SOME PROBLEMS OF MIXING SMALL PARTICLES

by

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INTRODUCTION

One process that involves the mixing of small particles is the preparation of animal feeds. Drugs, vitamins and minerals are often mixed in very small quantities with large quantities of feed. It is necessary that something be known about the dispersion of these items throughout the ration in order to make statements concerning the percentages of the daily requirements of the additives that are being met. This thesis describes some of the problems encountered when working with mixtures of small spherical particles.

The two main problems studied were the methods of obtaining a distribution of the weight of an additive per portion of the feed when (a) the particle diameters vary from batch to batch but have equal diameters within a given batch and (b) the particle diameters have a distribution. More directly this thesis was concerned with studying the affect of (a) the inter-batch variation of the diameters and (b) the intra-batch variation of the diameters on the weight of an additive per portion of the feed.

It is necessary that certain assumptions be made before attempting to solve the problems. As assumed above, all results presented in this paper are based on particles that are spherical in shape. The theory assumes that mixing is perfect, that is, the mixing process results in a random mixture so that the particle counts of an additive follow the Poisson distribution in portions of the feed. The number of particles in the total amount of additive used is assumed to be sufficient to allow using the normal approximation to the Poisson distribution. This allows, as will be shown later, for the distribution of weight per portion to be considered as a normal distribution.

In the first case the methods used in obtaining a weight per portion distribution were analytical. The marginal distributions of weight per portion were obtained by integration of a joint distribution and a simulation method where Pearson curves were fitted to an empirical distribution. The Pearson curves were selected and their equations were obtained.

The second problem led to tables which show the comparison of areas under distributions of weight per portion when all diameters are equal with the area under distributions when the diameters have a specific distribution. The results give some indication of the error committed by assuming all particles are the same size. Using the cube of the mean diameter in computing the weight distribution is equivalent to assuming all particles have equal diameters. Hence, to use the idea of distribution of particles affecting the weight distribution, one must use the mean of the cubed diameters as a basis for computing the weight distributions.

RATIONALE

History

Most of the work on problems involving mixtures of small particles is based on the assumption that all of the particles have the same size and are spherical in shape. The theory developed on such assumptions is limited in its usefulness.

The discovery and development of new drugs and additives for animal feeds have caused the idea of quality control on a mixing procedure to be more important. It is necessary to make confidence statements about the amount of additive that will be in each daily portion of the ration. Some vitamins, drugs and minerals are needed daily, while others are stored and can be mixed with wider tolerance limits. Since the absence of a drug from a daily ration can allow the start of a disease and sometimes a slight overdose of the same drug can be fatal, it is important that a very small quantity of the additive is thoroughly mixed with a much larger quantity of feed so as to meet the tolerance limits. This condition indicates that the number of particles of the additive must be sufficient to allow dispersion to all of the daily portions of feed.

Various methods have been used in trying to get a frequency distribution of particles. Two of these methods are counting the particles with the aid of a microscope and screening the particles by sieve analysis. The first method results in a distribution by count while the second method is often used to obtain a distribution by weight. It is possible to transform a number distribution into a weight distribution and vice versa if one is willing to make certain assumptions about the size and shape of the particles.

There are many articles on the distributions of particle sizes. These include arguments for and against various distributions. Kottler (1951) indicates that the lognormal distribution is representative of particle sizes, especially when the particles are produced by severe crushing or grinding. Other particles seem to be somewhat uniformly distributed. The normal distribution might be fitted provided the area under the normal curve below zero is small. Other distributions can be used to represent a particular particle distribution. Some experimenters believe that fewer problems are involved if distributions which are defined for non-negative values are used such as the two parameter gamma distribution.

The Use of the Poisson Distribution

Under the assumption of uniform particle size researchers have indicated that the number of particles in each portion of mix will follow the Poisson distribution if the mixing is perfect. Bloom and Livesey (1953) assume perfect mixing in their results regarding the particle size needed when a given quantity of additive is mixed. Their results and the normal approximation to the Poisson distribution were used in developing the distribution of the weight of an additive per portion of mix.

Suppose n spherical particles of additive, each with diameter v, are available to be used in mixing r portions of feed. Then the total weight of additive is

$$w = \frac{\pi\rho}{6} v^3 n$$

where ρ is the density of the additive. The expected number of particles per portion of mix is $\frac{n}{r}$. Let m be the number of particles observed in a given portion of feed, then m follows a Poisson distribution with mean and variance equal to $\frac{n}{r}$. As $\frac{n}{r}$ becomes larger the Poisson distribution can be approximated by the normal distribution. The results in this paper were obtained by assuming that $\frac{n}{r}$ is sufficiently large to make this approximation valid. This assumption implies that

$$z = \frac{m - \frac{n}{r}}{\sqrt{\frac{n}{r}}}$$

is approximately a standard normal deviate. Note that the weight of m particles is $\frac{\pi\rho}{6} v^3 m$ and that the expected weight per portion is $\frac{\pi\rho}{6} v^3 E(m)$ or $\frac{\pi\rho}{6} v^3 \frac{n}{r}$. Consider the result when z is multiplied and divided by $\frac{\pi\rho}{6} v^3$; that is

$$(1) \qquad z = \frac{\frac{\pi\rho}{6}v^3 m - \frac{\pi\rho v^3 n}{6r}}{\sqrt{\left[\frac{\pi\rho}{6}v^3\right]^2 \frac{n}{r}}}$$

is also approximately a standard normal deviate. The weight per portion is normally distributed with mean $\frac{\pi\rho v^3 n}{6r}$ and variance $\left[\frac{\pi\rho v^3}{6}\right]^2 \frac{n}{r}$. If the mean is defined to be a_p, then the variance is $\frac{\pi\rho}{6}v^3 a_p$.

