Identifying and Solving Multivariate Rational Expectations Models.

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Abstract

This article discuses the identification of Generalised Rational Expectations Models. It is shown that the necessary and sufficient conditions for local identification of the Quasi-Structural Form (Q-SF) derive from the first derivatives of the Non-Linear Instrumental Variables (NLIV) criterion. The necessary and sufficient conditions for local identification consist of an appropriately defined and informative instrument set and a Jacobian matrix with appropriate rank.

However, these conditions do not identify the full structural form (SF) linked to either the true expectations or the full solution. For the identification of SF, the parameters need to be associated with a model that satisfies the transversality condition. It is shown that the testing of this condition is impossible when relying exclusively on the existing instruments.

1 Introduction

This article considers the identification of multivariate, multi-period rational expectations (RE) models as discussed by, Broze and Szafarz (1991), Broze, Gourieroux and Szafarz (1995) and Binder and Pesaran (1995(BP95)). Firstly, an extension to proposition 1 in BP97 is developed via the Generalized Bézout Theorem to show, that under more general conditions, a regular solution to the RE problem exists. Secondly, necessary and sufficient conditions for local identification of the Quasi-Structural Form (Q-SF) require the existence of the appropriate rank of the product of the moment matrix of the data and the Jacobian matrix of the parameters. That is to identify the Q-SF, what is required is a set of valid instruments combined with a Jacobian that reflects conditions associated with the solution. These requirements for local identification stem from the application of the criterion developed by Sargan (1983a) for the identification of simultaneous equation systems with time dependent errors.

Conventionally, linearized RE models are estimated by Instrumental Variables (IV)/ Generalized Method of Moments estimators, as Phillips (2003) mentions, identification is often accepted purely on the basis of a test of over-identifying restrictions (Arellano et al (2000) and Sargan (1964)). Usually, the existence of sufficient exogenous and predetermined variables appears to be satisfactory for identification. However, there is some doubt as to whether tests for the validity of over-identifying restrictions, even corrected (Kleibergen (2003)), are able to provide reliable inference and as a result identify model parameters in either Q-SF and SF. Furthermore, Sargan (1983) suggests that failure of identification, for whatever reason, will affect the underlying limit distributions of the estimator. As Stock, Wright and Yogo (2002) show in the presence of weak instruments additional over-identifying restrictions associated with further moment conditions do not solve the problem.

Flôres and Szafarz (1994) and Hunter (1992) remark that identification of the Q-SF follows from the non-linear structure of the RE problem. Flôres and Szafarz (1994) develop a condition that they obtain from the Jacobian of the transformation from the deep or structural parameters to Quasi-Structural parameters. This condition does not rely on instrument selection or the moment matrix of the data, and as a result data dependencies, such as those due to cointegration, have no direct effect on this condition for identification.[1]

The local conditions presented here are influenced both by dependencies across the Jacobian matrix, which reflect the parametric structure of the RE problem, and the moment matrix of the data. However, the local conditions are necessary and sufficient for identification of the Q-SF only, because such parameters do not embed information associated with satisfaction of the transversality condition. The transversality condition must be satisfied for identification of the structural form. Should, one attempt to test the full set of over-identifying restrictions using a limited information estimator (IV/GMM), the restriction associated with the regular solution of the RE model, that consequently satisfies the transversality condition, is not testable without additional information regarding measures of forward expectations that are independent to the set of instruments included in the model. Testing requires separate measures of expectations based either on models that solve out the RE problem or survey data.

This article is structured as follows: in section 2 the canonical representation of the multivariate RE model is presented; identification is viewed in terms of instrument validity in section 3. Both instruments and parametric restriction on the Q-SF are derived in

[1] One form of non-linearity not considered here, relates to the impact of the discount rate on identification (Gregory et al (1993), Sargan (1982)), excepting the single equation case, the introduction of a fixed discount rate normally aids identification (Hunter (1992) and Hunter and Ioannidis (2000)).

section 4; Section 5 discusses the impact of the transversality condition on identification and then conclusions are drawn in the final section 6.

2 Generalized Multivariate Expectations Models

Binder and Pesaran (1995(BP95)) show that the generalized multivariate rational expectations model first considered by Broze and Szafarz (1991) can be expressed in canonical form:

$$\overline{x}_t = A\overline{x}_{t-1} + BE(\overline{x}_{t+1}|I_t) + w_t \tag{1}$$

following the notation in Binder and Pesaran (1997(BP97)) $\overline{x}_t = (x_t^{'}, x_{t-1}^{'}, ...x_{t-K+1}^{'})$, $x_t^{'} = (y_t^{'}, E(y_{t+1}^{'}|I_t), ...E(y_{t+H}^{'}|I_t))$, y_t is a G vector of decision variables. The remaining elements in (1) are defined as $A = -D_0^{-1}D_1,\ B = -D_0^{-1}D_{-1},\ w_t = -D_0^{-1}\bar{\vartheta}_t,\ \bar{\vartheta}_t = (\vartheta_t^{'}, 0_n^{'}, ...0_n^{'})^{'},\ \vartheta_t = (u_t^{'}, 0_G^{'}, ...0_G^{'})^{'}$; with $n = (H+1)G$. Where $\bar{\vartheta}_t$ and $\overline{x}_t$ are both $K(H+1)G \times 1$ matrices , u_t is a G vector of forcing variables, $0_n^{'}$ is an $n \times 1$ vector of zeros, ϑ_t is of dimension $n \times 1$ and the matrices D_i for $i = -1, 0, 1$ are defined as:

