A MANY-TO-ONE BOOLEAN TRANSFORMATION

by

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INTRODUCTION

A need for sophisticated methods of handling complicated contact networks, employing simple ON and OFF switches, has arisen since the great expansion of automatic and semiautomatic systems into everyday use at home, office, and industry. A century before, an English mathematician, George Boole, developed Boolean algebra for studying logical relations. It was later extended by other mathematicians, and finally was adapted by the engineer as a tool for handling complicated electrical logic systems employing relays and switches.

In this paper a set of Boolean functions shall be developed and treated with a double purpose in mind:

1. Investigation of some properties of these functions.
2. Physical interpretation and applications--i.e., a contact network realization of these Boolean functions.

Separate sections shall be concerned with a recursive relation, alternative forms, and approximations of $\prod_{k=1}^{n-1} \binom{n}{k}$, which is the number of equivalence classes of the many-to-one transformation of elementary symmetric functions.

ELEMENTARY SYMMETRIC FUNCTIONS (ESF's)

ESF Generation

Let P(x) be a polynomial such that

$$P(x) = \prod_{k=1}^{n} (x + a_k) = \sum_{k=0}^{n} p_k x^{n-k} \qquad a_k = 0,\ 1$$

Upon expansion, one obtains

$$P(x) = p_0 x^n + p_1 x^{n-1} + \ . \ . \ . + p_{n-1} x^1 + p_n x^0 \ .$$

Define $\sum_r$ for $r = 1, 2, \ldots, n$ as the sum over all the ways of selecting r distinct a_k from the n given a_k. Then

$$\begin{aligned}
p_0 &= 1 \\
p_1 &= \textstyle\sum_1 a_k = a_1 + a_2 + \ . \ . \ . + a_n \\
p_2 &= \textstyle\sum_2 a_k = a_1a_2 + a_1a_3 + \ . \ . \ . + a_1a_n + a_2a_3 + a_2a_4 \\
&\qquad + \ . \ . \ . + a_2a_n + \ . \ . \ . + a_{n-1}a_n \\
&\ \ \vdots \\
p_n &= \textstyle\sum_n a_k = a_1a_2 \ . \ . \ . \ a_n
\end{aligned}$$

Boolean expressions $\{p_k\}$ are called <u>e</u>lementary <u>s</u>ymmetric <u>f</u>unctions (ESF's).

Boolean variables $\{a_k\}$ are called variables of symmetry. The operations are cup and cap.

<u>Definition</u>. A Boolean function of n variables a_1, a_2, . . ., a_n is said to be symmetric in these variables if any permutation of these variables leaves the function unchanged (Ref. 9).

Properties of ESF's

Theorem 1. If any m variables of $\{a_k\}$ are ones where $0 \leqslant m < n$, and the remaining $(n-m)$ variables of $\{a_k\}$ are zeros, then $p_k = 1$ for $k = 0, 1, 2, \ldots, m$ and $p_k = 0$ for $k = m+1, m+2, \ldots, n$.

Proof. p_k is the sum of all possible combinations of products of n variables $\{a_k\}$ taken k at a time; hence there are $\binom{n}{k} = n!/k!(n-k)!$ distinct terms in p_k, and each term is a product of k variables a_k. Every term in p_m consists of m variables a_k. If and only if m of the $\{a_k\}$ variables are one, then one of the $\binom{n}{m}$ products in p_m is one and the rest of the $\binom{n}{m} - 1$ products are zeros, and p_m, which is the Boolean sum of the products, is one. $p_k = p_{m-1}$ has $\binom{m}{m-1}$ distinct products that are ones, and in general $p_k = p_{m-q}$ where $0 \leqslant q < m$ has $\binom{m}{m-q}$ products of value one and the rest of the products are zeros and $p_{m-q} = 1$.

For $k > m$, $p_k = p_{m+q}$, $0 < q \leqslant n-m$, is the sum of products of $m + q$ variables, but only m of them are ones and the remaining variables are zeros. Thus the value of every product is zero and $p_k = 0$.

Corollary. All n-tuples $(a_1, a_2, \ldots, a_n)$ containing m bits one which are subjected to ESF transformations result in a single n-tuple $(p_1, p_2, \ldots, p_n)$.

Definitions.

D1: $B_1 \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

D2: θ_n is a column vector of n 0 bits

D3: U_n is a column vector of n 1 bits.

Lemma. There exists a recursive algorithm which constructs a binary sequence.

Proof.

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \theta_2 & B_1 \\ U_2 & B_1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \theta_{2^2} & B_2 \\ U_{2^2} & B_2 \end{bmatrix}$$

In general, the desired recurrence relation is

$$B_n = \begin{bmatrix} \theta_{2^{n-1}} & B_{n-1} \\ U_{2^{n-1}} & B_{n-1} \end{bmatrix}$$

<u>Theorem</u> <u>2</u>. For a given binary sequence, containing n columns and all 2^n possible distinct rows, there are $\binom{n}{m}$ rows containing m ones, where m = 0, 1, . . ., n.

<u>Proof</u>. The proof is by induction on n.

1. A binary sequence is constructed for n = 2.

Row count

$$B_2 = \begin{bmatrix} \theta_2 & B_1 \\ U_2 & B_1 \end{bmatrix} = \begin{array}{cc} a_2 & a_1 \\ \end{array}$$
$$\begin{array}{c} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \binom{2}{0} \\ \binom{2}{1} \\ \\ \binom{2}{2} \end{array}$$

As can be seen from the above binary sequence, Theorem 2 holds for n = 2.

2. Construct a binary sequence for n + 1 = 3.

$$B_{n+1} = B_3 = \begin{bmatrix} \theta_{2^2} & B_2 \\ U_{2^2} & B_2 \end{bmatrix}$$

The first 2^n rows of B_{n+1} are obtained by providing additional column of zeros to B_n. Thus there are the same number $\binom{n}{m}$ of rows containing m ones as in B_n. The remaining 2^n rows are obtained by providing additional column of ones to B_n. Thus there are $\binom{n}{m-1}$ rows containing m ones.

3. In general,

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$$

If the above relation holds, then Theorem 2 is true for any n bits.

4. The closed form for $\binom{n}{m}$ is

$$\binom{n}{m} = \frac{n!}{m!(n - m)!}$$

Thus

$$\binom{n}{m} + \binom{n}{m-1} = \frac{n!}{m!(n-m)!} + \frac{n!}{(m-1)!(n-m+1)!}$$

$$= \frac{n!}{m!(n-m+1)!}\left[(n-m+1) + (m)\right]$$

$$= \frac{(n+1)!}{m!(n+1-m)!} = \binom{n+1}{m}$$

Hence the previous relation holds and Theorem 2 is proved.

Theorem 1 is demonstrated for n = 3 by subjecting the binary sequence to ESF transformations.

Table 1. Truth table for ESF transformations.

a_3	a_2	a_1	p_3	p_2	p_1
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	0	1	1
1	1	1	1	1	1

i.e.,

$(p_1, p_2, p_3) = (0, 0, 0)$ where none of the variables a_k are ones

$(p_1, p_2, p_3) = (1, 0, 0)$ where any one of the variables a_k is one

$(p_1, p_2, p_3) = (1, 1, 0)$ where any two of the variables a_k are ones

$(p_1, p_2, p_3) = (1, 1, 1)$ where all the variables a_k are ones.

The a_k's in Table 1 can be reordered in groups of $\binom{n}{m}$ rows containing m ones where m = 0, 1, 2. Such a reordered table is shown in Table 2. In general, every group of $\binom{n}{m}$ rows $(a_1, a_2, \ldots, a_n)$ containing m ones which is subject to ESF transformation results in one and only one row, $(p_1, p_2, \ldots, p_n)$.

Definitions.

D1: A_m^n is a in an array of n columns and $\binom{n}{m}$ distinct rows where m bits in each row are ones and the remaining bits are zeros.

D2: θ_m^n is a column vector of $\binom{n}{m}$ zero bits.

D3: U_m^n is a column vector of $\binom{n}{m}$ one bits.

D4: $\theta_m^n \; A_m^n \equiv [\theta A]_m^n$.

D5: $U_m^n \; A_m^n \equiv [UA]_m^n$.

D6: The operation $\nearrow$ is defined by $A_m^n \nearrow [\theta A]_m^n$.

Table 2. A reordered truth table.

