0.1$

A list of initial approximations is shown in Table 9.

Problem D

$c_A = 0.01$ $S(0) = 1.0$

All other parameters were the same as in problem C. Three different initial approximations were used for this problem. They are tabulated in Table 10.

Problem E

The only difference between problems E and C is in the initial condition of the sales. In the present problem, the initial condition for sales is

$S(0) = 0.1$

All others values are the same as in problem C. The values of the initial approximation used are given in Table 11.

The initial values used for the particular and homogeneous solutions are given in Table 12.

Table 7. Initial approximations for problem A.

Set No.	$T_{1,0}(t)$	$T_{2,0}(t)$	$S_0(t)$	$\lambda_{6,0}(t)$
1A	345	345	1	0
2A	345	345	50	-0.5
3A	345	345	30	-1.0
4A	330	330	1	0

Table 8. Initial approximations for problem B.

Set No.	$T_{1,0}(t)$	$T_{2,0}(t)$	$S_0(t)$	$\lambda_{6,0}(t)$
1B	330	330	1	0
2B	340	340	1	0
3B	345	345	1	0

Table 9. Initial approximations for problem C.

Set No.	$T_{1,0}(t)$	$T_{2,0}(t)$	$S_0(t)$	$\lambda_{6,0}(t)$
1C	330	330	1	0
2C	345	345	1	0

Table 10. Initial approximations for problem D.

Set No.	$T_{1,0}(t)$	$T_{2,0}(t)$	$S_0(t)$	$\lambda_{6,0}(t)$
1D	355	355	20	-0.5
2D	345	345	50	0
3D	350	350	25	-0.25

Table 11. Initial approximations for problem E.

Set No.	$T_{1,0}(t)$	$T_{2,0}(t)$	$S_0(t)$	$\lambda_{6,0}(t)$
1E	345	345	50	0

Table 12. Initial conditions used for obtaining particular and homogeneous solutions.

	P.I. for A, B	P.I. for C, E	P.I. for D	Homogeneous Solutions					
				Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
$x_1(0)$	0.53	0.53	0.53	0	0	0	0	0	0
$y_1(0)$	0.43	0.43	0.43	0	0	0	0	0	0
$x_2(0)$	0.53	0.53	0.53	0	0	0	0	0	0
$y_2(0)$	0.43	0.43	0.43	0	0	0	0	0	0
I(0)	1.0	8.0	8.0	0	0	0	0	0	0
S(0)	0.1	0.1	1.0	0	0	0	0	0	0
$_1(0)$	0	0	0	1	0	0	0	0	0
$_2(0)$	0	0	0	0	1	0	0	0	0
$_3(0)$	0	0	0	0	0	1	0	0	0
$_4(0)$	0	0	0	0	0	0	1	0	0
$_5(0)$	0	0	0	0	0	0	0	1	0
$_6(0)$	0	0	0	0	0	0	0	0	1

5.6 COMPUTATIONAL ASPECTS

Basically the same procedure was followed for all the five problems. This procedure was essentially the same as that used in Chapter 4. With the initial values given in Table 12, a set of particular solutions and six sets of homogeneous solutions were obtained by numerical integration using the Runge-Kutta method.

In order to solve for six integration constants, six equations for which the final conditions were known, were selected from Equations (114) through (125). To solve this 6 x 6 matrix on computer, matrix inversion subroutine SIMQ supplied by IBM was used. A printout of this subroutine is shown in Appendix 3. Using these integration constants with the newly obtained particular and homogeneous solutions, the final solutions for all twelve variables were obtained. Next, using Equation (93), values of advertisement at all grid points were obtained.

For simplicity the following approximation was used to calculate the total profit.

$$J = \sum_{t_i}^{t_f} [c_1 c_q S + c_2 q x_2 + c_3 q(1 - x_2 - y_2) - c_I(I_m - I)^2 - c_A A^2 S^2 - c_T\{(T_{1m} - T_1)^2 + (T_1 - T_2)^2\}]\Delta t \tag{128}$$

Computation of T_1 and T_2 were rather difficult because Equations (94) and (95) are implicit in T_1 and T_2. To overcome this difficulty, c_T was assumed to be zero and the last term in both the equations dropped out. As a result, explicit expressions were derived as

$$T_1 = \frac{(E_a - E_b)/R}{\ln[\frac{G_a\ x_1\ E_a(\lambda_2 - \lambda_1)}{G_b\ y_1\ E_b\ \lambda_2}]}$$

$$T_2 = \frac{(E_a - E_b)/R}{\ln[\frac{G_a\ x_1\ E_a\ (\lambda_2 - \lambda_1)}{G_b\ y_2\ E_b\ \lambda_4}]} \qquad (129)$$

These equations can be solved easily, but examining these equations carefully, at final time $t = t_f$, the denominator would involve $\ln[\frac{0}{0}]$ which is indeterminate.

Another approach to this difficulty was to apply the Newton-Raphson method of root finding. This method is described in Appendix 1. As indicated earlier, it is fundamentally the same as quasilinearization. To start the Newton-Raphson method, initial approximations for both T_1 and T_2 were assumed to be 350.0. With these values, T_1 and T_2 at the first grid points of the first iteration were solved. At other grid points, solutions of T_1 and T_2 at previous grid points of same iteration were used as the initial approximations. Same procedure was repeated for the next iteration.

This scheme encountered convergence problem in problems 3A, 3B, 1D and 1E. The logic was slightly modified to overcome this trouble. The procedure remained the same for the first iteration. For other iterations, solutions of T_1 and T_2 at the same grid point of previous iteration were used as the initial approximation. With this change the convergence difficulty was overcome in problems 3B and 1E. The accuracy to check

Newton-Raphson convergence was 0.1.

All these computations completed one iteration. Further iterations were allowed until convergence was obtained.

5.7 RESULTS

Problem A

The four sets of initial approximations tried in this case are listed in Table 7. Out of these four, three sets resulted in convergence to the solution of the problem.

The convergence rates of the three control variables and the six state variables for problem 1A are shown in Figs. 15 through 23. Fig. 24.shows the optimal profile of the profit function. The optimal profiles of Lagrange multipliers are shown in Figs. 25 and 26. The total profit was $79.75.

In order to visualize the convergence rates of the control variables in more detail, they are tabulated in Tables 13 through 15.

Problem 3A did not converge to the optimal solution. It encountered Newton-Raphson convergence problem in the first iteration.

Some modifications were made to study the convergence problem.

In problem 4A, T_{1m} was changed to 300, with other values remaining the same. This problem experienced the Newton-Raphson convergence problem in iteration 2.

Again in problem 4A, changing T_{1m} = 300 and c_A = 0.0, with all other values remaining the same, the advertisement curve became nearly discontinuous, but it did converge in the sense that there was little difference

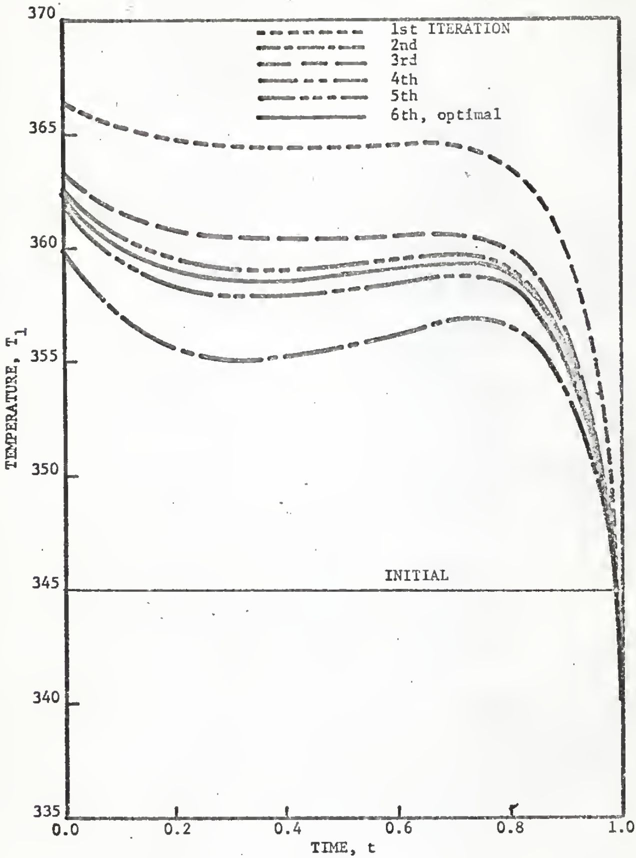

Fig. 15. Convergence Rate of Temperature T_1, Problem 1A.

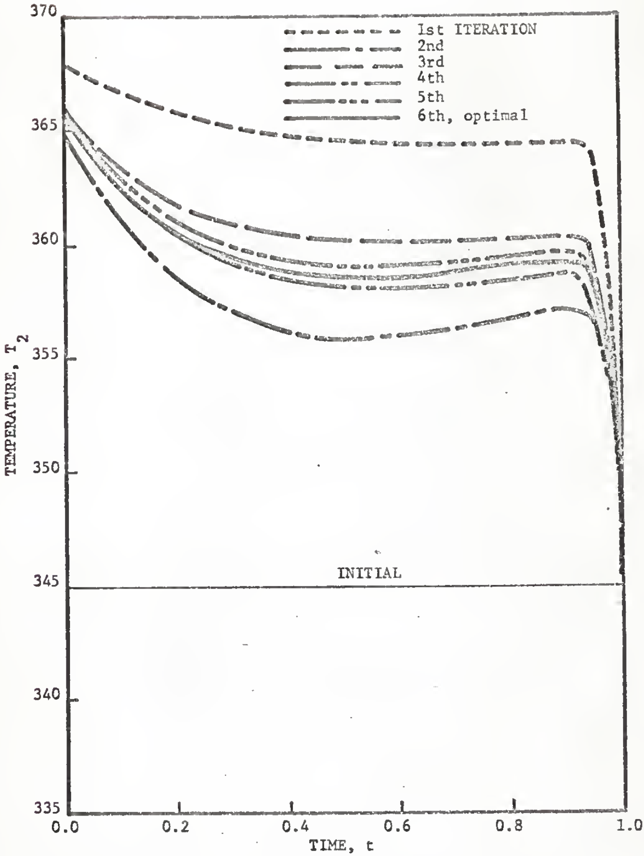

Fig. 16. Convergence Rate of Temperature T_2, Problem 1A.

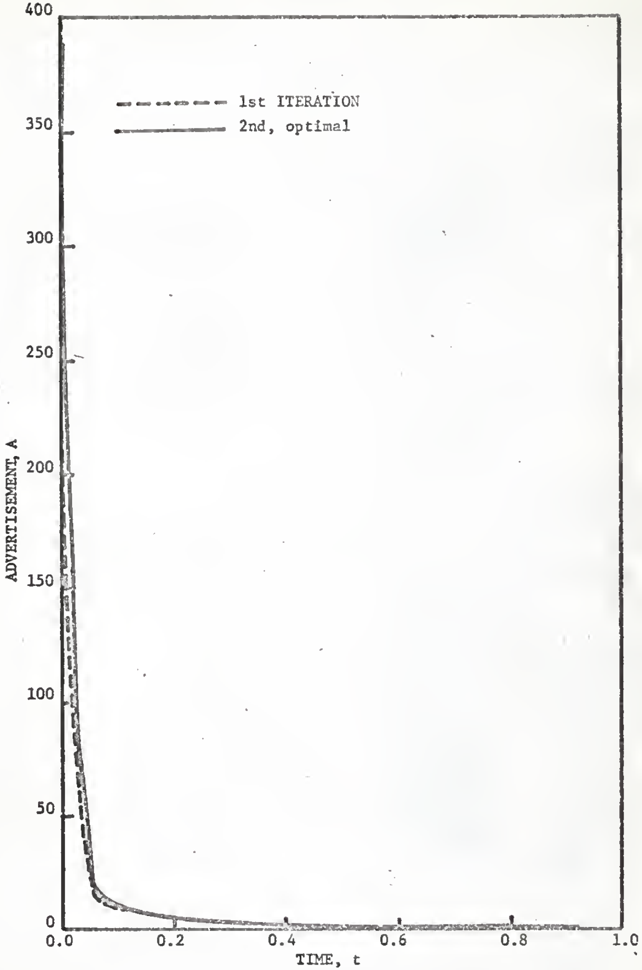

Fig. 17. Convergence Rate of Advertisement A, Problem 1A.

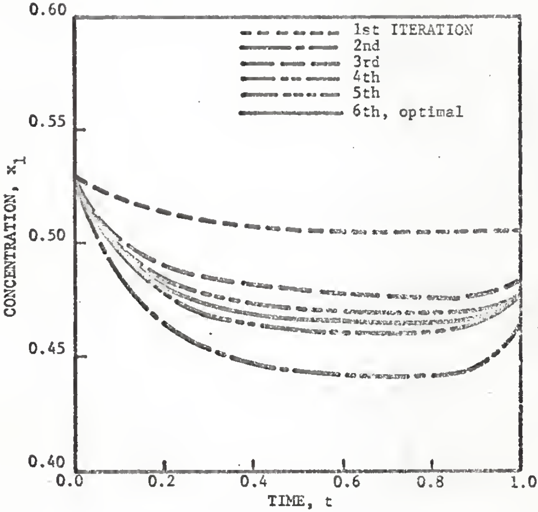

Fig. 18. Convergence Rate of Concentration x_1, Problem IA.

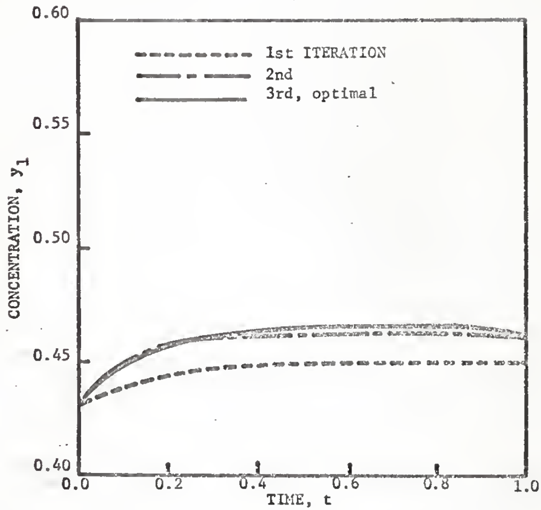

Fig. 19. Convergence Rate of Concentration y_1, Problem 1A.

