AN APPROXIMATE IDENTITY OPERATOR FOR CONTINUOUS SERVOMECHANISMS WITH TIME LAG

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INTRODUCTION

Rault (1) has outlined a method of servo design by considering the admissibility problem first. For the admissibility criterion he used the following definition based upon King's (2) zero error coefficient theory.

Definition. A feedback device is admissible into the servo class if it satisfies King's criterion. He further expands upon this admissibility criterion by the introduction of an algorithm for determining an approximate identity operator.

An approximate identity (or aidentity) of order k is defined as a servo that follows an input $t^k/k!$ with zero steady-state error. An example of a first-order aidentity is the transfer function

$$T(s) = \frac{1}{1 + s} \tag{1}$$

This function follows a unit step function with zero steady-state error.

King (2) and Rault (1) have developed a set of relationships which are sufficient to guarantee that a transfer function is an aidentity of some given order. They extended this theory to a relationship between the coefficients of the numerator and the denominator.

Rault observed that an increase of aidentity order increased the performance of the servo. However, the transfer functions under consideration were either ratios of polynomials of s alone or z alone. The present paper discusses the admissibility of continuous feedback devices with time lag into the servo class and finds a process by which this control system can be improved. A constraint that the servo must be stable will be imposed; an unstable servo is a self-contradictory system. To insure this last condition only positive real functions will be used. Stability considerations alone do not exclude the large class of partial positive real functions because some partial positive real functions are stable and others are unstable.

It is beyond the scope of this paper to study the transient response of these systems.

REQUIREMENTS FOR THE n^{th}-ORDER AIDENTITY

A control system whose transfer function is dependent on s and time lag e^{-Ts} can have the form:

$$T(s,z) = \frac{\sum_{n=0}^{N} \sum_{m=0}^{N} a_{n,m} s^n e^{-mTs}}{\sum_{n=0}^{N} \sum_{m=0}^{N} b_{n,m} s^n e^{-mTs}} \qquad (2)$$

This can be expressed as a rational bilinear form

$$T(s,z) = \frac{\sigma' A \zeta}{\sigma' B \zeta} \qquad (3)$$

where $z = e^{-Ts}$

$$\sigma = (s^0, s^1, s^2, \ldots, s^N)' \qquad (4)$$

$$\zeta = (z^0, z^1, z^2, \ldots, z^N)'$$

$A = (a_{m,n})$, a square matrix of order N + 1

= $(b_{m,n})$, a square matrix of order N + 1

' (primes) denote the transpose operation.

<u>Lemma</u>. A transfer function with time lag as defined by (2) is an n^{th}-order aidentity if it satisfies

$$
\left.
\begin{aligned}
&\left.\left.\sum_{k=0}^{N} a_{0,k} = \sum_{k=0}^{N} b_{0,k} \quad\right\}\ 1^{st}\text{-order aidentity} \\
&\sum_{k=0}^{N} k a_{0,k} = \sum_{k=0}^{N} k b_{0,k} \\
&\sum_{k=0}^{N} a_{1,k} = \sum_{k=0}^{N} b_{1,k} \quad\right\}\ 2^{nd}\text{-order aidentity} \\
&\quad\vdots \qquad\qquad\quad \vdots \\
&\sum_{k=0}^{N} k^{n-1} a_{0,k} = \sum_{k=0}^{N} k^{n-1} b_{0,k} \\
&\sum_{k=0}^{N} k^{n-2} a_{1,k} = \sum_{k=0}^{N} k^{n-2} b_{1,k} \\
&\quad\vdots \qquad\qquad\quad \vdots \\
&\sum_{k=0}^{N} k a_{n-2,k} = \sum_{k=0}^{N} k b_{n-2,k} \\
&\sum_{k=0}^{N} a_{n-1,k} = \sum_{k=0}^{N} b_{n-1,k}
\end{aligned}
\right\}\ n^{th}\text{-order aidentity} \qquad (5)
$$

These relations are independent of the nonzero sampling interval, T seconds.