$$D_{-1} = \begin{bmatrix} \Gamma_{-1} & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{bmatrix}, D_0 = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{K-1} \\ 0_n & I_n & \cdots & 0_n \\ & & \ddots & \\ 0_n & 0_n & \cdots & I_n \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0_n & 0_n & \cdots & 0_n & \Gamma_K \\ I_n & 0_n & \cdots & 0_n & 0_n \\ & & \ddots & & \\ 0_n & 0_n & \cdots & I_n & 0_n \end{bmatrix} \text{ with } \Gamma_k, k = -1, 0, 1, ...K,$$

$$\Gamma_{-1} = \begin{bmatrix} 0_G & 0_G & \cdots & 0_G & 0_G \\ -I_G & 0_G & \cdots & 0_G & 0_G \\ & & \ddots & & \\ 0_G & 0_G & \cdots & -I_G & 0_G \end{bmatrix}, \Gamma_0 = \begin{bmatrix} I_G & -A_{01} & \cdots & -A_{0H} \\ 0_G & I_G & \cdots & 0_G \\ & & \ddots & \\ 0_G & 0_G & \cdots & I_G \end{bmatrix}$$

$$\text{and } \Gamma_i = \begin{bmatrix} -A_{i0} & -A_{i1} & \cdots & -A_{iH} \\ 0_G & 0_G & \cdots & 0_G \\ & & \ddots & \\ 0_G & 0_G & \cdots & 0_G \end{bmatrix} \text{ for } i = 1, ...K.$$

It follows from proposition 1 in BP97 that when λ_i defines a solution to the scalar problem:

$$\phi(\lambda_i) = \det(B\lambda_i^2 - \lambda_i I + A) = 0$$

there are finitely many Jordan matrices J_i for $i = 1, 2, ...l$ that solve $\phi(C) = 0$, where C=SJ_iS^{-1} for some non-singular matrix S. Any matrix J_i that solves $\phi(C) = 0$ also satisfies:

$$BS_i^2 - SJ_i + AS = 0. \tag{2}$$

Vectorizing (2):

$$((J_i^2)^{'} \otimes B - J_i^{'} \otimes I_m + I_m \otimes A)vecS = 0.$$

When $rank\{(J_i^2)^{'} \otimes B - J_i^{'} \otimes I_m + I_m \otimes A\} = m - 1$, then S is a non-singular matrix, and $J_i = \Lambda$ is a solution to $\phi(C) = 0$, which also solves:

$$P(C) = BC^2 - C + A = 0;$$

where $C = S\Lambda S^{-1}$. If $P(C) = 0$ then it follows from the Generalized Bézout Theorem (Gantmacher (1960)) that the polynomial in z-transfom, $P(z) = (Bz^2 - zI + A)$ has a left hand divisor of the form $(zI - C)$. Mapping $P(z) = F(z)(zI - C)$ onto the time domain:

$$P(L^{-1}) = F(L^{-1})(L^{-1}I - C)$$

where $F(L^{-1}) = (I - BC)(FL^{-1} - I)$. $P(C) = 0$, when $(I - BC)$ is non-singular and $rank\{(J_j^2)^{'} \otimes B - J_j^{'} \otimes I_m + I_m \otimes A\} = m-1$. $P(L^{-1}) = (BL^{-1}\overline{x}_t - \overline{x}_t + AL\overline{x}_t)$ decomposes into backward and forward components:

$$\begin{aligned} -P(L^{-1})L\overline{x}_t &= -(I - BC)(FL^{-1} - I)(L^{-1}I - C)L\overline{x}_t = & (3) \\ &= (I - BC)(I - FL^{-1})(I - CL)\overline{x}_t = w_t & (4) \end{aligned}$$

and $(I - FL^{-1})$ inverts to produce the regular solution in BP97 without the requirement that $AB = BA$:[2]

$$\overline{x}_t - C\overline{x}_{t-1} = \sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} E(\bar{\vartheta}_{t+s}|I_t) \qquad (5)$$

If $u_t = \psi z_t + \varepsilon_t$ (BP95), with ψ dimensioned $G \times G_z$, G_z the number of exogenous variables, then solving for expectations gives rise to the Reduced Form (RF):

$$\overline{x}_t - C\overline{x}_{t-1} = \Upsilon(L)\overline{z}_t + \overline{\varepsilon}_t \qquad (6)$$

where $\Upsilon(L) = (\Upsilon_0 + \Upsilon_1 L + ...\Upsilon_{s-1}L^{s-1})$, $\overline{z}_t' = [z_t^{*\prime}, 0_{n^*}^{'}, ...0_{n^*}^{'}]$ is a state vector containing exogenous processes, with $z_t^{*\prime} = [z_t', 0_{G_z}^{'}, ...0_{G_z}^{'}]$ and $n^* = (1 + H)G_z$, and where $\overline{\varepsilon}_t' = [\varepsilon_t^{*\prime}, 0_n^{'}, ...0_n^{'}]$ is a state vector containing white noise residuals with $\varepsilon_t^{*\prime} = [\varepsilon_t', 0_G^{'}, ...0_G^{'}]$.