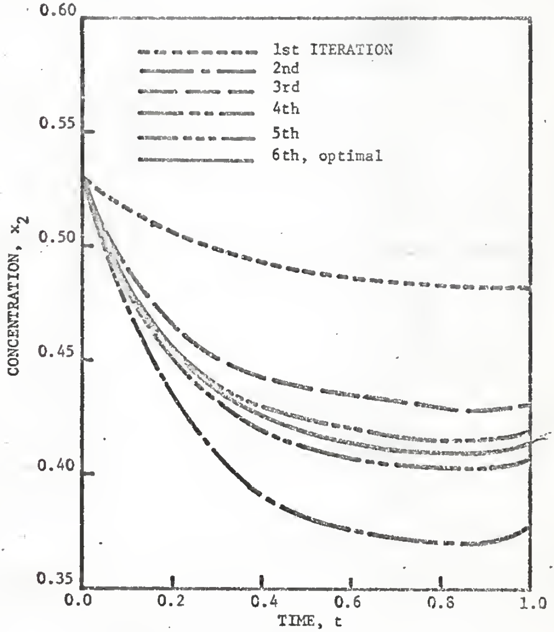

Fig. 20. Convergence Rate of Concentration x_2, Problem 1A.

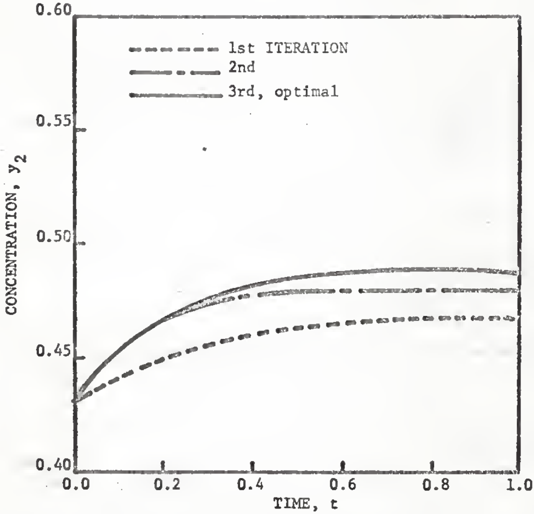

Fig. 21. Convergence Rate of Concentration y_2, Problem 1A.

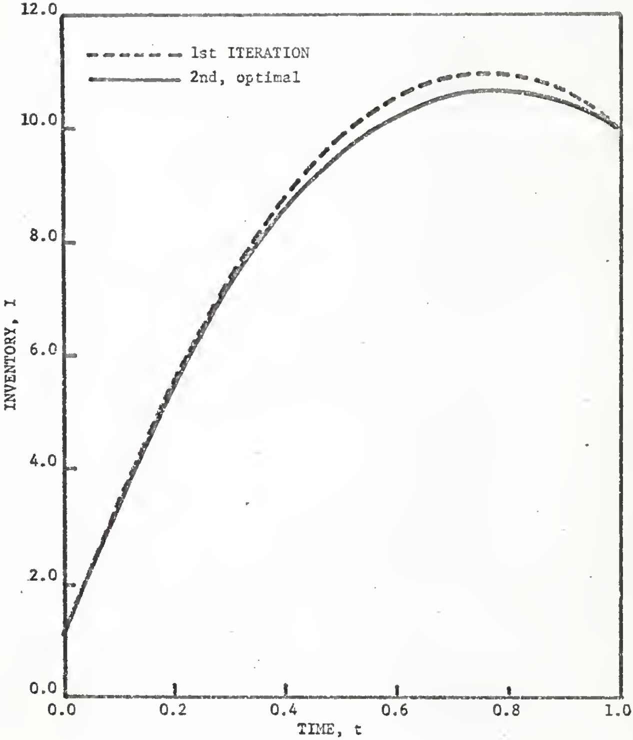

Fig. 22. Convergence Rate of Inventory I, Problem 1A.

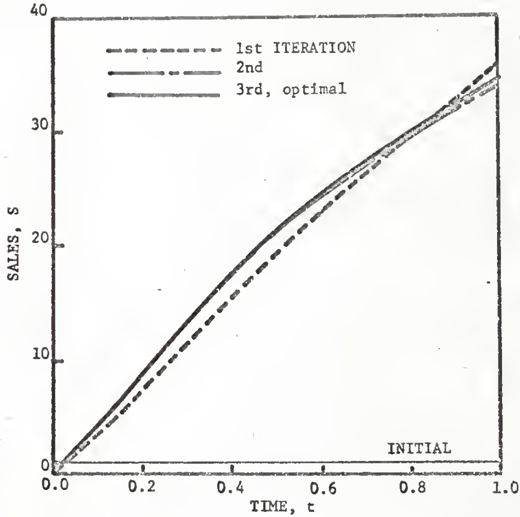

Fig. 23. Convergence Rate of Sales S in Problem 1A.

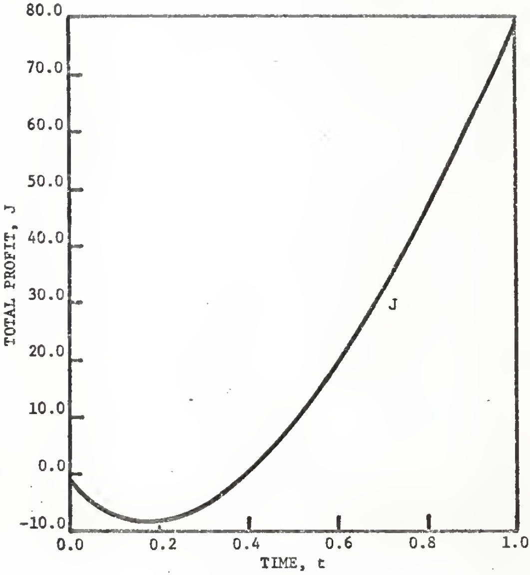

Fig. 24. Optimal Total Profit Curve, Problem 1A.

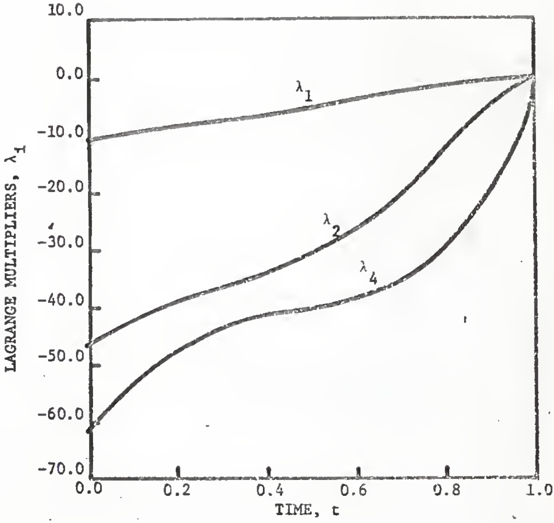

Fig. 25. Optimal Profiles of λ_1, λ_2, and λ_4, Problem 1A.

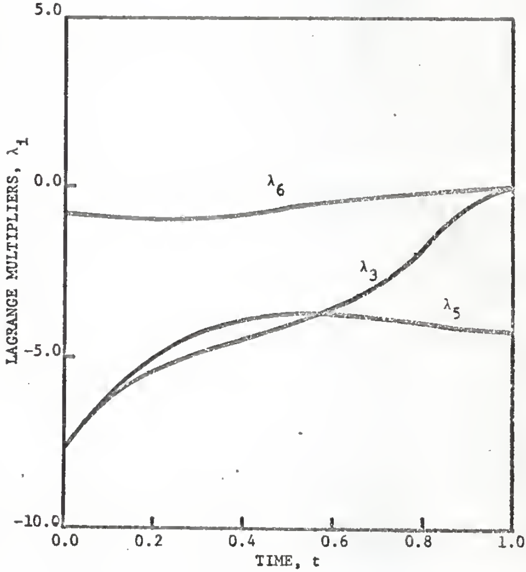

Fig. 26. Optimal Profiles of λ_3, λ_5, and λ_6, Problem 1A.

Table 13. Convergence rate of $T_1(t)$ in problem A.

iter / time	0	1	2	3	4	5	6	7	8
0.0	345.0	366.3	360.1	363.4	361.9	362.6	362.3	362.5	362.4
0.2	345.0	364.8	355.8	360.8	358.5	359.6	359.1	359.3	359.2
0.4	345.0	364.4	355.3	360.4	357.9	359.2	358.6	358.9	358.7
0.6	345.0	364.4	356.2	360.5	358.5	359.5	359.0	359.2	359.1
0.8	345.0	363.7	356.7	359.9	358.4	359.1	358.8	358.9	358.9
1.0	345.0	340.0	340.1	339.9	340.2	340.2	340.0	340.0	340.0

Table 14. Convergence rate of $T_2(t)$ in problem A.

iter / time	0	1	2	3	4	5	6	7	8
0.0	345.0	367.8	364.8	366.0	365.6	365.7	365.7	365.7	365.7
0.2	345.0	365.8	358.6	361.7	360.4	361.0	360.7	360.8	360.8
0.4	345.0	364.7	356.2	360.3	358.4	359.3	358.9	359.1	359.0
0.6	345.0	364.3	355.9	360.1	358.0	359.0	358.6	358.8	358.7
0.8	345.0	364.3	356.7	360.3	358.5	359.4	359.0	359.2	359.1
1.0	345.0	340.0	340.0	339.9	340.2	340.2	340.0	340.0	340.0

Table 15. Convergence rate of A(t) in problem A.

iter / time	0	1	2	3	4	5	6	7	8
0.0		260.40	358.80	366.40	370.60	369.90	370.50	370.30	370.40
0.2		4.47	4.65	4.68	4.67	4.67	4.67	4.67	4.67
0.4		1.38	1.70	1.72	1.72	1.72	1.72	1.72	1.72
0.6		0.43	0.67	0.69	0.69	0.69	0.69	0.69	0.69
0.8		0.09	0.22	0.23	0.22	0.22	0.22	0.22	0.22
1.0		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

in the results of iterations 8 and 9.

Problem B

All the three initial approximations listed in Table 8 converged to the optimal solution.

Problem 3B encountered Newton-Raphson convergence difficulty initially, but using the same grid point of the previous iteration as the starting value in the Newton-Raphson solution, the optimal solution was obtained in about 6 iterations. The optimal profiles of the six state variables and the three control variables are shown in Figs. 27, 28 and 28A. The total profit in this case was $95.79.

Problem C

Unfortunately, for both the initial approximations given in Table 9, the problem did not converge.

In problem 1C, the computer experienced exponential overflow while calculating the control variables in the first iteration. The following values of the state variables were obtained in the first iteration

$I(1) = -12.62 \qquad S(1) = 238.76$

All other values were reasonable.

There are two reasons for this convergence problem: the Newton-Raphson convergence difficulty or the quasilinearization difficulty. In this particular case, the author feels that it was the Newton-Raphson convergence difficulty.

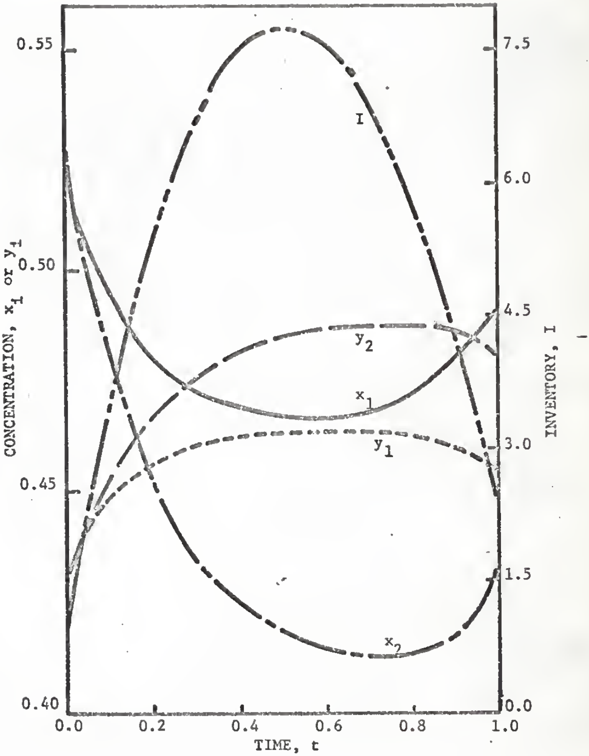

Fig. 27. Optimal Solutions of x_1, y_1, x_2, y_2, and I, Problem B.

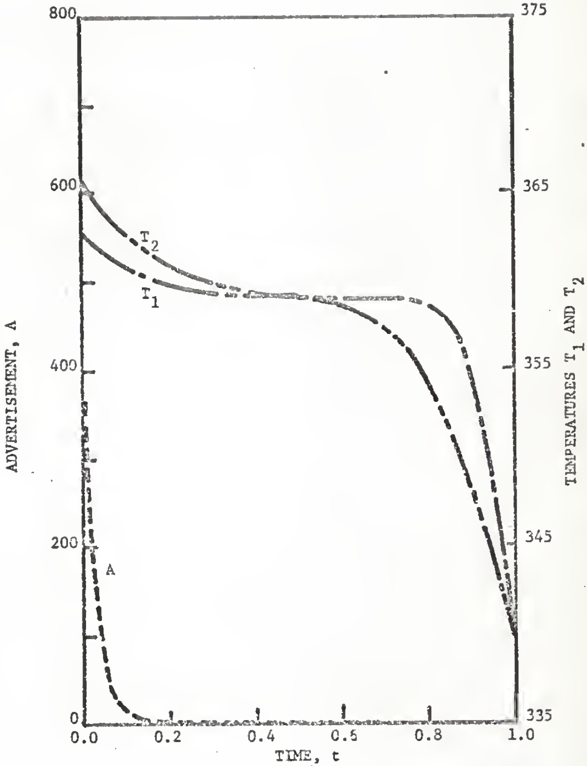

Fig. 28. Optimal Profiles of A, T_1, T_2, problem B.