<u>Proof</u>. The steady-state error of this system can be described as

$$E_n = \lim_{s \to 0} \left[\frac{1}{s^n} - \frac{\sigma' A \zeta}{\sigma' B \zeta} \frac{1}{s^n} \right] s \qquad (6)$$

where n-1 is the degree of the polynomial input function. If the system is an n^{th}-order aidentity, E_n must be zero. This equation may be expressed as

$$E_n = \lim_{s \to 0} \left[\frac{\sigma' B \zeta - \sigma' A \zeta}{(s^{n-1})(\sigma' B \zeta)} \right] = 0 \qquad (7)$$

By assuming that

$$\lim_{s \to 0} \sigma' B \zeta \neq 0 \qquad (8)$$

we obtain

$$E_n = \lim_{s \to 0} \left[\frac{\sigma' B \zeta - \sigma' A \zeta}{s^{n-1}} \right] = 0 \qquad (9)$$

Starting with a unit step function input, the first-order aidentity criterion is

$$E_1 = \lim_{s \to 0} (\sigma' B \zeta - \sigma' A \zeta) = 0 \qquad (10)$$

which yields

$$\lim_{s \to 0} (\sigma' A \zeta) = \lim_{s \to 0} (\sigma' B \zeta) \qquad (11)$$

For the second-order aidentity one obtains the previous formula and

$$E_2 = \lim_{s \to 0} \left[\frac{\sigma' B \zeta - \sigma' A \zeta}{s} \right] = 0 \qquad (12)$$

It is noted that E_2 is indeterminate in form and L'Hospital's Rule may be applied to obtain

$$E_2 = \lim_{s \to 0} (\nabla \sigma' B \zeta - \nabla \sigma' A \zeta) = 0 \qquad (13)$$

$$\lim_{s \to 0} (\nabla \sigma' A \zeta) = \lim_{s \to 0} (\nabla \sigma' B \zeta) \qquad (14)$$

where $\nabla = d/ds$.

This illustrates the fact that for the second-order aidentity to hold, the first must also hold.

By assuming that the first n-1 aidentities hold, an expression for the n^{th}-order aidentity can be obtained.

$$E_n = \lim_{s \to 0} \left[\frac{\sigma' B \zeta - \sigma' A \zeta}{s^{n-1}} \right] = 0 \qquad (15)$$

By using L'Hospital's Rule n-1 times

$$E_n = \lim_{s \to 0} (\nabla^{n-1} \sigma' B \zeta - \nabla^{n-1} \sigma' A \zeta) = 0 \qquad (16)$$

which implies

$$\lim_{s \to 0} (\nabla^{n-1} \sigma' A \zeta) = \lim_{s \to 0} (\nabla^{n-1} \sigma' B \zeta) \qquad (17)$$

The operator ∇^{n-1} is the $(n-1)^{th}$ derivative of the products $\sigma' A \zeta$, and $\sigma' B \zeta$. This operation can be expressed as Leibnitz's formula for the $(n-1)^{th}$ derivative of the product of two functions (4), or

$$\nabla^{n-1} \sigma' A \zeta = \sum_{r=0}^{n-1} C_{n-1,r} (\nabla^{n-r-1} \sigma' A)(\nabla^r \zeta) \qquad (18)$$

and

$$\nabla^{n-1} \sigma' B \zeta = \sum_{r=0}^{n-1} C_{n-1,r} (\nabla^{n-r-1} \sigma' B)(\nabla^r \zeta) \qquad (19)$$

where $C_{n-1,r}$ are the binomial coefficients.

These relationships between numerator and denominator will be further developed. Beginning with the first-order case, one obtains

$$\lim_{s \to 0} \sigma' A \zeta = u_0' A v_0 \tag{20}$$

where

$$u_0' = \lim_{s \to 0} \sigma' = (1, 0, 0, . . ., 0) \tag{21}$$

and

$$v_0 = \lim_{s \to 0} \zeta = (1, 1, 1, . . ., 1)' ;$$

Likewise for the denominator one obtains

$$\lim_{s \to 0} \sigma' B \zeta = u_0' B v_0 \tag{22}$$

and

$$u_0' A v_0 = u_0' B v \tag{23}$$

This equation may be reduced to a relationship between the coefficients of the transfer function by replacing the u_0's and v_0's by their appropriate vectors and multiplying.

$$u_0' A v_0 = (1,0,0,0,\ldots,0) \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N} \\ \vdots & & & \vdots \\ a_{N,0} & \cdot & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{24}$$

$$= \sum_{k=0}^{N} a_{0,k}$$

and

$$u_0' B v_0 = (1,0,0,0,\ldots,0) \begin{bmatrix} b_{0,0} & \cdots & b_{0,N} \\ \vdots & & \vdots \\ b_{N,0} & \cdots & b_{N,N} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{25}$$

$$= \sum_{k=0}^{N} b_{0,k}$$

For the second-order aidentity to hold, the above relationship as well as

$$\lim_{s \to 0} (\nabla \sigma' A \zeta) = \lim_{s \to 0} (\nabla \sigma' B \zeta) \tag{26}$$

must hold. This last expression can be expanded by executing the implied differentiation.