Consider the case where the diameters are not all the same size, but where the diameters are classified in k classes. If the expected weight per portion of particles with diameter v_i is a_{ip}, then the following table can be obtained when r portions are being mixed.

Table 1. Distribution of diameters.

Classes	Diameter	Number	Weight per class
1	v_1	n_1	ra_{1p}
2	v_2	n_2	ra_{2p}
3	v_3	n_3	ra_{3p}
...	...	...	...
k	v_k	n_k	ra_{kp}
Total		N	ra_p

Let m_i represent the number of particles from the ith class that is observed in a given portion of the final mixture. Then the preceding argument can be used to show that the weight contributed to each portion by the ith class will be normally distributed with mean a_{ip} and variance $\frac{\pi\rho}{6}v^3 a_{ip}$. If $x_1, x_2, \ldots, x_k$ are the weights contributed to a given portion by the particles of size $v_1, v_2, \ldots, v_k$, then the total weight per portion of mixture is $x_1 + x_2 + \ldots + x_k = X$. The variable X is normally distributed with mean

$$\sum_{i=1}^{k} a_{ip} = a_p$$

and variance

$$\frac{\pi\rho}{6}\sum_{i=1}^{k} v_i^3 a_{ip}.$$

Since usual mixing procedures would call for the addition of a given quantity of additive to a fixed number of portions, the mean

$$a_p = \frac{\text{weight of additive}}{\text{number of portions}} = E(X).$$

This shows that

$$z = \frac{X - a_p}{\sqrt{\frac{\pi\rho}{6}\sum v_i^3 a_{ip}}} \tag{2}$$

is approximately a standard normal deviate.

Moments of the Distributions of Diameters

Reference will be made to the moments of various distributions that are assumed to be representative of the diameters of the additive particles. Some of the moments can be obtained in closed form while others must be obtained from general relationships (3) and (4) stated below. The following notation will be used: μ_r' is the rth ordinary moment of the distribution and μ_r is the rth central moment of the distribution.

The following formulae, Wilks (1962), can be used for obtaining the ordinary moments when the central moments are known or for obtaining the central moments when the ordinary moments are known

(3) $$\mu'_r = \sum_{i=0}^{r} \binom{r}{i} \mu_{r-i}\mu^i$$

and

(4) $$\mu_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \mu'_{r-i}\mu^i \quad .$$

Suppose the diameters of a group of particles are uniformly distributed between a and b with the probability density function

(5) $$f(\nu) = \frac{1}{b-a} \qquad a \le \nu \le b$$

$$= 0 \qquad \nu < a \text{ and } \nu > b.$$

It can then be shown that the ordinary moments and central moments for (5) can be obtained by the formulae

(6) $$\mu'_r = \frac{b^{r+1} - a^{r+1}}{(r+1)\ (b-a)}$$

(7) $$\mu_r = \frac{(b-a)^r}{2^r(r+1)} \qquad \text{if } r \text{ is even,}$$

$$= 0 \qquad \text{if } r \text{ is odd.}$$

Consider a group of particles which have diameters that are lognormally distributed with mean μ, variance σ^2, and probability density function

(8) $$f(\nu) = \frac{1}{\sqrt{2\pi}\nu\sigma} \exp\left[-\frac{(\log \nu - \mu)^2}{2\sigma^2}\right] \quad \nu > 0$$

$$= 0 \text{ elsewhere.}$$

The ordinary moments for (8) are given by

(9) $$\mu'_r = \exp\left[r\mu + \frac{r^2\sigma^2}{2}\right].$$

Formula (4) can be used with (9) to obtain the central moments that are desired.

If a given group of particles with diameters that are distributed as a gamma distribution, with parameters α and p, and probability density function

$$(10) \qquad f(v) = \frac{\alpha^p}{\Gamma(p)} v^{p-1} e^{-\alpha v} \qquad v > 0$$

$$= 0 \text{ elsewhere,}$$

then the ordinary moments of (10) are found to be given by

$$(11) \qquad \mu_r' = \frac{\Gamma(p+r)}{\alpha^r \Gamma(p)} .$$

The use of (11) with (4) will give the central moments for the gamma distribution.

Consider a group of particles with diameters that are normally distributed with mean μ, variance σ^2, and probability density function

$$(12) \qquad f(v) = \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left[\frac{-(v - \mu)^2}{2\sigma^2}\right] \qquad \text{for all } v .$$

The central moments for (12) can be obtained from

$$(13) \qquad \mu_{2r} = \frac{(2r)!}{2^r r!} \sigma^{2r} \qquad \text{for } r = 0, 1, 2, \ldots$$

and

$$(14) \qquad \mu_{2r+1} = 0 \qquad \text{for } r = 0, 1, 2, \ldots .$$

The ordinary moments for (12) can be obtained by using formulae (13) and (14) along with formula (3).