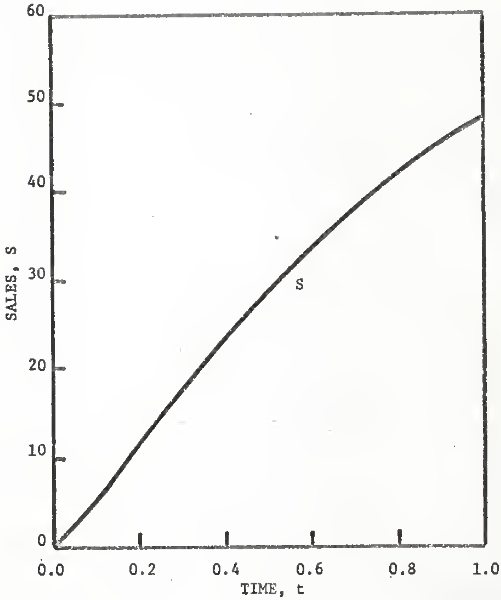

Fig. 28A. Optimal Profile of S, Problem B.

In problem 2C, computation was not possible in the fifth iteration because of the Newton-Raphson convergence problem. Results of the fourth iteration indicate that the sales goes to negative in the initial time and then rises up to 82.61 at the final time. Another peculiarity of this problem was the near discontinuity of the advertisement curve. A(0) is -5639.0 and A(0.01) is +90.54. Such a sharp change of the variable with time can make both the Newton-Raphson method and the quasilinearization method useless.

Examining the linearized performance Equations (108) and (113), we can see that c_A appears in the denominator. Hence very small value of c_A can make the problem unstable. In this case $c_A = 0.0002$. The author feels that this is a fairly low value and considering that Equations (108) and (113) are very sensitive to c_A, this was the main reason for not obtaining a solution to this problem.

In spite of this difficulty, with all other values remaining the same, S(0) was changed in hope of obtaining a solution. Initially S(0) was 0.1, and the following values of S(0) were tried.

a. S(0) = 15.0. The value of the advertisement was negative, A(0) = -145.6. This case encountered the Newton-Raphson convergence difficulty in the first iteration.

b. S(0) = 8.0. The value of the advertisement was negative and decreasing very rapidly. A solution was not possible because of the Newton-Raphson convergence problem in the second iteration.

c. S(0) = 4.0. The value of the advertisement was negative and decreasing rapidly. The same convergence difficulty was encountered but this time in the fourth iteration.

It can be seen in all these cases that

1. The value of the advertisement curve was nearly discontinuous, and

2. The Newton-Raphson method caused convergence difficulty.

This leads to the idea of increasing c_A.

Problem D

$c_a = 0.01$ and $S(0) = 1.0$

These two changes were made in the parameters of problem C. Table 10 shows the three initial approximations used in this problem. Sets 2D and 3D proved to be good guesses and convergence for these sets was obtained in about 5 iterations.

This problem had no final condition on the inventory. This is where it differs from problem A. For the purpose of comparison, the detailed results of this problem are given. Figs. 29 through 37 show the convergence rates of the three control variables and the six state variables for problem 2D. The convergence rate of the profit function is given in Fig. 38. The total profit was \$66.26. The advertisement curve as can be seen in Fig. 31 is not as sharp as the previous ones. It is a monotomically decreasing curve.

Set 1D did not converge. Examining the initial approximations, only $\lambda_{6,0}(t) = -0.5$ could be a wrong guess, but the final solution of $\lambda_6(t)$ in tbe first iterations look reasonable. It encountered convergence problem in the calculation of T_1 and T_2 in the first iteration.

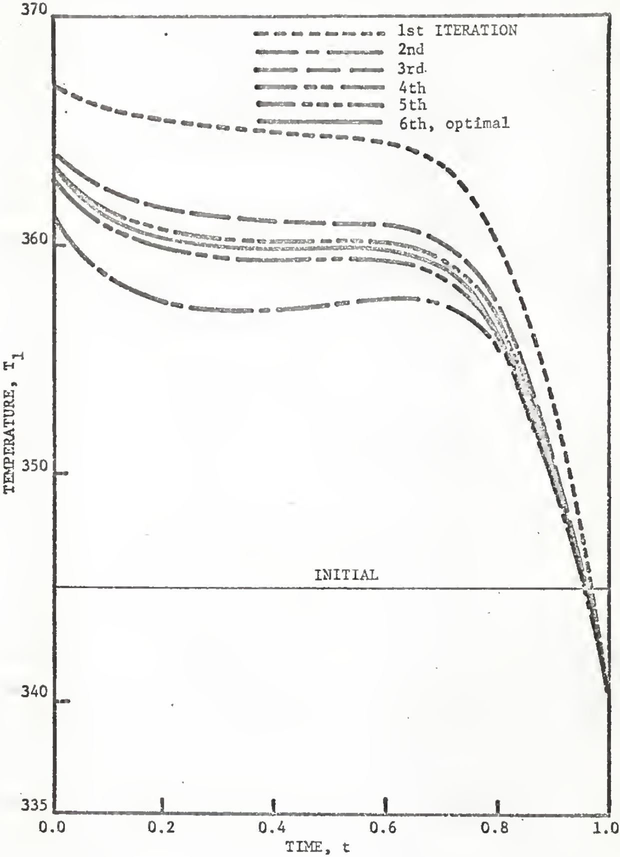

Fig. 29. Convergence Rate of Temperature T_1, Problem 2D.

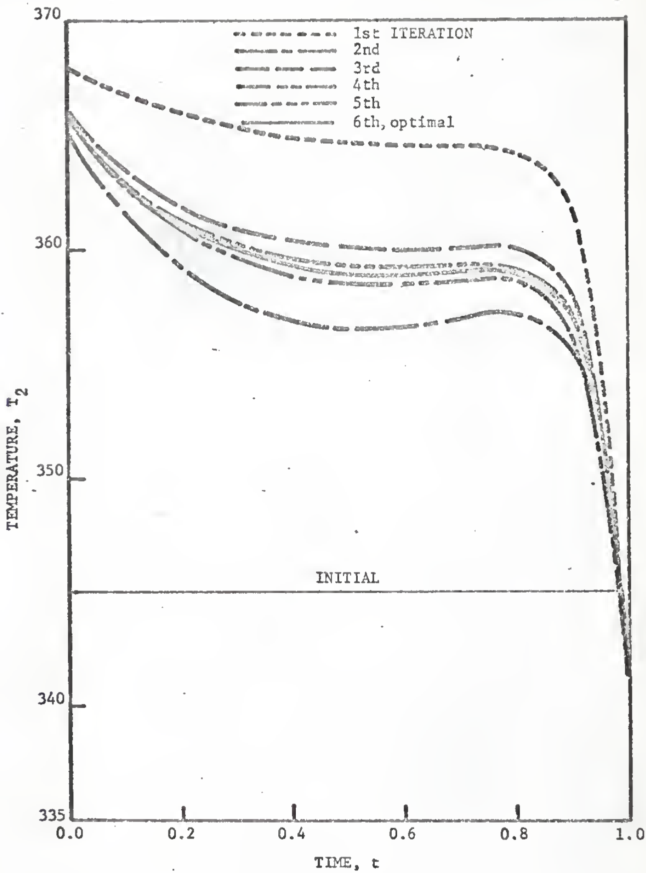

Fig. 30. Convergence Rate of Temperature T_2, problem 2D.

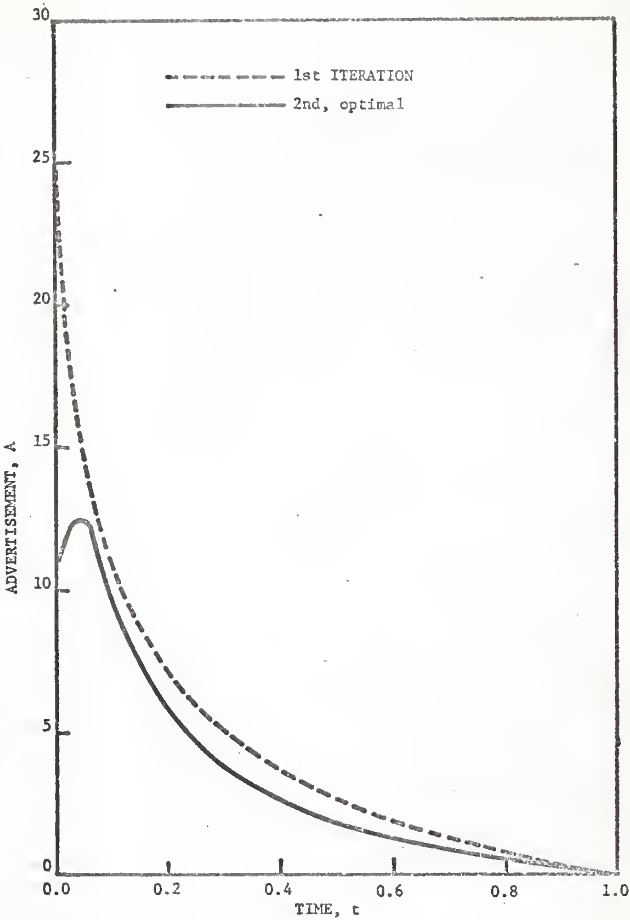

Fig. 31. Convergence Rate of Advertisement A, Problem 2D.

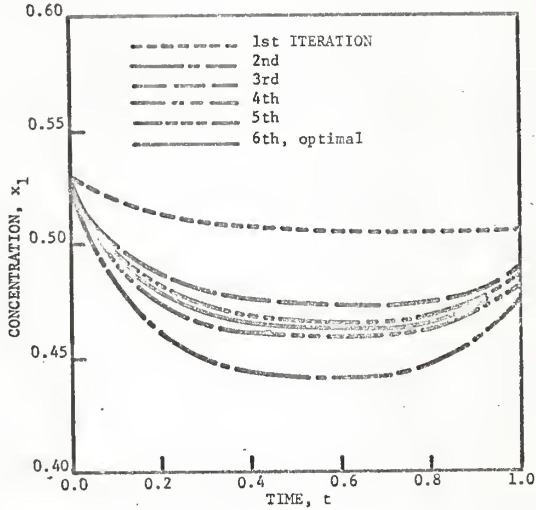

Fig. 32. Convergence Rate of Concentration x_1, Problem 2D.

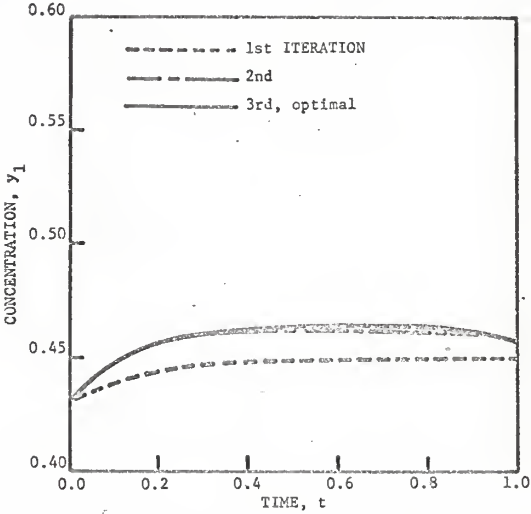

Fig. 33. Convergence Rate of Concentration y_1, Problem 2D.

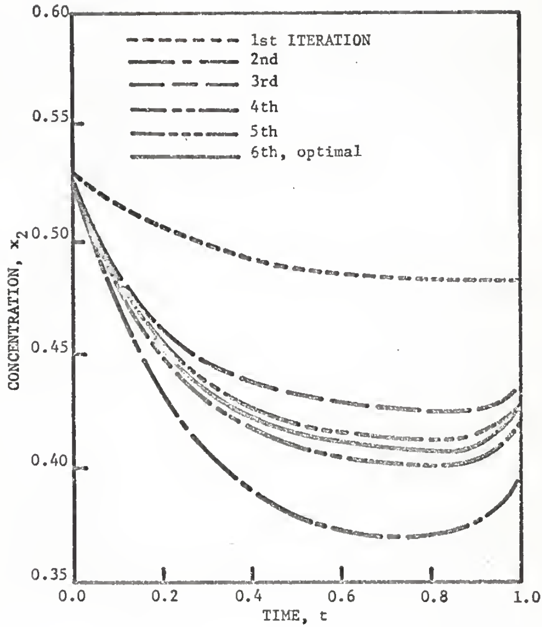

Fig. 34. Convergence Rate of Concentration x_2, Problem 2D.

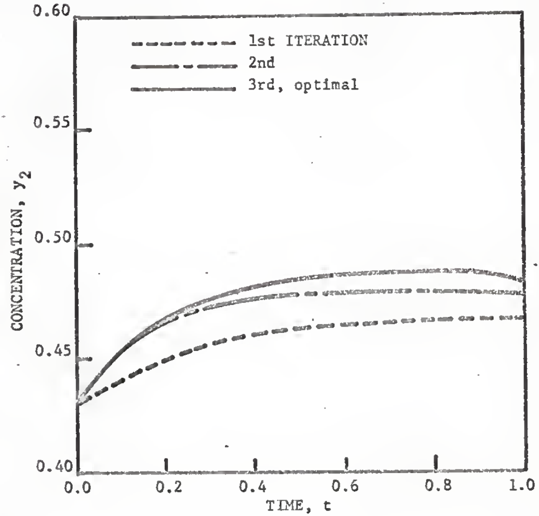

Fig. 35. Convergence Rate of Concentration y_2, Problem 2D.

