MULTIVARIABLE SYSTEMS

by

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A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

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INTRODUCTION

In the vast area of control theory much has been written concerning the linear single variable or one input-one output type system. However, this is most artificial for it is virtually impossible to visualize a practical system without disturbances distributed throughout the system. Nearly all physical systems when considered in great enough detail have more than one input and one or more outputs. Since such systems have more than one variable, they are called multivariable systems.

The importance of this field has grown considerably from its beginnings in the 1930's and 1940's to a place where its vast potentialities are just being realized. Table 1 lists the number of papers concerned with such systems published during each year from its birth to the fall of 1962. Numerous classified and proprietary reports on the subject exist but are excluded from the table and the bibliography.

To see the effect of disturbances on an artificial system, consider the linear single input-single output system diagramed in Fig. 1. From basic control theory it is known that its input-output relationship expressed in the complex frequency domain is

$$Y = \frac{PCX}{1 + PCF} \tag{1}$$

With noise or disturbances at the input of the plant as shown in Fig. 2, the output can be expressed as

$$Y = \frac{CPX_1}{1 + PCF} + \frac{PX_2}{1 + PCF} \tag{2}$$

Table 1. The number of readily available
papers concerning multivariable
control systems published each
year from 1934 to September, 1962.

Year	:	Papers	:	Year	:	Papers
1934		1^1		1949		1
1935		0		1950		1
1936		0		1951		2
1937		0		1952		2
1938		1^1		1953		5
1939		1^1		1954		3
1940		1^1		1955		6
1941		0		1956		9
1942		0		1957		16
1943		0		1958		18
1944		0		1959		20
1945		1		1960		65
1946		0		1961		27
1947		1		1962 to		
1948		2		Sept.		25

[1] Listed by Strejc (187).

If the system is rearranged by defining Y_1 as the resultant output
caused by input X_1 and Y_2 as the resultant output caused by input
X_2, then Eq. (2) can be broken down into two equations.

$$Y_1 = (1 + PCF)^{-1}PCX_1 \tag{3}$$

$$Y_2 = (1 + PCF)^{-1}PX_2 \tag{4}$$

The system can now be rediagramed to that given in Fig. 3.

Using matrix notation, Eq. (3) and (4) can be combined as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{PC}{1 + PCF} & 0 \\ 0 & \dfrac{P}{1 + PCF} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{5}$$

or
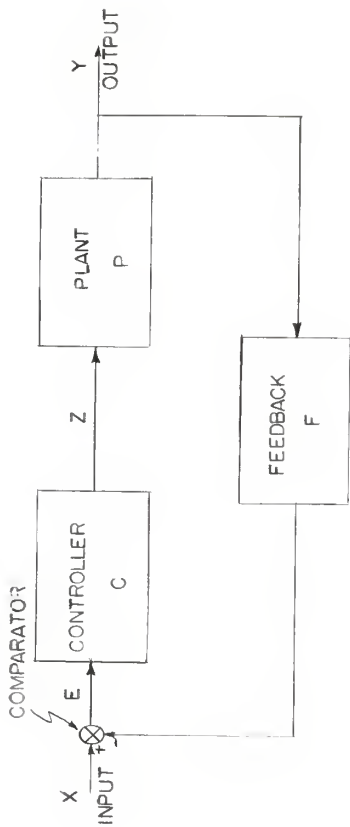$$\underline{Y} = \underline{WX} \tag{6}$$

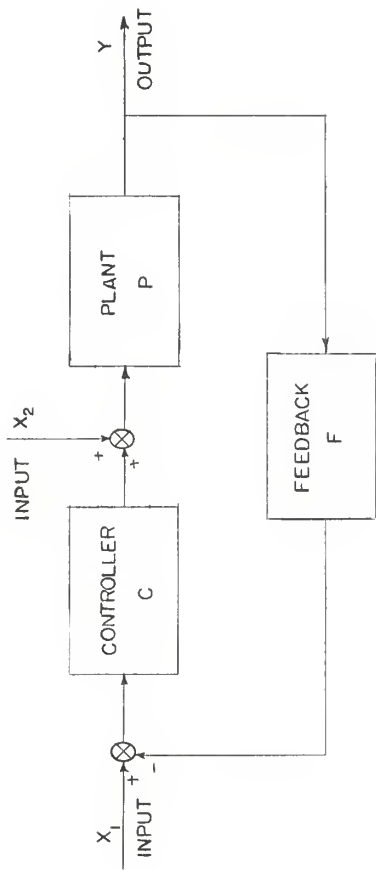FIG. I. SINGLE INPUT—SINGLE OUTPUT CONTROL SYSTEM.

Fig. 2. A one input—one output system with a disturbance.
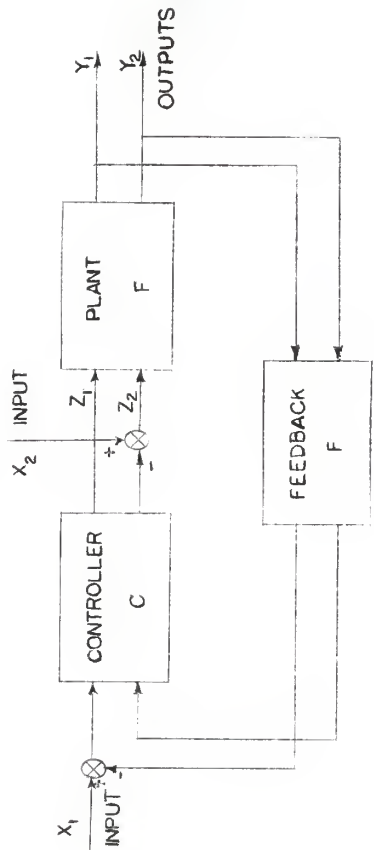
Fig. 3. A simple multivariable system.

where the bar under the letter indicates a matrix. From the definition of this system, $\underline{W}$ is a diagonal matrix allowing for a one-to-one correspondence between the inputs and the outputs. This type of system has the principle of noninteraction among the inputs. A more detailed discussion of the one-input and disturbance system is found in many elementary control texts such as Truxal (192)[1], Chestnut and Mayer (26), and others[2].

In most physical systems with which the control engineer will be dealing, the plants will have more than one input or output. Fogel (44) discusses a multivariable automatic control for high-speed aircraft. Frazer (45) reviews several multivariable systems appearing in airplanes and vibrating structures. A turbojet engine is a multivariable plant for it has such inputs as fuel flow, exhaust cross-section area, and propeller pitch, and such outputs as aircraft speed and tail pipe temperature. A chemical mixing process could have inputs such as flow of various raw materials, mixing speed, temperature, and pressure, and outputs such as blend of the aggregate, viscosity, and melting point. Other multivariable plants include guidance and control of space vehicles, rolling mills in a steel plant, and electrical power distribution systems. The list could be extended further to include every system if the investigation went into greatest detail.

In all cases the plant can be represented as in Fig. 4 by a black box with m inputs and n outputs. For unique control of the

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[1]Numbers in parenthesis refer to numbered references in bibliography.
[2]References 19, 56, 131, and 170.

FIG. 4. DIAGRAM OF AN MXN PLANT.

outputs, m must be greater than or equal to n. If m = n, a one-
to-one correspondence can be readily obtained. The m > n case
calls for the expansion of the mxn box into an mxm size box where
the m - n additional outputs are tied by some means to m - n in-
puts. These additional outputs are called synthetic or virtual
outputs.

With the mxn plant, the control system of Fig. 1 may be en-
larged so that the feedback box has n inputs and m outputs and the
controller has m inputs and m outputs. This complete system is
shown in Fig. 5. Using matrix notation, the various parts of the
system become

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} \tag{7}$$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_m \end{bmatrix} \tag{8}$$

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ p_{n1} & p_{n2} & \cdots & p_{nm} \end{bmatrix} \tag{9}$$
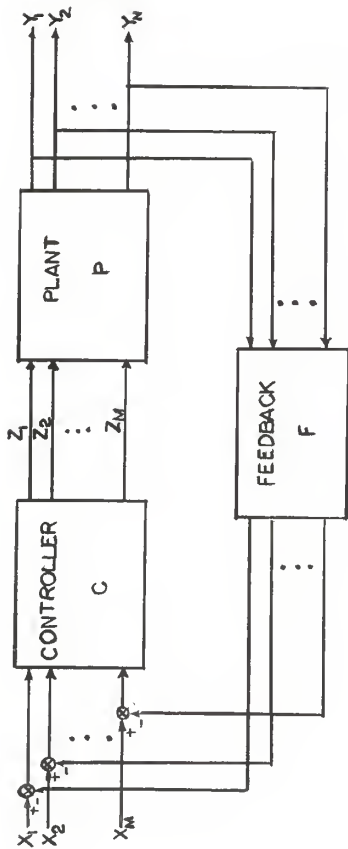
FIG. 5. THE MULTIVARIABLE CONTROL SYSTEM.

$$\underline{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{bmatrix} \tag{10}$$

$$\underline{F} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} \tag{11}$$

Remembering that care must be taken in the multiplication of matrices because matrix multiplication is not commutative, the system equation can be developed as

$$\underline{E} = \underline{X} - \underline{FY}$$
$$\underline{Y} = \underline{PCE}$$
$$\underline{Y} = \underline{PC}(\underline{X} - \underline{FY}) = \underline{PCX} - \underline{PCFY}$$
$$(\underline{I} + \underline{PCF})\underline{Y} = \underline{PCX}$$
$$\underline{Y} = (\underline{I} + \underline{PCF})^{-1}\underline{PCX} \tag{12}$$

or
$$\underline{Y} = \underline{WX} \tag{13}$$

where $\underline{I}$ is the unit matrix. Further discussion on the general multivariable system can be found in papers by Mesarović (122-126) and Zadeh (206). In his book, Mesarović shows how the system can be expressed in different "canonical" forms, each form being used to represent systems having different interrelations. Zadeh was among the first to describe and define the system but was mainly interested in filter theory. He used the word multipole instead of

multivariable. Present usage tends to find the word multipole
associated with mxn filter systems and multivariable with mxn con-
trol systems.