$$\lim_{s \to 0} (\nabla \sigma' A \zeta) = u_1' A v_0 + u_0' A v_1$$

$$= \lim_{s \to 0} (\nabla \sigma' B \zeta) = u_1' B v_0 + u_0 B v_1 \tag{27}$$

where u_0 and v_0 are as previously defined

$$u_1 = \lim_{s \to 0} (\nabla \sigma) = (0, 1, 0, 0, \ldots, 0)' \tag{28}$$

$$v_1 = \lim_{s \to 0} (\nabla \zeta) = -T(0, 1, 2, 3, \ldots, N)$$

Using these vectors the coefficient relationship is established.

$$(0,1,0,\ldots,0) \begin{bmatrix} a_{0,0} & \cdots & a_{0,N} \\ a_{1,0} & & \cdot \\ \vdots & & \vdots \\ a_{N,0} & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$-T(1,0,0,\ldots,0) \begin{bmatrix} a_{0,0} & \cdots & a_{0,N} \\ a_{1,0} & & \cdot \\ \vdots & & \vdots \\ a_{N,0} & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{bmatrix} \tag{29}$$

$$= \sum_{k=0}^{N} a_{1,k} - T \sum_{k=0}^{N} ka_{0,k}$$

The same process can be applied to the left side and yields

$$u_1'Bv_0 + u_0'Bv_1 = \sum_{k=0}^{N} b_{1,k} - T \sum_{k=0}^{N} kb_{0,k} \qquad (30)$$

or

$$\sum_{k=0}^{N} a_{1,k} - T \sum_{k=0}^{N} ka_{0,k} = \sum_{k=0}^{N} b_{1,k} - T \sum_{k=0}^{N} kb_{0,k} \qquad (31)$$

If aidentities of order one through n-1 are assumed to hold, the coefficient relationship for the n^{th}-order aidentity can be derived. It has already been shown that for the n^{th}-order aidentity to hold the relationship

$$\lim_{s \to 0} (\nabla^{n-1} \sigma' A \zeta) = \lim_{s \to 0} (\nabla^{n-1} \sigma' B \zeta) \qquad (32)$$

must hold.

Taking the limit of the implied differentiation yields

$$u_{n-1}' Av_0 + (n-1)u_{n-2}' Av_1 + \frac{(n-1)(n-2)}{2!} u_{n-3}' Av_2$$

$$+ \ . \ . \ . + (n-1)u_1' Av_{n-2} + u_0' Av_{n-1}$$

$$= u_{n-1}' Bv_0 + (n-1)u_{n-2}' Bv_1 + \frac{(n-1)(n-2)}{2!} u_{n-3}' Bv_2$$

$$+ \ . \ . \ . + (n-1)u_1' Bv_{n-2} + u_0' Bv_{n-1} \qquad (33)$$

where u_0, u_1, v_0, v_1 are as previously defined.

$$u_2 = \lim_{s \to 0} (\nabla^2 \sigma) = (0,0,2,0,\ \ldots,\ 0)'$$

$$u_3 = \lim_{s \to 0} (\nabla^3 \sigma) = (0,0,0,6,0,\ \ldots,\ 0)'$$

$$\vdots \qquad (34)$$

$$u_{n-1} = \lim_{s \to 0} (\nabla^{n-1} \sigma) = (0,0,0,\ \ldots,\ 0,(n-1)!,0,\ \ldots,\ 0)$$

$$v_2 = \lim_{s \to 0} (\nabla^2 \zeta) = T^2(0,1,4,9,\ \ldots,\ N^2)'$$

$$v_3 = \lim_{s \to 0} (\nabla^3 \zeta) = -T^3(0,1,8,27,\ \ldots,\ N^3)'$$

$$\vdots \qquad (35)$$

$$v_{n-1} = \lim_{s \to 0} (\nabla^{n-1} \zeta) = (-1)^{n-1}\, T^{n-1}\, (0,1,2^{n-1},\ \ldots,\ N^{n-1})'$$

Replacing u's and v's by their appropriate vectors and executing the implied multiplication gives the required coefficient relationship for the n^{th}-order aidentity.