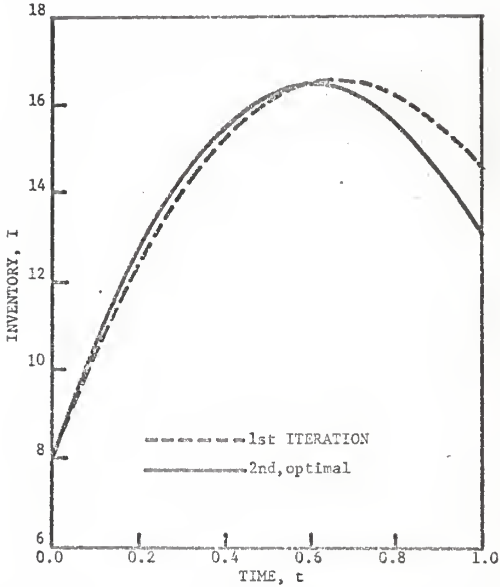

Fig. 36. Convergence Rate of Inventory I, Problem 2D.

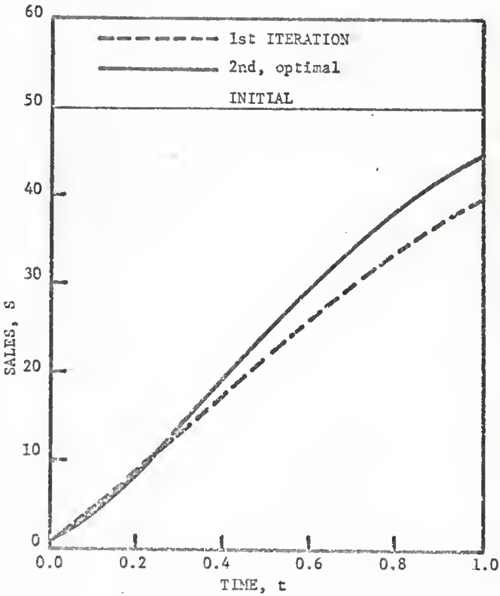

Fig. 37. Convergence Rate of Sales S, Problem 2D.

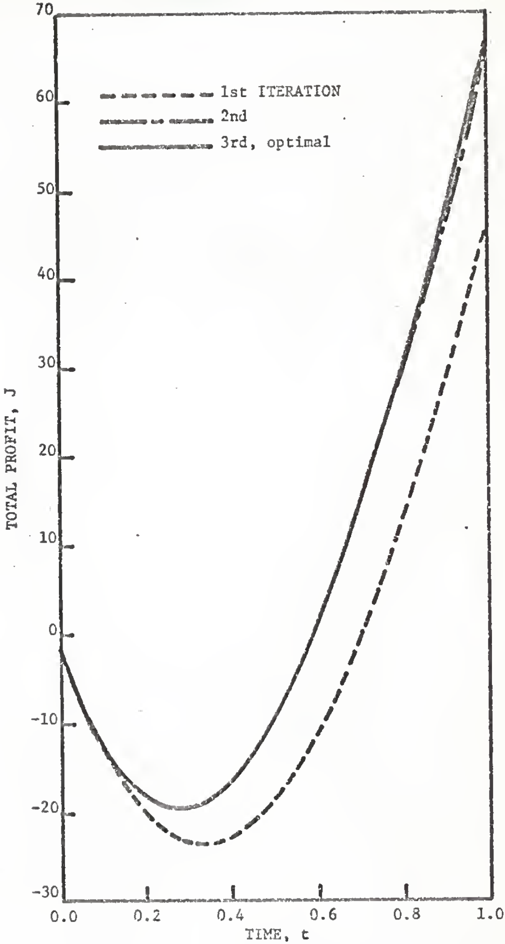

Fig. 38. Convergence Rate of Profit Function, Problem 2D.

As a slight modification, S(0) = 5.0 was tried. All other parameters remain unchanged. This problem encountered the Newton-Raphson convergence difficulty in the fourth iteration. Observing the optimal profiles of all the variables, the author feels that this was the most stable problem out of the five problems tried.

Problem E

Only change in the parameters was S(0) = 0.1, the other parameters remain unchanged. The values of the initial approximation used is given in Table 11.

Set 1E converged to the optimal solution. The optimal profiles of the six state variables and the three control variables are shown in Figs. 40, and 41. The total profit was $65.98. The advertisement profile was very sharp again. Obviously, this is because of the change in the initial sales. There is not much difference from the other profiles.

In order to get an overall view, A(0) and J for all the problems solved are compared in Tables 16 and 17.

On the average, this problem converged in 6 iterations with 3 digits accuracy. For 9 iterations, IBM 360/50 computer took about 16 minutes with FORTRAN IV H LEVEL compiler. The computer program is given in Appendix 3.

5.8 DISCUSSION

The results for all the problems indicate that the optimal profiles of the six state variables and the two control variables, T_1 and T_2 are

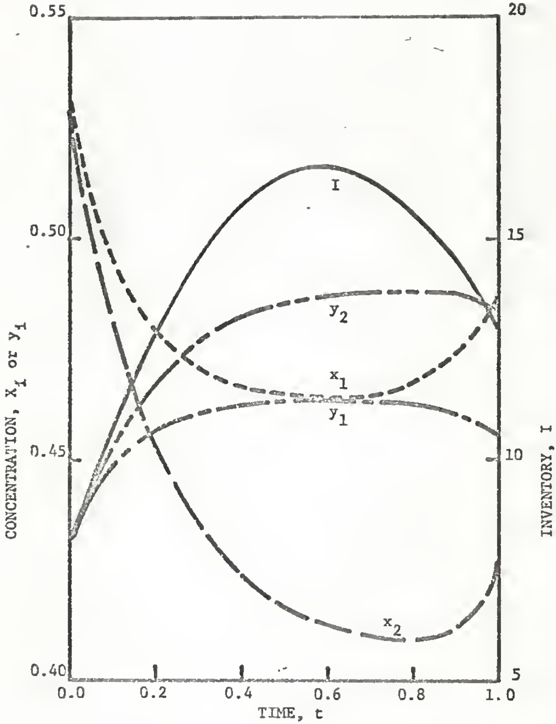

Fig. 39. Optimal Solutions of x_1, y_1, x_2, y_2, and I, Problem E.

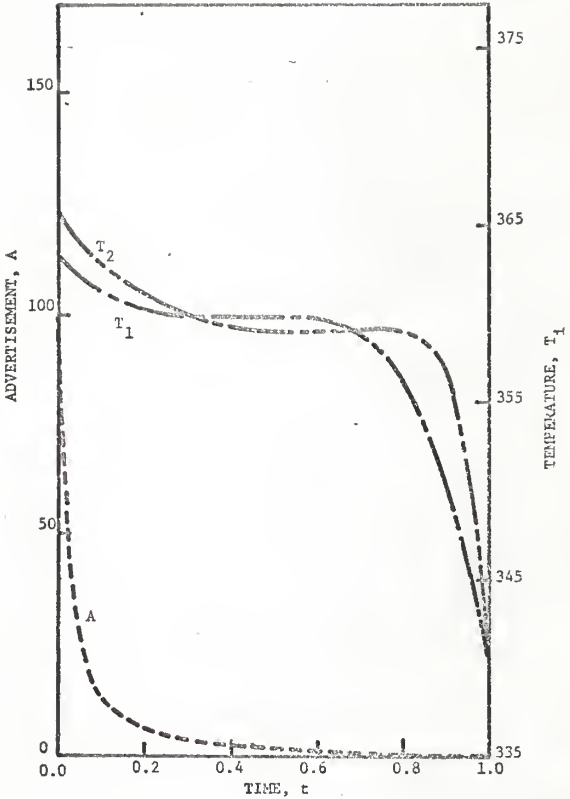

Fig. 40. Optimal Profiles of A, T_1 and T_2, problem E.

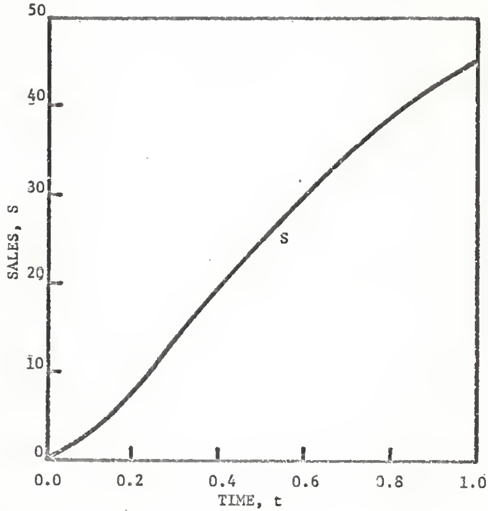

Fig. 41. Optimal Profile of S, Problem E.

Table 16.. Convergence rates of A(0)

problem / iteration	1A	2A	4A	1B	2B	3B	2D	3D	1E
1	260.4	-24.31	228.7	212.3	230.8	244.0	25.57	14.58	362.6
2	358.8	369.8	344.7	452.5	461.7	467.5	10.87	11.11	159.7
3	366.4	366.3	363.0	474.6	476.3	477.4	11.39	11.48	161.7
4	370.6	370.6	370.4	480.7	480.8	480.9	11.62	11.63	164.0
5	369.9	369.8	369.5	480.3	480.4	480.5	11.61	11.61	164.0
6	370.5	370.5	370.6	480.9	480.9	480.9	11.63	11.63	164.2
7	370.3	370.3	370.2	480.7	480.7	480.8	11.62	11.63	164.1
8	370.4	370.4	370.4	480.8	480.8	480.8	11.63	11.63	164.1

Table 17. Convergence rates of total profit J

problem / iteration	1A	2A	4A	1B	2B	3B	2D	3D	1E
1	77.847	80.979	72.806	108.634	111.870	114.041	46.108	61.290	42.914
2	78.406	78.484	76.471	93.721	94.581	95.108	64.796	65.526	64.465
3	79.471	79.469	79.171	95.425	95.537	95.611	66.083	66.159	65.811
4	79.681	79.680	79.599	95.710	95.736	95.751	66.223	66.240	65.948
5	79.727	79.727	79.707	95.774	95.778	95.782	66.254	66.257	65.977
6	79.743	79.744	79.740	95.789	95.790	95.791	66.258	66.260	65.984
7	79.744	79.743	79.743	95.791	95.791	95.791	66.259	66.260	65.984
8	79.746	79.746	79.746	95.792	95.792	95.792	66.260	66.260	65.984

either increasing or decreasing slowly. The optimal profile of the advertisement A decreases very rapidly making it almost discontinuous. Surprisingly, quasilinearization did not encounter any trouble with this type of curve. The gradient technique [15] and the second variation technique [13] seemed to have failed because of this curve.

In general, convergence was obtained in 5 to 6 iterations. Tables 16 and 17 give the comparison of the convergence rates for A(0) and J for all the problems solved.

Comparing the optimal curves of all the problems, it was observed that there was no significant difference in the production and temperature profiles. It may be concluded that a change in certain parameters have little effect on these profiles. However, a significant change in the advertisement curve was noted. This can be explained as follows.

With certain starting values of production and inventory, there is a definite range of initial sales the market can absorb with reasonable advertisement. If the initial sales is too low, the market needs a very high advertisement to bring up the sales to the market capacity. On the other hand, if the initial sales is too high, the market needs negative advertisement to bring down the sales. For this reason initial sales was a critical value.

Another critical value was c_A. Too low a value of c_A means that the cost of advertisement is very little. From the cost point of view, heavy fluctuations in A would not affect the optimal solution seriously. Hence this profile was observed to be either negative or discontinuous or unstable in cases where $c_A = 0.0002$. Stable curves were obtained with $c_A = 0.01$.

There were two main difficulties in solving this problem. Convergence difficulties in 1. the Newton-Raphson method, and 2. the Quasilinearization method. It was because of the former difficulty that some of the problems did not give any solution. No problem failed because of the latter difficulty. If a better method for solving T_1 and T_2 could be used the author is optimistic that all the problems discussed here would converge to the solution.

CHAPTER 6

CONCLUSION

The numerical examples presented in this work suggest that the quasi-linearization technique may be a useful tool for obtaining the solutions of nonlinear mixed boundary value problems. Because of the intimate association between the boundary value problems and optimization and control, this technique is also a useful tool for solving optimization problems and systems analysis.

The object of this work has been to illustrate the effectiveness of this method in overcoming the non-linearity difficutly in two point boundary value problems encountered in optimization. The advantage of this approach lies in its rapid rate of convergence, provided that the initial approximations are within the interval of convergence of the problem. This interval is fairly large for a number of problems. Furthermore, this interval can be enlarged by using devices such as data perturbation.

Convergence rate was found very rapid in the problems solved. This implies computational efficiency in terms of computer time for a prescribed accuracy. This is an important advantage of this method over other optimization techniques such as the gradient techniques where the convergence rate is slow particularly near the optimal solution.

The convergence of this method is contingent on the choice of starting functions. In this work choosing correct initial approximation was not difficult. In general the basic physical knowledge of the system is enough to make a correct guess.

In the formulation of the problem the control variables can generally be eliminated. For this reason, this method is found superior to other optimization techniques such as dynamic programming and the gradient techniques. In the second problem, two control variables could not be eliminated from the performance equations, but still quasilinearization method worked well. It has always been the difficulty in solving these control variables, that some of the problems did not give a solution.

This method was observed to be more accurate than either the first variational or the second variational techniques. In the latter methods, the average control variable is used throughout the calculation of one step size. For a fast increasing or fast decreasing curve, this is not likely to give accurate results. Quasilinearization, on the other hand, uses the control variable for the calculation at the same point. For this reason the accuracy is higher in this method.

For illustrative purposes, only two problems have been considered. Obviously, this method can be applied to a variety of other complex problems arising in industrial management systems. In addition, it can be combined with other optimization techniques such as dynamic programming and non-linear programming to optimize various topologically complex processes encountered in the industry. Recently, this method has been proved to be an efficient tool for reducing the dimensionality difficulty in dynamic programming.