DETERMINATION OF WEIGHT PER PORTION DISTRIBUTION

Introduction

Vitamins, drugs and other additives are not produced with uniform diameters from one batch to another. Hence, it was assumed that the production process produced the batches in such a way that the population of diameters, v, followed some known distribution. The following four distributions were assumed to be representative of the various small particle distributions: the uniform distribution; the lognormal distribution; the gamma distribution; and the normal distribution.

This section deals with the problem of finding a distribution of the weight of some additive per portion of feed (quantity of mixture) when the additive batch is taken randomly from the production process. Two methods of approach were used in analyzing this problem. The first method was to find a marginal distribution of weight per portion, X, of mixture by integration of a joint distribution of X and v. Since in most cases it is difficult to obtain such marginal distributions, the second method was to approximate these distributions by a Pearson type distribution. For special cases attempts were made to evaluate the goodness of fit of such distributions.

Analytic Approach

<u>General Case</u>. Let $f(v)$ be the density function of the diameters of particles of a given additive used in a mix. If X is the weight of additive per quantity of mix, then $f(X|v)$ is the conditional distribution of weight per portion of feed when v, the diameter, is given. It follows from previous assumptions that

$$(15) \qquad f(X|v) = \frac{1}{\sqrt{2\pi}\,\sigma(v)} \exp\left[-\frac{(X - a_p)^2}{2\sigma^2(v)}\right]$$

where a_p is the expected weight per portion of mix and $\sigma(v)$ is the standard deviation given by $\frac{\pi\rho}{6}v^3\, a_p$.

By definition the joint distribution of X and v is equal to the marginal distribution of v times the conditional distribution of X given v, that is

$$f(X,v) = f(v)\, f(X|v).$$

Then the marginal distribution $f(X)$ can be obtained by integrating the joint distribution over ν. Thus,

$$(16) \qquad f(X) = \int_\nu f(X,\nu)d\nu .$$

Special Case. Assume that ν has a rectangular distribution between 0 and b. Formulae (5) and (16) give

$$f(X,\nu) = \frac{1}{b\sqrt{2\pi}\,\sigma(\nu)} \exp\left[-\frac{(X - a_p)^2}{2\sigma^2(\nu)}\right] .$$

Hence,

$$f(X) = \frac{1}{b\sqrt{2\pi}} \int_0^b \frac{1}{\sigma(\nu)} \exp\left[-\frac{(X - a_p)^2}{2\sigma^2(\nu)}\right] d\nu .$$

This can be transformed into

$$(17) \qquad f(X) = \frac{\sqrt{2}\ \Gamma(1/6)}{3^{1/6}\, b[\pi^4(X-a_p)]^{1/3}\,(\rho a_p)^{5/6}} \left[1 - \frac{1}{\Gamma(1/6)} \int_0^A U^{1/6\,-1} e^{-U}\, dU\right]$$

where

$$A = \frac{3(X-a_p)^2}{b^3\pi\rho a_p} , \; U = \frac{3(X-a_p)^2}{\pi\rho a_p \nu^3} , \text{ and } \frac{1}{\Gamma(1/6)} \int_0^A U^{1/6\,-1} e^{-U}\, dU$$

is an incomplete gamma function. Formula (17) can be evaluated by using tables of the incomplete gamma distribution. Clearly the evaluation of the distribution function of X is quite difficult even under the assumption of a simple ν distribution.

Moments of the Weight Per Portion Distribution

Formula (2) gives the mean and the variance of a weight per portion distribution obtained from particles having diameters with intra-batch variation.

The mean is a_p and the variance is

$$\frac{\pi\rho}{6}\sum_{i=1}^{k} \nu_i^3 \, a_{ip}.$$

This variance can be expressed as $\frac{\pi\rho}{6} a_p \mu_3'(\nu)$ where $\mu_3'(\nu)$ is the third ordinary moment for the distribution of ν.

Consideration of the quantity $\mu_3'(\nu)$ when working with a weight per portion distribution produced by particles having diameters with inter-batch variation gives

$$(18) \qquad \mu_{2r}(X) = \frac{(2r)!}{2^r r!} \left[\frac{\pi\rho a_p}{6}\right]^r \mu_{3r}'(\nu) \qquad \text{for } r = 0, 1, 2, \ldots$$

and

$$\mu_{2r+1}(X) = 0 \qquad \text{for } r = 0, 1, 2, \ldots .$$

These formulae can be used to find the parameters for a given weight per portion distribution when the density function cannot be easily obtained.

Fitting Pearson Curves to the Empirical Distributions

Introduction. Pearson and Hartley (1954) state that the type of Pearson curve to be used is determined by the size of β_1 and β_2. These values are defined as

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} .$$

Since all of the theoretical distributions are symmetrical in shape $\beta_1 = 0$ for all the distributions. This indicates that either type II or type VII curves should be used. For $\beta_2 > 3$, the type VII curve would be appropriate, while a $\beta_2 < 3$ would require a type II curve. If $\beta_2 = 3$, a normal curve should be used in fitting the distribution.