$$(n-1)! \sum_{k=0}^{N} a_{n-1,k} - (n-1)!\, T \sum_{k=0}^{N} k a_{n-2,k} + \frac{(n-1)!}{2!} T^2 \sum_{k=0}^{N} k^2 a_{n-3,k}$$

$$- \ . \ . \ . + \frac{(-1)^m (n-1)!}{m!} T^m \sum_{k=0}^{N} k^m a_{n-(1+m),k} + \ . \ . \ .$$

$$+ \frac{(-1)^{n-1}(n-1)!}{(n-1)!} T^{n-1} \sum_{k=0}^{N} k^{n-1}\, a_{0,k}$$

$$= (n-1)! \sum_{k=0}^{N} b_{n-1,k} - (n-1)!\, T \sum_{k=0}^{N} k b_{n-2,k} \qquad (36)$$

$$+ \frac{(n-1)!}{2!} T^2 \sum_{k=0}^{N} k^2 b_{n-3,k} + \ldots + \frac{(-1)^m (n-1)!}{m!} T^m \sum_{k=0}^{N} k^m b_{n-(m+1),k}$$

$$+ \frac{(-1)^{n-1}(n-1)!}{(n-1)!} T^{n-1} \sum_{k=0}^{N} k^{n-1} b_{0,k}$$

The aidentity equation in this expanded form can be looked upon as polynomials in T. These polynomials are equal if and only if coefficients of like powers of T are equal. It is therefore permissible to write the above equation as a set of equations by equating like powers of T. This results in the set of equations

$$\begin{aligned} \sum_{k=0}^{N} a_{n-1,k} &= \sum_{k=0}^{N} b_{n-1,k} \\ \sum_{k=0}^{N} k a_{n-2,k} &= \sum_{k=0}^{N} k b_{n-2,k} \\ &\vdots \\ \sum_{k=0}^{N} k^{n-1} a_{0,k} &= \sum_{k=0}^{N} k^{n-1} b_{0,k} \end{aligned} \qquad (37)$$

This completes the Lemma's proof.

AN AIDENTITY ALGORITHM

The aidentity criterion for the rational bilinear form can be expressed as a matrix algorithm of the form

$$\text{Diagonal } (AQ_n) = \text{diagonal } (BQ_n) \qquad (38)$$

The matrix Q_n is the n^{th}-order aidentity operator matrix. This matrix is used only after showing that the first order through the $(n-1)^{th}$-order aidentity criterion are satisfied by the given transfer function.

$$
\begin{aligned}
\mathrm{diag}(AQ_1) &= \mathrm{diag}(BQ_2) \\
\mathrm{diag}(AQ_2) &= \mathrm{diag}(BQ_2) \\
\mathrm{diag}(AQ_3) &= \mathrm{diag}(BQ_3) \\
&\vdots \\
\mathrm{diag}(AQ_n) &= \mathrm{diag}(BQ_n)
\end{aligned} \qquad (39)
$$

for the complete n^{th}-order aidentity criterion.

Definition. Q_n is a square matrix of order N+1 where N+1 is the order of the square matrices A and B in the rational bilinear form. Specific forms of Q_n are:

$$
Q_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \qquad (40)
$$

$$
Q_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 2 & 1 & 0 & \cdots & 0 \\ 3 & 1 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ N & 1 & 0 & \cdots & 0 \end{bmatrix} \qquad (41)
$$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$$\underset{(N+1)x(N+1)}{Q_n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1^{(n-1)} & 1^{(n-2)} & \cdots & 1^{(n-p)} & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 2^{(n-1)} & 2^{(n-2)} & \cdots & 2^{(n-p)} & \cdots & 2 & 1 & 0 & \cdots & 0 \\ 3^{(n-1)} & . & & . & & . & . & . & & . \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ N^{(n-1)} & N^{(n-2)} & \cdots & N^{(n-p)} & \cdots & N & 1 & 0 & \cdots & 0 \end{bmatrix} \qquad (42)$$

METHODS OF AIDENTITY IMPROVEMENT

The criterion for a servo that will follow an n^{th}-order input function has been defined. The next problem is to generate this n^{th}-order servo recursively given some simpler system as a starting point.

Rault has shown that given a transfer function A (a function of s alone or z alone) which is at least a first-order aidentity; a higher ordered aidentity will be generated by the transformation

$$\frac{B_n - 1}{B_n + 1} = \left[\frac{A - 1}{A + 1}\right]^n \qquad (43)$$

This can be extended to the form

$$\frac{B_{n+1} - 1}{B_{n+1} + 1} = \left[\frac{A - 1}{A + 1}\right]^n \left[\frac{A - 1}{A + 1}\right] \qquad (44)$$

which reduces to

$$B_{n+1} = \frac{B_n + A}{1 + AB_n} \tag{45}$$

This is a special case of Richards' form for a positive real function

$$B = \frac{a + b}{\zeta^2 + ab} \tag{46}$$

Halijak (3) has shown that if ζ^2 is a positive real constant, a and b are positive real functions; then B is also positive real. If only positive real functions are considered, B_{n+1} will then always be positive real, and hence stable. This method of improvement of the function B_n has the canonical (i.e., most economical) block diagram of Fig. 1.