APPENDIX 1

NEWTON-RAPHSON METHOD OF ROOT FINDING

Basically, this method is the Taylor series expansion with second and higher order terms neglected. Expanding $f(u_{n+1})$ around u_n

$$f(u_{n+1}) = f(u_n) + (u_{n+1} - u_n)\, f'(u_n) + \ldots$$

$$f(u_n) + (u_{n+1} - u_n)\, f'(u_n) = 0$$

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)}$$

This is the Newton-Raphson equation of root finding.

Since it is essential to solve two equations (for T_1 and T_2) simultaneously in chapter 5, we derive Newton-Raphson equations for such case.

$$f(T_1^{n+1}, T_2^{n+1}) = f(T_1^n, T_2^n) + (T_1^{n+1} - T_1^n)\frac{\partial f}{\partial T_1} + (T_2^{n+1} - T_2^n)\frac{\partial f}{\partial T_2} = 0$$

$$g(T_1^{n+1}, T_2^{n+1}) = g(T_1^n, T_2^n) + (T_1^{n+1} - T_1^n)\frac{\partial g}{\partial T_1} + (T_2^{n+1} - T_2^n)\frac{\partial g}{\partial T_2} = 0$$

Rearranging the terms,

$$T_1^{n+1} \frac{\partial f}{\partial T_1} + T_2^{n+1} \frac{\partial f}{\partial T_2} = T_1^n \frac{\partial f}{\partial T_1} + T_2^n \frac{\partial f}{\partial T_2} - f(T_1^n, T_2^n)$$

$$T_1^{n+1} \frac{\partial g}{\partial T_1} + T_2^{n+1} \frac{\partial g}{\partial T_2} = T_1^n \frac{\partial g}{\partial T_1} + T_2^n \frac{\partial g}{\partial T_2} - g(T_1^n, T_2^n)$$

Since T_1^n and T_2^n are known (initial approximation needed for first iteration), only unknowns in these equations are T_1^{n+1}, and T_2^{n+1}. This iterative procedure is carried out until desired accuracy

$$T_1^{n+1} - T_1^n < \varepsilon$$

$$T_2^{n+1} - T_2^n < \varepsilon$$

is obtained.

A correct choice of initial approximation should be emphasized. Any error in this selection can make the procedure diverge.

APPENDIX 2

```
$JOB            SHAH,RUN=CHECK,TIME=9,PAGES=100,LINES=55
C
C      THIS PROGRAM SOLVES A SET OF FOUR DIFFERENTIAL
C      EQUATIONS TWO POINT SPLIT BOUNDARY VALUE TYPE USING
C      SUPERPOSITION PRINCIPLE AND RUNGE KUTTA TECHNIQUE
C
C
C
C      MAIN  PROGRAM
C
       COMMON SX2,SX4,P,DT,A,B,CC,CA,AN,CI,AIM,C,J,X1,X2,X3,X4
       DIMENSION X1(105),X2(105),X3(105),X4(105),SX1(19,102),
      1SX2(19,102),SX3(19,102),SX4(19,102),PX1(102),PX2(102),
      2PX3(102),PX4(102),H1X1(102),H1X2(102),H1X3(102),H1X4(102),
      3H2X1(102),H2X2(102),H2X3(102),H2X4(102),PR1(105),PR2(105),
      4PR3(105),PRFT(105),ADVT(105)
C
C      READING IN DATA
C
   100 FORMAT(8F9.4)
       READ 100,A,B,C,CC,CA,AN,CI,AIM
   101 FORMAT(3F9.4)
       READ 101,DT,W1,W2
   110 FORMAT('1VALUE OF THE CONSTANTS')
       PRINT 110
   120 FORMAT(' A=',F8.3,'B=',F8.3,'C=',F8.3,'CC=',F8.3,'CA=',
      1F7.3,'AN=',F8.3,'CI=',F7.4,'AIM=',F9.4,'S,O(T)=',F9.4,
      2'L6,C(T)=',F9.4,'DT=',F6.3)
       PRINT 120,A,B,C,CC,CA,AN,CI,AIM,W1,W2,DT
       DO 130 I=1,101
       SX2(1,I)=W1
       SX4(1,I)=W2
   130 CONTINUE
   140 FORMAT(' X1(0)=',F10.4,'  X2(0)=',F10.4,'  X3(0)=',F10.4,
      1'  X4(0)=',F10.4)
   141 FORMAT(1H ,I4,2X,4E20.7)
C
C      QUASILINEARIZATION ITERATIONS START
C
       DO 300 J=2,19,1
C
C      PARTICULAR SOLUTION
C
   150 FORMAT(4F10.4)
   160 FORMAT('-PARTICULAR SOLUTION')
       PRINT 160
       P=1.
       READ 150,X1(1),X2(1),X3(1),X4(1)
       PRINT 140,X1(1),X2(1),X3(1),X4(1)
       CALL RKT
       DO 170 I=1,101
       PX1(I)=X1(I)
       PX2(I)=X2(I)
       PX3(I)=X3(I)
       PX4(I)=X4(I)
```

```
  170 CONTINUE
      PRINT 141,(I,X1(I),X2(I),X3(I),X4(1),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION  FIRST SET
C
      P=0.0
  180 FORMAT('-HOMOGENEOUS SOLUTION   FIRST SET')
      PRINT 180
      READ 150,X1(1),X2(1),X3(1),X4(1)
      PRINT 140,X1(1),X2(1),X3(1),X4(1)
      CALL RKT
      DO 190 I=1,101
      H1X1(I)=X1(I)
      H1X2(I)=X2(I)
      H1X3(I)=X3(I)
      H1X4(I)=X4(I)
  190 CONTINUE
      PRINT 141,(1,X1(I),X2(I),X3(I),X4(1),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION   SECOND SET
C
      P=0.C
  200 FORMAT('-HOMOGENEOUS SOLUTION   SECOND SET')
      PRINT 200
      READ 150,X1(1),X2(1),X3(1),X4(1)
      PRINT 140,X1(1),X2(1),X3(1),X4(1)
      CALL RKT
      DO 210 I=1,101
      H2X1(I)=X1(I)
      H2X2(I)=X2(I)
      H2X3(I)=X3(I)
      H2X4(I)=X4(I)
  210 CONTINUE
      PRINT 141,(I,X1(I),X2(1),X3(I),X4(I),I=1,101,20)
C
C     SOLUTION OF INTEGRATION CONSTANTS
C
  220 FORMAT(2F9.4)
      READ 220,BB1,BB2
      B1=BB1-PX3(101)
      B2=BB2-PX4(101)
      DET=H1X3(101)*H2X4(101)-H1X4(101)*H2X3(101)
      A1=(B1*H2X4(101)-B2*H2X3(101))/DET
      A2=(B2*H1X3(101)-B1*H1X4(101))/DET
  230 FORMAT(1H ,'A1=',F9.4,'     A2= ',F9.4)
      PRINT 230,A1,A2
C
C     RECOVERY OF SOLUTION      SUPERPOSITION PRINCIPLE
C
      DO 250 I=1,101
      SX1(J,I)=PX1(I)+A1*H1X1(I)+A2*H2X1(I)
      SX2(J,I)=PX2(I)+A1*H1X2(I)+A2*H2X2(I)
      SX3(J,I)=PX3(I)+A1*H1X3(I)+A2*H2X3(I)
      SX4(J,I)=PX4(I)+A1*H1X4(I)+A2*H2X4(I)
  250 CONTINUE
```

```
  260 FORMAT(' FINAL SOLUTION       ITERATION NO    ',I4,//)
      JJ=J-1
      PRINT 260,JJ
  270 FORMAT(1H ,I4,2X,4E19.7,6X,E15.5,6X,E15.5)
C
C     CALCULATION OF CONTROL VARIABLE AND PROFIT
C
      P1R=0.0
      P2R=0.0
      P3R=0.0
      DO 280 I=1,101
      ADVT(I)=(SX4(J,I)/(2.*CA))*(SX2(J,I)/AN-1.)
      PR1(I)=P1R+DT*(C*SX2(J,I))
      P1R=PR1(I)
      PR2(I)=P2R+DT*(C1*((AIM-SX1(J,I))**2))
      P2R=PR2(I)
      PR3(I)=P3R+DT*(CA*SX2(J,I)*(ADVT(I)**2))
      P3R=PR3(I)
      PRFT(I)=PR1(I)-PR2(I)-PR3(I)
      PRFT(I)=0.0
  280 CONTINUE
      PRINT 270,(I,SX1(J,I),SX2(J,I),SX3(J,I),SX4(J,I),ADVT(I),
     1PRFT(I),I=1,101)
  290 FORMAT('-TOTAL PROFIT          ',F15.3,15X,3F10.3)
      PRINT 290,PRFT(101),PR1(101),PR2(101),PR3(101)
C
C     END OF ONE ITERATION
C
  300 CONTINUE
C
C     END OF DO LOOP FOR QUASILINEARIZATION
C
      STOP
      END
```

```
      SUBROUTINE RKT
C
C     THIS SUBROUTINE IS USED TO INTEGRATE THE FOUR LINEARIZED
C     EQUATIONS SIMULTANEOUSLY BY RUNGE KUTTA METHOD
C
      COMMON SX2,SX4,P,DT,A,B,CC,CA,AN,CI,AIM,C,J,X1,X2,X3,X4
      DIMENSION X1(105),X2(105),X3(105),X4(105),A1(105),
     1A2(105),A3(105),A4(105),B1(105),B2(105),B3(105),
     2B4(105),C1(105),C2(105),C3(105),C4(105),D1(105),
     3D2(105),D3(105),D4(105),SX2(19,102),SX4(19,102)
      DO 500 I=1,100
      V=SX2(J-1,I)
      W=SX4(J-1,I)
      TA=I-1
      T=TA*DT
      A1(I)=DT*(P*A+P*B*T-X2(I))
      B1A=CC*V+W*(V**2)/(CA*AN)-W*V/(2.*CA)-CC*(V**2)/AN-
     1W*(V**3)/(2.*CA*(AN**2))
      B2A=CC+2.*V*W/(CA*AN)-W/(2.*CA)-2.*CC*V/AN-3.*W*(V**2)
     1/(2.*CA*(AN**2))
      B3A=V**2/(CA*AN)-V/(2.*CA)-V**3/(2.*CA*(AN**2))
      B1(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+X2(I)*B2A+X4(I)*B3A)
      C1(I)=DT*(P*2.*CI*AIM-2.*CI*X1(I))
      D1A=C-W*CC+3.*(W**2)*(V**2)/(4.*CA*(AN**2))+W**2/(4.*CA)
     1-(W**2)*V/(CA*AN)+2.*CC*V*W/AN
      D2A=3.*V*(W**2)/(2.*CA*(AN**2))-W**2/(CA*AN)+2.*CC*W/AN
      D3A=W/(2.*CA)-CC+3.*W*(V**2)/(2.*CA*(AN**2))-2.*W*V/
     1(CA*AN)
     1+2.*CC*V/AN
      D1(I)=DT*(X3(I)+P*D1A-P*V*D2A-P*W*D3A+X2(I)*D2A+X4(I)*
     1D3A)
      A2(I)=DT*(P*A+P*B*(T+DT/2.)-(X2(I)+B1(I)/2.))
      B2(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B1(I)/2.)*B2A+
     1(X4(I)+D1(I)/2.)*B3A)
      C2(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A1(I)/2.))
      D2(I)=DT*((X3(I)+C1(I)/2.)+P*D1A-P*V*D2A-P*W*D3A+(X2(I)
     1+B1(I)/2.)*D2A+(X4(I)+D1(I)/2.)*D3A)
      A3(I)=DT*(P*A+P*B*(T+DT/2.)-(X2(I)+B2(I)/2.))
      B3(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B2(I)/2.)*B2A+
     1(X4(I)+D2(I)/2.)*B3A)
      C3(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A2(I)/2.))
      D3(I)=DT*((X3(I)+C2(I)/2.)+P*D1A-P*V*D2A-P*W*D3A+(X2(I)
     1+B2(I)/2.)*D2A+(X4(I)+D2(I)/2.)*D3A)
      A4(I)=DT*(P*A+P*B*(T+DT)-(X2(I)+B3(I)))
      B4(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B3(I))*B2A+
     1(X4(I)+D3(I))*B3A)
      C4(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A3(I)))
      D4(I)=DT*((X3(I)+C3(I))+P*D1A-P*V*D2A-P*W*D3A+(X2(I)+
     1B3(I))*D2A+(X4(I)+D3(I))*D3A)
      X1(I+1)=X1(I)+(A1(I)+2.*A2(I)+2.*A3(I)+A4(I))/6.
      X2(I+1)=X2(I)+(B1(I)+2.*B2(I)+2.*B3(I)+B4(I))/6.
      X3(I+1)=X3(I)+(C1(I)+2.*C2(I)+2.*C3(I)+C4(I))/6.
      X4(I+1)=X4(I)+(D1(I)+2.*D2(I)+2.*D3(I)+D4(I))/6.
  500 CONTINUE
      RETURN
```