Row count	a_3	a_2	a_1	p_3	p_2	p_1
$\binom{3}{0}$	0	0	0	0	0	0
	0	0	1			
$\binom{3}{1}$	0	1	0	0	0	1
	1	0	0			
	0	1	1			
$\binom{3}{2}$	1	0	1	0	1	1
	1	1	0			
$\binom{3}{3}$	1	1	1	1	1	1

D7: The operation $\searrow$ is defined by $A^n_m \searrow [UA]^n_m$.

Definitions D2 and D3 differ from previous U_n and θ_n.

The following assertions may be deduced:

A1: $[\theta A]^n_0 = A^{n+1}_0$

A2: $[UA]^n_0 = A^{n+1}_{n+1}$

A3: $\begin{bmatrix} [UA]^n_{m-1} \\ [\theta A]^n_m \end{bmatrix} = A^{n+1}_m$

Implicit in A3 is a conformability condition:

$$\begin{bmatrix} R^m_n \\ \\ S^p_q \end{bmatrix} \quad \text{has meaning if and only if } m = p.$$

Interchange of rows in A^n_m is an operation which leaves the count of ones invariant.

<u>Lemma</u>. There exists a recursive algorithm which constructs $\left\{A^{n+1}_m\right\}$ for all integers m such that $0 \leqslant m \leqslant n + 1$.

<u>Proof</u>. A binary sequence which contains n columns and all the 2^n possible rows can be reordered into arrays A^n_m for all m such that $0 \leqslant m \leqslant n$. Given A^n_m for all integers m such that $0 \leqslant m \leqslant n$, a recursive algorithm for determining A^{n+1}_m where $0 \leqslant m \leqslant n + 1$ is shown in Fig. 1. Note that there are $\binom{n}{m}$ rows in A^n_m ; thus the total number of rows in $\left\{A^n_m\right\}$, for all integers m where $0 \leqslant m \leqslant n$, is $\sum_{m=0}^{n} \binom{n}{m} = 2^n$.

An example of a recursive algorithm which constructs all A^{n+1}_m arrays for n = 1, 2, 3, 4 is shown in Fig. 2.

<u>Theorem 3</u>. If the binary sequence is subjected to the many-to-one transformation, ESF's, then there exist $\prod_{k=1}^{n-1} \binom{n}{k}$ equivalence classes. The principal class is arbitrarily defined to be the upper member of the reordering transformation. If there exists but one member, then it is the upper member.

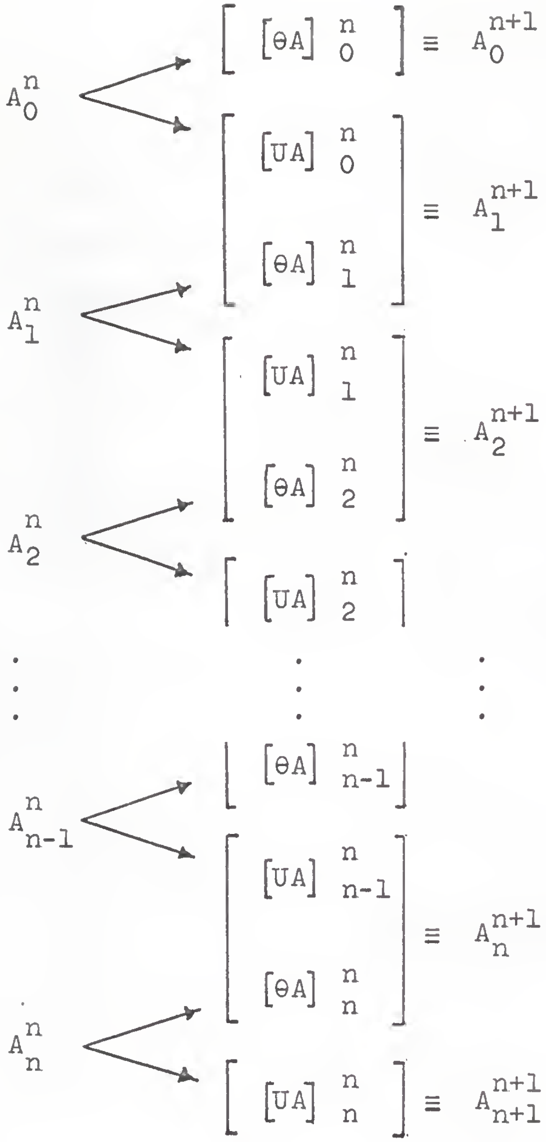

Fig. 1. A recursive algorithm which constructs $\left\{A_m^{n+1}\right\}$ for all integers m such that $0 \leqslant m \leqslant n$.

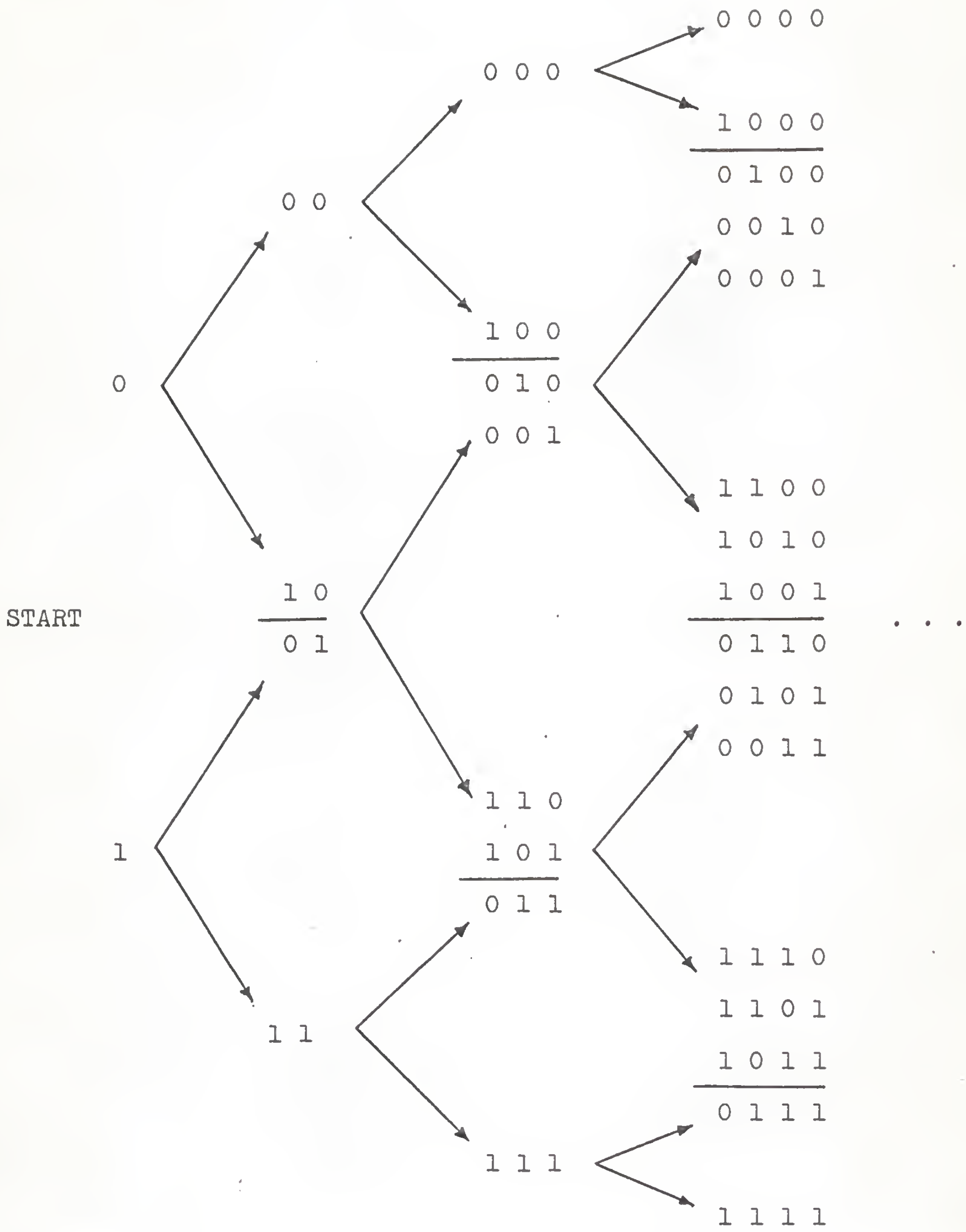

Fig. 2. An example of the recursive algorithm which constructs all A_m^{n+1} arrays for n = 1, 2, 3, 4.