Now $\overline{x}_{t-1}$ contains perception variables $(E(y_{t-i}^{'}|I_{t-j})$ when $j > i$.[3]. Given measures for all elements in $\overline{x}_{t-1}$a solved form of the first order condition is derived in Appendix A; the Quasi-Structural Form (Q-SF):

$$(I+FC)\overline{x}_t - FE(\overline{x}_{t+1}|I_{t+1}) - C\overline{x}_{t-1} - (I-BC)^{-1}D_0^{-1}(\Psi\overline{z}_t + \overline{\varepsilon}_t) = F(\Psi R\bar{\zeta}_{t+1} + \overline{\varepsilon}_{t+1}') \quad (7)$$

where $\bar{\vartheta}_t = \Psi\overline{z}_t + \overline{\varepsilon}_t$, I is a Kn dimensional identity matrix and the block diagonal matrix, $\Psi_{(Kn \times Kn^*)} = diag[\psi, 0_{G_z \times G}, ...\ 0_{G_z \times G}]$ and $R = \sum_{s=0}^{\infty} F^s (BC - I)^{-1} D_0^{-1} R_s$. The Kn^* vector $\bar{\zeta}_{t+1}$ contains innovations in the exogenous variable processes. The following linearization of (7) can be estimated consistently by Instrumental Variables (IV) when an optimal instrument set exists (Sargan (1983)):

$$Q_o\overline{x}_t - Q_1\overline{x}_{t+1} - Q_2\overline{x}_{t-1} - \Pi\overline{z}_t = \varsigma_{t+1} \qquad (8)$$

[2]The result presented replaces Proposition 2 in BP(97). It follows from the application of Frobenius Theorem or the appropriate matrices having Property P (Motzkin and Taussky (1954)), that $P(C) = BC^2 - C + A = 0$ has the following roots $\phi = \lambda_b\lambda^2 - \lambda + \lambda_a = 0$ when $AB = BA$. In (1), $AB = BA$ only occurs when the problem is first order and $B = \beta A$.

Alternatively, it is either required that $\lambda - \lambda_b\lambda^2 = \lambda_a$ and property P is satisfied or $P(C) = 0$ and $C - BC^2 = A$. It follows from the similarity of $C - BC^2$ and A that the roots (μ) of $C - BC^2$ equal λ_a.

Any matrix pair (C, B) with roots (λ, λ_b) is said to satisfy property P (see Motzkin and Taussky (1954)) when $F(C, B)$ has as its roots $f(\lambda, \lambda_b)$. More generally, Schneider (1955) shows that property P follows for every pair of matrices (A_i, A_j) in an ordered matrix polynomial when $(A_iA_j - A_jA_i)R_i = 0$.Hence, $C - BC^2$ has roots $\lambda - \lambda_b\lambda^2$ when $(BC - CB)R_i = 0$ for some matrix polynomial R_i. Property P will be satisfied when $BC = CB$ and $AB = BA$ Otherwise, when $\mu = \lambda_a$ and a Jordon form $C = SJ_iS^{-1}$ satisfies $P(C) = 0$, then $\mu = \lambda - \lambda_b\lambda^2$. Conditions (ii)-(iv) in BP(97) follow from these results without the requirement that A and B commute.

[3]Such variables are either directly measurable at time t, can be computed from the RF or using the Quadratic Determinantal Equation Method developed by BP95.

where $\varsigma_{t+1}= (I_n - BC)^{-1}D_0^{-1}\overline{\varepsilon}_t - F\overline{\varepsilon}'_{t+1} - F\Psi R\bar{\zeta}_{t+1}$, is an MA(1) error in state space form, $Qo = (I - BC)^{-1}$, $Q_1 = (I - BC)^{-1}B$, $Q_2 = C$ and $\Pi = (BC - I)^{-1}D_0^{-1}\Psi$. Multiplying through by $(I - BC)$ yields the Q-RF

$$\overline{x}_t - P_1\overline{x}_{t+1} - P_2\overline{x}_{t-1} - P_3\overline{z}_t = \varsigma^*_{t+1} \tag{9}$$

Where $\varsigma^*_{t+1} = \overline{\varepsilon}^+_t - FD_0\overline{\varepsilon}^+_{t+1} - F\Psi R\bar{\zeta}_{t+1}$, $\overline{\varepsilon}^+_t = D_0^{-1}\overline{\varepsilon}_t$, $P_1 = B = D_0^{-1}D_{-1}$, $P_2 = C - BC^2$ and $P_3 = D_0^{-1}\Psi$. Having derived the Q-RF, the identification of the linearized version of (8) is considered in the following section.