The purpose of the control system can be manifold. One of the
most important capabilities of a multivariable control system is
its ability to obtain noninteraction among the inputs. If a change
in one particular output is desired, it is much better if only one
input variable be changed rather than several. If any other rela-
tionship is desired between the inputs and outputs than is presented
by the plant, this type of system will enable its realization.
Optimization of some specified function in the system may be de-
sired. The optimal control can take the form of minimizing the
error between the outputs and a set of desired outputs, minimizing
the effect of plant parameter drift, minimizing the response time,
or minimizing the cost of operation as the system changes from one
point in an n dimensional space to another point in the same space.
The design techniques for all these types of systems have been
developed.

In reviewing the progress in the field of multivariable con-
trol systems, the most valuable approach for the reader would be
one which gives a detailed description of the major areas so that
they may be easily applied to the reader's own particular control
problem. Thus in this paper discussion will be limited to the
problem of the noninteraction system, minimization of the output
error, minimization of the effect of plant parameter variation, and
two forms of optimization. Each is written so the careful reader
need not read the preceding sections.

Other areas of multivariable systems not mentioned in detail in this report are the Bode plot and Nyquist diagram approaches to synthesis[1], adaptive multivariable systems[2], and digital computer control[3].

NONINTERACTION

Describing the System

The first significant work in multivariable system study was in the area of obtaining noninteraction from multiple input plants. Strejc (187) cites four references involving autonomous control written before 1941. Freeman (47) lists an unpublished report by Harris (70) as one of the first papers in this area. Boksenbom and Hood (16) presented a detailed paper concerned with the non-interaction of inputs as applied to a turbojet engine. Tsien (194) repeated their work while Feder and Hood (41) gave a detailed study of tests resulting from the use of their ideas.

Noninteraction of inputs has been discussed by Kavanagh (87-91) and many others[4]. Bohn (14, 15) presents the idea of non-interaction as that control system which causes the error signals to be zero. The presentation here closely follows Freeman (46-49) where noninteraction is expanded to the idea of independent output

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[1] References 14, 15, 25, 77, 92, 95, 113, 143, 181, and 190.
[2] References 10, 13, 55, 69, 109, 114, 129, and 168.
[3] References 7, 12, 50, 51, 63, 64, 65, 86, 94, 107, 108, 110, 130, 159, 161, 179, 185, and 195.
[4] References 29, 31, 35, 59, 82, 85, 97, 118, 135, 136, 140, 141, 152, 155, 157, 158, and 189.

restoration, i.e., the fact that a change in one output should not affect another output as this change is acted upon by the feedback, control, and plant elements.

Starting with the desired stable overall system equation of

$$\underline{Y} = \underline{W}\underline{X} \tag{14}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{n1} & w_{n2} & \cdots & w_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} \tag{15}$$

it is required, as mentioned earlier, that for uniqueness of a solution the system must have $m \geq n$, and, if $m > n$, $m - n$ virtual outputs must be defined by expanding the system. Expanding and partitioning equation (15), one obtains

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \\ \hline y_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix} = \left[\begin{array}{cccc|cccc} w_{11} & w_{12} & \cdots & w_{1n} & w_{1,n+1} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2n} & w_{2,n+1} & \cdots & w_{2m} \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ w_{n1} & w_{n2} & \cdots & w_{nn} & w_{n,n+1} & \cdots & w_{nm} \\ \hline w_{n+1,1} & w_{n+1,2} & \cdots & w_{n+1,n} & w_{n+1,n+1} & \cdots & w_{n+1,m} \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ w_{m1} & w_{m2} & \cdots & w_{mn} & w_{m,n+1} & \cdots & w_{mm} \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ \hline x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} \tag{16}$$

or upon defining submatrices as

$$\underline{X}_a = \text{col.} \begin{bmatrix} x_1, & x_2, & \cdots & x_n \end{bmatrix} \qquad \text{the variable inputs} \tag{17}$$

$$\underline{X}_b = \text{col.} \begin{bmatrix} x_{n+1}, & x_{n+2}, & \cdots & x_m \end{bmatrix} \quad \text{the arbitrary inputs} \quad (18)$$

$$\underline{Y}_a = \text{col.} \begin{bmatrix} y_1, & y_2, & \cdots & y_n \end{bmatrix} \quad \text{the actual outputs} \quad (19)$$

$$\underline{Y}_b = \text{col.} \begin{bmatrix} y_{n+1}, & y_{n+2}, & \cdots & y_m \end{bmatrix} \quad \text{the virtual outputs} \quad (20)$$

$$\underline{W}_a = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{n1} & w_{n2} & & w_{nn} \end{bmatrix}$$

$$\underline{W}_b = \begin{bmatrix} w_{n+1,n+1} & w_{n+1,n+2} & \cdots & w_{n+1,m} \\ w_{n+2,n+1} & w_{n+2,n+2} & \cdots & w_{n+2,m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{m,n+1} & w_{m,n+2} & \cdots & w_{m,m} \end{bmatrix}$$

$$(21)$$

$$\underline{W}_c = \begin{bmatrix} w_{1,n+1} & w_{1,n+2} & \cdots & w_{1,m} \\ w_{2,n+1} & w_{2,n+2} & \cdots & w_{2,m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{n,n+1} & w_{n,n+2} & \cdots & w_{nm} \end{bmatrix}$$

$$\underline{W}_d = \begin{bmatrix} w_{n+1,1} & w_{n+1,2} & \cdots & w_{n+1,n} \\ w_{n+2,1} & w_{n+2,2} & \cdots & w_{n+2,n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{m,1} & w_{m,2} & \cdots & w_{mn} \end{bmatrix}$$

Eq. (16) becomes

$$\begin{bmatrix} \underline{Y}_a \\ \underline{Y}_b \end{bmatrix} = \begin{bmatrix} \underline{W}_a & \underline{W}_c \\ \underline{W}_d & \underline{W}_b \end{bmatrix} \begin{bmatrix} \underline{X}_a \\ \underline{X}_b \end{bmatrix} \quad (22)$$

With these expanded forms, the diagram of the system can be redrawn as that in Fig. 6. The system equation from Eq. (12) still is

$$\underline{Y} = (\underline{I} + \underline{PCF})^{-1}\underline{PCX} \tag{23}$$

As was done with the overall transfer matrix, $\underline{P}$, $\underline{C}$, and $\underline{F}$ are expanded and partitioned so as to obtain

$$\underline{P} = \begin{bmatrix} \underline{P}_a & \underline{P}_c \\ \underline{P}_d & \underline{P}_b \end{bmatrix} \tag{24}$$

$$\underline{C} = \begin{bmatrix} \underline{C}_a & \underline{C}_c \\ \underline{C}_d & \underline{C}_b \end{bmatrix} \tag{25}$$

$$\underline{F} = \begin{bmatrix} \underline{F}_a & \underline{F}_c \\ \underline{F}_d & \underline{F}_b \end{bmatrix} \tag{26}$$

where $\underline{P}_a$, $\underline{C}_a$, and $\underline{F}_a$ are all nxn matrices.

From the definition of the plant there are n actual outputs resulting from m inputs, and the m - n additional outputs will be set equal to m - n inputs. Thus the expanded matrix $\underline{P}$ in Eq. (24) must have $\underline{P}_d = 0$ and $\underline{P}_b = I$, or

$$\underline{P} = \begin{bmatrix} \underline{P}_a & \underline{P}_c \\ 0 & I \end{bmatrix} \tag{27}$$

Since the m - n virtual outputs were arbitrarily selected, they should have no feedback effect upon the n inputs which have been chosen to control the n actual outputs. For this reason $\underline{F}_c = \underline{F}_d = 0$, and

$$\underline{F} = \begin{bmatrix} \underline{F}_a & 0 \\ 0 & F_b \end{bmatrix} \tag{28}$$

Using Eqs. (24), (27), and (28), an exact expression for the overall transfer function matrix, $\underline{W}$, is desired. Referring to Eq. (23), first $(\underline{I} + \underline{PCF})^{-1}$ will be determined and then $\underline{PC}$.
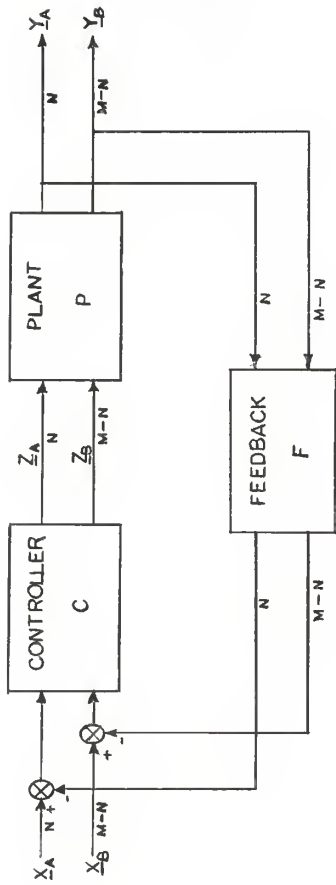
FIG. 6. EXPANDED MULTIVARIABLE CONTROL SYSTEM.