A future section will be concerned with approximations of

$$\prod_{k=1}^{n-1} \binom{n}{k} .$$

Realization of ESF's by Contact Networks

Operations in the previous section were cup and cap and the two states of the variables were represented by 0 and 1. Thus transformation of Boolean variables into switching variables can be made. The variable a_k will designate a switch or a relay in a contact network. The state of $a_k = 1$ would represent a switch in ON position, while the opposite state $a_k = 0$ would represent a switch in OFF position. Boolean function $p_k = 1$ or 0 would represent a closed- or an open-circuit condition between two terminals. For convenience the output terminal of the function p_k will be designated by P_k. The input terminal would be common to all functions p_k, and can be denoted by P_0 since $p_0 = 1$ and its input and output terminals are electrically the same. The overbar will denote "the complementary state"--if $a_k = 0$ then $\overline{a}_k = 1$, and vice versa. A switch designated by a_k would represent an open contact when in OFF position, and closed contact when in ON position. In the same manner a switch designated by $\overline{a}_k$ would represent a closed contact when in OFF position and open contact when in ON position. A function such as $p_1 = a_1 + a_2 + \ . \ . \ . + a_n$ represents a contact network of n switches in parallel while a function such as $p_n = a_1 a_2 \ . \ . \ . \ a_n$ represents a contact network of n switches in series.

An interesting consequence of Theorem 1 is that in a contact network realization of a set of ESF's, one obtains transmission on terminal P_1 and no transmission on terminals P_2, P_3, . . ., P_n by operating any one out of n switches. Transmission on terminals P_1 and P_2 and no transmission on terminals P_3, P_4, . . ., P_n are obtained by operating any two out of n switches; and in general one obtains transmission on terminals P_1, P_2, . . ., P_m and no transmission on terminals P_{m+1}, P_{m+2}, . . ., P_n by operating any m out of n switches.

The ESF's, $\{p_k\}$, are sums of products of a_k; thus it is always possible to represent such contact networks by simply assigning a switch or relay to each of the variables a_k and obtain a simple series-parallel network. An example of such a contact network for $\{p_k\}$ for k = 0, 1, 2, 3, 4 is shown in Fig. 3. Note that a source is connected between P_0 and each light, the other terminal of each light is connected to one of the corresponding P_k's. For simplicity, there is a common source to all the lights in Figs. 3, 4, and 5.

Theorem 4. Series-parallel realization of $\{p_k\}$, for k = 0, 1, . . ., n can be obtained by using $n \cdot 2^{n-1}$ elements.

Proof. Let g_n be the number of elements (contacts of a relay or a switch) to be used. Then

$$g_n = \binom{n}{1} \cdot 1 + \binom{n}{2} \cdot 2 + \dots + \binom{n}{n-1} \cdot (n-1) + \binom{n}{n} \cdot n \quad .$$

Substitution of $\binom{n}{k} \cdot k = \binom{n-1}{k-1} \cdot n$ into the previous expression yields

$$g_n = \binom{n-1}{0} \cdot n + \binom{n-1}{1} \cdot n + \ldots + \binom{n-1}{n-2} \cdot n + \binom{n-1}{n-1} \cdot n.$$

Since $\sum_{k=0}^{m} \binom{m}{k} = 2^m$, taking $m = n - 1$, one obtains

$$g_n = n \cdot 2^{n-1}$$

End of proof.

Series-parallel realization of ESF's can be economized by reassociation.

Illustrative example. The network shown in Fig. 3 represents the following ESF's:

$$p_1 = a_1 + a_2 + a_3 + a_4$$
$$p_2 = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4$$
$$p_3 = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4$$
$$p_4 = a_1a_2a_3a_4$$

By reassociation of the variables in p_2 and p_3, one obtains

$$p_1 = a_1 + a_2 + a_3 + a_4$$
$$p_2 = (a_1 + a_2)(a_3 + a_4) + a_1a_2 + a_3a_4$$
$$p_3 = (a_1 + a_2)a_3a_4 + (a_3 + a_4)a_1a_2$$
$$p_4 = a_1a_2a_3a_4$$

A contact network for representation of this set of ESF's is shown in Fig. 4.

Though simple to synthesize, the series-parallel representation of $\{p_k\}$ can be reduced to a simpler and more economical equivalent circuit. An example of such a contact network for $\{p_k\}$ for $k = 0, 1, 2, 3, 4$ and an extension to $n > 4$ is shown in Fig. 5.

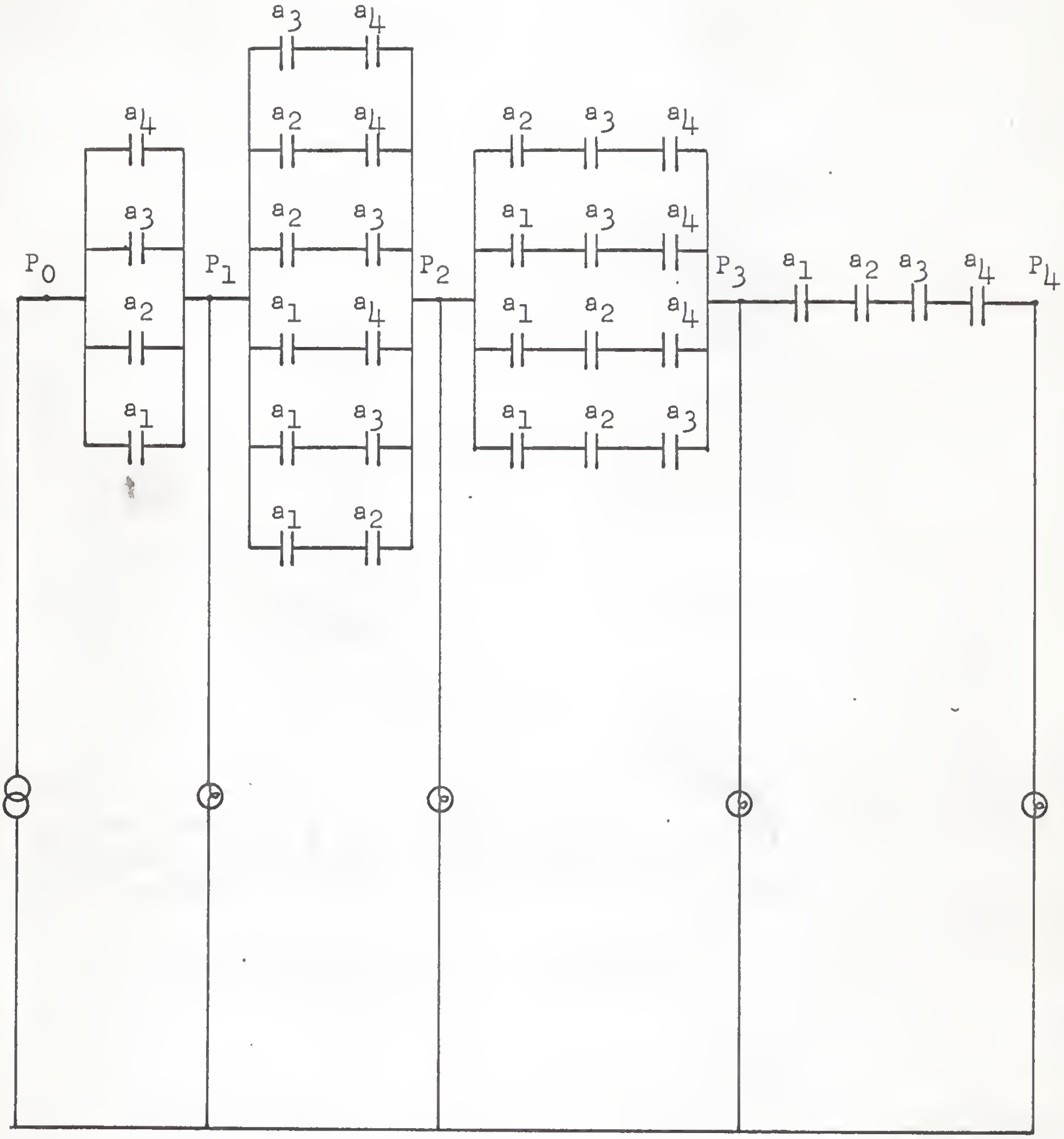

Fig. 3. A series-parallel realization of ESF's.