APPENDIX 3

```
      $JOB            SHAH,RUN=CHECK,TIME=9,PAGES=100,LINES=55
      C
      C     THIS PROGRAM SOLVES A SET OF TWELVE DIFF. EQUATIONS TWO POINT
      C     SPLIT TYPE USING THE SUPERPOSITION PRINCIPLE AND RUNGA_KUTTA
      C     TECHNIQUE.
      C
      C     MAIN PROGRAM
      C
1           COMMCN X1,Y1,X2,Y2,A1,Q,Z1,Z2,Z3,Z4,Z5,Z6,TP,TQ,DT,XO,
           1GA,GB,EA,R,AQ,V1,V2,YO,EB,AN,CA,CI,AIM,CB,CC,CQ,C,AC,
           2CT,P,W1,W2,W3,W4,TEM1,TEM2,J,SAQ,SZ6
2           DIMENSION X1(105),Y1(105),X2(105),Y2(105),AI(105),
           1Q(105),Z1(105),Z2(105),Z3(105),Z4(105),Z5(105),
           2Z6(105),SX1(10,102),SY1(10,102),SX2(10,102),SY2(10,102),
           3SAI(10,102),SAQ(10,102),SZ1(10,102),SZ2(10,102),
           4SZ3(10,102),SZ4(10,102),SZ5(10,102),SZ6(10,102),
           5PX1(102),PY1(102),PX2(102),PY2(102),PI(102),PQ(102),
           6PZ1(102),PZ2(102),PZ3(102),PZ4(102),PZ5(102),PZ6(102),
           7XO(1C5),YO(105),TEM1(10,102),TEM2(10,102)
3           DIMENSION H1X1(102),H1Y1(102),H1X2(102),H1Y2(102),
           1H1A1(102),H1Q(102),H1Z1(102),H1Z2(102),H1Z3(102),
           2H1Z4(102),H1Z5(102),H1Z6(102),H2X1(102),H2Y1(102),
           3H2X2(102),H2Y2(102),H2A1(102),H2Q(102),H2Z1(102),
           4H2Z2(102),H2Z3(102),H2Z4(102),H2Z5(102),H2Z6(102),
           5H3X1(102),H3Y1(102),H3X2(102),H3Y2(102),H3AI(102),
           6H3Q(102),H3Z1(102),H3Z2(102),H3Z3(102),H3Z4(102),
           7H3Z5(102),H3Z6(102),H4X1(102),H4Y1(102),H4X2(102),
           8H4Y2(102),H4A1(102),H4Q(102),H4Z1(102),H4Z2(102),
           9H4Z3(102),H4Z4(102),H4Z5(102),H4Z6(102),H5XI(102),
           1H5Y1(102),H5X2(102),H5Y2(102),H5A1(102),H5Q(102),
           2H5Z1(102),H5Z2(102),H5Z3(102),H5Z4(102),H5Z5(102),
           3H5Z6(102),H6X1(102),H6Y1(102),H6X2(102),H6Y2(102),
           4H6AI(102),H6Q(102),H6Z1(102),H6Z2(102),H6Z3(102),
           5H6Z4(102),H6Z5(102),H6Z6(102)
4           DIMENSION ADV(105),PR1(105),PR2(105),PR3(105),PR4(105),
           1PR5(105),PR6(105),PRFT(105)
      C
      C     READING IN DATA
      C
5       401 FORMAT(2E10.3,2F7.1,2F5.1,5F4.1,F5.1,F3.1)
6           READ 401,GA,GB,EA,EB,TP,TQ,R,AQ,V1,V2,AIM,AN,C
7       402 FORMAT(F7.5,F4.2,F7.5,5F5.3,4F6.3,F5.1)
8           READ 402,CT,DT,CA,AC,CB,CC,CQ,CI,W1,W2,W3,W4,TIM
9       406 FORMAT(1H1,'VALUE OF THE CONSTANTS')
10          PRINT 406
11      403 FORMAT(4H-GA=,E10.3,'  GB=',E10.3,'  EA=',F7.1,'  EB=',
           1F7.1,'  R=',F4.1,'  FLOW RATE=',F5.1,'  V1=',F4.1,
           2'  V2=',F4.1,'  MEAN INV.=',F4.1,'  MEAN TEMP=',F6.2)
12          PRINT 403,GA,GB,EA,EB,R,AQ,V1,V2,AIM,TIM
13      404 FORMAT(1H-,'N=',F5.1,'  C=',F3.1,'  CT=',F7.5,'  DT=',
           1F4.2,'  CA=',F7.5,'  C1=',F5.3,'  C2=',F5.3,'  C3=',
           2F5.3,'  CQ=',F5.3,'  CI=',F5.3,' XO(T)=',F6.3,
           3' YO(T)=',F6.3)
14          PRINT 404,AN,C,CT,DT,CA,AC,CB,CC,CQ,CI,W1,W2
15      405 FORMAT(1H-,'T1=',E12.4,'  T2=',E12.4,'  QO(T)=',E10.2,
```

```
     1'  L6(T)=',E10.2)
      PRINT 405,TP,TQ,W3,W4
  129 FORMAT(1H ,I4,2X,12E10.2)
  110 FORMAT(1H-,'X1(0)=',F4.2,'Y1(0)=',F4.2,'X2(0)=',F4.2,
     1'Y2(0)=',F4.2,'I(0)=',F4.2,'Q(0)=',F4.2,'L1(0)=',F4.2,
     2'L2(0)=',F4.2,'L3(0)=',F4.2,'L4(0)=',F4.2,'L5(0)=',F4.2,
     3'L6(0)=',F4.2)
  725 FORMAT(4E15.5)
  726 FORMAT(1H ,'T1=',E15.5,'T2=',E15.5,'Q=',E15.5,'L6=',E15.5)
      DO 162 I=1,101
      TEM1(1,I)=TP
      TEM2(1,I)=TQ
      SAQ(I,I)=W3
      SZ6(1,I)=W4
  162 CONTINUE
C
C     QUASILINEARIZATION ITERATIONS START
C
      DO 300 J=2,10,1
C
C     PARTICULAR SOLUTION
C
  601 FORMAT(12(F5.2))
  600 FORMAT(1H-,'PARTICULAR SOLUTION')
      PRINT 600
      P=1.
      READ 601,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),Z1(1),
     1Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
      PRINT 110,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),Z1(1),
     2Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
      CALL RKT
      DO 201 I=1,101
      PX1(I)=X1(I)
      PY1(I)=Y1(I)
      PX2(I)=X2(I)
      PY2(I)=Y2(I)
      PI(I)=AI(I)
      PQ(I)=Q(I)
      PZ1(I)=Z1(I)
      PZ2(I)=Z2(I)
      PZ3(I)=Z3(I)
      PZ4(I)=Z4(I)
      PZ5(I)=Z5(I)
      PZ6(I)=Z6(I)
  201 CONTINUE
      PRINT 129,(I,X1(I),Y1(I),X2(I),Y2(I),AI(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION  FIRST SET
C
      P=0.0
  602 FORMAT(1H-'HOMOGENEOUS SOLUTION FIRST SET')
      PRINT 602
      READ 601,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),Z1(1),
     1Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
```

```
      PRINT 110,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),Z1(1),
     2Z2(1),Z3(1),Z4(1),Z5(I),Z6(1)
      CALL RKT
      DO 2C2 I=1,101
      H1X1(I)=X1(I)
      H1Y1(I)=Y1(I)
      H1X2(I)=X2(I)
      HIY2(I)=Y2(I)
      H1AI(I)=AI(I)
      H1Q(I)=Q(I)
      HIZ1(I)=Z1(I)
      H1Z2(I)=Z2(I)
      H1Z3(I)=Z3(I)
      H1Z4(I)=Z4(I)
      H1Z5(I)=Z5(I)
      H1Z6(I)=Z6(I)
  202 CONTINUE
      PRINT 129,(I,X1(I),YI(I),X2(I),Y2(I),AI(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION SECOND SET
C
      P=0.0
  603 FORMAT(1H-,'HOMOGENEOUS SOLUTION SECONO SET')
      PRINT 603
      READ 601,X1(1),Y1(1),X2(1),Y2(I),AI(1),Q(1),Z1(1),
     1Z2(1),Z3(1),Z4(1),Z5(1),76(I)
      PRINT 110,X1(1),YI(1),X2(1),Y2(1),AI(I),Q(I),Z1(1),
     2Z2(1),Z3(1),Z4(1),Z5(1),Z6(I)
      CALL RKT
      OO 2C3 I=1,101
      H2XI(I)=X1(I)
      H2Y1(I)=Y1(I)
      H2X2(I)=X2(I)
      H2Y2(I)=Y2(I)
      H2AI(I)=AI(I)
      H2Q(I)=Q(I)
      H2Z1(I)=Z1(I)
      H2Z2(I)=Z2(I)
      H2Z3(I)=Z3(I)
      H2Z4(I)=Z4(I)
      H2Z5(I)=Z5(I)
      H2Z6(I)=Z6(I)
  203 CONTINUE
      PRINT I29,(I,X1(I),Y1(I),X2(I),Y2(I),A1(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION THIRD SET
C
      P=0.0
  604 FORMAT(1H-,'HOMOGENEOUS SOLUTION THIRD SET')
      PRINT 604
      READ 601,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),ZI(I),
     1Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
      PRINT 110,X1(1),YI(1),X2(1),Y2(1),AI(1),Q(1),Z1(1),
```

```
     2Z2(1),Z3(1),Z4(1),Z5(I),Z6(I)
      CALL RKT
      DO 204 I=I,101
      H3X1(I)=X1(I)
      H3Y1(I)=Y1(I)
      H3X2(I)=X2(I)
      H3Y2(I)=Y2(I)
      H3AI(I)=AI(I)
      H3Q(I)=Q(I)
      H3Z1(I)=Z1(I)
      H3Z2(I)=Z2(I)
      H3Z3(I)=Z3(I)
      H3Z4(I)=Z4(I)
      H3Z5(I)=Z5(I)
      H3Z6(I)=Z6(I)
  204 CONTINUE
      PRINT 129,(I,XI(I),YI(I),X2(I),Y2(I),AI(I),Q(I),ZI(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=I,101,20)
C
C     HOMOGENEOUS SOLUTION FOURTH SET
C
      P=0.0
  605 FORMAT(1H-,'HOMOGENEOUS SOLUTION FOURTH SET')
      PRINT 605
      READ 601,X1(I),YI(1),X2(I),Y2(1),AI(1),Q(1),ZI(1),
     1Z2(1),Z3(1),Z4(I),Z5(1),Z6(I)
      PRINT 110,X1(1),Y1(1),X2(1),Y2(I),AI(1),Q(1),ZI(I),
     2Z2(1),Z3(1),Z4(I),Z5(I),Z6(1)
      CALL RKT
      DO 205 I=1,101
      H4X1(I)=X1(I)
      H4Y1(I)=YI(I)
      H4X2(I)=X2(I)
      H4Y2(I)=Y2(I)
      H4AI(I)=AI(I)
      H4Q(I)=Q(I)
      H4Z1(I)=Z1(I)
      H4Z2(I)=Z2(I)
      H4Z3(I)=Z3(I)
      H4Z4(I)=Z4(I)
      H4Z5(I)=Z5(I)
      H4Z6(I)=Z6(I)
  205 CONTINUE
      PRINT 129,(I,X1(I),Y1(I),X2(I),Y2(I),AI(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION FIFTH SET
C
      P=0.C
  606 FORMAT(IH-,'HOMOGENEOUS SOLUTION FIFTH SET')
      PRINT 606
      READ 601,X1(1),YI(1),X2(I),Y2(1),AI(I),Q(1),Z1(1),
     1Z2(1),Z3(1),Z4(1),Z5(1),Z6(I)
      PRINT 110,X1(1),Y1(1),X2(1),Y2(1),AI(1),Q(1),Z1(I),
     2Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
```

```
      CALL RKT
      DO 2C6 I=1,101
      H5X1(I)=X1(I)
      H5Y1(I)=Y1(I)
      H5X2(I)=X2(I)
      H5Y2(I)=Y2(I)
      H5AI(I)=AI(I)
      H5Q(I)=Q(I)
      H5Z1(I)=Z1(I)
      H5Z2(I)=Z2(I)
      H5Z3(I)=Z3(I)
      H5Z4(I)=Z4(1)
      H5Z5(I)=Z5(I)
      H5Z6(I)=Z6(I)
  206 CONTINUE
      PRINT 129,(I,X1(I),YI(I),X2(I),Y2(I),AI(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     HOMOGENEOUS SOLUTION  SIXTH SET
C
      P=0.0
  607 FORMAT(1H-,'HOMOGENEOUS SOLUTION SIXTH SET')
      PRINT 607
      READ 60I,XI(I),Y1(1),X2(1),Y2(1),AI(1),Q(1),ZI(1),
     1Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
      PRINT 110,X1(1),Y1(I),X2(1),Y2(1),AI(1),Q(1),Z1(1),
     2Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
      CALL RKT
      DO 2C7 I=I,I01
      H6X1(I)=X1(I)
      H6Y1(I)=Y1(I)
      H6X2(I)=X2(I)
      H6Y2(I)=Y2(I)
      H6AI(I)=AI(I)
      H6Q(I)=Q(I)
      H6Z1(I)=Z1(I)
      H6Z2(I)=Z2(I)
      H6Z3(I)=Z3(I)
      H6Z4(I)=Z4(I)
      H6Z5(I)=Z5(I)
      H6Z6(I)=Z6(I)
  207 CONTINUE
      PRINT 129,(I,X1(I),Y1(I),X2(I),Y2(I),AI(I),Q(I),Z1(I),
     1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,101,20)
C
C     SOLUTION INTEGRATION CONSTANTS
C
      DIMENSION B(6),A(36),BB(6)
      N=6
  150 FORMAT(6(F5.2))
      READ 150,(BB(I),I=1,6)
  800 FORMAT(1H ,'FINAL CONDITIONS',10X,6FI5.4)
      PRINT 800,(BB(I),I=1,6)
      B(1)=BB(1)-PZ5(101)
      B(2)=BB(2)-PZ1(101)
```

```
      B(3)=BB(3)-PZ2(101)
      B(4)=BB(4)-PZ3(101)
      B(5)=BB(5)-PZ4(101)
      B(6)=BB(6)-PZ6(101)
      A(1)=H1Z5(101)
      A(2)=H1Z1(101)
      A(3)=H1Z2(101)
      A(4)=H1Z3(101)
      A(5)=H1Z4(101)
      A(6)=H1Z6(101)
      A(7)=H2Z5(101)
      A(8)=H2Z1(101)
      A(9)=H2Z2(101)
      A(10)=H2Z3(101)
      A(11)=H2Z4(101)
      A(12)=H2Z6(101)
      A(13)=H3Z5(101)
      A(14)=H3Z1(101)
      A(15)=H3Z2(101)
      A(16)=H3Z3(101)
      A(17)=H3Z4(101)
      A(18)=H3Z6(101)
      A(19)=H4Z5(101)
      A(20)=H4Z1(101)
      A(21)=H4Z2(101)
      A(22)=H4Z3(101)
      A(23)=H4Z4(101)
      A(24)=H4Z6(101)
      A(25)=H5Z5(101)
      A(26)=H5Z1(101)
      A(27)=H5Z2(101)
      A(28)=H5Z3(101)
      A(29)=H5Z4(101)
      A(30)=H5Z6(101)
      A(31)=H6Z5(101)
      A(32)=H6Z1(101)
      A(33)=H6Z2(101)
      A(34)=H6Z3(101)
      A(35)=H6Z4(101)
      A(36)=H6Z6(101)
      KS=0
      CALL SIMQ(A,B,N,KS)
  151 FORMAT(1H-,'A1=',F15.5,'A2=',F15.5,'A3=',F15.5,'A4=',
     1F15.5,'A5=',F15.5,'A6=',F15.5)
      PRINT 151,(B(I),I=1,6)
C
C     RECOVERY OF SOLUTION     SUPERPOSITION PRINCIPLE
C
      DO 160 I=1,101
      SX1(J,I)=PX1(I)+B(1)*H1X1(I)+B(2)*H2X1(I)+B(3)*H3X1(I)+
     1B(4)*H4X1(I)+B(5)*H5X1(I)+B(6)*H6X1(I)
      SY1(J,I)=PY1(I)+B(1)*H1Y1(I)+B(2)*H2Y1(I)+B(3)*H3Y1(I)+
     1B(4)*H4Y1(I)+B(5)*H5Y1(I)+B(6)*H6Y1(I)
      SX2(J,I)=PX2(I)+B(1)*H1X2(I)+B(2)*H2X2(I)+B(3)*H3X2(I)+
     1B(4)*H4X2(I)+B(5)*H5X2(I)+B(6)*H6X2(I)
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      SY2(J,I)=PY2(I)+B(1)*H1Y2(I)+B(2)*H2Y2(I)+B(3)*H3Y2(I)+
     1B(4)*H4Y2(I)+B(5)*H5Y2(I)+B(6)*H6Y2(I)
      SAI(J,I)=PI(I)+B(I)*H1AI(I)+B(2)*H2AI(I)+B(3)*H3AI(I)+
     1B(4)*H4AI(I)+B(5)*H5AI(I)+B(6)*H6AI(I)
      SAQ(J,I)=PQ(I)+B(I)*HIQ(I)+B(2)*H2Q(I)+B(3)*H3Q(I)+
     1B(4)*H4Q(I)+B(5)*H5Q(I)+B(6)*H6Q(I)
      SZ1(J,I)=PZI(I)+B(1)*H1ZI(I)+B(2)*H2ZI(I)+B(3)*H3ZI(I)+
     1B(4)*H4ZI(I)+B(5)*H5Z1(I)+B(6)*H6Z1(I)
      SZ2(J,I)=PZ2(I)+B(1)*H1Z2(I)+B(2)*H2Z2(I)+B(3)*H3Z2(I)+
     1B(4)*H4Z2(I)+B(5)*H5Z2(I)+B(6)*H6Z2(I)
      SZ3(J,I)=PZ3(I)+B(I)*HIZ3(I)+B(2)*H2Z3(I)+B(3)*H3Z3(I)+
     IB(4)*H4Z3(I)+B(5)*H5Z3(I)+B(6)*H6Z3(I)
      SZ4(J,I)=PZ4(I)+B(I)*HIZ4(I)+B(2)*H2Z4(I)+B(3)*H3Z4(I)+
     1B(4)*H4Z4(I)+B(5)*H5Z4(I)+B(6)*H6Z4(I)
      SZ5(J,I)=PZ5(I)+B(I)*HIZ5(I)+B(2)*H2Z5(I)+B(3)*H3Z5(I)+
     1B(4)*H4Z5(I)+B(5)*H5Z5(I)+B(6)*H6Z5(I)
      SZ6(J,I)=PZ6(I)+B(1)*HIZ6(I)+B(2)*H2Z6(I)+B(3)*H3Z6(I)+
     IB(4)*H4Z6(I)+B(5)*H5Z6(I)+B(6)*H6Z6(I)
  I60 CONTINUE
C
C     PRINTING THE FINAL SOLUTION
C
  161 FORMAT(1H1,'FINAL SOLUTION    ITERATION NO ',I3)
      JJ=J-1
      PRINT I61,JJ
  I31 FORMAT(1H ,I4,3X,6E18.5)
      PRINT 13I,(I,SX1(J,I),SY1(J,I),SX2(J,I),SY2(J,I),
     ISAI(J,I),SAQ(J,I),I=I,1OI)
  132 FORMAT(1H-,'ADDITIONAL STATE VARIABLES',//)
      PRINT 132