$$\underline{I} + \underline{PCF} = \underline{I} + \begin{bmatrix} \underline{P}_a & \underline{P}_c \\ 0 & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{C}_a & \underline{C}_c \\ \underline{C}_d & \underline{C}_b \end{bmatrix} \begin{bmatrix} \underline{F}_a & 0 \\ 0 & \underline{F}_b \end{bmatrix}$$

$$= \begin{bmatrix} \underline{I} + (\underline{P}_a\underline{C}_a + \underline{P}_c\underline{C}_d)\underline{F}_a & (\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b \\ \underline{C}_d\underline{F}_a & \underline{I} + \underline{C}_b\underline{F}_b \end{bmatrix} \quad (29)$$

The inversion of this matrix is a tedious process but using the
simple equation

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} U_a \\ U_b \end{bmatrix} \quad (30)$$

it is known that

$$\begin{bmatrix} U_a \\ U_b \end{bmatrix} = \begin{bmatrix} (K - LN^{-1}M)^{-1} & -(K - LN^{-1}M)^{-1}LN^{-1} \\ -N^{-1}M(K - LN^{-1}M)^{-1} & N^{-1} + N^{-1}M(K - LN^{-1}M)^{-1}LN^{-1} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} (31)^1$$

Noting the common factor $(K - LN^{-1}M)^{-1}$, this expression is formed
in terms of the submatrices of Eq. (29) and is

$$\underline{\curvearrowleft}^{-1} = \left[\underline{I} + (\underline{P}_a\underline{C}_a + \underline{P}_c\underline{C}_d)\underline{F}_a - (\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b \\ (\underline{I} + \underline{C}_b\underline{F}_b)^{-1} \underline{C}_d\underline{F}_a \right]^{-1} \quad (32)$$

Using the substitutions of Eqs. (30) and (31), the inverse of
Eq. (29) becomes

$$\begin{bmatrix} \underline{\curvearrowleft}^{-1} & -\underline{\curvearrowleft}^{-1}(\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b(\underline{I} + \underline{C}_b\underline{F}_b)^{-1} \\ -(\underline{I} + \underline{C}_b\underline{F}_b)^{-1}\underline{C}_d\underline{F}_a\underline{\curvearrowleft}^{-1} & (\underline{I} + \underline{C}_b\underline{F}_b)^{-1} + (\underline{I} + \underline{C}_b\underline{F}_b)^{-1} \underline{C}_d\underline{F}_a\underline{\curvearrowleft}^{-1} \\ & (\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b(\underline{I} + \underline{C}_b\underline{F}_b)^{-1} \end{bmatrix} \quad (33)$$

The other part of Eq. (23) is

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[1] Gabriel Kron, _Tensor Analysis of Networks_. London: John
Wiley and Sons, 1942, p. 258.

Basically, Eq. (30) is expanded into a system of two
equations for V_a and V_b in terms of U_a and U_b. These are then
solved for U_a and U_b.

$$\underline{PC} = \begin{bmatrix} \underline{P}_a & \underline{P}_c \\ 0 & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{C}_a & \underline{C}_c \\ \underline{C}_d & \underline{C}_b \end{bmatrix} = \begin{bmatrix} \underline{P}_a\underline{C}_a + \underline{P}_c\underline{C}_d & \underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b \\ \underline{C}_d & \underline{C}_b \end{bmatrix} \quad (34)$$

Multiplying Eqs. (33) and (34) together and substituting their product into Eq. (23), the complete input-output matrix equation for the control system is shown in Eq. (35).

From Eq. (22)

$$\begin{bmatrix} \underline{Y}_a \\ \underline{Y}_b \end{bmatrix} = \begin{bmatrix} \underline{W}_a & \underline{W}_c \\ \underline{W}_d & \underline{W}_b \end{bmatrix} \begin{bmatrix} \underline{X}_a \\ \underline{X}_b \end{bmatrix} \quad (36)$$

Using Eq. (35), $\underline{W}_a$, $\underline{W}_b$, $\underline{W}_c$, and $\underline{W}_d$ can be expressed.

Depending upon the particular plant and the system requirements, Eq. (35) may be simplified considerably. If the plant can be expressed as an nxn matrix, then $\underline{P}_b = \underline{P}_c = \underline{P}_d = 0$, and the b, c, and d parts of $\underline{C}$, $\underline{F}$, and $\underline{W}$ may be omitted.

For noninteraction, the criterion is that there shall be a one-to-one correspondence between inputs and outputs, i.e., a disturbance signal sent through the feedback from one of the output channels should not affect any other output, a disturbance from one of the virtual outputs should not affect any of the actual outputs, and likewise disturbances in one of the actual outputs should not affect any of the virtual outputs. Thus from the diagram in Fig. 7 the equation concerning the feedback disturbances is

$$\underline{Y}_o = \underline{PCFY}_1 \quad (37)$$

and from Eqs. (24), (27), and (28)

$$\begin{bmatrix} \underline{Y}_{a_o} \\ \underline{Y}_{b_o} \end{bmatrix} = \begin{bmatrix} \underline{P}_a & \underline{P}_c \\ 0 & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{C}_a & \underline{C}_c \\ \underline{C}_d & \underline{C}_b \end{bmatrix} \begin{bmatrix} \underline{F}_a & 0 \\ 0 & \underline{F}_b \end{bmatrix} \begin{bmatrix} \underline{Y}_{a_1} \\ \underline{Y}_{b_1} \end{bmatrix}$$

$$
\begin{bmatrix} \underline{Y}_A \\ \underline{Y}_B \end{bmatrix} =
\begin{bmatrix}
\underline{\Lambda}^{-1}[(P_cC_A + P_cC_D) - (P_AC_c + P_cC_B)F_B(\underline{I} + C_BF_B)^{-1}C_D] \\[4pt]
-(\underline{I} + C_BF_B)^{-1}\{C_DF_A\Lambda^{-1}(P_cC_A + P_cC_D) - [\underline{I} + C_DF_A\Lambda^{-1}(P_AC_c \\ + P_cC_B)F_B(\underline{I} + C_BF_B)^{-1}]C_D\}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\Lambda^{-1}(P_AC_c + P_cC_B)[\underline{I} - F_B(\underline{I} + C_BF_B)^{-1}C_B] \\[4pt]
-(\underline{I} + C_BF_B)^{-1}\{C_DF_A\Lambda^{-1}(P_AC_A \quad P_cC_B) - [\underline{I} \\ + C_DF_A\Lambda^{-1}(P_AC_c + P_cC_B)F_B(\underline{I} + C_BF_B)^{-1}]C_B\}
\end{bmatrix}
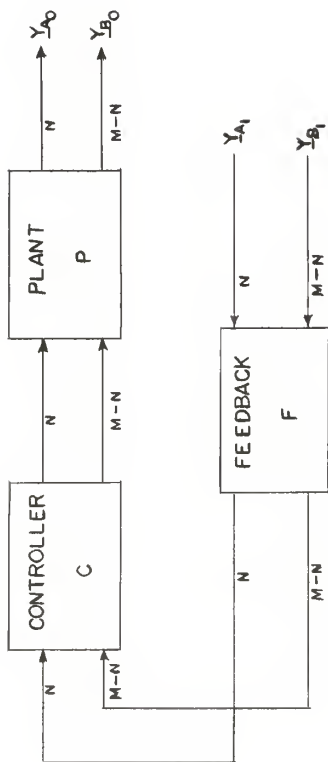\begin{bmatrix} \underline{X}_A \\ \underline{X}_B \end{bmatrix}
$$

(35)

Fig. 7. System configuration used to establish noninteraction.

$$= \begin{bmatrix} (\underline{P}_a\underline{C}_a + \underline{P}_c\underline{C}_d)\underline{F}_a & (\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b \\ \underline{C}_d\underline{F}_a & \underline{C}_b\underline{F}_b \end{bmatrix} \begin{bmatrix} \underline{Y}_{a_i} \\ \underline{Y}_{b_i} \end{bmatrix} \quad (38)$$

From the above discussion it is seen that

$$\underline{C}_d\underline{F}_a = 0 \quad (39)$$

$$(\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b)\underline{F}_b = 0 \quad (40)$$

$$(\underline{P}_a\underline{C}_a + \underline{P}_c\underline{C}_d)\underline{F}_a = \text{a diagonal matrix} \quad (41)$$

$$\underline{C}_b\underline{F}_b = \text{a diagonal matrix} \quad (42)$$

In Eq. (39) either $\underline{C}_d = 0$ or $\underline{F}_a = 0$[1]. Since $\underline{F}_a$ is the nxn feed-back submatrix of the n actual outputs, it cannot be zero. Therefore it is concluded that

$$\underline{C}_d = 0 \quad (43)$$

Since there must also be a feedback matrix for the virtual outputs, Eq. (40) leads to

$$\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b = 0\text{[2]} \quad (44)$$

Using Eq. (43), Eq. (41) becomes

$$\underline{P}_a\underline{C}_a\underline{F}_a = \text{a diagonal matrix} \quad (45)$$

Eq. (32) becomes

$$\underline{\Omega}^{-1} = (\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)^{-1} \quad (46)$$

and Eq. (35), the overall transfer function, becomes

$$\underline{W} = \begin{bmatrix} (\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)^{-1}\underline{P}_a\underline{C}_a & (\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)^{-1}(\underline{P}_a\underline{C}_c + \underline{P}_c\underline{C}_b) \\ & [\underline{I} - \underline{F}_b(\underline{I} + \underline{C}_b\underline{F}_b)^{-1}\underline{C}_b] \\ 0 & (\underline{I} + \underline{C}_b\underline{F}_b)^{-1}\underline{C}_b \end{bmatrix} \quad (47)$$

---

[1]The unique case where the elements of $\underline{C}_d$ and $\underline{F}_a$ are such that in the matrix multiplication each element of the product matrix, $\sum_{k=1}^{n} c_{ik}f_{kj}$, equals zero will not be considered.