3 Identification and Instrument Validity

The conditions presented in Pesaran (1987) can be extended to identify the linear Q-SF parameters $b = [Q_0, Q_1, \Pi, C]$ of (8).Identification of SF parameters follows from the existence of a well defined RF and identification of (6) stems from the existence of sufficient lagged information.[4]

To identify b, when Q_o is non-singular, $\overline{x}_t = C\overline{x}_{t-1} + \Upsilon(L)\overline{z}_t + \overline{\varepsilon}_t$ and $E(\overline{x}_t|I_t)$ depends on $\overline{x}_{t-1}$ and $\overline{z}_{t-i}$, for $i = 0, 1, 2..s - 1$, .it is required that the matrix UQ below has rank $Kn + Kn^* + rank(UQ^*)$:

$$UQ = \begin{bmatrix} b\Phi & O & O & ... & O \\ O & I_{Kn} & O & ... & O \\ O & O & I_{Kn^*} & ... & O \\ O & C^2 & \Xi_0 & ... & \Xi_{S-1} \end{bmatrix},$$

$$U = \begin{bmatrix} Q_0 & -Q_1 & -\Pi & -Q_2 \\ O & I_{Kn} & O & O \\ O & O & I_{Kn^*} & O \\ O & O & O & I_{Kn} \end{bmatrix} \text{ and } Q = \begin{bmatrix} \Phi_i & C & \Upsilon_0 & ... & \Upsilon_{S-1} \\ O & I_{Kn} & O & ... & O \\ O & O & I_{Kn^*} & ... & O \\ O & C^2 & \Xi_0 & ... & \Xi_{S-1} \end{bmatrix}.$$

Where $Q_0C - Q_1 - Q_2C^2 = 0$, $Q_0\Upsilon_0 - \Pi - Q_2\Xi_0 = 0$, $Q_0\Upsilon_i - Q_2\Xi_i = 0\ for\ i = 2, 3, ...s - 1$, $\Xi_i = C\Upsilon_i$, $\Phi = [\Phi_i : O]$ and following Pesaran (1987):

$$UQ^* = \begin{bmatrix} \Phi B & O & ... & O \\ O & \Xi_1 & ... & \Xi_{S-1} \end{bmatrix}.$$

If UQ is dimensioned $2Kn \times (r + Kn^*(s - 1))$ with r conventional identification restrictions on B(Sargan (1988)) and $rank(\Xi_i = C\Upsilon_i) \leq \min(Kn, Kn^*)$, then a unique solution to b, implies $rank(UQ^*) = 2Kn - 1$. It follows from the dimension of UQ, that the order condition is $2K(1 + H)G - 1 < r + K(1 + H)G_z(s - 1)$. Exact identification requires, in addition to r restrictions, enough pre-determined information via lags on the RF (s) and/or number of exogenous variables (G_z).[5] Given sufficient a priori restrictions, rank($\begin{bmatrix} \Xi_1 & ... & \Xi_{S-1} \end{bmatrix}) = Kn$ is required for identification. It may be possible that even when $rank(\begin{bmatrix} \Xi_1 & ... & \Xi_{S-1} \end{bmatrix}) = Kn$, when a long enough dynamic process is estimated, identification though technically feasible may be undetectable due to weak instruments. The generalisation of the rank condition derived from Pesaran (1987) is also necessary to identify (9) when $Q_0 = I$.

The Pesaran condition is appropriate to identify linear models, but when identification of the structural parameters embedded in (7) is considered then the impact of the non-linear restrictions needs to be taken into account. In the next section conditions based on the impact of these restrictions and the moments of the data are derived.

[4]Such conditions derive from Rothenberg(1971) and relate to the instrument matrix having sufficient rank.

[5]If $s = 2$ and $G = G_z$ the order condition has a more usual form $r > K(1 + H)G - 1$.

4 Non-linear Identification of the Quasi-Structural Form

In this section necessary and sufficient conditions for local identification are derived following the approach developed by Sargan (1983a).[6] When compared with Flôres and Szafarz (1994), the results presented here combines the Jacobian condition with the moment matrix of the data and the models incorporate perceptions ($K > 1$).

Stacking (8) the system is written as:

$$V(\theta)X^{*\prime} = E' \tag{10}$$

where $V(\theta) = [D_0 :\text{-}\Psi : D_{-1} : -D_0C + D_{-1}C^2\ ]^7$, $X^* = [X\ Z\ X_{+1}\ X_{-1}\ ]$ where X is an $N \times Kn$ matrix of observations on $\bar{x}_t$, Z is an $N \times Kn^*$ stacked matrix of observations on $\overline{z}_t$, $E'$is an $N \times Kn$ stacked matrix of observations on ς^*_{t+1} and N is the number of time observations. The data matrices subscripted by $_{+1}$ relate to observations for the period $t + 1$ and $_{-1}$ to period $t - 1$.

Flôres and Szafarz (1994) derive conditions which depend on the rank of the Jacobian matrix, considered without expectations by Rothenberg (1971). The Jacobian of the transformation from the Q-SF to SF parameters ($\theta = [vec(D_0)' : vec(\Psi)' : vec(C)'])^8$ is:

$$\frac{dvec(V(\theta))}{d\theta} = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \\ -(C' \otimes I) & 0 & -(I \otimes D_0) + ((I \otimes D_0C) + (C' \otimes D_0) \end{bmatrix}, . \tag{11}$$

while $rank(\frac{dvec(V(\theta))}{d\theta}) < 2Kn + Kn^*$ is the condition required.[9] However, Sargan (1983a) shows for a class of dynamic model that the condition on the rank of the Jacobian is only sufficient for identification.