In the context of equation (43), B_n is merely the n^{th} iteration of A, the original transfer function. It is possible, however, to use two separate functions for A and B_n and get an increased order aidentity. With the idea of two independent functions this aidentity improvement method can be looked at in two ways: (1) as an iterative procedure using simple structures for A to give improvement; and (2) as a one-step procedure using a complex structure for A which will achieve the same improvement.

Example of Iterative Process

Choose:

$$A = \frac{1}{s + 1} \tag{47}$$

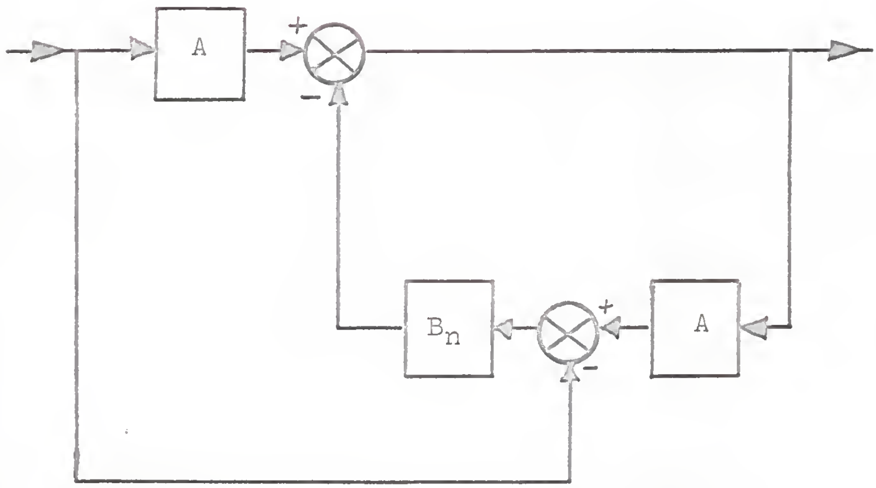

Fig. 1. Canonical block diagram of

$$B_{n+1} = \frac{A + B_n}{1 + AB_n} .$$

and
$$B_1 = \frac{1}{1 + 2(1-z)} \tag{48}$$

Both of these functions are p.r.f. and approximate identities. By using the Richards' form, a new transfer function is generated which is an approximate identity of order two.

$$B_2 = \frac{\dfrac{1}{1+s} + \dfrac{1}{1+2(1-z)}}{1 + \dfrac{1}{(1+s)(1+2(1-z))}}$$

$$= \frac{4 + s - 2z}{4 + 3s - 2z - 2sz}$$

$$= \frac{(1,s)\begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix}}{(1,s)\begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix}} \tag{49}$$

The improvement of the response may be shown by applying the aidentity algorithm derived earlier.

For the first-order aidentity one obtains

$$\operatorname{diag}\left[\begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right] = \operatorname{diag}\left[\begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right] \tag{50}$$

$$= (2,0)$$

The system under consideration was already a first-order aidentity. To show an improvement the new system must be at least a second-order aidentity. Applying the second-order aidentity criterion,

$$\text{Diag}\left[\begin{bmatrix}4 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\right] = \text{diag}\left[\begin{bmatrix}4 & -2\\ 3 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\right] \qquad (51)$$

$$= (-2,\ 1)$$

This shows that aidentity improvement has been realized. It is readily seen that this is the highest order aidentity contained in the given transfer function. To achieve a higher order aidentity the same process may be used a second time; the results are

$$B_3 = \frac{\dfrac{1}{1+s} + \dfrac{4+s-2z}{4+3s-2z-2sz}}{1 + \dfrac{4+s-2z}{(1+s)(4+3s-2z-2sz)}}$$

$$= \frac{8+8s-4z-4sz+s^2}{8+8s-4z-4sz+3s^2-2s^2z}$$

$$= \frac{(1,s,s^2)\begin{bmatrix}8 & -4 & 0\\ 8 & -4 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ z\\ z^2\end{bmatrix}}{(1,s,s^2)\begin{bmatrix}8 & -4 & 0\\ 8 & -4 & 0\\ 3 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\ z\\ z^2\end{bmatrix}} \qquad (52)$$

Testing this function it is seen that the transfer function is again improved by an increment of one.

The Single Step Process

In the previous example aidentity order was increased by one for each iteration. To get an increment of two in a single operation the initial function A is chosen such that it is a second-order aidentity. An example illustrates this point.