Determining the Type of Curve to be Fitted. The use of formula (18) shows that

$$\beta_2 = \frac{3\mu_6'(v)}{[\mu_3'(v)]^2} .$$

From the consideration of the variance of a new variable, $y = v^3$, it follows that $\mu_6'(v) \geq [\mu_3'(v)]^2$. This indicates that $\beta_2 \geq 3$ for all distributions of v. Therefore, a type VII Pearson curve should fit the weight per portion distribution regardless of the distribution of particle diameters. For the special distributions of diameters considered in this thesis, it follows that

$$\beta_2 = \frac{48}{7} \qquad \text{when } v \text{ is uniformly distributed between 0 and b,}$$

$$\beta_2 = 3e^{9\sigma^2} \qquad \text{when } v \text{ is lognormally distributed,}$$

$$\beta_2 = 3\left[1 + \frac{9p^2 + 45p + 60}{p^3 + 3p^2 + 2p}\right] \qquad \text{when } v \text{ has a gamma distribution,}$$

and

$$\beta_2 = 3\left[1 + \frac{9\sigma^2\mu^4 + 36\sigma^4\mu^2 + 15\sigma^6}{\mu^6 + 6\mu^4\sigma^2 + 9\mu^2\sigma^4}\right] \qquad \text{when } v \text{ is normally distributed.}$$

Fitting a Type VII Pearson Curve to the Empirical Data. The probability density function for the type VII curve as given by Pearson and Hartley (1954) is

$$f(y) = y_o\left[1 + \frac{y^2}{a^2}\right]^{-m} \qquad \text{for } -\infty < y < \infty.$$

To fit this curve it is necessary to evaluate y_o, m, and a. These quantities are the simultaneous solutions obtained by setting the integral of f(y) equal to 1, $E(y^2)$ equal to the theoretical second central moment given by formula (18), and $E(y^4)$ equal to the fourth theoretical central moment given by formula (18). This simultaneous solution gives

$$(19) \qquad y_o = \frac{\Gamma(m)}{a\,\Gamma(1/2)\,\Gamma(m - 1/2)}$$

$$(20) \qquad a = \left[(2m - 3)\,\frac{\pi\rho_a}{6\rho_p}\,\mu_3'(v)\right]^{1/2}$$

and

$$(21) \qquad m = \frac{3[\mu_3'(v)]^2 - 5\mu_6'(v)}{2[\mu_3'(v)]^2 - 2\mu_6'(v)} \quad .$$

For convenience of computation the quantity m can be expressed in terms of the parameters of the diameter distribution. If v is uniformly distributed between α and β, then

$$(21a) \qquad m = \frac{59(\alpha^8 + \beta^8) - 2\alpha\beta(40\beta^6 - 21\beta^3\alpha^3 + 40\alpha^6)}{18(\alpha^8 + \beta^8) - 4\alpha\beta(8\beta^6 - 7\beta^3\alpha^3 + 8\alpha^6)} \quad .$$

Note that when α is equal to 0, formula (21a) reduces to $m = \frac{59}{18}$.

If v is lognormally distributed with mean μ and variance σ^2, then

$$(21b) \qquad m = 2.5 + \frac{1}{e^{9\sigma^2} - 1} \quad .$$

If v has a gamma distribution with parameters α and p, then

$$(21c) \qquad m = \frac{2p^3 + 51p^2 + 149p + 300}{18p^2 + 90p + 120} \quad .$$

If v is normally distributed with mean μ and variance σ^2, then

$$(21d) \qquad m = \frac{75\sigma^6 + 198\sigma^4\mu^2 + 57\sigma^2\mu^4 + 2\mu^6}{6(5\sigma^6 + 12\sigma^4\mu^2 + 3\sigma^2\mu^4)} \quad .$$

If v is uniformly distributed between 0 and .08 centimeters, then formulae (19), (20) and (21a) give $m = 3.28$, $a = .00021333$ and $y_o = 4210.4$.

When v is lognormally distributed with $\mu = 4.244$ and $\sigma^2 = .28346$ formulae (19), (20) and (21b) give $m = 2.58$, $a = .000047922$ and $y_o = 15997.2$.

If v is distributed as a gamma distribution with α equal to 250 and p equal to ten, then formulae (19), (20) and (21c) give $m = 3.44$, $a = .00018087$ and $y_o = 5126.7$.

When v is normally distributed with μ equal to .04 and σ equal to .01 formulae (19), (20) and (21d) give $m = 4.645$, $a = .00021337$ and $y_o = 5223.6$.

These values when substituted into the equation of the type VII curve give the equations that fit the distributions in Table 3. A transformation is used to locate each distribution at the theoretical mean of the distributions.

The equations are

$$(22) \qquad f(X) = 4210.4 \left[1 + \frac{(X - .000175)^2}{(.00021333)^2}\right]^{-3.28}$$

when v is uniformly distributed,

$$(23) \qquad f(X) = 15997.2 \left[1 + \frac{(X - .000175)^2}{(.000047922)^2}\right]^{-2.58}$$

when v is lognormally distributed,

$$(24) \qquad f(X) = 5126.7 \left[1 + \frac{(X - .000175)^2}{(.00018087)^2}\right]^{-3.44}$$

when v follows a gamma distribution and

$$(25) \qquad f(X) = 5223.6 \left[1 + \frac{(X - .000175)^2}{(.00021337)^2}\right]^{-4.645}$$

when v is normally distributed.