Defining Z^*as $\left[\hat{X}\ Z\ \hat{X}_{+1}\ X_{-1}\ \right]$ where $\hat{X}$ and $\hat{X}_{+1}$ are matrices of predictions or forecasts of X and X_{+1}, .and post multiplying (10) by Z^* gives rise to:

$$V(\theta)X^{*\prime}Z^* = E'Z^* \tag{12}$$

Consistent estimation of $\theta = [vec(D_0)' : vec(\Psi)' : vec(C)']$ requires that:

$$V(\theta)\frac{plim(X^{*\prime}Z^*)}{N} = 0 \tag{13}$$

(Sargan (1983a)). The criterion is made operational by replacing $\hat{X}$ and $\hat{X}_{+1}$by their instruments Z^+ =$[X_{-1}, Z, Z_{-1}, Z_{-2}, ...Z_{-s}]$ so the moment matrix of the data can be written:

$$\begin{aligned} p\lim(\frac{Z^{+\prime}X^*}{N}) &= M = plim\left[\frac{Z^{+\prime}X}{N} : \frac{Z^{+\prime}Z}{N} : \frac{Z^{+\prime}X_{+1}}{N} : \frac{Z^{+\prime}X_{-1}}{N}\right] \\ &= [M_0 : M_1 : M_2 : M_3]. \end{aligned}$$

[6]Broze and Szafarz (1991) consider identification of the structural form, $D_0E(\bar{x}_t|I_{t-1}) + D_1\bar{x}_{t-1} + D_{-1}E(\bar{x}_{t+1}|I_{t-1}) = \Psi E(\bar{z}_t|I_{t-1}) + \varepsilon_t.$, when $K = 1$ and $H = 1$. From property 5.96 in Broze and Szafarz(1991), identification of the structural form requires that the expectations are uniquely recovered by the estimator. However, as is shown in section 5 of this article, identifying the parameters from a limited information estimator is not feasible without further knowledge of the expectations. Thus the conclusions offered in Broze and Szafarz (1991) may be overly optimisitic vis-a-vis the extent to which a model can be identified without knowledge of the solution.

[7]For ease of exposition, all the terms in $b = [Q_0, Q_1, \Pi, C]$ are pre-multiplied by $D_0(I - BC)$.

[8]Given, the definition of the model in section 2, D_1 is a fixed matrix and is identified a priori.

[9]The results presented here do not consider conditions on the variance-covariance matrix, but what distinguishes this article from that of Flôres and Safarz is the interaction between the moment conditions and the Jacobian. The conditions on the variance-covariance matrix induce further non-linearities, which only makes identification more complicated.

Vectorizing $(V(\theta)M)$:

$$vec(V(\theta)M') = (M \otimes I_{Kn})vec(V(\theta)) = 0. \tag{14}$$

Following Sargan(1983a) the necessary and sufficient conditions for the local identification of dynamic autoregressive models estimated by IV is derived from the first derivative of (12) with respect to θ:

$$\frac{dvec(V(\theta)M')}{d\theta} = (M \otimes I_{Kn})\frac{dvec(V(\theta))}{d\theta}. \tag{15}$$

Expanding (15):

$$\begin{aligned}
(M \otimes I_{Kn})\frac{dvec(V(\theta))}{d\theta} &= [(M_0 \otimes I_{Kn}) - (M_3 \otimes I_{Kn})(C' \otimes I_{Kn}) : \\
-(M_1 \otimes I_{Kn}) &: (M_3 \otimes I_{Kn})(-(I \otimes D_0) + ((I \otimes D_0C) + (C' \otimes D_0))] \\
&= [(M_0 \otimes I_{Kn}) - (M_3C' \otimes I_{Kn}) : -(M_1 \otimes I_{Kn}) : \\
&\quad (M_3 \otimes I_{Kn})((C' - I) \otimes D_0 + (I \otimes D_0C))] \\
&= [((M_0 - M_3C') \otimes I) : -(M_1 \otimes I_{Kn}) : (M_3(C' - I) \otimes D_0) \\
&\quad +(M_3 \otimes D_0C))]
\end{aligned} \tag{16}$$

gives rise to the rank condition:

$$rank\{(M \otimes I_{Kn})\frac{dvec(V(\theta))}{d\theta}\} < m = 2Kn + Kn^*, \tag{17}$$

which is both necessary and sufficient for the local identification of the Q-SF parameters. The vector θ is not identified when the above rank condition fails or:

$$\begin{aligned}
rank(M_0 - M_3C') &< Kn && \text{(I)} \\
rank(M_1) &< Kn^* && \text{(II)} \\
rank(M_3(C' - I) \otimes D_0) + (M_3 \otimes D_0C)) &< Kn && \text{(III)}
\end{aligned}$$

It follows that D_0 is not-identified when $M_0 = M_3C'$ or certain rows and columns in M_0 and M_3C' are subject to some cancellation. A special case of this occurs when there are unit roots in the process driving the exogenous variables. Failure of (II) occurs when M_1 is rank deficient and as a result Ψ is not identified. This failure occurs when there are insufficient instruments or z is cointegrating exogenous for a subset of the SF parameters (Hunter (1989)). Cointegrating exogeneity occurs when the long-run processes forcing y do not apply to z. Failure of (III) occurs when either D_0 or M_3 are rank deficient. The matrix C is not fully identified when there are dependencies between $M_3(C' - I) \otimes D_0)$ and $(M_3 \otimes D_0C)$ and more specifically $C = 0$ when $M_3(C' - I) \otimes D_0) = -(M_3 \otimes D_0C)$.[10]