      PRINT 131,(I,SZI(J,I),SZ2(J,I),SZ3(J,I),SZ4(J,I),
     1SZ5(J,I),SZ6(J,I),I=I,10I)
C
C     CALCULATION OF CONTROL VARIABLES AND TOTAL PROFIT
C
  I64 FORMAT(1H ,I4,5X,'ADVT=',EI2.4,17X,'TI=',EI4.6,3X,'T2=',
     1E14.6,12X,'TOTAL PROFIT',F1D.5)
      PIR=0.D
      P2R=0.D
      P3R=C.0
      P4R=0.D
      P5R=C.D
      P6R=C.D
      DO 5CD I=1,1D1
      ADV(I)=(SZ6(J,I)/(2.*CA))*((1./AN)-(1./SAQ(J,I)))
      PR1(I)=PIR+AC*CQ*SAQ(J,I)*DT
      PIR=PRI(I)
      PR2(I)=P2R+CB*AQ*SX2(J,I)*DT
      P2R=PR2(I)
      PR3(I)=P3R+CC*AQ*(I.-SX2(J,I)-SY2(J,I))*DT
      P3R=PR3(I)
      PR4(I)=P4R+CI*((AIM-SAI(J,I))**2)*DT
      P4R=PR4(I)
      PR5(I)=P5R+CA*(ADV(I)**2)*(SAQ(J,I)**2)*DT
      P5R=PR5(I)
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```
      PR6(I)=P6R+CT*((TIM-TEM2(J-1,I))**2+(TEM1(J-I,I)
     1-TEM2(J-1,I))**2)*DI
      P6R=PR6(I)
      PRFT(I)=PR1(I)+PR2(I)+PR3(I)-PR4(I)-PR5(I)-PR6(I)
  500 CONTINUE
  502 FORMAT(1H-,'TOTAL PROFIT=',F15.7,10X,6F15.5)
      PRINT 502,PRFT(101),PR1(101),PR2(101),PR3(101),PR4(101)
     1,PR5(101),PR6(101)
  163 FORMAT(1H1,'VALUES OF THE CONTROL VARIABLES')
      PRINT 163
  501 FORMAT(1H ,'   ADVT FOR PREVIOUS ITERATION         TEMP
     1 FOR CURRENT ITERATION',12X,'PROFIT FOR CURRENT ITERATION')
      PRINT 501
C
C     CALCULATION OF TEMP 1 AND TEMP 2 BY NEWTON RAPHSON METHOD
C
      DIMENSION TM1(202),TM2(202)
      IF (J.GE.3) GO TO 905
      TM1(1)=350.0
      TM2(1)=350.0
      GO TO 906
  905 TM1(1)=TEM1(J-1,1)
      TM2(1)=TEM2(J-1,1)
  906 CONTINUE
      DO 165 I=1,101
      DO 166 N=1,201
      D1T1A=((SZ1(J,I)-SZ2(J,I))*GA*SX1(J,I)*EA)/(R*(TM1(N)**4))
      D1T1B=(EA/R-2.*TM1(N))*EXP(-EA/(R*TM1(N)))
      D1T1C=(SZ2(J,I)*GB*SY1(J,I)*EB)/(R*(TM1(N)**4))
      D1T1D=(EB/R-2.*TM1(N))*EXP(-EB/(R*TMI(N)))
      D1T1=D1T1A*D1T1B+D1T1C*D1T1D-4.*CT
      D1T2=2.*CT
      D2T1=2.*CT
      D2T2A=((SZ3(J,I)-SZ4(J,I))*GA*SX2(J,I)*EA)/(R*(TM2(N)**4))
      D2T2B=(EA/R-2.*TM2(N))*EXP(-EA/(R*TM2(N)))
      D2T2C=(SZ4(J,I)*GB*SY2(J,I)*EB)/(R*(TM2(N)**4))
      D2T2D=(EB/R-2.*TM2(N))*EXP(-EB/(R*TM2(N)))
      D2T2=D2T2A*D2T2B+D2T2C*D2T2D-2.*CT
      FUN1A=((SZ1(J,I)-SZ2(J,I))*GA*SX1(J,I)*EA)/(R*(TM1(N)**2))
      FUN1B=EXP(-EA/(R*TM1(N)))
      FUN1C=(SZ2(J,I)*GB*SY1(J,I)*EB)/(R*(TM1(N)**2))
      FUN1D=EXP(-EB/(R*TM1(N)))
      FUN1E=2.*CT*(2.*TM1(N)-TM2(N)-TIM)
      FUN1=FUN1A*FUN1B+FUN1C*FUN1D-FUN1E
      FUN2A=((SZ3(J,I)-SZ4(J,I))*GA*SX2(J,I)*EA)/(R*(TM2(N)**2))
      FUN2B=EXP(-EA/(R*TM2(N)))
      FUN2C=(SZ4(J,I)*GB*SY2(J,I)*EB)/(R*(TM2(N)**2))
      FUN2D=EXP(-EB/(R*TM2(N)))
      FUN2E=2.*CT*(TM1(N)-TM2(N))
      FUN2=FUN2A*FUN2B+FUN2C*FUN2D+FUN2E
      RHS1=TM1(N)*D1T1+TM2(N)*D1T2-FUN1
      RHS2=TM1(N)*D2T1+TM2(N)*D2T2-FUN2
      DETR=D1T1*D2T2-D1T2*D2T1
      TM1(N+1)=(RHS1*D2T2-RHS2*D1T2)/DETR
      TM2(N+1)=(RHS2*D1T1-RHS1*D2T1)/DETR
```

```
      DEL1=TM1(N+1)-TM1(N)
      DEL2=TM2(N+1)-TM2(N)
      TMNW1=TM1(N+1)
      TMNW2=TM2(N+1)
      IF (ABS(DEL1).GT.0.1) GO TO 166
      IF (ABS(DEL2).LE.0.1) GO TO 170
  166 CONTINUE
  168 FORMAT(70X,'TEMP DID NOT CONVERGE',2X,I3,'  DEL1=',F9.3,
     1'  DEL2=',F9.3)
      PRINT 168,N,DEL1,DEL2
  170 TEM1(J,I)=TMNW1
      TEM2(J,I)=TMNW2
      IF (J.GE.3) GO TO 900
      TM1(1)=TEM1(J,I)
      TM2(1)=TEM2(J,I)
      GO TO 901
  900 TM1(1)=TEM1(J-1,I+1)
      TM2(1)=TEM2(J-1,I+1)
  901 CONTINUE
      PRINT 164,I,ADV(I),TEM1(J,I),TEM2(J,I),PRFT(I)
  165 CONTINUE
C
C     END OF ONE ITERATION
C
  300 CONTINUE
C
C     QUASILINEARIZATION DO LOOP ENDS HERE
C
      STOP
      END
```

```
      SUBROUTINE RKT
C
C     THIS SUBROUTINE IS USED TO INTEGRATE 12 LINEARIZED
C     EQUATIONS SIMULTANEOSLY BY RUNGE KUTTA METHOD
C
      COMMON X1,Y1,X2,Y2,AI,Q,Z1,Z2,Z3,Z4,Z5,Z6,TP,TQ,DT,XO,
     1GA,GB,EA,R,AQ,V1,V2,YO,EB,AN,CA,CI,AIM,CB,CC,CQ,C,AC,
     2CT,P,W1,W2,W3,W4,TEM1,TEM2,J,SAQ,SZ6
      DIMENSION X1(105),Y1(105),X2(105),Y2(105),AI(105),
     1Q(105),Z1(105),Z2(105),Z3(105),Z4(105),Z5(105),Z6(105),
     2A1(105),A2(105),A3(105),A4(105),B1(105),B2(105),B3(105),
     3B4(105),C1(105),C2(105),C3(105),C4(105),D1(105),D2(105),
     4D3(105),D4(105),E1(105),E2(105),E3(105),E4(105),F1(105),
     5F2(105),F3(105),F4(105),G1(105),G2(105),G3(105),G4(105),
     6H1(105),H2(105),H3(105),H4(105),R1(105),R2(105),R3(105),
     7R4(105),S1(105),S2(105),S3(105),S4(105),T1(105),T2(105),
     8T3(105),T4(105),U1(105),U2(105),U3(105),U4(105),XO(105),
     9YO(105),TEM1(10,102),TEM2(10,102),SAQ(10,102),SZ6(10,102)
      SP=AC/V1
      SQ=AC/V2
      DO 130 I=1,100
      TT1=R*TEM1(J-1,I)
      TT2=R*TEM2(J-1,I)
      XO(I)=W1
      YO(I)=W2
      V=SAQ(J-1,I)
      W=SZ6(J-1,I)
      A1(I)=DT*(P*SP*XO(I)-SP*X1(I)-GA*EXP(-EA/TT1)*X1(I))
      B1(I)=DT*(P*SP*YO(I)-SP*Y1(I)-GB*EXP(-EB/TT1)*Y1(I)+
     1GA*EXP(-EA/TT1)*X1(I))
      C1(I)=DT*(SQ*(X1(I)-X2(I))-GA*EXP(-EA/TT2)*X2(I))
      D1(I)=DT*(SQ*(Y1(I)-Y2(I))-GB*EXP(-EB/TT2)*Y2(I)+
     1GA*EXP(-EA/TT2)*X2(I))
      E1(I)=DT*(AQ*Y2(I)-CQ*Q(I))
      F1A=(C*V-C*(V**2)/AN+V*W/(CA*AN)-W/(2.*CA)-W*(V**2)/
     1(2.*CA*(AN**2)))
      F2A=(C-2.*C*V/AN+W/(CA*AN)-W*V/(CA*(AN**2)))
      F3A=(V/(CA*AN)-1./(2.*CA)-V**2/(2.*CA*(AN**2)))
      F1(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+Q(I)*F2A+Z6(I)*F3A)
      G1(I)=DT*(SP*Z1(I)-SQ*Z3(I)+(Z1(I)-Z2(I))*GA*EXP(-EA/TT1))
      H1(I)=DT*(SP*Z2(I)-SQ*Z4(I)+Z2(I)*EXP(-EB/TT1)*GB)
      R1(I)=DT*(SQ*Z3(I)+(Z3(I)-Z4(I))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
      S1(I)=DT*(SQ*Z4(I)+Z4(I)*GB*EXP(-EB/TT2)-Z5(I)*AQ-P*CC*AQ)
      T1(I)=DT*(P*2.*CI*AIM-2.*CI*AI(I))
      U1A=(AC*CQ-C*W+2.*C*V*W/AN+(W**2)*V/(2.*CA*(AN**2))
     1-W**2/(2.*CA*AN))
      U2A=(2.*C*W/AN+W**2/(2.*CA*(AN**2)))
      U3A=(2.*C*V/AN-C+W*V/(CA*(AN**2))-W/(CA*AN))
      U1(I)=DT*(CQ*Z5(I)+P*U1A-P*V*U2A-P*W*U3A+Q(I)*U2A+Z6(I)
     1*U3A)
      A2(I)=DT*(P*SP*XO(I)-SP*(X1(I)+A1(I)/2.)-GA*EXP(-EA/TT1)
     1*(X1(I)+A1(I)/2.))
      B2(I)=DT*(P*SP*YO(I)-SP*(Y1(I)+B1(I)/2.)-GB*EXP(-EB/TT1)
     1*(Y1(I)+B1(I)/2.)+GA*EXP(-EA/TT1)*(X1(I)+A1(I)/2.))
      C2(I)=DT*(SQ*((X1(I)+A1(I)/2.)-(X2(I)+C1(I)/2.))-
```

```
      1GA*EXP(-EA/TT2)*(X2(I)+C1(I)/2.))
74     D2(I)=DT*(SQ*((Y1(I)+B1(I)/2.)-(Y2(I)+D1(I)/2.))-GB*EXP
      1(-EB/TT2)*(Y2(I)+B1(I)/2.)+GA*EXP(-EA/TT2)*(X2(I)+
      2C1(I)/2.))
75     E2(I)=DT*(AQ*(Y2(I)+D1(I)/2.)-CQ*(Q(I)+F1(I)/2.))
76     F2(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F1(I)/2.)*F2A+
      1(Z6(I)+U1(I)/2.)*F3A)
77     G2(I)=DT*(SP*(Z1(I)+G1(I)/2.)-SQ*(Z3(I)+R1(I)/2.)+((
      1Z1(I)+G1(I)/2.)-(Z2(I)+H1(I)/2.))*GA*EXP(-EA/TT1))
78     H2(I)=DT*(SP*(Z2(I)+H1(I)/2.)-SQ*(Z4(I)+S1(I)/2.)+
      1(Z2(I)+H1(I)/2.)*EXP(-EB/TT1)*GB)
79     R2(I)=DT*(SQ*(Z3(I)+R1(I)/2.)+((Z3(I)+R1(I)/2.)-(Z4(I)
      1+S1(I)/2.))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
80     S2(I)=DT*(SQ*(Z4(I)+S1(I)/2.)+(Z4(I)+S1(I)/2.)*GB*
      1EXP(-EB/TT2)-(Z5(I)+T1(I)/2.)*AQ-P*CC*AQ)
81     T2(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E1(I)/2.))
82     U2(I)=DT*(CQ*(Z5(I)+T1(I)/2.)+P*U1A-P*V*U2A-P*W*U3A+
      1(Q(I)+F1(I)/2.)*U2A+(Z6(I)+U1(I)/2.)*U3A)
83     A3(I)=DT*(P*SP*X0(I)-SP*(X1(I)+A2(I)/2.)-GA*EXP(-EA/TT1)
      1*(X1(I)+A2(I)/2.))
84     B3(I)=DT*(P*SP*Y0(I)-SP*(Y1(I)+B2(I)/2.)-GB*EXP(-EB/TT1)
      1*(Y1(I)+B2(I)/2.)+GA*EXP(-EA/TT1)*(X1(I)+A2(I)/2.))
85     C3(I)=DT*(SQ*((X1(I)+A2(I)/2.)-(X2(I)+C2(I)/2.))-
      1GA*EXP(-EA/TT2)*(X2(I)+C2(I)/2.))
86     D3(I)=DT*(SQ*((Y1(I)+B2(I)/2.)-(Y2(I)+D2(I)/2.))-GB*EXP
      1(-EB/TT2)*(Y2(I)+B2(I)/2.)+GA*EXP(-EA/TT2)*(X2(I)+
      2C2(I)/2.))
87     E3(I)=DT*(AQ*(Y2(I)+D2(I)/2.)-CQ*(Q(I)+F2(I)/2.))
88     F3(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F2(I)/2.)*F2A+
      1(Z6(I)+U2(I)/2.)*F3A)
89     G3(I)=DT*(SP*(Z1(I)+G2(I)/2.)-SQ*(Z3(I)+R2(I)/2.)+((
      1Z1(I)+G2(I)/2.)-(Z2(I)+H2(I)/2.))*GA*EXP(-EA/TT1))
90     H3(I)=DT*(SP*(Z2(I)+H2(I)/2.)-SQ*(Z4(I)+S2(I)/2.)+
      1(Z2(I)+H2(I)/2.)*EXP(-EB/TT1)*GB)
91     R3(I)=DT*(SQ*(Z3(I)+R2(I)/2.)+((Z3(I)+R2(I)/2.)-(Z4(I)
      1+S2(I)/2.))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
92     S3(I)=DT*(SQ*(Z4(I)+S2(I)/2.)+(Z4(I)+S2(I)/2.)*GB*
      1EXP(-EB/TT2)-(Z5(I)+T2(I)/2.)*AQ-P*CC*AQ)
93     T3(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E2(I)/2.))
94     U3(I)=DT*(CQ*(Z5(I)+T2(I)/2.)+P*U1A-P*V*U2A-P*W*U3A+
      1(Q(I)+F2(I)/2.)*U2A+(Z6(I)+U2(I)/2.)*U3A)
95     A4(I)=DT*(P*SP*X0(I)-SP*(X1(I)+A3(I))-GA*EXP(-EA/TT1)*
      1(X1(I)+A3(I)))
96     B4(I)=DT*(P*SP*Y0(I)-SP*(YI(I)+B3(I))-GB*EXP(-EB/TT1)*
      1(Y1(I)+B3(I))+GA*EXP(-EA/TT1)*(X1(I)+A3(I)))
97     C4(I)=DT*(SQ*((X1(I)+A3(I))-(X2(I)+C3(I)))-GA*EXP(-EA/
      1TT2)*(X2(I)+C3(I)))
98     D4(I)=DT*(SQ*((Y1(I)+B3(I))-(Y2(I)+D3(I)))-GB*EXP(-EB/
      1TT2)*(Y2(I)+B3(I))+GA*EXP(-EA/TT2)*(X2(I)+C3(I)))
99     E4(I)=DT*(AQ*(Y2(I)+D3(I))-CQ*(Q(I)+F3(I)))
00     F4(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F3(I))*F2A+(Z6(I)
      1+U3(I))*F3A)
01     G4(I)=DT*(SP*(Z1(I)+G3(I))-SQ*(Z3(I)+R3(I))+((Z1(I)+
      1G3(I))-(Z2(I)+H3(I)))*GA*EXP(-EA/TT1))
02     H4(I)=DT*(SP*(Z2(I)+H3(I))-SQ*(Z4(I)+S3(I))+(Z2(I)+
```

```
     1H3(I))*EXP(-EB/TT1)*GB)
      R4(I)=DT*(SQ*(Z3(I)+R3(I))+((Z3(I)+R3(I))-(Z4(I)+S3(I)
     1))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
      S4(I)=DT*(SQ*(Z4(I)+S3(I))+(Z4(I)+S3(I))*GB*EXP(-EB/
     1TT2)-(Z5(I)+T3(I))*AQ-P*CC*AQ)
      T4(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E3(I)))
      U4(I)=DT*(CQ*(Z5(I)+T3(I))+P*U1A-P*V*U2A-P*W*U3A+(Q(I)
     1+F3(I))*U2A+(Z6(I)+U3(I))*U3A)
      X1(I+1)=X1(I)+1./6.*(A1(I)+2.*A2(I)+2.*A3(I)+A4(I))
      Y1(I+1)=Y1(I)+1./6.*(B1(I)+2.*B2(I)+2.*B3(I)+B4(I))
      X2(I+1)=X2(I)+1./6.*(C1(I)+2.*C2(I)+2.*C3(I)+C4(I))
      Y2(I+1)=Y2(I)+1./6.*(D1(I)+2.*D2(I)+2.*D3(I)+D4(I))
      AI(I+1)=AI(I)+1./6.*(E1(I)+2.*E2(I)+2.*E3(I)+E4(I))
      Q(I+1)=Q(I)+1./6.*(F1(I)+2.*F2(I)+2.*F3(I)+F4(I))
      Z1(I+1)=Z1(I)+1./6.*(G1(I)+2.*G2(I)+2.*G3(I)+G4(I))
      Z2(I+1)=Z2(I)+1./6.*(H1(I)+2.*H2(I)+2.*H3(I)+H4(I))
      Z3(I+1)=Z3(I)+1./6.*(R1(I)+2.*R2(I)+2.*R3(I)+R4(I))
      Z4(I+1)=Z4(I)+1./6.*(S1(I)+2.*S2(I)+2.*S3(I)+S4(I))
      Z5(I+1)=Z5(I)+1./6.*(T1(I)+2.*T2(I)+2.*T3(I)+T4(I))
      Z6(I+1)=Z6(I)+1./6.*(U1(I)+2.*U2(I)+2.*U3(I)+U4(I))
  130 CONTINUE
      RETURN
      END
```

```
      SUBRCUTINE SIMQ(A,B,N,KS)
C
C     THIS SUBROUTINE IS USEO TO INVERT A SIX BY SIX
C     MATRIX ENCOUNTERED IN THE CALCULATION OF SIX
C     INTEGRATION CONSTANTS. IHIS IS SUPPLIEO BY IBM.
C
      OIMENSION A(I),B(1)
C
C        FORWARO SOLUTION
C
      TOL=0.0
      KS=0
      JJ=-N
      DO 65 J=I,N
      JY=J+1
      JJ=JJ+N+I
      BIGA=0
      IT=JJ-J
      OO 30 I=J,N
C
C        SEARCH FOR MAXIMUM COEFFICIENT IN COLUMN
C
      IJ=IT+I
      IF(ABS(BIGA)-ABS(A(IJ))) 20,30,30
   20 BIGA=A(IJ)
      IMAX=I
   30 CONTINUE
C
C        TEST FOR PIVOT LESS THAN TOLERANCE (SINGULAR MATRIX)
C
      IF(ABS(BIGA)-TOL) 35,35,40
   35 KS=1
      RETURN
C
C        INTERCHANGE ROWS IF NECESSARY
C
   40 I1=J+N*(J-2)
      IT=IMAX-J
      DO 50 K=J,N
      I1=I1+N
      I2=I1+IT
      SAVE=A(I1)
      A(I1)=A(I2)
      A(I2)=SAVE
C
C        DIVIDE EQUATION BY LEAOING COEFFICIENT
C
   50 A(I1)=A(I1)/BIGA
      SAVE=B(IMAX)
      B(IMAX)=B(J)
      B(J)=SAVE/BIGA
C
C        ELIMINATE NEXT VARIABLE
C
      IF(J-N) 55,70,55
```

```
   55 IQS=N*(J-1)
      DO 65 IX=JY,N
      IXJ=IQS+IX
      IT=J-IX
      DO 60 JX=JY,N
      IXJX=N*(JX-1)+IX
      JJX=IXJX+IT
   60 A(IXJX)=A(IXJX)-(A(IXJ)*A(JJX))
   65 B(IX)=B(IX)-(B(J)*A(IXJ))
C
C        BACK SOLUTION
C
   70 NY=N-1
      IT=N*N
      DO 80 J=1,NY
      IA=IT-J
      IB=N-J
      IC=N
      DO 80 K=1,J
      B(IB)=B(IB)-A(IA)*B(IC)
      IA=IA-N
   80 IC=IC-1
      RETURN
      END
```