[2]Ibid., for $(\underline{P}_a\underline{C}_c + \underline{P}_c\underline{P}_b)$ and $\underline{F}_b$.

Equation (44) contains a relationship between the control elements and the plant elements. In some systems the plant elements are fixed, time invariant, but in others the plant elements are subject to change. Thus for Eq. (44) to hold when the system includes a time dependent plant, a constant monitoring of the relationship and appropriate adjustment of the controller elements must occur. Assuming a computer, analog or digital, forms the elements of the controller with the restriction that Eq. (44) holds, then for both the time variant and time invariant plant the system equation becomes

$$\begin{bmatrix} \underline{Y}_a \\ \underline{Y}_b \end{bmatrix} = \begin{bmatrix} (\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)^{-1}\underline{P}_a\underline{C}_a & 0 \\ 0 & (\underline{I} + \underline{C}_b\underline{F}_b)^{-1}\underline{C}_b \end{bmatrix} \begin{bmatrix} \underline{X}_a \\ \underline{X}_b \end{bmatrix} \quad (48)$$

Without the computer monitoring, the noninteracting multivariable system becomes that shown in Fig. 8, and the special case of $m = 3$ and $n = 2$ is shown in Fig. 9.

System Synthesis

The noninteracting system is now described by Eq. (48). The next problem is that of synthesis or the determination of the elements of each block of the diagram of Fig. 6. Generally the plant elements can be obtained by experimental measurements from the outside terminals. The procedures for obtaining these elements have been discussed by Goldberg and Felsenthal (58) and others[1]. The feedback part is usually represented by the set of transfer functions of the measuring instruments being used to detect the
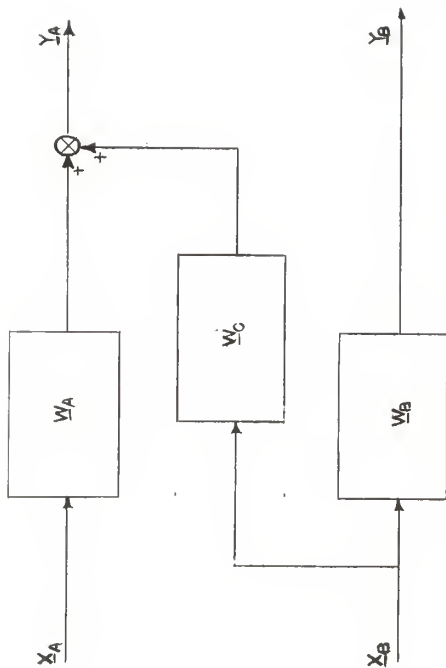
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[1]References 60, 105, 106, and 191.

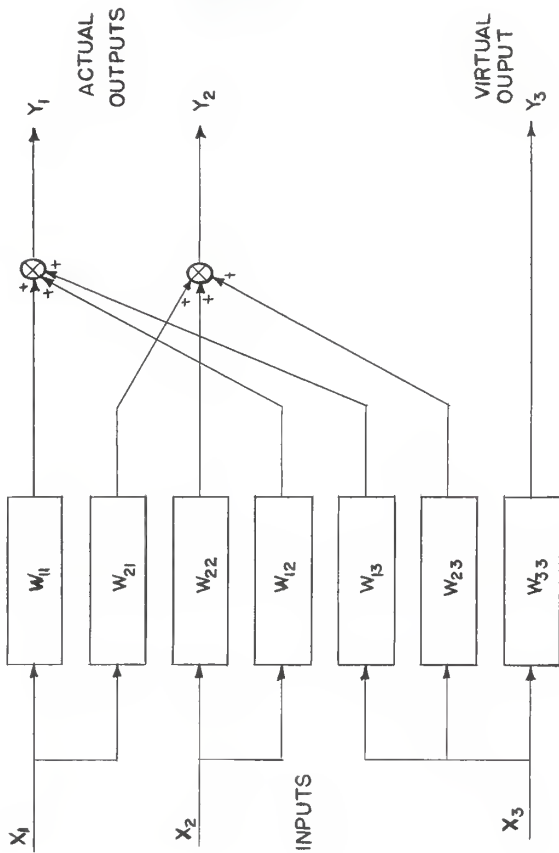FIG. 8. THE GENERAL MULTIVARIABLE SYSTEM FOR WHICH EQ.(44) IS NOT VALID.

Fig. 9. The resultant 3x2 multivariable system.

output. There still are restrictions imposed upon these elements as noted below. The desired final overall transfer function is set by the system requirements. Thus the elements of the control matrix which will transform the interacting mxn plant into a noninteracting multivariable system are the only unknowns in Eqs. (43), (44), and (48).

From Eq. (48),

$$\underline{W}_a = (\underline{I} + \underline{P}_a \underline{C}_a \underline{F}_a)^{-1} \underline{P}_a \underline{C}_a \tag{49}$$

$$\underline{W}_b = (\underline{I} + \underline{C}_b \underline{F}_b)^{-1} \underline{C}_b \tag{50}$$

$\underline{C}_a$ and $\underline{C}_b$ are obtained as

$$\underline{C}_a = \underline{P}_a^{-1} \underline{W}_a (\underline{I} - \underline{F}_a \underline{W}_a)^{-1} \tag{51}$$

$$\underline{C}_b = \underline{W}_b (\underline{I} - \underline{F}_b \underline{W}_b)^{-1} \tag{52}$$

From Eq. (44)

$$\underline{C}_c = -\underline{P}_a^{-1} \underline{P}_c \underline{C}_b = -\underline{P}_a^{-1} \underline{P}_c \underline{W}_b (\underline{I} - \underline{F}_b \underline{W}_b)^{-1} \tag{53}$$

and from Eq. (43)

$$\underline{C}_d = 0 \tag{54}$$

An illustrative example of a 3x2 plant is included in Appendix A.

It was assumed at the beginning of this section that the desired overall transfer function was stable. Since the controller matrix will be formed by either active or passive elements, a general requirement which is used is that all of its elements must be stable also. Thus $\underline{W}_a$, $\underline{W}_b$, $\underline{C}_a$, $\underline{C}_b$, and $\underline{C}_c$ must have poles only in the left half s plane. With these restrictions it is observed from Eq. (49) that for $\underline{W}_a$ to be stable $\underline{P}_a$ must have its poles in the left half plane, from Eq. (53) that for $\underline{C}_c$ to be stable $\underline{P}_c$ must have its poles in the left half plane, from Eq. (51) that for $\underline{C}_a$ to be stable the determinant of $\underline{P}_a$ must have its zeros in the left

half plane, from Eq. (49) that the determinant of $(\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)$ will be in the denominator and since it is composed only of diagonal elements each zero of $\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a$, and thus each zero of each diagonal element must be in the left half plane, and as a side note there are restrictions on the feedback matrix, namely, that all zeros of the determinants of $(\underline{I} + \underline{C}_b\underline{F}_b)$, $(\underline{I} - \underline{F}_a\underline{W}_a)$, and $(\underline{I} - \underline{F}_b\underline{W}_b)$ must be in the left half plane. These restrictions may be listed as:

1. The poles of the plant elements must lie in the left half of the s plane.

2. The determinant of the nxn matrix $\underline{P}_a$ must have all of its zeros in the left half plane.

3. The zeros of $(\underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a)_{jj}$; $j = 1, 2, \ldots, n$ $\left[\text{this includes all of the elements of } \underline{I} + \underline{P}_a\underline{C}_a\underline{F}_a \text{ for it is a diagonal matrix from Eq. (45)}\right]$ must lie in the left half plane.

4. The zeros of the determinants of $(\underline{I} + \underline{C}_b\underline{F}_b)$, $(\underline{I} - \underline{F}_a\underline{W}_a)$, and $(\underline{I} - \underline{F}_b\underline{W}_b)$ must lie in the left half plane.

Detailed examples of this approach have been given by Pack and Phillips (145) and Novik (142) for the turbojet engine, Stout (185, 186), Shumilovskii etal (173), and Eckmann and Lefsowitz (37) for chemical processes, Ergin and Ling (40) and Chien, Ergin, and Ling (27) for boilers or steam generating units, Courcoulas and Ham (30) for steel rolling mills, and others[1]. Much has been written concerning specific 2x2 systems while Čerak, Hanuš, Peterka, and Štěpán (22) have concentrated on a special 4x4 system.[2]

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[1] References 24, 34, 96, 99, 144, 163, 164, and 165.
[2] References 9, 43, 57, 61, 62, 66, 68, 71, 72, 92, 112, 120, 132, 135, 136, 139, 148, 149, 167, 177, 188, 197, and 205.