In the existing literature, failure of identification is associated with $rank(\frac{dvec(V(\theta))}{d\theta}) < m$ (Flôres and Szafarz(1994)) and $rank((M \otimes I_{Kn})) < m$ (Pesaran(1987)). The conditions presented in this article when compared with Flôres and Szafarz(1994) are affected by the existence of unit roots, cointegration and annihilations that may occur between the moment and the Jacobian matrix. When looking at the moment matrix of the data alone, then it is common practice to accept identification based simply on the imposition of enough over-identifying restrictions and these are rarely tested (Phillips (2003)).

[10] There may also be rank dependencies across the columns of (16) associated with cointegration between the x variables alone This will lead to further restrictions which imply that not all the parameters in C are identified

Sargan (1983) showed that for NLIV, loss of identification is often associated with the failure of conventional asymptotic theory, implying that conventional test statistics at best converge slowly to their limit distributions. This point is further elaborated by Stock, Wright and Yogo (2002) who point out that both the IV and GMM estimators may have non–normal limit distributions and that inference based on sample evidence is unreliable[11].

Were one to undertake such tests, then there are serious questions over their application (Dufour (1997)) and their performance (Stock and Wright (2000)). Dufour is critical of the power of the underlying tests based on what are limited information estimators, while Stock and Wright consider the question of weak instruments. Empirically, the loss of identification associated with conditions (I)-(III) is likely to occur in the presence of weak instruments and/or the incapacity to detect the supposed RE structure.

The latter observation leads to the final proposition of this article, that identification, thus far described, relates to the parametric identification of the Q-RF and Q-SF parameters alone and not to the forward looking representation of the model. Furthermore, estimates of parameters based on (1) (5) and (6) need to be observationally equivalent to those derived from (7), (8) and (9) for full identification of the structure. As the (Q-RF) and (Q-SF) do not impose the solution, then equivalence with (1) only occurs when the instruments accurately approximate the true expectations.

5 Identification and the Transversality condition

The local conditions developed in the previous section are necessary and sufficient to identify the parameters of both (8) and (9), but they are only necessary for the identification of the parameters of (1) that contains the true expectations.[12] In models that contain future variables or their expectations, full identification of the structure implies that (1) and (9) are isomorphic. These equations are said to be isomorphic when the two sets of parameters are observationally equivalent. However, in the case of RE, (9) is a model with a Moving Average (MA) error structure that does not use estimates of expectations that bind the parameters to the solution, while (1) contains the true expectations.

Comparing (1) and (9):

$$A\overline{x}_{t-1} + B(E(\overline{x}_{t+1}|I_t) + w_t - P_1\overline{x}_{t+1} - P_2\overline{x}_{t-1} + P_3\overline{z}_t - \varsigma^*_{t+1} = 0$$

with some re-arrangement:

$$A\overline{x}_{t-1} - P_2\overline{x}_{t-1} + B(E(\overline{x}_{t+1}|I_t) - P_1\overline{x}_{t+1} + +P_3\overline{z}_t + w_t - \varsigma^*_{t+1} = 0$$

Therefore:

$$\begin{aligned} A\overline{x}_{t-1} - P_2\overline{x}_{t-1} + B(E(\overline{x}_{t+1}|I_t) - P_1\overline{x}_{t+1} + P_3\overline{z}_t + w_t &= \varsigma^*_{t+1} \\ (A - P_2)\overline{x}_{t-1} + B(E(\overline{x}_{t+1}|I_t) - \overline{x}_{t+1}) + (P_1 - B)\overline{x}_{t+1} + P_3\overline{z}_t + w_t &= \varsigma^*_{t+1} \end{aligned}$$

where $w_t = -D_0^{-1}\Psi\overline{z}_t - D_0^{-1}\overline{\varepsilon}_t$, then it follows:

$$(A - P_2)\overline{x}_{t-1} + B(E(\overline{x}_{t+1}|I_t) - \overline{x}_{t+1}) + (P_1 - B)\overline{x}_{t+1} + (P_3 - D_0^{-1}\Psi)\overline{z}_t - D_0^{-1}\overline{\varepsilon}_t = \varsigma^*_{t+1}.$$

[11]Despite the small sample corrections introduced by Kleibergen(2003), the reliability of the test statistics will be questionable because when there is failure of identification, the rate of convergence to their limiting distribution is non-standard

[12]Comparison can equally well be made between (1) and (6) or between $D_0\overline{x}_t = -D_1\overline{x}_{t-1} - D_{-1}E(\overline{x}_{t+1}|I_t) + \bar{\vartheta}_t$ (SF) and (5).