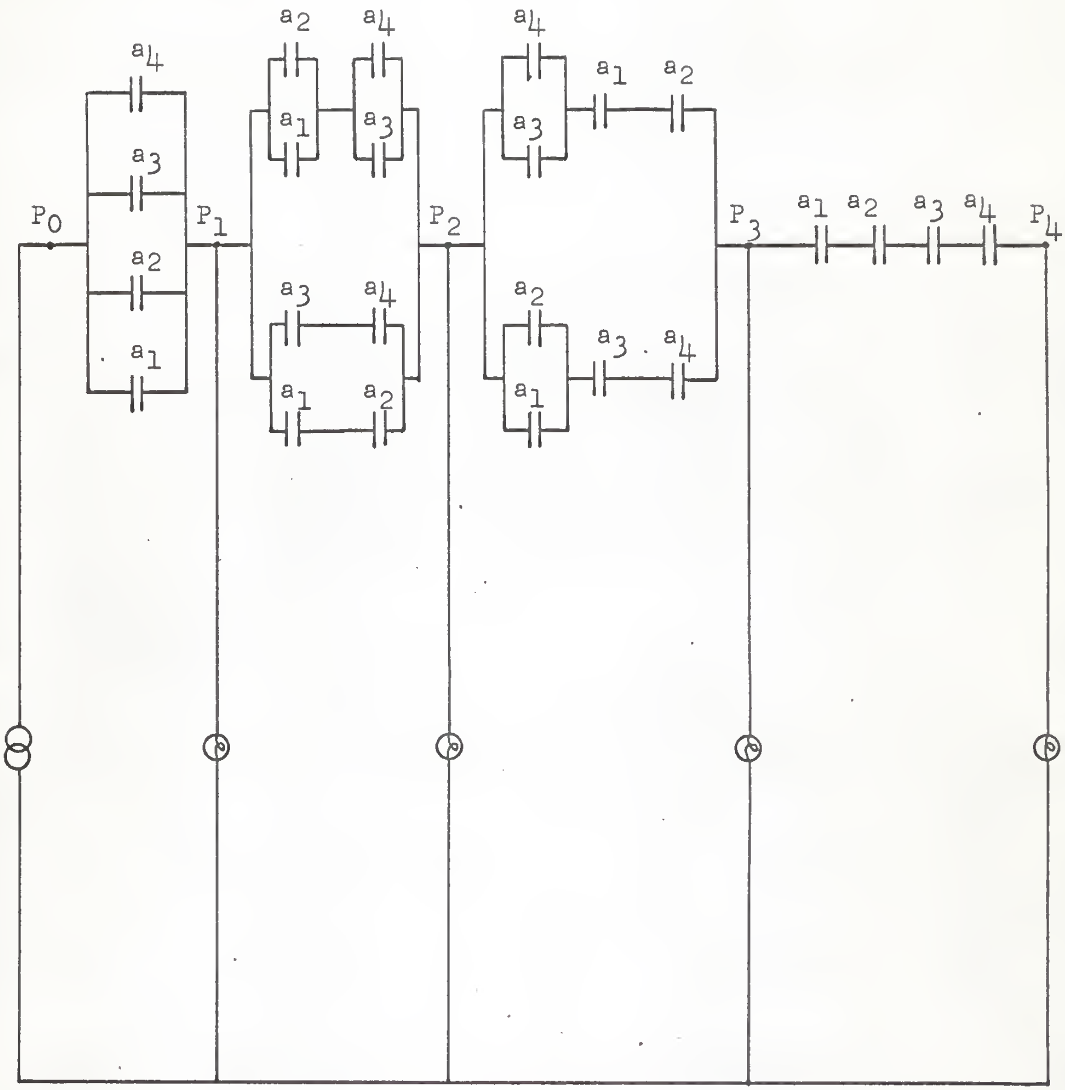

Fig. 4. A reassociated series-parallel
realization of ESF's.

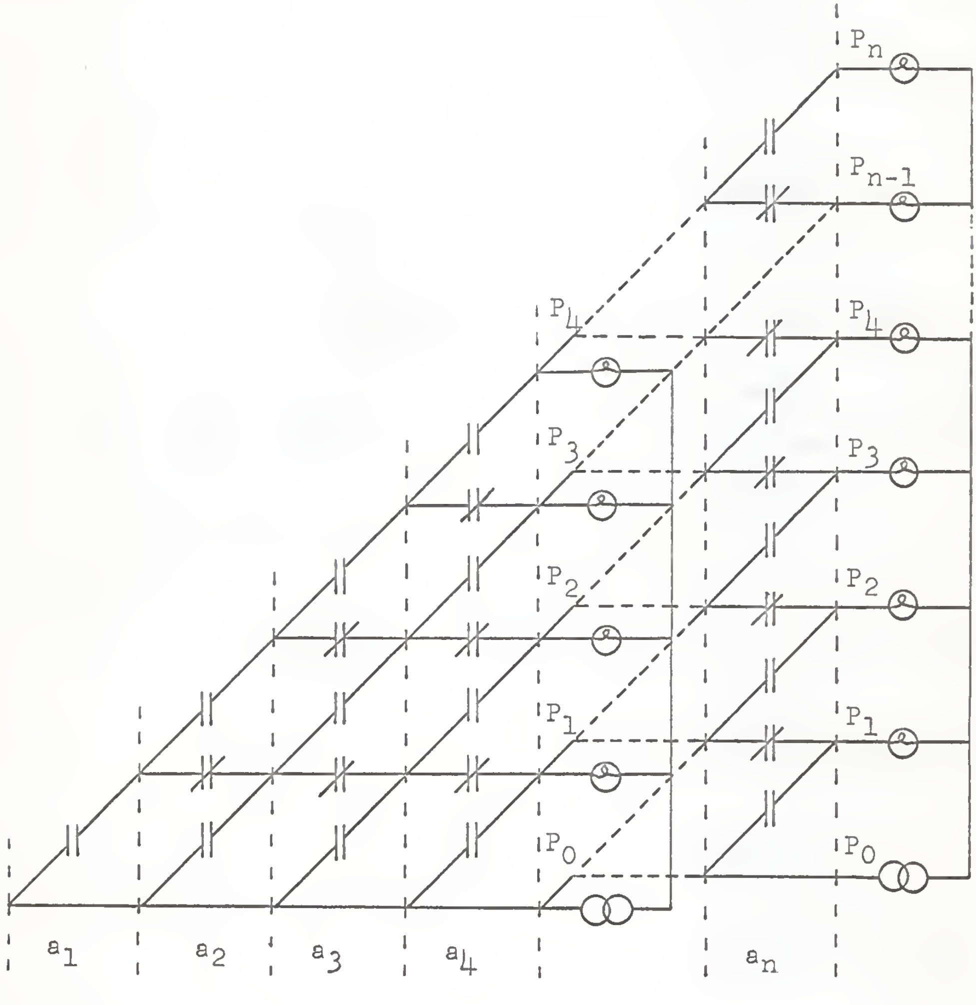

Fig. 5. Most economical realization of ESF's in contact network.

Theorem 5. The ESF's can be realized by the diagram of Fig. 5.

Proof. This realization obviously works for two switches and two lights. Presume that it works for n switches. If any m of n switches are ON, then m first lights would be ON. Adjoining a stage of OPERATED transfer switches will shift these m lights UP and the light at the zero level will also be ON. Therefore adding one stage of transfer switches has caused (m + 1) lights to be ON when (m + 1) switches are ON. End of proof.

Theorem 6. The number of elements required in the contact network realization of a set of ESF's as represented in the diagram of Fig. 5, are n^2.

Proof. Let g_n be the number of elements required. This satisfies the recurrence relation (see Fig. 5)

$$g_{n+1} = g_n + 2n + 1$$

with initial condition $g_1 = 1$. The closed form of this recurrence relation is

$$g_n = n^2 .$$

End of proof.

Applications

Applications of previously discussed contact networks will be discussed. In general, these networks can be used as sequential switching control devices. As a particular example, assume there is an electronic device whose filament power should always be turned on first, grid bias voltage turned on second, and the plate voltage turned on last. All that one should pay attention to in operating such a contact network, is to turn any one switch at a time. A switching control device for such an operation is illustrated in Fig. 6.