MINIMIZING THE OUTPUT ERROR

The Problem

A major control problem is that of minimizing the error between an actual output and a desired output. The means of expressing this error can take different forms but generally it is expressed as the square of the difference between the actual and the desired outputs. The problem has been described by Amara[1] as

> ...given a set of random inputs and arbitrary cross correlations and a plant matrix which specifies fixed and unalterable elements to be controlled, what is the transfer function of the compensating matrix that, in conjunction with the plant matrix, transforms the set of system inputs as closely as possible to a set of desired output functions?

The method of solution presented here will closely follow Amara's (1-4) work. Papers which have been written in this area are by Hsieh and Leondes (79, 80), Kuroda (100), Narendra and Goldwyn (137), and others[2]. The astute reader will notice a similarity between the problem presented and that found in the communication area where a series of multiplexed channels have interrelated signals as inputs. Much of the control systems theory in this area has evolved from work done on signals in multiple filters. A number of papers[3] concerning multipole filters are listed in the references at the end of this paper.

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[1] Reference 1, p. 51.
[2] References 8, 20, 117, 128, 166, 172, and 182.
[3] References 39, 73, 78, 81, 84, 162, 178, 183, 200, and 201.

Expressing the Error

The control system as shown in Fig. 5, which when rearranged so as to have n inputs to the controller, m outputs from the controller to the m inputs of the plant, and with a unity feedback matrix, has the input-output relationship of

$$\underline{Y} = (\underline{I} + \underline{PC})^{-1}\underline{PCX} \tag{55}$$

but by recognizing this new system as a regulator, one obtains

$$\underline{Y} = \underline{PGX} \tag{56}$$

where

$$\underline{PG} = (\underline{I} + \underline{PC})^{-1}\underline{PC}$$

and

$$\underline{C} = \underline{G}(\underline{I} - \underline{PG})^{-1} \tag{57}$$

The system of Fig. 5 with the above substitutions now becomes that of Fig. 10. Figure 10 also shows the output error, the difference between the actual output, $\underline{Y}_{a_i}$, and the desired output, $\underline{Y}_{d_i}$.

Working in the time domain, the total mean square error $\overline{E_T^2(t)}$ is

$$\overline{E_T^2(t)} = \sum_{h=1}^{n} \overline{e_h^2(t)} \tag{58}$$

where

$$\overline{e_h^2(t)} = \overline{\left[y_{a_h}(t) - y_{d_h}(t)\right]^2} \tag{59}$$

Thus

$$\overline{E_T^2(t)} = \sum_{h=1}^{n} \overline{y_{a_h}^2(t)} - \overline{2y_{a_h}(t)y_{d_h}(t)} + \overline{y_{d_h}^2(t)} \tag{60}$$

where the over bar indicates the time average.

To form the first term, by use of convolution and linear superposition one obtains as the input to the plant

$$z_i(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} g_{ij}(\sigma) \, x_j(t - \sigma) \, d\sigma$$
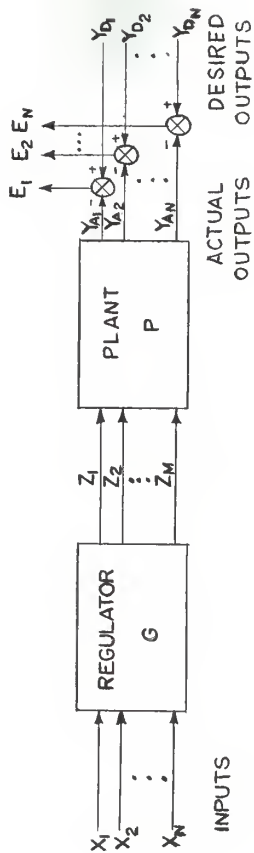
$$i = 1, 2, \ldots, m \tag{61}$$

FIG. 10. A NEW FORM OF THE MULTIVARIABLE SYSTEM SHOWING THE OUTPUT ERROR.

and the output as

$$y_{a_h}(t) = \sum_{i=1}^{m} \int_{-\infty}^{\infty} p_{hi}(\alpha) z_i(t - \alpha) d\alpha$$
$$h = 1, 2, \ldots, n \quad (62)$$

Combining Eqs. (61) and (62) and reordering

$$y_{a_h}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) g_{ij}(\sigma)$$
$$x_j(t - \sigma - \alpha) d\sigma d\alpha \quad h = 1, 2, \ldots, n \quad (63)$$

Since $y_{a_h}^2(t)$ is desired, Eq. (63) may be formed again using different variables and indices, obtaining

$$y_{a_h}(t) = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hk}(\beta) g_{k\ell}(\gamma)$$
$$x_\ell(t - \beta - \gamma) d\gamma d\beta \quad h = 1, 2, \ldots, n \quad (64)$$

Now Eqs. (63) and (64) are multiplied together and averaged with respect to time noting that since the system is assumed stationary and ergodic, the correlation function is

$$\overline{x_j(t - \xi) x_\ell(t - \Lambda)} \triangleq \phi_{x_j x_\ell}(\xi - \Lambda) \quad (65)$$

where $\xi = \sigma + \alpha$ and $\Lambda = \gamma + \beta$

$$\overline{y_{a_h}^2(t)} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha)$$
$$p_{hk}(\beta) g_{ij}(\sigma) g_{k\ell}(\gamma) \phi_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta)$$
$$d\gamma \, d\sigma \, d\beta \, d\alpha \, ^1 \quad h = 1, 2, \ldots, n \quad (66)$$

The result of the second term of Eq. (60) may be written as

---

[1]Since m and n are finite the uniform convergence criteria for interchange of summation and integration does not apply and the interchange is permissible.

$$\overline{2y_{a_h}(t)y_{d_h}(t)} = 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) g_{ij}(\sigma)$$

$$\phi_{x_j y_{dh}}(\sigma + \alpha) \; d\sigma d\alpha \qquad h = 1, 2, \ldots, n \qquad (67)$$

where

$$\overline{x_j(t - \sigma - \alpha)y_{d_h}(t)} \overset{\Delta}{=} \phi_{x_j y_{d_h}}(\sigma + \alpha)$$

The third term becomes

$$\phi_{y_{d_h}y_{d_h}}(0) \qquad (68)$$

Thus Eq. (60) becomes

$$\overline{E_T^2(t)} = \sum_{h=1}^{n} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) \right.$$

$$p_{hk}(\beta) g_{k\ell}(\gamma) g_{ij}(\sigma) \phi_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta) d\gamma d\sigma \, d\beta d\alpha$$

$$- 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) g_{ij}(\sigma) \phi_{x_j y_{dh}}(\sigma + \alpha) d\sigma d\alpha$$

$$\left. + \phi_{y_{d_h}y_{d_h}}(0) \right\} \qquad (69)$$

Minimizing the Error

The error is to be minimized with respect to the elements g_{ij} and $g_{k\ell}$, so assume that the solutions are g_{ij_m} and $g_{k\ell_m}$ and that the general element can be expressed as

$$g_{ij}(\sigma) = g_{ij_m}(\sigma) + \epsilon \, g_{ij_\epsilon}(\sigma)$$

$$g_{k\ell}(\gamma) = g_{k\ell_m}(\gamma) + \epsilon \, g_{k\ell_\epsilon}(\gamma) \qquad (70)$$

where $g_{ij_\epsilon}(\sigma)$ and $g_{k\ell_\epsilon}(\gamma)$ are fixed and defined as nonzero for σ and $\gamma \geq 0$ and vanish for σ and $\gamma < 0$. When Eq. (70) is substituted into Eq. (69) the only variable present will be ϵ which can be positive, negative, or zero.

From standard calculus, the extremum will be obtained when

the derivative of the function is set equal to zero. Here the value of the single variable is known when the extremum occurs, namely, $\epsilon = 0$, for then $g_{ij}(\sigma) = g_{ij_m}(\sigma)$ and $g_{k\ell}(\gamma) = g_{k\ell_m}(\gamma)$, the solutions as defined. Substituting Eq. (70) into Eq. (69), taking the partial derivative with respect to ϵ, equating the result evaluated at $\epsilon = 0$ to zero, and observing that

$$\emptyset_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta) = \emptyset_{x_j x_\ell}(\gamma + \beta - \sigma - \alpha); \qquad j = \ell$$

$$\emptyset_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta) = \emptyset_{x_\ell x_j}(\gamma + \beta - \sigma - \alpha); \qquad j \neq \ell \tag{71}$$

one obtains

$$\frac{1}{2} \left. \frac{\partial \overline{E_T^2(t)}}{\partial \epsilon} \right|_{\epsilon=0} = 0 = \sum_{h=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha)$$

$$p_{hk}(\beta) g_{ij_\epsilon}(\sigma) g_{k\ell_m}(\gamma) \emptyset_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta) d\gamma d\beta d\alpha d\sigma$$

$$- \sum_{h=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) g_{ij_\epsilon}(\sigma) \emptyset_{x_j y_{d_h}}(\sigma + \alpha) d\alpha d\sigma$$

$$i = 1, 2, \ldots, m$$

$$j = 1, 2, \ldots, n \tag{72}$$

The $g_{ij_\epsilon}(\sigma)$ is in both terms and may be factored out, giving

$$\int_{-\infty}^{\infty} g_{ij_\epsilon}(\sigma) \left\{ \sum_{h=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) p_{hk}(\beta) \right.$$

$$g_{k\ell_m}(\gamma) \emptyset_{x_j x_\ell}(\sigma + \alpha - \gamma - \beta) d\gamma d\beta d\alpha$$

$$\left. - \sum_{h=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\alpha) \emptyset_{x_j y_{d_h}}(\sigma + \alpha) d\alpha \right\} d\sigma = 0$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \tag{73}$$

This procedure gives a true minimum for when Eq. (72) is substituted into Eq. (69), $\left. \dfrac{\partial^2 \overline{E_T^2(t)}}{\partial \epsilon^2} \right|_{\epsilon=0} > 0$.[1]

[1]Reference 1, p. 58.