From the definition of the parameters of (9) $P_2 = C - BC^2$ and it follows that:

$$(A-C-BC^2)\overline{x}_{t-1}+B(E(\overline{x}_{t+1}|I_t)-\overline{x}_{t+1})+(P_1-B)\overline{x}_{t+1}+(P_3-D_0^{-1}\Psi)\overline{z}_t-D_0^{-1}\overline{\varepsilon}_t=\varsigma^*_{t+1}.$$

Equations (1) and (9) are isomorphic when the following parametric restrictions hold: $P_1 - B = 0$, $P_3 - D_0^{-1}\Psi = 0$ and $P(C) = BC^2 - C + A = 0$. Irrespective of the nature of expectations, when imposing the above restrictions:

$$\varsigma^*_{t+1} = B(E(\overline{x}_{t+1}|I_t) - \overline{x}_{t+1}) - D_0^{-1}\overline{\varepsilon}_t.$$

When the RE problem is solved then $P(C) = 0$ and the parameters of (1) and (9) are equivalent by definition as $A = C - BC^2$. It follows from the solution to the RE problem:

$$\varsigma^*_{t+1} = \overline{\varepsilon}^+_t - FD_0\overline{\varepsilon}^+_{t+1} - F\Psi R\bar{\zeta}_{t+1}$$

is an MA(1) error process with a future innovation in the exogenous variables.

The joint hypothesis:

$$H_0 : P_1 - B = 0,\ P_3 - D_0^{-1}\Psi = 0, BC^2 - C + A = 0$$

can apparently be tested from regression estimates of:

$$\hat{\varsigma}^*_{t+1} = \Pi_1\overline{x}_{t-1} + \Pi_2\overline{x}_{t+1} + \Pi_3\overline{z}_t + B(E(\overline{x}_{t+1}|I_t) - \overline{x}_{t+1}) + \overline{\varepsilon}^*_t$$

under the null hypothesis $\overline{\varepsilon}^*_t = -D_0^{-1}\overline{\varepsilon}_t$, $\Pi_1 = -BC^2 + C - A = 0$, $\Pi_2 = P_1 - B$,and $\Pi_3 = P_3 - D_0^{-1}\Psi = 0$. However, testing requires estimates of the innovation $(\widehat{E(\overline{x}_{t+1}|I_t)} - \overline{x}_{t+1})$. If the innovations are estimated from the residuals of the unrestricted RF (6)[13], they will be linear combinations of $\overline{x}_{t-i}$ for $i = 1, ..., s-1$ and $\overline{z}_t$. The test is not operational, because the same instruments were used to generate $\hat{\varsigma}^*_{t+1}$. It is also not feasible to use estimates of

$$\hat{\varsigma}^*_{t+1} = \Pi_1\overline{x}_{t-1} + \Pi_2\overline{x}_{t+1} + \Pi_3\overline{z}_t + \varsigma^*_{t+1}$$

by Ordinary Least Squares[14] since $\hat{\varsigma}^*_{t+1}$ and $\bar{x}_{t-1}$ $\widehat{\overline{x}_{t+1}}$, $\overline{z}_t$ are orthogonal by definition when (9) is estimated by IV or GMM. Hence, this proposition will not be testable from the residuals of the Q-RF or Q-SF.

6 Conclusions

This article presents conditions for the identification of a broad class of models that include forward expectations and past perceptions. The literature thus far has broadly considered the identification of Q-RF and Q-SF parameters (Pesaran (1987), Flôres and Szafarz (1994))..Here a distinction is made between weak identification of Q-SF and Q-RF models that include estimates of expectations as compared with identification of models that include either independent measures of expectations or solve the forward looking problem.

The local identification conditions presented in section 4 distinguish between a failure of identification induced by weak instruments, the case considered by Pesaran (1987) and those associated with the existence of inappropriate restrictions, the case considered by Flôres and Szafarz (1994). Poor or weak instruments will affect identification (Stock

[13]Were such estimates used to explain the forward error in (6), then this would define the first step of the Vector MA(1) estimator defined by (Spliid (1983)).

[14]It is also of interest to note that any model normalised on $\overline{x}_{t+1}$ does not alleviate this problem, because the expected error in either case is zero

and Wright (2000)), because conventional tests of over-identifying restrictions or moment conditions, have been shown to be unreliable (Stock, Wright and Yogo (2002)). Although tests of over-identifying restrictions may form the basis of an identification strategy, they should not be the sole criterion upon which it is decided that a model with expectations has been identified. Further moment conditions do not solve this problem as test performance may further deteriorate when GMM is applied (Stock, Wright and Yogo (2002))

In section 2 it is shown that the condition $P(C) = 0$ is required for the existence of a regular solution to the RE problem and is the basis for a form of the Q-SF that employs the RE restrictions. As a result, identification depends on both the rank of the moment matrices of the data and a set of non-linear restrictions that relate the linear form of the model which contains true expectations, into a model where the expectations have been replaced by instruments. On there own, satisfaction of the moment conditions and the Jacobian conditions are necessary, but not sufficient for identification of the Q-SF and Q-RF. The existence of different types of data dependency associated with cointegration results in the insufficiency of the Jacobian condition. The presence of weak instruments reduces the reliability of data moments. The imposition of non-linear cross equation restrictions by reducing the dimension of the parameter space ameliorates the situation.