Tabulation of the Probabilities Associated with (22), (23), (24) and (25). The cumulative distribution is defined as

$$(26) \qquad F(y') = y_o \int_{-\infty}^{y'} \left[1 + \frac{y^2}{a^2}\right]^{-m} dy$$

or

$$F(y') = \frac{1}{2} + \int_0^{y'} f(y)dy \qquad \text{for } y' > 0.$$

This can be transformed to give

$$(27) \qquad F(y') = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{B(\frac{1}{2}, m - \frac{1}{2})} \int_0^{\frac{y'^2}{a^2 + y'^2}} (1 - Z)^{(m - \frac{1}{2}) - 1} Z^{\frac{1}{2} - 1} dZ\right]$$

where

$$Z = \frac{y^2}{a^2 + y^2} \quad .$$

The quantity in brackets is the incomplete beta function. Hence, these tables can be used in evaluating $F(y')$.

The symmetry of $f(y)$ can be used to obtain values for y' less than zero when

$$(28) \qquad F(y') = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{1}{B(m - \frac{1}{2}, \frac{1}{2})} \int_0^{1 - \frac{y'^2}{a^2+y'^2}} Z^{(m - \frac{1}{2})-1} (1 - Z)^{\frac{1}{2} - 1} dZ\right]$$

for $y' \geq 0$.

Formulae (27) and (28) could be used with tables of the incomplete beta function to obtain the probabilities to the right of the mean for each fitted curve. However, Table 2 was obtained by directly integrating the type VII curves given in (22) to (25) by using computer methods of applying Simpson's rule for approximating the area under a curve.

Table 2. Theoretical frequency for 2500 samples from (22), (23), (24) and (25).

Class Boundaries Multiplied by 10^5	v Distributions			
	Uniform	Lognormal	Gamma	Normal
-50 to -45	.25	.00	.08	.01
-45 to -40	.41	.00	.13	.03
-40 to -35	.67	.01	.22	.05
-35 to -30	1.16	.01	.39	.11
-30 to -25	2.06	.02	.74	.26
-25 to -20	3.82	.03	1.47	.62
-20 to -15	7.41	.07	3.11	1.58
-15 to -10	15.04	.15	7.01	4.31
-10 to - 5	31.69	.38	16.85	12.40
- 5 to 0	68.10	1.17	42.66	36.63
0 to 5	143.50	4.86	109.46	105.02
5 to 10	276.10	32.49	260.51	264.15
10 to 15	438.14	392.25	493.59	505.12
15 to 20	518.52	1637.36	627.12	639.41
20 to 25	438.14	392.25	493.59	505.12
25 to 30	276.10	32.49	260.51	264.15
30 to 35	143.50	4.86	109.45	105.02
35 to 40	68.10	1.17	42.66	36.63
40 to 45	31.69	.38	16.85	12.40
45 to 50	15.04	.15	7.01	4.31
50 to 55	7.41	.07	3.11	1.58
55 to 60	3.82	.03	1.47	.62
60 to 65	2.06	.02	.74	.26
65 to 70	1.16	.01	.39	.11

Experimental Approach

A sieve analysis of vitamin A was used to obtain computation constants so that most of the density for the diameter distribution was between 0 and .08 centimeters. This data was used as a basis for generating random samples of a distribution of weight per portion. The sampling procedures are given in the appendix. Table 3 gives the frequency distributions observed when 2500 samples were generated from each distribution of diameters.

Table 3. Empirical distributions of weight per portion.

Class Boundaries Multiplied by 10^5	ν Distributions			
	Uniform	Lognormal	Gamma	Normal
-50 to -45	1	0	0	0
-45 to -40	0	0	0	0
-40 to -35	1	0	4	0
-35 to -30	3	0	0	0
-30 to -25	4	0	1	0
-25 to -20	11	0	3	3
-20 to -15	18	0	14	5
-15 to -10	27	0	14	14
-10 to - 5	38	4	30	21
- 5 to 0	49	4	63	36
0 to 5	91	6	109	122
5 to 10	159	36	231	221
10 to 15	327	250	434	452
15 to 20	1026	1929	641	780
20 to 25	347	227	499	454
25 to 30	164	35	228	231
30 to 35	93	5	129	83
35 to 40	57	1	54	49
40 to 45	37	1	24	14
45 to 50	22	1	11	10
50 to 55	10	1	9	1
55 to 60	10	0	2	1
60 to 65	4	0	0	3
65 to 70	1	0	0	0
Total	2500	2500	2500	2500

Table 4 shows that the means and variances of the simulated distributions are near the means and variances of the theoretical distributions. Perhaps better results would be realized if more intervals were used. Further, the higher moments of the random normal deviates obtained by the method described in the appendix differ from the moment of the exact normal deviates. Hence, the simulated distributions differ from the expected distributions.

Table 4. A comparison of means and variances of weight per portion distributions.

v Distribution	Expected		Observed	
	Mean	Variance	Mean	Variance
Uniform	.000175	$.0^{7}128$	.0001736	$.0^{7}12783$
Lognormal	.000175	$.0^{8}10586$	.0001742	$.0^{8}10785$
Gamma	.000175	$.0^{8}8448$	.0001741	$.0^{7}10795$
Normal	.000175	$.0^{8}78$	.0001727	$.0^{8}82352$

The Kolmogorov - Smirnov test indicates that the simulated weight per portion distribution produced by the particles having diameters with a gamma distribution was not significantly different from the theoretical weight per portion distribution. However, the other three simulated distributions were significantly different from their respective theoretical distributions by the same test.