Since $g_{ij\epsilon}(\sigma)$ was zero for $\sigma < 0$ and not zero for $\sigma \geq 0$, the quantity inside the braces of Eq. (73) must be zero for $\sigma \geq 0$. Substituting Eq. (73) into Eq. (69), one obtains the minimum mean square error as

$$\overline{E_T^2(t)} = \sum_{h=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\emptyset_{y_{d_h} y_{d_h}}(0) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\propto) g_{ij}(\sigma) \right.$$

$$\left. \emptyset_{y_j y_{d_n}}(\sigma + \propto) d\sigma\, d\propto \right] \qquad \sigma \geq 0 \qquad (74)$$

For Eq. (74) to be the minimum mean square error, Eq. (73) must hold and the expression in braces of Eq. (73) must be

$$\sum_{h=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{hi}(\infty) p_{hk}(\beta) g_{k\ell_m}(\gamma)$$

$$\emptyset_{x_j x_\ell}(\sigma + \propto - \gamma - \beta) d\gamma\, d\beta\, d\propto - \sum_{h=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$$

$$p_{hi}(\propto) \emptyset_{x_j y_{d_h}}(\sigma + \propto) d\propto = 0 \quad \text{for} \quad \sigma \geq 0$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \qquad (75)$$

It is noted that Eq. (75) can be used to form the elements of an mxn matrix, $\underline{F}(\sigma)$. Interchanging the order of summation and integration in Eq. (75) and carrying out the indicated multiplications in conformable order, one is able to revert to matrix notation to obtain

$$\underline{F}(\sigma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{P}^T(\propto) \underline{P}(\beta) \underline{g}(\gamma) \, \underline{\Phi}_{xx}(\sigma + \propto - \gamma - \beta)$$

$$d\gamma\, d\beta\, d\propto - \int_{-\infty}^{\infty} \underline{P}^T(\propto) \, \underline{\Phi}_{x y_d}(\propto + \sigma) d\propto = 0 \quad {}^1$$

$$\sigma \geq 0 \qquad (76)$$

---

[1]Note $\underline{\Phi}_{x y_d}$ is the matrix containing the $\emptyset_{x_j y_{d_h}}$ elements where h indicates the column and j the row. It is not to be confused with the spectral density function commonly used.

where the superscript T indicates the transpose operation. The integration with respect to γ in the first term is the same as the convolution of $\underline{G}(\gamma)$ with $\underline{\Phi}_{xx}(\tau - \gamma)$, where $\tau = \sigma + \alpha - \beta$. The integration with respect to β is the same as the convolution between $\underline{P}(\beta)$ and the result of the previous convolution. These two integrations will produce a new matrix, $\underline{U}$, which will be in terms of $(\sigma + \alpha)$ only and will change Eq. (76) into

$$\underline{F}(\sigma) = \int_{-\infty}^{\infty} \underline{P}^T(\alpha)\underline{U}(\sigma+\alpha)d\alpha - \int_{-\infty}^{\infty} \underline{P}^T(\alpha)\underline{\Phi}_{xy_d}(\sigma+\alpha)d\alpha = 0$$
$$\sigma \geq 0 \qquad (77)$$

Equation (77) reduces to a series of equations of the form

$$f_{ij}(\sigma) = \sum_{h=1}^{n} \int_{-\infty}^{\infty} p_{hi}(\alpha)u_{hj}(\sigma+\alpha)d\alpha$$
$$- \sum_{h=1}^{n} \int_{-\infty}^{\infty} p_{hi}(\alpha)\phi_{x_j y_{d_h}}(\sigma+\alpha) d\alpha = 0$$
$$i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n$$
$$(78)$$

The Fourier transform of Eq. (78) by use of Parseval's theorem[1], i.e.,

$$r(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau+t)d\tau$$

$$\int_{-\infty}^{\infty} r(t)e^{-j\omega t} dt = F(-j\omega)G(j\omega)$$

becomes

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[1]Wilbur R. LePage, _Complex Variables and the Laplace Transform for Engineers_. New York: McGraw-Hill Book Co., 1961, p. 280.

$$f_{ij}(s) = \sum_{h=1}^{n} p_{hi}(-s)u_{hj}(s) - \sum_{h=1}^{n} p_{hi}(-s)\phi_{x_j y_{d_h}}(s) = 0$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$$

where $s \overset{\Delta}{=} j\omega$, or returning to matrix form becomes

$$\underline{F}(s) = \underline{P}(-s)\underline{U}(s) - \underline{P}(-s)\underline{\Phi}_{xy_d}(s) = 0 \tag{79}$$

Since $\underline{X}$ was the vector input, it must be well-behaved for if an ill-behaved input were used the system could leave the assumed linear range. The same can be said for $\underline{G}$ because the variation considered, $g_{ij} = g_{ij_m} + \epsilon g_{ij_\epsilon}$, must lie in the stable region. Thus since $\underline{F}(s)$ was found when $\underline{G}$ was varied in the left half plane, the fact that $\underline{F}(s) = 0$ is really only valid in the left half plane. Upon defining $\begin{bmatrix} \ \ \end{bmatrix}^+$ as the operation which yields the partial fraction expansion of the quantity inside the brackets in terms of the left half plane poles only, and $\overline{\underline{P}} = \overline{\underline{P}}(s) = \underline{P}(-s)$, one obtains

$$\left[\overline{\underline{P}}^T \underline{U}\right]^+ = \left[\overline{\underline{P}}^T \underline{\Phi}_{xy_d}\right]^+ \tag{80}$$

Equation (80) will be recognized as a form of the solution of the classical Wiener-Hopf equation. The general solution of this equation for the elements of $\underline{U}$ which contain the elements of the unknown $\underline{G}$ has been found by Wiener and Masani (201), and further work has been done by Youla (204) and Wong and Thomas (202). Recently Brockett (17, 18) presented a straightforward method of solving this equation.

MINIMIZING THE EFFECT OF PARAMETER VARIATION

Realization of the Transfer Function

As mentioned previously, in many plants the elements are

subject to variations; these changes can be periodic but generally are caused by natural drift. Horwitz (77) presents the analysis of this problem by dividing it into two parts: first, the realization of the desired transfer function, and, second, the minimization of the parameter variation effect. The transfer function will be found in this part and the minimization discussed in the following part. Chiu and Webb (39) have also considered this problem.

The matrix control equation for the multivariable system shown in Fig. 5 but with $m = n$ is

$$\underline{Y} = (\underline{I} + \underline{PCF})^{-1}\underline{PCX} = \underline{WX} \tag{81}$$

from which

$$\underline{F} = \underline{W}^{-1} - (\underline{PC})^{-1} \tag{82}$$

The error vector can be expressed as

$$\underline{E} = \underline{X} - \underline{FPCE} \tag{83}$$

and

$$\underline{E} = (\underline{I} + \underline{FPC})^{-1}\underline{X} \tag{84}$$

It is clear from Fig. 5 that

$$\underline{Y} = \underline{PCE} = \underline{PC}(\underline{I} + \underline{FPC})^{-1}\underline{X} = \underline{WX} \tag{85}$$

Now two different expressions may be written for $\underline{W}$, namely,

$$\underline{W} = (\underline{I} + \underline{PCF})^{-1}\underline{PC} \tag{86}$$

$$= \underline{PC}(\underline{I} + \underline{FPC})^{-1} \tag{87}$$

From Eq. (82) the feedback matrix can be found if the elements of $\underline{C}$ are set so as to minimize the parameter variation effect and $\underline{W}$ and $\underline{P}$ are known a priori.