Another application of the contact network is step control of starting current. Let it be required to accelerate a direct-current shunt motor. As the motor accelerates, its armature current drops with a corresponding drop in torque due to a rise of its countervoltage. It is then necessary to reduce its resistance a step at a time. Normally the resistances reduced in each of the steps are of different values. Let R_1 be the resistance to be cut out first, and R_2 last. An arrangement for such a step control is shown in Fig. 7. Note that the sequence in which these switches are operated is of no significance as far as the required sequence of starting the motor and reducing resistances is concerned.

RECURSIVE RELATIONS AND ALTERNATIVE FORMS FOR $\prod_{k=1}^{n-1} \binom{n}{k}$

Theorem 7. If $\phi_n = \prod_{k=1}^{n-1} \binom{n}{k}$, then there exists a recursive relation such that

$$\phi_n = \phi_{n-1}(n^n/n!) \ ;$$

and there exist three alternative forms

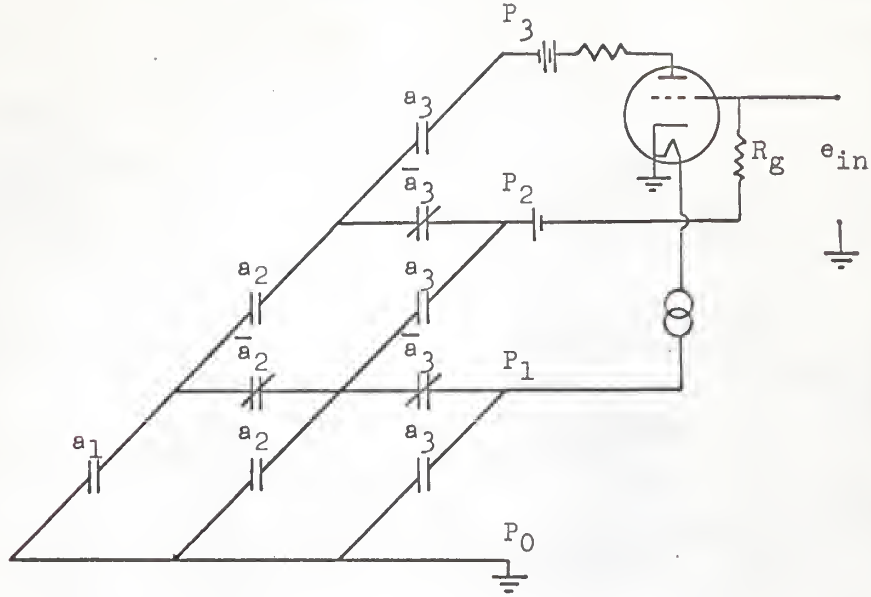

Fig. 6. Sequential operation of triode amplifier.

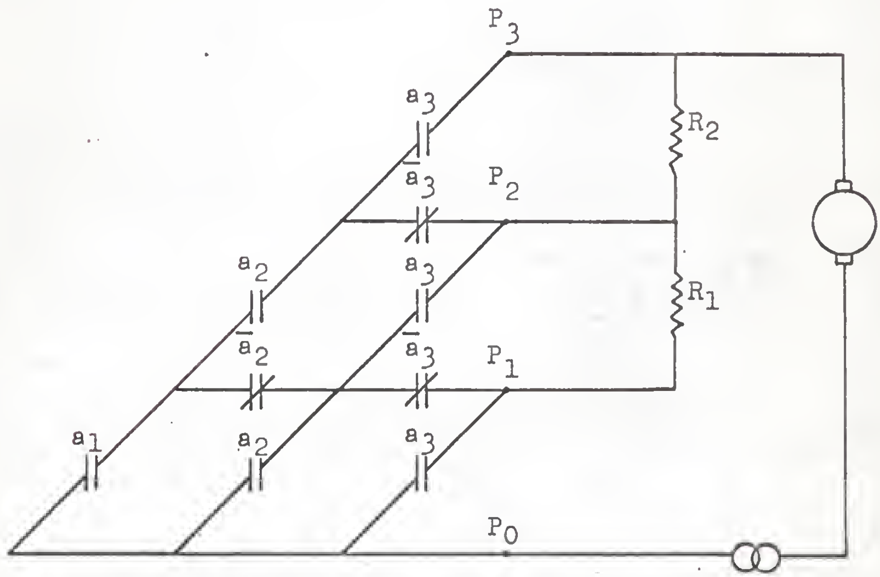

Fig. 7. Step control of starting current.

$$\phi_n = \prod_{k=2}^{n} (k^k/k!)$$

$$\phi_n = \prod_{k=2}^{n} k^{2k-n-1}$$

$$\phi_n = \prod_{k=1}^{n-1} (1 + 1/k)^{k(n-k)}$$

Proof. The form $\prod_{k=1}^{n-1} \binom{n}{k}$ can be written as

$$\phi_n = \frac{n!}{(n-1)!1!} \frac{n!}{(n-2)!2!} \cdots \frac{n!}{[n - (n-2)]!(n-2)!} \cdot \frac{n!}{[n-(n-1)]!(n-1)!}$$

$$= \frac{n!}{(n-1)!1!} \cdot \frac{n!}{(n-2)!2!} \cdots \frac{n!}{2!(n-2)!} \cdot \frac{n!}{1!(n-1)!}$$

$$= \frac{(n!)^{n-1}}{[1!2!\ldots(n-2)!(n-1)!]^2}$$

Substitution of (n-1) for n yields:

$$\phi_{n-1} = \frac{[(n-1)!]^{n-2}}{[1!2!\ldots(n-3)!(n-2)!]^2}$$

The ratio (ϕ_n/ϕ_{n-1}) becomes

$$\frac{\phi_n}{\phi_{n-1}} = \frac{(n!)^{n-1}}{[1!2!\ldots(n-2)!(n-1)!]^2} \cdot \frac{[1!2!\ldots(n-3)!(n-2)!]^2}{[(n-1)!]^{n-2}}$$

$$= \frac{[(n-1)!n]^{n-1}}{[(n-1)!]^2[(n-1)!]^{n-2}} = \frac{[(n-1)!]^{n-2}(n-1)! \cdot n^{n-1}}{[(n-1)!]^2\ [(n-1)!]^{n-2}}$$

$$= \frac{n^{n-1}}{(n-1)!} = \frac{n^n}{n!}$$

Therefore the required recursion relation is

$$\phi_n = \phi_{n-1}(n^n/n!) \ .$$

Since $\phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$, one can deduce the first alternative form

$$\phi_n = \prod_{k=2}^{n} (k^k/k!)$$

The second alternative form can now be derived. The previous product can be rearranged as

$$\phi_n = \frac{2^2}{2!} \cdot \frac{3^3}{3!} \cdots \frac{(n-1)^{n-1}}{(n-1)!} \cdot \frac{n^n}{n!}$$

After collecting integers from each factorial, one obtains

$$\phi_n = \frac{2^2 \cdot 3^3 \ . \ . \ . \ (n-1)^{n-1} \cdot n^n}{2^{n-1} \cdot 3^{n-2} \cdots (n-1)^{n-(n-2)} \cdot n^{n-(n-1)}}$$

$$= 2^{2-(n-1)} \cdot 3^{3-(n-2)} \cdots (n-1)^{(n-1)-[n-(n-2)]} \cdot n^{n-[n-(n-1)]}$$

$$= 2^{3-n} \cdot 3^{5-n} \cdots (n-1)^{n-3} \cdot n^{n-1}$$

This yields the second alternative form,

$$\phi_n = \prod_{k=2}^{n} k^{2k-n-1}$$

In order to derive the third alternative form let $S_n = \dfrac{n^n}{n!}$.

The ratio (S_n/S_{n-1}) becomes

$$\frac{S_n}{S_{n-1}} = \frac{n^n}{n!} \cdot \frac{(n-1)!}{(n-1)^{n-1}} = \left(\frac{n}{n-1}\right)^{n-1}$$

$$S_n = S_{n-1}\left(\frac{n}{n-1}\right)^{n-1}$$

with initial condition $S_1 = 1$.