After obtaining the weight per portion distribution one can then compute the error committed by assuming that all of the particles in all of the batches are equal, when in reality the diameters are varying from batch to batch. This can be done by computing the area under the normal curve that would result from particles with equal diameters and the area under the corresponding theoretical curve given by formula (22), (23), (24) or (25). If the batches are distributed as a gamma distribution and an error of 5 per cent is claimed when assuming equal diameters, then the actual error committed is approximately 10 per cent. This indicates the importance of having the appropriate weight per portion distribution before attempting to make inferences about the mixing process.

A COMPARISON OF $(\bar{\nu})^3$ AND $\overline{(\nu)^3}$

Introduction

The main purpose of this section is to calculate the amount of error committed in making statements about the weight per portion when intra-batch variation in the particle diameters is ignored. Research workers have been ignoring such variations by calculating the average diameter and then assuming that all of the particles are of equal size having diameters equal to the mean diameter. Under these assumptions the weight per portion, X, is normally distributed with mean a_p and variance $\frac{\pi\rho}{6} a_p[\mu_1'(\nu)]^3$. However, if intra-batch variations exist, then X is normally distributed with mean a_p and variance $\frac{\pi\rho}{6} a_p\mu_3'(\nu)$. In this section these two approaches are compared by using the normal deviates based on each type of assumption about ν.

Comparing the Normal Deviates

Formula (2) states that the weight per portion distribution can be transformed into a standard normal deviate by

$$Z_1 = \frac{X - a_p}{\sqrt{\frac{\pi\rho}{6} \Sigma\, v_i^3\, a_{ip}}} \ .$$

This can be rearranged to give

$$(29) \qquad Z_1 = \frac{X - a_p}{\sqrt{\frac{\pi\rho}{6}\, a_p[\mu_3'(\nu)]}} \ .$$

This standard normal deviate is based on the assumption that the weight per portion distribution was obtained from particles having a specific distribution of diameters.

If the average diameter is used as the diameter of each particle, then the weight per portion distribution can be transformed into a standard normal deviate by

$$Z_2 = \frac{X - a_p}{\sqrt{\frac{\pi\rho}{6} a_p(\bar{v})^3}} .$$

This can be rearranged to give

$$(30) \qquad Z_2 = \frac{X - a_p}{\sqrt{\frac{\pi\rho}{6} a_p[\mu_1'(v)]^3}} .$$

A special case of formula (30) is when the diameters of all particles are equal.

As a basis for comparison of the two methods, the ratio Z_1/Z_2 was considered and calculated for each distribution of diameters. The ratio can be reduced to give

$$(31) \qquad \frac{Z_1}{Z_2} = \left[\frac{[\mu_1'(v)]^3}{\mu_3'(v)}\right]^{1/2} .$$

The reduction of this expression gives

$$(32) \qquad \frac{Z_1}{Z_2} = \left[\frac{1}{2} + \frac{ab}{b^2 + a^2}\right]^{1/2}$$

when v is uniformly distributed between a and b,

$$(33) \qquad \frac{Z_1}{Z_2} = \left[e^{-3\sigma^2}\right]^{1/2}$$

when v is lognormally distributed with mean μ and variance σ^2,

$$(34) \qquad \frac{Z_1}{Z_2} = \left[\frac{p^2}{(p+2)(p+1)}\right]^{1/2}$$

when v has a gamma distribution with parameters α and p, and

(35) $$\frac{Z_1}{Z_2} = \left[\frac{\mu^2}{\mu^2+3\sigma^2}\right]^{1/2}$$

when v is normally distributed with mean μ and variance σ^2.

The values of (32), (33), (34) and (35) are limited by the terms which determine them. Formula (32) is maximum when a is equal to b and minimum when a is zero. The maximum of formula (33) is when σ^2 is zero, while it is minimum when σ^2 is infinity. The maximum of formula (34) is realized when p is zero, while it is minimum when p is infinity. Formula (35) is maximum when σ^2 is zero and minimum when σ^2 is infinity. Note that the maximum for each quantity is one and the minimum is zero for formulae (33), (34) and (35) and the square root of one-half for formula (32). This implies that Z_2 is always greater than or equal to Z_1 for these four distributions of diameters. The maximum of Z_1/Z_2 is obtained and the equality of Z_1 and Z_2 holds only when the diameter distribution is degenerate. If inference is made on the assumption that the diameters of all particles are equal when they have a distribution, the type I error committed will be greater than the type I error claimed. The amount of increase in error will depend on the characteristics of the distribution of diameters.

The value of Z_1 can be determined for a given Z_2 and a given distribution of diameters. From formula (31) it follows that

(36) $$Z_1 = \left[\frac{[\mu_1'(v)]^3}{\mu_3'(v)}\right]^{1/2} Z_2.$$

Formula (36) and tables of the standard normal distribution were used to obtain Table 5, Table 6, Table 7, and Table 8.

Table 5. Type I error for Z_1 distribution when ν has a uniform distribution.

$\frac{a}{b}$	Type I error for Z_2 distribution		
	10%	5%	1%
.00	24.5%	16.6%	6.9%
.05	22.2	14.6	5.6
.10	20.3	12.9	4.6
.15	18.6	11.5	3.8
.20	17.1	10.3	3.2
.25	15.9	9.3	2.7
.30	14.8	8.5	2.3
.35	13.8	7.8	2.0
.40	13.1	7.2	1.8
.45	12.4	6.7	1.6
.50	11.9	6.3	1.4
.55	11.4	6.0	1.3
.60	11.0	5.7	1.2
.65	10.8	5.5	1.2
.70	10.5	5.3	1.1
.75	10.4	5.3	1.1
.80	10.2	5.1	1.0
.85	10.1	5.1	1.0
.90	10.1	5.0	1.0
.95	10.0	5.0	1.0
1.00	10.0	5.0	1.0

Table 6. Type I error for Z_1 distribution when v has a lognormal distribution.