Minimizing the Parameter Variation Effect

Let $\delta \underline{P}$ describe the matrix of the plant variation and $\delta \underline{W}$ its effect on the overall system. Thus the resulting new matrices are

$$\underline{P}' = \underline{P} + \delta \underline{P} \tag{88}$$

$$\underline{W}' = \underline{W} + \delta \underline{W} \tag{89}$$

From Eqs. (86) and (87)

$$\underline{W} = \underline{PC}(\underline{I} + \underline{FPC})^{-1} \tag{90}$$

$$\underline{W}' = (\underline{I} + \underline{P}'\underline{CF})^{-1}\underline{P}'\underline{C} \tag{91}$$

and remembering that $(\underline{AB})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$

$$\begin{aligned}
(\underline{W}')^{-1} &= (\underline{P}'\underline{C})^{-1}(\underline{I} + \underline{P}'\underline{CF}) \\
&= \underline{C}^{-1}(\underline{P}')^{-1}(\underline{I} + \underline{P}'\underline{CF})
\end{aligned} \tag{92}$$

Defining $\underline{S} \triangleq (\underline{W}')^{-1}\delta\underline{W}$

$$\begin{aligned}
\underline{S} = (\underline{W}')^{-1}\delta\underline{W} &= (\underline{W}')^{-1}(\underline{W}' - \underline{W}) = \left[\underline{C}^{-1}(\underline{P}')^{-1}(\underline{I} + \underline{P}'\underline{CF})\right] \\
&\qquad \left[(\underline{I} + \underline{P}'\underline{CF})^{-1}\underline{P}'\underline{C} - (\underline{I} + \underline{PCF})^{-1}\underline{PC}\right] \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\left\{\underline{P}'\underline{C} - \left[(\underline{P} + \delta\underline{P})\underline{CF} + \underline{I}\right]\left[(\underline{PCF} + \underline{I})^{-1}\underline{PC}\right]\right\} \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\left\{\underline{P}'\underline{C} - \left[(\underline{PCF} + \underline{I}) + \delta\underline{PCF}\right]\left[(\underline{PCF} + \underline{I})^{-1}\underline{PC}\right]\right\} \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\left[\underline{P}'\underline{C} - \underline{PC} - \delta\underline{PCF}(\underline{PCF} + \underline{I})^{-1}\underline{PC}\right] \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\left[\delta\underline{PC} - \delta\underline{PCFW}\right] \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\delta\underline{PC}\left[\underline{I} - \underline{FW}\right]
\end{aligned} \tag{93}$$

Using Eq. (82)

$$\begin{aligned}
\underline{S} &= \underline{C}^{-1}(\underline{P}')^{-1}\delta\underline{PC}\left\{\underline{I} - \left[\underline{W}^{-1} - (\underline{PC})^{-1}\right]\underline{W}\right\} \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\delta\underline{PC}\left[\underline{I} - \underline{I} + \underline{C}^{-1}\underline{P}^{-1}\underline{W}\right] \\
&= \underline{C}^{-1}(\underline{P}')^{-1}\delta\underline{PP}^{-1}\underline{W} \\
&= (\underline{W}')^{-1}\delta\underline{W}
\end{aligned} \tag{94}$$

Now let the elements in $\delta\underline{P}$ be designated by r_{ij}, in $\delta\underline{W}$ by

$\mathcal{T}_{ij}$, $P_{\alpha\beta}$ be the cofactor of the α, β element of $\underline{P}$. etc., Δ_p the determinant of $\underline{P}$, etc., and Δ_W be the determinant of the adjoint of $\underline{W}$, etc. Thus

$$\underline{S} = \underline{C}^{-1}(\underline{P}')^{-1} \, \underline{\underline{S}} \, \underline{PP}^{-1}\underline{W} = \frac{1}{\Delta_c \; \Delta_p \; \Delta_{p'}}$$

$$\begin{bmatrix} \sum{}^1 \, P'_{ui}C_{i1}r_{uj}P_{vj}w_{v1} & \sum P'_{ui}C_{i1}r_{uj}P_{vj}w_{v2} \; \cdots \; \sum P'_{ui}C_{i1}r_{uj}P_{vj}w_{vn} \\ \sum P'_{ui}C_{i2}r_{uj}P_{vj}w_{v1} & \sum P'_{ui}C_{i2}r_{uj}P_{vj}w_{v2} \; \cdots \; \sum P'_{ui}C_{i2}r_{uj}P_{vj}w_{vn} \\ \cdot & \cdot \qquad\qquad\qquad\quad \cdot \\ \cdot & \cdot \qquad\qquad\qquad\quad \cdot \\ \cdot & \cdot \qquad\qquad\qquad\quad \cdot \\ \sum P'_{ui}C_{in}r_{uj}P_{vj}w_{v1} & \sum P'_{ui}C_{in}r_{uj}P_{vj}w_{v2} \; \cdots \; \sum P'_{ui}C_{in}r_{uj}P_{vj}w_{vn} \end{bmatrix}$$

$$= \frac{1}{\Delta_{w'}} \begin{bmatrix} \sum w'_{i1}\mathcal{T}_{i1} & \sum w'_{i1}\mathcal{T}_{12} & \cdots & \sum w'_{i1}\mathcal{T}_{in} \\ \sum w'_{i2}\mathcal{T}_{i1} & \sum w'_{i2}\mathcal{T}_{12} & \cdots & \sum w'_{i2}\mathcal{T}_{in} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sum w'_{in}\mathcal{T}_{i1} & \sum w'_{in}\mathcal{T}_{12} & \cdots & \sum w'_{in}\mathcal{T}_{in} \end{bmatrix}$$

$$= (\underline{W}')^{-1} \, \underline{S} \, \underline{W} \tag{95}$$

The first column of the matrix of $\Delta_{w'}\underline{S}$ will be

$$\Delta_{w'}S_{\ell 1} = \sum_{i=1}^{n} w'_{i\ell} \, \mathcal{T}_{i\ell} \quad \text{for } \ell = 1, 2, \ldots, n$$

By using Cramer's rule[2]

$$\mathcal{T}_{11} = \frac{\Delta_{w'}}{\Delta_{w'}} \, (S_{11}M_{11} + S_{21}M_{21} + \cdots + S_{n1}M_{n1}) \tag{96}$$

where M_{ij} is the cofactor of the ij'th element of $\Delta_{w'}(W')^{-1}$.

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[1]Summations actually include $\sum_{i=1}^{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \sum_{j=1}^{n}$ but the single $\sum$ symbol will be used to conserve space and to indicate summation over all indices from 1 to $m = n$.

[2]Proof of Eqs. (96), (97), and (98) is given in Appendix B.

using Eq. (82).

As a special case assume the desired $\underline{C}$ is diagonal and that the overall deviations are to be small enough that $w'_{\alpha\beta}$ and $w_{\alpha\beta}$ can be considered equal. With these assumptions Eq. (103) becomes

$$\tau_{\alpha\beta} = \sum_{i=1}^{n} \sum_{v=1}^{n} \frac{w_{\alpha i} w_{v\beta}}{c_{ii}} \left(\frac{P_{vi}}{\Delta_p} - \frac{P'_{vi}}{\Delta_{p'}} \right) \qquad (104)$$

It is observed that c_{ii} may be chosen as large as possible to make the τ's as small as desired.

OPTIMIZATION

General Background

For many plants the relationships between the inputs and outputs are in the form of simple derivatives or just straightforward algebraic equations. Space coordinate transformations are a part of many multivariable systems such as the control and position of a space vehicle.[1] The equations of motion may be expressed in terms of the six variables x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$, where the dot indicates a derivative with respect to time. The equations will generally contain expressions involving second and higher order derivatives but by expanding the number of dependent variables all of the equations can be expressed by a set

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[1]References 21, 32, 33, 52, 67, 102, 111, 134, 146, 153, 160, 169, 171, 176, 193, 196, and 199.

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_{11}x_1 + b_{12}x_2 + \cdots + b_{1m}x_m$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot$$

$$\dot{y}_n = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nm}x_m$$

$$(105)$$

where the dependent variables or outputs are y_1, y_2, $\cdots$ y_n, and the independent variables or inputs are x_1, x_2, $\cdots$ x_n. An example is

$$\dot{y}_1 = 3y_1 + y_2 + 6x_1 + 9x_2 + 3x_3$$

$$\dot{y}_2 = 2y_2 + 4y_3 + 8x_1 + x_4$$

$$\dot{y}_3 = y_1 + 2y_2 + 6y_3 + 5x_1 + 9x_3 + 7x_4$$

Equation (105) can be condensed using matrix notation into

$$\dot{\underline{Y}} = \underline{A}\,\underline{Y} + \underline{B}\,\underline{X} \qquad (106)$$

where

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \qquad\qquad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$\underline{A}$ is an nxn matrix, and $\underline{B}$ is an nxm matrix. $\underline{Y}$ is called the state vector, $\underline{X}$ the forcing vector, $\underline{A}$ the coefficient matrix, and $\underline{B}$ the forcing matrix. Now the differential equation, Eq. (106), must be solved.

Considering the homogeneous system first,

$$\dot{\underline{Y}} = \underline{A}\,\underline{Y} \qquad\qquad \underline{Y}(t_o) = \underline{C} \qquad (107)$$

where $\underline{C}$ is a constant vector. Let the vector

$$\underline{q}_i(t) = \begin{bmatrix} q_{1_i} \\ q_{2_i} \\ \cdot \\ \cdot \\ \cdot \\ q_{n_i} \end{bmatrix} \qquad i = 1, 2, \ldots, n \qquad (108)$$

be such that

$$\frac{d}{dt} q_i = \underline{A} q_i \qquad (109)$$

meaning that $\underline{q}_i$ is a solution to equation (107) and that the $\underline{q}_i$'s are linearly independent. Now define $\underline{R}$ as the fundamental matrix[1] of Eq. (107), where $\underline{R}(t_o) = \underline{K}$, a constant nonsingular matrix, and where

$$\frac{d\underline{R}}{dt} = \underline{A}R \qquad\qquad \underline{R}(t_o) = \underline{K} \qquad (110)$$

As a special case let $\underline{Q}$ be the unique solution of Eq. (107), where $\underline{Q}(t_o) = \underline{I}$.[2] Now consider the matrix form

$$\underline{P} = \underline{Q}\underline{K} \qquad (111)$$

then

$$\dot{\underline{P}} = \frac{d}{dt} (\underline{Q}\underline{K}) = (\frac{d}{dt} \underline{Q})K + \underline{Q}(\frac{d}{dt} \underline{K})$$

$$= \underline{A}\underline{Q}\underline{K} = \underline{A}\underline{P} \qquad (112)$$

$$\underline{P}(t_o) = \underline{Q}(t_o)\underline{K} = \underline{I}\underline{K} = \underline{K} \qquad (113)$$