Calculation of S_n yields

$$S_2 = \left(\frac{2}{1}\right)^1$$

$$S_3 = \left(\frac{2}{1}\right)^1 \cdot \left(\frac{3}{2}\right)^2$$

$$\vdots$$

$$S_n = \left(\frac{2}{1}\right)^1 \cdot \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n}{n-1}\right)^{n-1}.$$

But

$$\phi_n = S_2 \cdot S_3 \cdots S_n$$

and this culminates in the final alternative form,

$$\phi_n = \prod_{k=1}^{n-1}\left(\frac{k+1}{k}\right)^{k(n-k)}$$

$$= \prod_{k=1}^{n-1}(1 + 1/k)^{k(n-k)}.$$

CLOSE BOUNDS FOR $\prod_{k=1}^{n-1} \binom{n}{k}$

Theorem 8. If $\phi_n = \prod_{k=1}^{n-1} \binom{n}{k}$, then

$$\phi_n = \frac{e^{(n^2+2n-2)/2} \, e^{-(R_2^n + 0.5 r_n)}}{(2\pi)^{((2n-1)/4)} \, n^{((2n+1)/4)}}$$

where the remainder R_p^n is given by

$$R_p^n \equiv \sum_{k=p}^{n} r_k$$

and simple bounds for r_n are

$$\frac{1}{12n + (3/2(2n+1))} < r_n < \frac{1}{12n} \qquad (1)$$

Proof. From Theorem 7, $\phi_n = \prod_{k=2}^{n} (k^k/k!)$

Stirling's formula for n! is

$$n! = (n/e)^n \, (2\pi n)^{1/2} \, e^{r_n}$$

and these simple bounds for r_n are given in Reference (6). The ratio $(n^n/n!)$ becomes

$$(n^n/n!) = e^{(n-r_n)}/(2\pi n)^{1/2} \qquad (2)$$

Upon substitution of Equation (2) into $\prod_{k=2}^{n} (k^k/k!)$, one obtains

$$\phi_n = \prod_{k=2}^{n} \left[e^{(k-r_k)}/(2\pi k)^{1/2} \right]$$

The previous term can be rearranged to form

$$\phi_n = \frac{e^{(2-r_2)}}{(2\pi \ . \ 2)^{1/2}} \cdot \frac{e^{(3-r_3)}}{(2\pi \ . \ 3)^{1/2}} \cdot \cdot \cdot \frac{e^{(n-r_n)}}{(2\pi n)^{1/2}}$$

$$= \frac{e^{\left(\sum_{k=2}^{n} k\right)} \cdot e^{\left(-\sum_{k=2}^{n} r_k\right)}}{(2\pi)^{((n-1)/2)}(n!)^{1/2}} = \frac{e^{((n^2+2n-2)/2)} \cdot e^{\left(-\sum_{k=2}^{n} r_k\right)}}{(2\pi)^{((n-1)/2)}(n!)^{1/2}}$$

Now, $(n!)^{1/2}$ can be obtained from Stirling's formula:

$$(n!)^{1/2} = (n/e)^{n/2} \ . \ (2\pi n)^{1/4} \ . \ e^{0.5r_n}$$

Substituting $(n!)^{1/2}$ into the above form of ϕ_n, one obtains

$$\phi_n = \frac{e^{(n^2+n-2)/2} \ . \ e^{\left(-\sum_{k=2}^{n} r_k\right)} \ e^{n/2}}{(2\pi)^{((n-1)/2)} n^{(n/2)} (2\pi n)^{(1/4)} \ . \ e^{0.5r_n}}$$

$$\phi_n = \frac{e^{(n^2+2n-2)/2} \ e^{\left(-0.5r_n - \sum_{k=2}^{n} r_k\right)}}{(2\pi)^{((2n-1)/4)} n^{((2n+1)/4)}}$$

Recalling that $R_2^n \equiv \sum_{k=2}^{n} r_k$, one obtains

$$\phi_n = \frac{e^{(n^2+2n-2)/2} \ e^{(-R_2^n - 0.5r_n)}}{(2\pi)^{((2n-1)/4)} n^{((2n+1)/4)}} \qquad (3)$$

This completes the proof.

An approximation of R_2^n is developed later.

<u>Theorem 9</u>. If $R_p^n \equiv \sum_{k=p}^{n} r_k$, and simple bounds for r_n are

$$\frac{1}{12n + (3/2(2n+1))} < r_n < \frac{1}{12n}$$

then

$$\frac{1}{12}\left[\ln\left(\frac{4n+5}{4p+1}\right) + 3\left(\frac{1}{4p+1} - \frac{1}{4n+5}\right) + 2\left(\frac{1}{(4p+1)^2} - \frac{1}{(4n+5)^2}\right)\right]$$
$$< R_p^n < \frac{1}{24}\ln\left(\frac{n(n+1)}{(p-1)p}\right)$$

<u>Proof</u>. The upper inequality will be considered first. Since $r_n < \frac{1}{12n}$, then

$$r_p + r_{p+1} + \dots + r_{n-1} + r_n < \frac{1}{12p} + \frac{1}{12(p+1)} + \dots$$
$$+ \frac{1}{12(n-1)} + \frac{1}{12n}$$

Therefore

$$R_p^n \equiv \sum_{k=p}^{n} r_k < \frac{1}{12}\sum_{k=p}^{n}\frac{1}{k} \qquad (4)$$

The sum $\sum_{k=p}^{n}\frac{1}{k}$ can be estimated by considering the integral of $1/x$. Since $1/x$ decreases monotonically as x increases, it is clear from Fig. 8 that

$$\int_{n-1}^{n}\frac{1}{x}\,dx - \frac{1}{n} > \frac{1}{n} - \int_{n-1}^{n}\frac{1}{x+1}\,dx$$

Adding these inequalities for k = p, (p+1), ..., (n-1), n, one obtains

$$\int_{p-1}^{n} \frac{1}{x}\,dx - \sum_{k=p}^{n} \frac{1}{k} > \sum_{k=p}^{n} \frac{1}{k} - \int_{p-1}^{n} \frac{1}{x+1}\,dx$$

$$2\cdot\sum_{k=p}^{n} \frac{1}{k} < \int_{p-1}^{n} \left(\frac{1}{x} + \frac{1}{x+1}\right)dx$$

Evaluation of this integral leads to the inequality

$$2\cdot\sum_{k=p}^{n} \frac{1}{k} < \ln\left(\frac{n(n+1)}{(p-1)p}\right)$$

Knowing that

$$R_p^n \equiv \sum_{k=p}^{n} r_k < \frac{1}{12} \cdot \sum_{k=p}^{n} \frac{1}{k} < \frac{1}{24} \ln\left(\frac{n(n+1)}{(p-1)p}\right)$$

one can conclude that

$$R_p^n < \frac{1}{24} \ln\left(\frac{n(n+1)}{(p-1)p}\right)$$

The lower inequality will be considered next. Since

$$\frac{1}{12n + (3/2(2n+1))} < r_n$$

and

$$\frac{1}{12n + (3/2(2n + 1))} = \frac{4n + 2}{48n^2 + 24n + 3} = \frac{1}{3} \cdot \frac{(4n + 2)}{(4n + 1)^2}$$

$$= \frac{1}{3}\left[\frac{1}{4n + 1} + \frac{1}{(4n + 1)^2}\right]$$

therefore

$$\frac{1}{3} \cdot \sum_{k=p}^{n} \left[\frac{1}{4k + 1} + \frac{1}{(4k + 1)^2} \right] < \sum_{k=p}^{n} r_k \qquad (5)$$

The sum $\sum_{k=p}^{n} \left[\frac{1}{4k + 1} + \frac{1}{(4k + 1)^2} \right]$ can be estimated by considering the integral of $\left[\frac{1}{4x + 1} + \frac{1}{(4x + 1)^2} \right]$. Since $\frac{1}{4x + 1}$ decreases as x increases, it is clear from Fig. 9 that:

$$\frac{1}{4n+1} > \int_{n}^{n+1} \frac{1}{4x+1}\, dx + \frac{1}{2} \left[\frac{1}{4n+1} - \frac{1}{4(n+1) + 1} \right]$$

Evaluation of this integral leads to the inequality:

$$\sum_{k=p}^{n} \frac{1}{4k+1} > \frac{1}{4} \cdot \ln\left(\frac{4n+5}{4p+1}\right) + \frac{1}{2}\left(\frac{1}{4p+1} - \frac{1}{4n+5}\right)$$