Given A and B such that

$$A = \frac{2 + 2s}{2 + 2s + s^2} \tag{53}$$

$$B_1 = \frac{1}{1 + 2(1 - z)} \tag{54}$$

One can find from Richards' form that B_2 is

$$B_2 = \frac{\dfrac{2 + 2s}{2 + 2s + s^2} + \dfrac{1}{1 + 2(1 - z)}}{1 + \dfrac{2 + 2s}{(2 + 2s + s^2)(1 + 2(1 - z)}}$$

$$= \frac{8 + 8s - 4z - 4sz + s^2}{8 + 8s - 4z - 4sz + 3s^2 - 2s^2z}$$

$$= \frac{(1, s, s^2) \begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}}{(1, s, s^2) \begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}} \tag{55}$$

This is the same result as B_3 in the previous example. This example leads to the following theorem.

Theorem on Aidentity Improvement

If
$$B_n = \frac{\sigma' A \zeta}{\sigma' B \zeta} = \frac{P_n}{Q_n} \tag{56}$$

is an approximate identity of order $n \geq 0$; if A is a k^{th}-order aidentity of the form

$$A = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_{k-1} s^{k-1}}{a_0 + a_1 s + a_2 s^2 + \ldots + a_{k-1} s^{k-1} + s^k}$$
$$= \frac{H(s)}{H(s) + s^k} \tag{57}$$

then

$$B_{n+1} = \frac{A + B_n}{1 + AB_n} \tag{58}$$

is an approximate identity of order $k + n$.

Proof.

$$B_{n+1} = \frac{\dfrac{H(s)}{H(s) + s^k} + \dfrac{P_n}{Q_n}}{1 + \dfrac{H(s)\ P_n}{Q_n(H(s) + s^k)}}$$

$$= \frac{H(s)\ Q_n + H(s) P_n + s^k P_n}{H(s)\ Q_n + H(s) P_n + s^k Q_n}$$

$$= \frac{\bar{\sigma}' V \bar{\zeta} + s^k P_n}{\bar{\sigma}' V \bar{\zeta} + s^k Q_n} \tag{59}$$

where

$$\bar{\sigma} = \text{col}\left[\sigma, (s^{N+1}, s^{N+2}, \ldots, s^{N+k})\right]$$

and

$$\bar{\zeta} = \text{col}\left[\zeta, (z^{N+1}, z^{N+2}, \ldots, z^{N+k})\right] \tag{60}$$

V is defined as an (N+1+k by N+1+k) matrix whose elements are $v_{i,j}$. Also s^kP_n can be rewritten as an (N+1+k by N+1+k) matrix of the form

$$
s^kP_n = \begin{bmatrix} \sigma \\ \begin{bmatrix} s^{N+1} \\ s^{N+2} \\ \vdots \\ s^{N+k} \end{bmatrix} \end{bmatrix}' \begin{bmatrix} 0 & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & 0 & & & & & \\ a_{0,0} & a_{0,1} & \cdots & a_{0,N} & 0 & \cdots & 0 \\ a_{1,0} & & & \vdots & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{N,0} & \cdot & \cdots & a_{N,N} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \begin{bmatrix} z^{N+1} \\ z^{N+2} \\ \vdots \\ z^{N+k} \end{bmatrix} \end{bmatrix} \tag{61}
$$

and $\bar{\sigma}' V \bar{\zeta} + s^kP_n$

$$
= \begin{bmatrix} \sigma \\ \begin{bmatrix} s^{N+1} \\ s^{N+2} \\ \vdots \\ s^{N+k} \end{bmatrix} \end{bmatrix}' \begin{bmatrix} v_{0,0} & \cdots & \cdots & \cdots v_{0,N+k} \\ \vdots & & & \vdots \\ v_{k-1,0} & v_{k-1,N} & v_{k-1,N+1} & v_{k-1,N+k} \\ v_{k,0}+a_{0,0} & v_{k,N}+a_{0,N} & v_{k,N+1} & v_{k,N+k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k+N,0}+a_{N,0} \cdots & v_{k+N,N}+a_{N,N} & v_{k+N,N+1} & \cdots v_{N+k,N+k} \end{bmatrix} \begin{bmatrix} \zeta \\ \begin{bmatrix} z^{N+1} \\ z^{N+2} \\ \vdots \\ z^{N+k} \end{bmatrix} \end{bmatrix} \tag{62}
$$

By the same method $\bar{\sigma}' V \bar{\zeta} + s^kQ_n$ is obtained.