σ^2	Type I error for Z_2 distribution		
	10%	5%	1%
∞	100 %	100 %	100 %
2.00	93.5	92.2	89.8
1.50	86.2	83.6	78.6
1.00	71.4	66.2	56.5
.95	69.2	63.7	53.6
.90	67.0	61.1	50.4
.85	64.6	58.4	47.2
.80	62.0	55.5	43.8
.75	58.7	52.5	40.3
.70	56.5	49.3	36.7
.65	53.5	46.0	33.1
.60	50.4	42.6	29.5
.55	47.1	39.0	25.9
.50	43.7	35.4	22.4
.45	40.2	31.8	19.0
.40	36.7	28.2	15.7
.35	33.0	24.6	12.8
.30	29.4	21.1	10.1
.25	25.8	17.8	7.7
.20	22.3	14.6	5.6
.15	18.9	11.8	4.0
.10	15.7	9.2	2.7
.08	14.5	8.2	2.2
.06	13.3	7.3	1.9
.04	12.1	6.5	1.5
.02	11.0	5.7	1.2
.00	10.0	5.0	1.0

Table 7. Type I error for Z_1 distribution when ν has a gamma distribution.

p	Type I error for Z_2 distribution		
	10%	5%	1%
.00	100 %	100 %	100 %
.05	95.5	94.6	93.0
.10	91.4	89.7	86.5
.15	87.5	85.2	80.6
.20	84.0	80.9	75.1
.25	80.6	77.0	70.1
.30	77.5	73.4	65.5
.35	74.7	70.0	61.3
.40	72.0	66.9	57.4
.45	69.4	64.0	53.8
.50	67.1	61.3	50.6
.60	62.8	56.4	44.9
.70	59.1	52.2	40.0
.80	55.8	48.5	35.9
.90	52.8	45.4	32.3
1.00	50.2	42.4	29.3
1.10	47.8	39.8	26.7
1.20	45.7	37.5	24.4
1.30	43.8	35.5	22.4
1.40	42.0	33.7	20.7
1.50	40.4	32.0	19.2
1.75	37.0	28.6	16.0
2.00	34.2	25.8	13.7
2.50	30.0	21.7	10.5
3.00	27.0	18.9	8.4
3.50	24.7	16.8	7.0
4.00	23.0	15.2	6.0
4.50	21.6	14.0	5.3
5.00	20.4	13.1	4.7
6.00	18.7	11.6	3.9
7.00	17.5	10.6	3.4
8.00	16.5	9.8	3.0
9.00	15.8	9.3	2.7
10.00	15.2	8.8	2.5
11.00	14.7	8.4	2.3
12.00	14.3	8.1	2.2
13.00	14.0	7.9	2.1
14.00	13.7	7.7	2.0
15.00	13.5	7.5	1.9
20.00	12.6	6.8	1.7
25.00	12.1	6.4	1.5
30.00	11.7	6.2	1.4
40.00	11.3	5.9	1.3
50.00	11.0	5.7	1.2
100.00	10.5	5.3	1.1
500.00	10.1	5.1	1.0
∞	10.0	5.0	1.0

Table 8. Type I error for Z_1 distribution when ν has a normal distribution.

$\frac{\sigma}{\mu}$	Type I error for Z_2 distribution		
	10%	5%	1%
∞	100 %	100 %	100 %
20.00	96.2	95.5	94.1
15.00	95.0	94.0	92.1
10.00	92.4	91.0	88.2
9.00	91.6	90.0	86.9
8.00	90.6	88.8	85.3
7.00	89.2	87.2	83.2
6.00	87.5	85.1	80.5
5.50	86.4	83.8	78.8
5.00	85.0	82.2	76.8
4.50	83.4	80.3	74.3
4.00	81.4	77.9	71.3
3.50	78.9	75.0	67.5
3.00	75.6	71.1	62.6
2.50	71.1	65.9	56.2
2.40	70.0	64.7	54.7
2.30	68.9	63.3	53.1
2.20	67.6	61.9	51.3
2.10	66.3	60.3	49.5
2.00	64.8	58.7	47.5
1.90	63.2	56.9	45.4
1.80	61.5	54.9	43.1
1.70	59.7	52.8	40.7
1.60	57.7	50.6	38.2
1.50	55.5	48.1	35.5
1.40	53.1	45.5	32.6
1.30	50.4	42.6	29.6
1.20	47.6	39.5	26.4
1.10	44.5	36.2	21.1
1.00	41.1	32.7	19.8
.95	39.3	30.9	18.1
.90	37.4	29.0	16.4
.85	35.5	27.1	14.8
.80	33.6	25.1	13.2
.75	31.6	23.2	11.6
.70	29.5	21.2	10.1
.65	27.5	19.3	8.6
.60	25.4	17.4	7.4
.55	23.4	15.6	6.2
.50	21.4	13.8	5.2
.45	19.5	12.2	4.2
.40	17.6	10.7	3.4
.35	15.9	9.4	2.8
.30	14.4	8.2	2.2

Table 8 (Cont.).