Identification of the Q-SF of an RE model requires that: (a) the problem is appropriately dimensioned, in terms of there being enough instruments and cross equation restriction; (b) the instrument set is valid based on a test of over-identifying restrictions; and (c) the non-linear restrictions are valid based either on the Jacobian matrix or the information matrix having appropriate rank. More appropriately, the rank conditions (I-III) presented in section 4 ought to be tested. In addition, full identification of the parameters associated with RE models requires that $P(C) = 0$ or that the transversality condition holds. As it is demonstrated in section 5, the hypotheses associated with the proposition that $P(C) = 0$ cannot be tested from the residuals of either a Q-SF or Q-RF, unless estimates of the expectations are available, which include information distinct from the instrument set suggested by the RF. For example, the test can be applied to the residuals of a model estimated by GMM or IV when separate survey data on the expectations is available. It is of interest to note that the conditions derived above for the identification of the Q-SF can be used to identify the structural parameters where the model estimated is based on the full solution to the RE problem, as in (5).

7 Appendix A

Applying the forward Koych transformation $(I - FL^{-1})$ to (5) assuming that there exist measures for all lagged expectations in $\overline{x}_{t-1}$:

$$\begin{aligned}(I - FL^{-1})(\overline{x}_t - C\overline{x}_{t-1}) \quad = \quad & \sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} E(\bar{\vartheta}_{t+s}|I_t) - \\ & F\sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} E(\bar{\vartheta}_{t+s+1}|I_{t+1}) \qquad (18)\end{aligned}$$

where $L^{-i}E(\overline{x}_{t+s}|I_t) = E(\overline{x}_{t+s+1}|I_{t+1})$. Re-ordering the indices on the summation signs and re-ordering terms gives rise to:

$$\begin{aligned}(I - FL^{-1})(\overline{x}_t - C\overline{x}_{t-1}) = (I_n - BC)^{-1} D_0^{-1} \bar{\vartheta}_t + \\ F\sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} (E(\bar{\vartheta}_{t+s}|I_t) - E(\bar{\vartheta}_{t+s+1}|I_{t+1}))\end{aligned}$$

When the exogenous processes have a Wold representation, $z_t = \sum_{r=1}^{\infty} \theta_i L^i v_t$, where θ_i is a $G_z \times G_z$ matrix of fixed parameters, v_t is a G_z vector of white noise residuals and L is the lag operator then:

$$\begin{aligned}(E(\bar{\vartheta}_{t+s}|I_t) - E(\bar{\vartheta}_{t+s}|I_{t+1})) &= (E(\Psi\overline{z}_{t+s} + \overline{\varepsilon}_{t+s}|I_t) - E(\Psi\overline{z}_{t+s} + \overline{\varepsilon}_{t+s}|I_{t+1})) \\ &= (\Psi E(\overline{z}_{t+s}|I_t) + E(\overline{\varepsilon}_{t+s}|I_t) - \Psi E(\overline{z}_{t+s}|I_{t+1}) - E(\overline{\varepsilon}_{t+s}|I_{t+1})) \\ &= (\Psi E(\overline{z}_{t+s}|I_t) - \Psi E(\overline{z}_{t+s}|I_{t+1})) + (E(\overline{\varepsilon}_{t+s}|I_t) - E(\overline{\varepsilon}_{t+s}|I_{t+1})) \\ &= -\Psi R_s \bar{\zeta}_{t+1} - \overline{\varepsilon}_{t+1}\end{aligned}$$

where, $E(\overline{\varepsilon}'_{t+s}|I_{t+i}) = 0$ for $s > i$, $\bar{\zeta}_t = (\zeta'_t, 0'_{n\times 1}, ...0'_{n\times 1})'$, $\zeta_t = (v'_t, 0'_{G\times 1}, ...0'_{G\times 1})'$; $\bar{\zeta}_t$ is an $Kn^* \times 1$ matrix $0'_{n^*\times 1}$ is an $n^* \times 1$ vector of zeros and ζ_t is of dimension $n^* \times 1$, and R_s is a square matrix defined as

$$R_s = \begin{bmatrix} \Theta_s & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{bmatrix} \text{ and } \Theta_s = \begin{bmatrix} \theta_s & 0_G & \cdots & 0_G \\ 0_G & 0_G & \cdots & 0_G \\ & & \ddots & \\ 0_G & 0_G & \cdots & 0_G \end{bmatrix} \text{ for } i = 1, ...\infty$$

A forward solution follows by substituting out for updated expectations:

$$(I - FL^{-1})(\overline{x}_t - C\overline{x}_{t-1}) = (I_n - BC)^{-1} D_0^{-1} \bar{\vartheta}_t - F\sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} (\Psi R_s \bar{\zeta}_{t+1} + \overline{\varepsilon}'_{t+1})$$

Multiplying through by the forward term on the Koych lead and replacing $\bar{\vartheta}_t$ by $\Psi\overline{z}_t + \overline{\varepsilon}_t$,reveals a Quasi-Structural Form (Q-SF):

$$(I + FC)\overline{x}_t - FE(\overline{x}_{t+1}|I_{t+1}) - C\overline{x}_{t-1} - (I_n - BC)^{-1} D_0^{-1}(\Psi\overline{z}_t + \overline{\varepsilon}_t) = F(\Psi R\bar{\zeta}_{t+1} + \overline{\varepsilon}'_{t+1})$$

where $F = (I_n - BC)^{-1}B$ and $R = \sum_{s=0}^{\infty} F^s (I_n - BC)^{-1} D_0^{-1} R_s$.

8 References

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