$$\begin{bmatrix} \sigma \\ \begin{bmatrix} s^{N+1} \\ s^{N+2} \\ \vdots \\ s^{N+k} \end{bmatrix} \end{bmatrix}' \begin{bmatrix} v_{0,0} & v_{0,1} & \cdots & & \cdots v_{0,N+k} \\ \vdots & & & & \vdots \\ v_{k-1,0} & & v_{k-1,N} & v_{k-1,N+1} & v_{k-1,N+k} \\ v_{k,0}+b_{0,0} & & v_{k-1,N}+b_{0,N} & v_{k,N+1} & v_{k,N+k} \\ \vdots & & \vdots & \vdots & \vdots \\ v_{N+k,0}+b_{N,0} & \cdots & v_{N+k,N}+b_{N,N} & v_{N+k,N+1} & \cdots v_{N+k,N+k} \end{bmatrix} \begin{bmatrix} \zeta \\ \begin{bmatrix} z^{N+1} \\ z^{N+2} \\ \vdots \\ z^{N+k} \end{bmatrix} \end{bmatrix} \tag{63}$$

The aidentity criterion may now be applied. For the first-order aidentity it is true that

$$\sum_{j=0}^{N+k} v_{0,j} = \sum_{j=0}^{N+k} v_{0,j} \tag{64}$$

The second-order aidentity includes the criterion for the first order and

$$\begin{aligned} \sum_{j=0}^{N+k} j\, v_{0,j} &= \sum_{j=0}^{N+k} j\, v_{0,j} \\ \sum_{j=0}^{N+k} v_{1,j} &= \sum_{j=0}^{N+k} v_{1,j} \end{aligned} \tag{65}$$

The k^{th}-order aidentity is proven in the same manner.

$$\begin{aligned} \sum_{j=0}^{N+k} j^{k-1} v_{0,j} &= \sum_{j=0}^{N+k} j^{k-1} v_{0,j} \\ \sum_{j=0}^{N+k} j^{k-2} v_{1,j} &= \sum_{j=0}^{N+k} j^{k-2} v_{1,j} \\ &\vdots \end{aligned} \tag{66}$$

$$\sum_{j=0}^{N+k} j\ v_{k-2,j} = \sum_{j=0}^{N+k} j\ v_{k-2,j}$$

$$\sum_{j=0}^{N+k} v_{k-1,j} = \sum_{j=0}^{N+k} v_{k-1,j}$$

The $(k+1)^{th}$-order aidentity depends upon the assumption that B_n is an aidentity of some order greater than zero. Applying the $(k+1)^{th}$-order aidentity criterion,

$$\sum_{j=0}^{N+k} j^k\ v_{0,j} = \sum_{j=0}^{N+k} j^k v_{0,j}$$

$$\vdots \qquad\qquad \vdots$$

$$\sum_{j=0}^{N+k} j\ v_{k-1,j} = \sum_{j=0}^{N+k} j\ v_{k-1,j} \tag{67}$$

$$\sum_{j=0}^{N} (v_{k,j} + a_{0,j}) + \sum_{j=N+1}^{N+k} v_{k,j}$$

$$= \sum_{j=0}^{N} (v_{k,j} + b_{0,j}) = \sum_{j=N+1}^{N+k} v_{k,j}$$

The last equation may be rewritten as:

$$\sum_{j=0}^{N+k} v_{k,j} + \sum_{j=0}^{N} a_{0,j} = \sum_{j=0}^{N+k} v_{k,j} + \sum_{j=0}^{N} b_{0,j} \tag{68}$$

Of all equations for the $(k+1)^{th}$-order aidentity only the last equation need be examined. If it be assumed the B_n is an aidentity of order $n > 0$, the equation

$$\sum_{j=0}^{N} a_{0,j} = \sum_{j=0}^{N} b_{0,j} \tag{69}$$

is true since it fulfills the condition of B_n being a first-order aidentity. This in turn satisfies the last equation for the $(k+1)^{th}$-order aidentity for B_{n+1}.