Following the same procedure, $\sum_{k=p}^{n} \frac{1}{(4k+1)^2}$ can be found to satisfy the inequality

$$\sum_{k=p}^{n} \frac{1}{(4k+1)^2} > \frac{1}{4}\left(\frac{1}{4p+1} - \frac{1}{4n+5}\right) + \frac{1}{2}\left(\frac{1}{(4p+1)^2} - \frac{1}{(4n+5)^2}\right)$$

$$\sum_{k=p}^{n} \left[\frac{1}{4k+1} + \frac{1}{(4k+1)^2} \right] > \frac{1}{4} \cdot \ln\left(\frac{4n+5}{4p+1}\right) + \frac{3}{4}\left(\frac{1}{4p+1} - \frac{1}{4n+5}\right) + \frac{1}{2}\left(\frac{1}{(4p+1)^2} - \frac{1}{(4n+5)^2}\right)$$

Knowing that

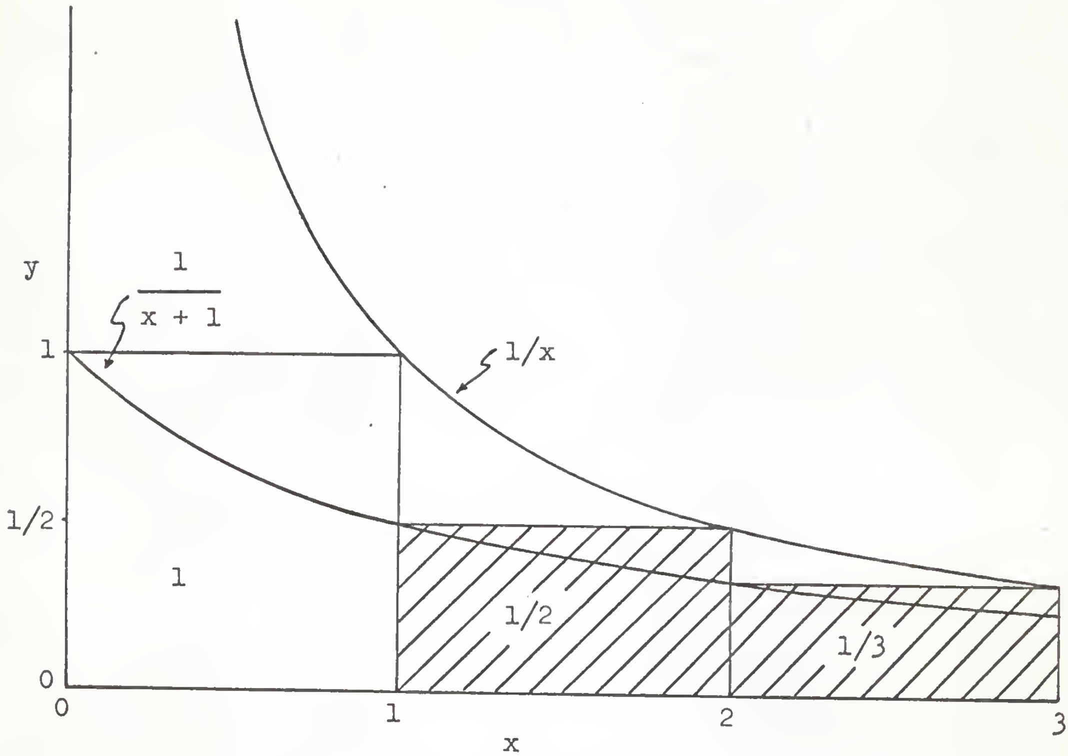

Fig. 8. Determination of upper bound of $\Sigma\ 1/k$.

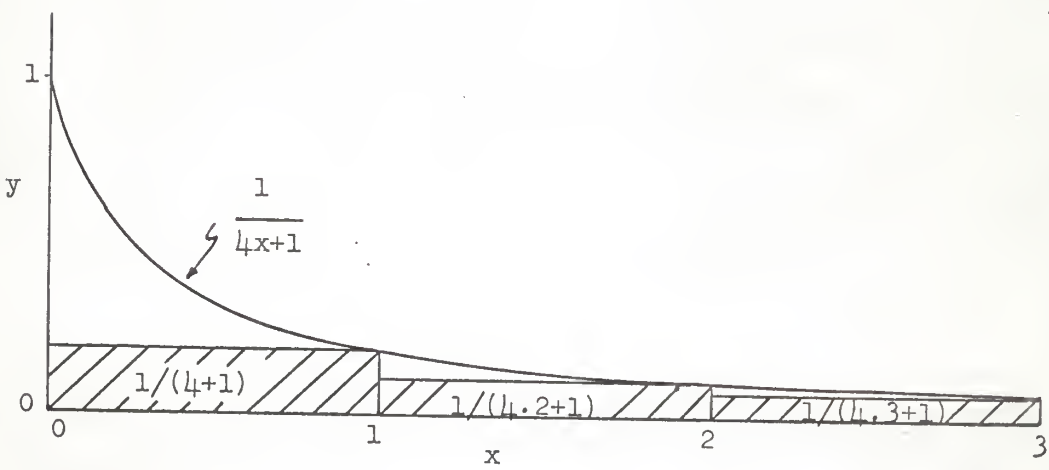

Fig. 9. Determination of lower bound of $\Sigma\ 1/(4k+1)$.

$$R_p^n \equiv \sum_{k=p}^{n} r_k > \frac{1}{3} \cdot \sum_{k=p}^{n} \left[\frac{1}{(4k+1)} + \frac{1}{(4k+1)^2} \right] \tag{6}$$

one can conclude that

$$R_p^n > \frac{1}{12} \left[\ln\left(\frac{4n+5}{4p+1}\right) + 3\left(\frac{1}{4p+1} - \frac{1}{4n+5}\right) + 2\left(\frac{1}{(4p+1)^2} - \frac{1}{(4n+5)^2}\right)\right] \tag{7}$$

End of proof.

Corollary. If $p = 2$ is substituted into R_p^n, bounds for $R_2^n + 0.5r_n$ are found to be

$$\frac{1}{12}\left[\ln\left(\frac{4n+5}{9}\right) + \frac{29}{81} + \frac{8n+4}{(4n+1)^2} - \frac{12n+17}{(4n+5)^2}\right] < R_2^n + 0.5r_n$$

$$< \frac{1}{24}\left[\frac{1}{n} + \ln\left(\frac{n(n+1)}{2}\right)\right] \tag{8}$$

Values of ϕ_n for $n = 1, 2, \ldots, 10$ are given in Table 3. Lower and upper bounds for ϕ_n as calculated by Equation (3) are given in Tables 4 and 5. In Table 4, bounds for R_2^n are found by the summation $\sum_{k=2}^{n} r_k$, while in Table 5 bounds are found from Equation (8).

Per cent error of the approximated ϕ_n increases with n, but it converges and approaches a limit as $n \rightarrow \infty$. The limit as $n \rightarrow \infty$ of the difference between the upper and the lower bounds of $R_2^n + 0.5r_n$ in Equation (8) yields $\frac{1}{24}\left[\left(\ln \frac{32}{81}\right) + \frac{58}{81}\right] = -0.00886100$; therefore the ratio between upper and lower bounds of ϕ_n found from Equation (3) cannot exceed

$e^{0.00886100} = 1.008907$. Thus if $\phi_{nL} < \phi_n < \phi_{nU}$ and $1 < \dfrac{\phi_{nU}}{\phi_{nL}} < 1.00897$, then $1 < \dfrac{\phi_{nU}}{\phi_n} < 1.00897$, and $1 < \dfrac{\phi_n}{\phi_{nL}} < 1.00897$ and the per cent error would always be smaller than 0.897 per cent.

It is clear from the above discussion that equation (8) can be simplified to read:

$$\frac{1}{24}\left[\frac{1}{n} + \ln\left(\frac{n(n+1)}{2}\right)\right] - 0.00861 < R_2^n + 0.5r_n$$
$$< \frac{1}{24}\left[\frac{1}{n} + \ln\left(\frac{n(n+1)}{2}\right)\right]$$

Since summation of the first few terms of r_n accounts for the major portion of the error, one can obtain a better approximation of ϕ_n by calculating the exact value of ϕ_m for $m < n$, then approximate the product $\prod_{k=m+1}^{n} (k^k/k!)$. Finally, ϕ_n can be obtained from the relation $\phi_n = \phi_m \cdot \prod_{k=m+1}^{n} (k^k/k!)$. A formula for the above relation is given in Theorem 10. When ϕ_n is obtained by $\phi_n = \phi_5 \cdot \prod_{k=6}^{n} (k^k/k!)$; the per cent error would be smaller than 0.073 per cent.