$\frac{\sigma}{\mu}$	Type I error for Z_2 distribution		
	10%	5%	1%
.25	13.1%	7.2%	1.8%
.20	12.0	6.4	1.5
.15	11.1	5.8	1.3
.10	10.5	5.3	1.1
.05	10.1	5.1	1.0
.00	10.0	5.0	1.0

CONCLUSIONS AND SUGGESTIONS

The distribution of the weight per portion variable is needed for making inference about any mixing problem. It is important that the correct distribution is obtained, since the inference based on the wrong distribution can be costly. For example, a drug may be needed in the daily ration of an animal for protection against disease but an overdose of the same drug could be fatal. Tables 5, 6, 7 and 8 indicate that inference can be in error by critical proportions unless the distribution of the diameters is used in obtaining the weight per portion distribution.

Another use of the weight per portion distribution is the determination of the particle sizes of additive needed to insure that a sufficient number of particles is available to meet the desired control limits.

The simulation method of obtaining an approximate distribution of weights per portion for a given distribution of diameters seems to be effective. Additional work needs to be done on the efficiency of estimates based on distributions obtained by simulation techniques.

The chi-squares for fitting the distributions in Table 2 to the distributions in Table 3 were highly significant. This implies that more samples should be studied. It might be helpful to use more intervals since the distributions are quite peaked in shape. A large portion of the chi-square value for each fitted curve was from the middle class intervals.

The Kolmogorov - Smirnov test of goodness of fit indicated that the distributions obtained from a uniform distribution of diameters, a lognormal distribution of diameters, and a normal distribution of diameters differed significantly from the theoretical distributions. However, the empirical distribution based on a gamma distribution of diameters was not significantly different from the theoretical distribution at the 5% level for this test.

Future work might be carried out to find the sampling distribution of estimates for the weight per portion distribution. More work is needed in improving the simulation techniques. The integration approach to obtaining the theoretical distribution could be studied more rigorously. These distributions would serve as a guide for various simulation methods.

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APPENDIX

APPENDIX

The empirical sampling was set up in such a way that the v distribution would be distributed with a large portion of the frequency between .001 and .080 centimeters. A sieve analysis of vitamin A and a ration of chick starter were furnished by the milling department. These were the basis for obtaining the theoretical mean, a_p = .000175, and the theoretical variance

$$\frac{\pi \rho a_p}{6} v^3 = .0001\, v^3 .$$

The first part of the sampling procedure was to select diameters at random from the various distributions of diameters. The power residue method was used to generate numbers uniformly distributed between 0 and 1. The normally distributed diameters were obtained from randomly generated standard normal deviates. Tables of the incomplete gamma distribution were used to obtain diameters at random from the gamma distribution. The lognormally distributed diameters were obtained by using normally distributed exponents of e. Transformations were then used to put these values in the proper size range.

The second step of the sampling procedure was to generate normally distributed weight per portion variables based on the randomly selected diameters. Standard normal deviates were selected by applying the central limit theorem to twelve randomly selected rectangular numbers. These were transformed to a normal variate with mean .000175 and variance $.0001\, v^3$.

The distributions obtained by this method are given in Table 3.

SOME PROBLEMS OF MIXING SMALL PARTICLES

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The development of new additives for animal feeds has caused concern about the mixing of minute quantities of additives with large quantities of feed. It is necessary that the mixing be accomplished in such a way that each small portion of mix will include some of the additive. If confidence statements are to be made about the final mix, then a distribution of the weight of additive per portion is needed. The purpose of this thesis was to study some of the problems encountered in obtaining and using these distributions of weight per portion.

Consider a process in which particles are being produced in batches. The diameters of the particles can vary either (1) from batch to batch and be equal within a given batch or (2) within a given batch with each batch having the same distribution of diameters. These conditions were the basis for determining the weight per portion distribution of an additive.

The batch diameters were assumed to be uniformly distributed under the first condition and the weight per portion distribution was obtained by integration. The resulting distribution function can not be easily evaluated. Therefore, an attempt was made to fit one of the Pearson type curves to this and other distributions where the batch diameters follow normal, lognormal and gamma distributions. It was found that type VII distribution will provide the required fit. Using Kolmogorov's test for goodness of fit it was confirmed that a simulated distribution when the inter-batch variations follow a gamma distribution did not differ significantly from the corresponding type VII distribution.

The second condition results in normal distributions of weight per portion. These distributions of weight per portion can be used in making inference about meeting the tolerance limits placed on a mixing process.

A comparison of weight per portion distributions based on the mean diameter cubed and on the mean cubed diameter was used to show the error committed when making inferences from the wrong distribution. A distribution determined by cubing the mean diameter is the same as the distribution obtained by assuming all of the particles have equal diameters. The use of the mean cubed diameter to obtain the weight per portion distribution allows the distribution of diameters to influence the weight per portion distribution. A study of the ratio of the normal deviates based on these two means resulted in Tables 5, 6, 7 and 8. These tables give the actual error committed by assuming that the diameters are equal when in reality they follow some distribution. The results of the study indicated that the mean cubed diameter should be used when working with distributions of weight per portion of mix.

The conclusions from this thesis show that more work is needed in perfecting the simulation method of obtaining the weight per portion distribution. More detailed studies can be completed for various additives that are used in mixing feeds.