In general, any aidentity of order less than or equal to k+n can be found for this process. Assuming that all conditions hold for order k+n-1, the aidentity of order k+n can be shown to exist. Applying the criterion for this case,

$$\begin{aligned}
&\sum_{j=0}^{N+k} j^{n+k-1} v_{0,j} = \sum_{j=0}^{N+k} j^{n+k-1} v_{0,j} \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
&\sum_{j=0}^{N+k} j^{n} v_{k-1,j} = \sum_{j=0}^{N+k} j^{n} v_{k-1,j} \\
&\sum_{j=0}^{N} j^{n-1}(v_{k,j}+a_{0,j}) + \sum_{j=N+1}^{N+k} j^{n-1} v_{k,j} \\
&= \sum_{j=0}^{N} j^{n-1}(v_{k,j}+b_{0,j}) + \sum_{j=N+1}^{N+k} j^{n-1} v_{k,j} \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
&\sum_{j=0}^{N} (v_{k+n-1,j} + a_{n-1,j}) + \sum_{j=N+1}^{N+k} v_{k+n-1,j} \\
&= \sum_{j=0}^{N} (v_{k+n-1,j} + b_{n-1,j}) + \sum_{j=N+1}^{N+k} v_{k+n-1,j}
\end{aligned} \tag{70}$$

This last set of equations may be rewritten as:

$$\sum_{j=0}^{N+k} j^{n+k-1} v_{0,j} = \sum_{j=0}^{N+k} j^{n+k-1} v_{0,j}$$

$$\vdots \qquad\qquad \vdots$$

$$\sum_{j=0}^{N+k} j^{n-1} v_{k,j} + \sum_{j=0}^{N} j^{n-1} a_{0,j} = \sum_{j=0}^{N+k} j^{n-1} v_{k,j} + \sum_{j=0}^{N} j^{n-1} b_{0,j}$$

$$\vdots \qquad\qquad \vdots$$

$$\sum_{j=0}^{N+k} v_{n+k-1,j} + \sum_{j=0}^{N} a_{n-1,j} = \sum_{j=0}^{N+k} v_{n+k-1,j} + \sum_{j=0}^{N} b_{n-1,j} \tag{71}$$

Summation of the $v_{i,j}$'s causes cancellation and leaves

$$\sum_{j=0}^{N} j^{n-1} a_{0,j} = \sum_{j=0}^{N} j^{n-1} b_{0,j}$$

$$\sum_{j=0}^{N} j^{n-2} a_{1,j} = \sum_{j=0}^{N} j^{n-2} b_{1,j}$$

$$\vdots \qquad\qquad \vdots$$

$$\sum_{j=0}^{N} a_{n-1,j} = \sum_{j=0}^{N} b_{n-1,j} \tag{72}$$

This set of equations, however, is the same set that B_n must satisfy if it is to be an n^{th}-order aidentity. Since by hypothesis these are true, the proof for the $(k+n)^{th}$-order aidentity for B_{n+1} is complete.

A corollary to the above theorem will now be stated.

Corollary. If $A(s,z)$ is an aidentity but $B_n(s,z)$ is not an aidentity, then $(A + B_n)/(1 + AB_n)$ will still be an aidentity of order k.

Proof. The aidentity criterion for the aidentities one through k do not involve any terms of B_n but only $v_{i,j}$'s. Therefore the first k-order aidentities are independent of B_n.

CONCLUSION

Linear feedback devices with time delay have been discussed in this thesis. The relations between numerator and denominator coefficients of this device's transfer function that insure zero steady-state error for polynomial inputs have been derived. These relations hold for all systems in this class, i.e., continuous systems with no time delay, continuous systems with time delay, or sampled-data systems.

It has been shown that Richards' form for a positive real function can increase the order of an aidentity. This procedure allows the original system and the compensating system to be completely independent of one another. Using this result, it is possible to achieve any order of aidentity improvement that is desirable.

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REFERENCES

1. Rault, Andre Jean
   Approximate Identity Operator in Linear Continuous and Sample-data Servomechanisms. M. S. Report, Kansas State University, 1964.

2. King, L. H.
   Reduction of Forced Error in Closed Loop Systems. Proc. IRE, August, 1953, pp. 1037-1042.

3. Halijak, Charles A.
   Classical Network Analysis and Positive Real Function Synthesis. Eighth Midwest Symposium on Circuit Theory Conference Paper, 1965, pp. 21-1, 21-2.

4. Echols, William H.
   Differential and Integral Calculus. Henry Holt & Co., New York, New York, 1902, p. 69.

5. Fuller, Leonard E.
   Basic Matrix Theory, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962.

AN APPROXIMATE IDENTITY OPERATOR FOR CONTINUOUS SERVOMECHANISMS WITH TIME LAG

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A method of control system design based upon admissibility of the control system into the servo class has been presented. A criterion for admissibility of control systems with time lag has been described. This criterion, based upon King's zero error coefficient theory, is expressed as a set of relations between coefficients of the control system transfer function.

Using Rault's work as a starting point, a method of realizing an improvement in the order of aidentity is presented. This development gives a systematic design for control systems.