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APPLICATION OF QUASILINEARIZATION TO INDUSTRIAL MANAGEMENT SYSTEMS

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The importance of quantitative techniques in decision making emphasizes the need of efficient techniques as a tool for solving management problems. However, fairly powerful algorithms are not yet available in solving dynamic management problems involving differential equations.

The two point boundary value problem with non-linear differential equations provides such an example. The nonlinearity in the performance equations does not allow the application of superposition principle.

Quasilinearization helps overcome this difficulty. It linearizes the non-linear equations and provides an algorithm which would give the solution by an iterative procedure.

The purpose of this work is to investigate the effectiveness of this recently developed tool in solving various industrial management problems.

First a brief introduction and computational procedure of quasilinearization is given. Then its application to an advertisement problem with two state variables and one control variable is discussed in detail.

Next is discussed the application of quasilinearization to an advertisement and production problem. This model has six state variables and three control variables. In addition, the profiles are fairly unstable due to the rapid change of variables with time.

It was concluded:

1. Choosing the initial approximations to start the solution is not difficult in most cases.
2. The convergence rate is almost independent of the choice of the initial approximations.
3. This algorithm converges quadratically, if it does converge.

4. For rapidly increasing or rapidly decreasing profiles, first variational and second variational techniques seemed to have failed. On the other hand, this method encountered no problem in converging to the optimal solution.

5. Because of the intimate association between the boundary value problems and the optimization and control problems, this technique may provide a useful tool for the systems analysts.