<u>Theorem</u> <u>10</u>. If $\phi_m = \prod_{k=1}^{m-1} \binom{m}{k}$, then

$$\phi_n = \phi_m \cdot \frac{e^{((n^2+2n-m^2-2m)/2)}\ e^{-R^n_{m+1}+((r_m-r_n)/2)}}{(2\pi)^{((n-m)/2)} n^{((2n+1)/4)} m^{-(2m+1)/4}} \ , \ 2 \leqslant m < n$$

where R^n_{m+1}, r_n, and r_m are defined in Theorem 8.

Proof. From Theorem 7, the following relation is obtained:

$$\phi_n = \prod_{k=2}^{n} (k^k/k!)$$

The above product can be split into two partial products such as

$$\phi_n = \prod_{k=2}^{m} (k^k/k!) \prod_{k=m+1}^{n} (k^k/k!)$$

The first partial product $\prod_{k=2}^{m} (k^k/k!)$ is equal to $\prod_{k=1}^{m-1} \binom{m}{k}$.

Thus

$$\phi_m = \prod_{k=2}^{m} (k^k/k!)$$

and

$$\phi_n = \phi_m \cdot \prod_{k=m+1}^{n} (k^k/k!) \ .$$

Upon substitution of Equation (2) into $\prod_{k=m+1}^{n} (k^k/k!)$, one obtains

$$\frac{\phi_n}{\phi_m} = \frac{e^{m+1} \cdot e^{-r_{m+1}}}{[2\pi(m+1)]^{1/2}} \cdot \frac{e^{m+2} \cdot e^{-r_{m+2}}}{[2\pi(m+2)]^{1/2}} \cdots$$

$$\frac{e^{n-1}\ e^{-r_{n-1}}}{[2\pi(n-1)]^{1/2}} \cdot \frac{e^{n} \cdot e^{-r_n}}{[2\pi n]^{1/2}}$$

$$= \frac{e^{\left(\sum_{k=m+1}^{n} k\right)} \cdot e^{-\left(\sum_{k=m+1}^{n} r_k\right)}}{(2\pi)^{((n-m)/2)}(n!/m!)^{1/2}}$$

$$(n!)^{1/2} = (n/e)^{n/2} (2\pi n)^{1/4} e^{0.5r_n}$$

$$(m!)^{1/2} = (m/e)^{m/2} (2\pi m)^{1/4} e^{0.5r_m}$$

Substitution of $(n!)^{1/2}$ and $(m!)^{1/2}$ into the previous expression of (ϕ_n/ϕ_m) yields:

$$\frac{\phi_n}{\phi_m} = \frac{e^{\left(\sum_{k=m+1}^{n} k\right)} e^{-\left(\sum_{k=m+1}^{n} r_k\right)} e^{(n-m-r_n+r_m)/2}}{(2\pi)^{((n-m)/2)} n^{(n/2)} (n/m)^{(1/4)} m^{-m/2}}$$

$$= \frac{e^{((n^2+n-m^2-m)/2)} e^{-\left(\sum_{k=m+1}^{n} r_k\right)} e^{((n-m)/2)} e^{(r_m-r_n)/2}}{(2\pi)^{((n-m)/2)} n^{((2n+1)/4)} m^{-(2m+1)/4}}$$

$$= \frac{e^{(n^2+2n-m^2-2m)/2} e^{\left(-\sum_{k=m+1}^{n} r_k\right)} e^{(r_m-r_n)/2}}{(2\pi)^{((n-m)/2)} (n)^{((2n+1)/4)} m^{-(2m+1)/4}}$$

Recalling that $R_{m+1}^{n} \equiv \sum_{k=m+1}^{n} r_k$, one obtains

$$\frac{\phi_n}{\phi_m} = \frac{e^{((n^2+2n-m^2-2m)/2)} e^{(-R_{m+1}^{n} + (r_m-r_n)/2)}}{(2\pi)^{((n-m)/2)} n^{((2n+1)/4)} m^{-((2m+1)/4)}}$$

Therefore

$$\not{o}_n = \not{o}_m \cdot \frac{e^{((n^2+2n-m^2-2m)/2)} \; e^{(-R^n_{m+1}+(r_m-r_n)/2)}}{(2\pi)^{((n-m)/2)} \; n^{((2n+1)/4)} \; m^{-(2m+1)/4}}$$

End of proof.

An approximation of R^n_{m+1} is in Theorem 9.

Table 3. Table for values of $\not{o}_n$.

n	$\not{o}_n$
1	1
2	2
3	9
4	96
5	2500
6	162000
7	26471025
8	11014635520
9	11759522374656
10	32406091200000000

Table 4. Bounds for ϕ_n as calculated from Equation (3).

$$1999 \cdot 10^{-3} < \phi_2 < 2001 \cdot 10^{-3}$$
$$8987 \cdot 10^{-3} < \phi_3 < 9002 \cdot 10^{-3}$$
$$95953 \cdot 10^{-3} < \phi_4 < 96028 \cdot 10^{-3}$$
$$24988 \cdot 10^{-1} < \phi_5 < 25008 \cdot 10^{-1}$$
$$16191 \cdot 10^{1} < \phi_6 < 16205 \cdot 10^{1}$$
$$26456 \cdot 10^{3} < \phi_7 < 26480 \cdot 10^{3}$$
$$11008 \cdot 10^{6} < \phi_8 < 11013 \cdot 10^{6}$$
$$11754 \cdot 10^{9} < \phi_9 < 11763 \cdot 10^{9}$$
$$32392 \cdot 10^{12} < \phi_{10} < 32416 \cdot 10^{12}$$

Table 5. Bounds for $\emptyset_n$ as calculated from Equation (3). (R_2^n in the above equation is approximated by Equation (8)).

$1990 \cdot 10^{-3} <$	$\emptyset_2$	$< 2002 \cdot 10^{-3}$
$8949 \cdot 10^{-3} <$	$\emptyset_3$	$< 9013 \cdot 10^{-3}$
$95411 \cdot 10^{-3} <$	$\emptyset_4$	$< 96159 \cdot 10^{-3}$
$24840 \cdot 10^{-1} <$	$\emptyset_5$	$< 25042 \cdot 10^{-1}$
$16097 \cdot 10^{1} <$	$\emptyset_6$	$< 16228 \cdot 10^{1}$
$26292 \cdot 10^{3} <$	$\emptyset_7$	$< 26510 \cdot 10^{3}$
$10943 \cdot 10^{6} <$	$\emptyset_8$	$< 11030 \cdot 10^{6}$
$11681 \cdot 10^{9} <$	$\emptyset_9$	$< 11782 \cdot 10^{9}$
$32186 \cdot 10^{12} <$	$\emptyset_{10}$	$< 32461 \cdot 10^{12}$

CONCLUSION

Some properties of Boolean elementary symmetric functions were investigated. Boolean variables were transformed into switching variables and contact network realizations of these functions were obtained. It was shown that networks representing these many-to-one transformations, ESF's, are not unique, and several equivalent networks were developed to represent these functions. Two physical applications of these networks were shown.

A formula for determining the number of equivalence classes of the many-to-one transformation, ESF's, was obtained. A recursive relation for this product, $\prod_{k=1}^{n-1} \binom{n}{k}$, and alternative forms were found. Finally, close bounds for this product of binomial coefficients were derived.

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A MANY-TO-ONE BOOLEAN TRANSFORMATION

by

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AN ABSTRACT OF A MASTER'S THESIS

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A set of two-valued elementary symmetric functions is generated, and some properties are investigated. The states of these variables are represented by 0 and 1. A realization of a set of elementary symmetric functions in a contact network is obtained through transformation of Boolean variables into switching variables, and reduced to more economical equivalent networks. Basic applications of the networks are given.

The number of equivalence classes of the many-to-one transformation of the elementary symmetric functions is found to be the product of the binomial coefficients $\prod_{k=1}^{n-1} \binom{n}{k}$. The last two sections of the paper are concerned with a recursive relation, alternative forms, and approximations of this product of binomial coefficients.