Therefore

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[1]Reference 184, p. A-6.
[2]Earl A. Coddington and Norman Levinson, Theory of Ordinary Differential Equations. New York: McGraw-Hill Book Co., 1955, p. 67.

$$\frac{d}{dt} \underline{P} = \underline{A}\underline{P} \qquad\qquad \underline{P}(t_o) = \underline{K} \qquad (114)$$

and hence from Eq. (110)

$$\underline{R} = \underline{P} = \underline{Q}\underline{K} \qquad (115)$$

Considering the original differential equation, let $\underline{B}\underline{X}(t) = V(t)$, so Eq. (106) becomes

$$\frac{d}{dt} \underline{Y}(t) = \underline{A}\underline{Y}(t) + \underline{V}(t) \qquad (116)$$

Let $\underline{Y} = \underline{Q}\,\underline{U}$, and from the definition of $\underline{Q}$, Eq. (116) becomes

$$\frac{d}{dt} \underline{Y} = \frac{d}{dt} (\underline{Q}\underline{U}) = \underline{Q} \frac{d}{dt} \underline{U} + (\frac{d}{dt} \underline{Q})\underline{U} = \underline{Q} \frac{d}{dt} \underline{U} + \underline{A}\underline{Q}\underline{U}$$

$$= \underline{A}\underline{Q}\underline{U} + \underline{V}(t) \qquad (117)$$

Therefore

$$\underline{Q} \frac{d}{dt} \underline{U} = \underline{V}(t)$$

$$\frac{d}{dt} \underline{U} = \underline{Q}^{-1}\underline{V}(t)$$

$$\underline{U} = \int_{t_o}^{t} \underline{Q}^{-1}(t')\underline{V}(t')dt' + \underline{U}(t_o) \qquad (118)$$

but

$$\underline{U}(t_o) = \underline{Q}^{-1}(t_o)\underline{Y}(t_o) = \underline{I}\underline{Y}(t_o) = \underline{C} \qquad (119)$$

Since from the definition of $\underline{U}$,

$$\underline{Y}(t) = \underline{Q}(t)\underline{U}(t) \qquad (120)$$

we now have a solution to the complete system

$$\underline{Y} = \underline{Q}(t)\underline{C} + \int_{t_o}^{t} Q(t)Q^{-1}(t')\,B\,X(t')dt' \qquad (121)$$

For a more detailed account of the operations concerning the

finding of the solution to Eq. (106), the reader is referred to Appendix A of Stone, et al (184) and to Friedland (153).

Now that the system can be described, some optimization criterion must be established. The form of this criterion may be a single function or a function with side restrictions. Generally this function is expressed in terms of the fact that the output vector, $\underline{Y}$, is different from some ideal or desired state, $\underline{D}$. It could be as simple as $L(\underline{Y}, \underline{D}) = 0$ for $Y = D$ and $L(\underline{Y}, \underline{D}) = 1$ for $Y \neq D$. Calling this quantity the loss function, $L(\underline{X}, \underline{Y}, \underline{D}, t)$, a cost expression, N, which for optimal conditions should be minimized, can be expressed as the integral of the loss function.

$$N = \int_0^t L(\underline{X}, \underline{Y}, \underline{D}, t)dt \qquad (122)$$

The minimization can be accomplished by use of the Euler-Lagrange formula from Calculus of Variations.[1] For this case it becomes

$$\frac{d}{dX_i} L - \frac{d}{dt}\left[\frac{d}{d\dot{X}_i} L\right] = 0 \qquad i = 1, 2, \ldots, m \qquad (123)$$

where L is the integrand of Eq. (122).

The Minimum Energy Controller

The cost in dollars and cents of operating a plant is usually

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[1] Gilbert Bliss, *Calculus of Variations*, 4th ed. LaSalle, Ill.: Carus Mathematical Monographs, The Open Court Publishing Co., 1949, p. 48. Robert Weinstock, *Calculus of Variations*. New York: McGraw-Hill Book Co., 1952, P. 20.

expressed in terms of the amount of energy used by the plant. For this reason a primary design item in choosing the proper controller is to minimize the energy used. Let the form of the energy function be $\underline{X}^T\underline{H}\underline{X}$, where $\underline{H}$ is symmetric and positive definite, i.e., the principle minors of its determinant are positive[1],

$$\left| h_{11} \right|, \quad \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & \cdots & h_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ h_{m1} & h_{m2} & & h_{mm} \end{vmatrix}$$

are all greater than zero.

A general constraint which will be used is that the system must reach a certain state over a given time interval, e.g.,

$$\underline{Y}_o = \underline{Y}(T) = 0 = \underline{Q}(T)\underline{C} + \int_0^T \underline{Q}(T)\underline{Q}^{-1}(t')\underline{B}\underline{X}(t')dt' \tag{124}$$

with $\underline{Q}(T)$ a constant matrix. Thus the optimum problem becomes: minimize

$$\int_0^T \underline{X}^T\underline{H}\underline{X} \ dt = \text{a scalar} \tag{125}$$

with the constraint

$$-\underline{C} = \int_0^T \underline{Q}^{-1}(t')\underline{B}\underline{X}(t')dt' \tag{126}$$

(Note the premultiplication of Eq. (124) by $\underline{Q}^{-1}(T)$.) Introducing the 1xn constant matrix Lagrange multiplier, $\underline{\lambda}$,[2] the loss function becomes

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[1]Marvin Marcus, Basic Theorems in Matrix Theory. Washington: U.S. Government Printing Office, National Bureau of Standards, Applied Mathematics Series 57, Jan. 22, 1960, p. 3.
[2]Weinstock, op. cit., p. 6.

$$L = \underline{X}^T \underline{H} \underline{X} + \underline{\lambda} \ \underline{Q}^{-1}(t) \underline{B} \underline{X}(t) \quad (127)$$

and this leads to an Euler-Lagrange equation of

$$\frac{d}{dX_i} \left[\underline{X}^T \underline{H} \underline{X} + \underline{\lambda} \ \underline{Q}^{-1} \underline{B} \underline{X} \right] = 0 \qquad i = 1, 2, \ldots, m \quad (128)$$

since L is independent of $\underline{\dot{X}}$.

Performing the derivative operation on the matrices leads to

$$\underline{M} \underline{H} \underline{X} + \underline{X}^T \underline{H} \underline{N} + \underline{\lambda} \ \underline{Q}^{-1} \underline{B} \underline{N} = 0 \quad (129)$$

where $\underline{M}$ is a 1xn matrix with all of its elements equal to one and $\underline{N}$ is an nx1 matrix with the same unity elements. Since $\underline{H}$ is symmetric it is noted that

$$\underline{M} \underline{H} \underline{X} = \underline{X}^T \underline{H} \underline{N}$$

so Eq. (129) becomes

$$2 \ \underline{X}^T \underline{H} \underline{N} + \underline{\lambda} \ \underline{Q}^{-1} \underline{B} \underline{N} = 0 \quad (130)$$

Since $\underline{H} \underline{N}$ and $\underline{Q}^{-1} \underline{B} \underline{N}$ are vectors with each element corresponding to a different orthogonal axis of the m dimensional space describing the m inputs, the operation of premultiplying the former by $2\underline{X}^T$ and the latter by $\underline{\lambda}$ will not change the orthogonality relationship. Thus the resultant scalars will be composed of the sum of quantities each corresponding to a different orthogonal axis. Since there are m such axes, Eq. (130) can be broken into m equations. The i'th equation will have as a common factor the i'th row element of $\underline{N}$ which can be divided out. Performing this same operation on all m equations and then recombining them into a matrix equation is the same as eliminating the $\underline{N}$'s in Eq. (130). With the $\underline{N}$'s so eliminated, Eq. (130) becomes

$$2\underline{X}^T \underline{H} + \underline{\lambda} \ \underline{Q}^{-1} \underline{B} = 0 \quad (131)$$

and upon taking the transpose of the entire equation

$$(2\underline{X}^T\underline{H})^T + (\underline{\lambda}\ \underline{Q}^{-1}\underline{B})^T = 0 \tag{132}$$

Remembering that $(\underline{AB})^T = \underline{B}^T\underline{A}^T$ and $\underline{H}^T = \underline{H}$, one obtains

$$\underline{X} = -\tfrac{1}{2}\left[\underline{H}^{-1}\underline{B}^T(\underline{Q}^{-1})^T\ \underline{\lambda}^T\right] \tag{133}$$

Substituting Eq. (133) into Eq. (126)

$$-\underline{C} = -\tfrac{1}{2}\int_0^T \underline{Q}^{-1}\underline{BH}^{-1}\underline{B}^T(\underline{Q}^{-1})^T\ \underline{\lambda}^T dt \tag{134}$$

$$= -\left[\tfrac{1}{2}\int_0^T \underline{Q}^{-1}\underline{BH}^{-1}\underline{B}^T(\underline{Q}^{-1})^T dt\right]\ \underline{\lambda}^T$$

$$\overset{\Delta}{=} -\underline{J}(T)\ \underline{\lambda}^T \tag{135}$$

and if $\underline{J}(T)$ is nonsingular, i.e., the determinant of $\underline{J}(T) \neq 0$, then

$$\underline{\lambda}^T = \underline{J}^{-1}(T)\underline{C} \tag{136}$$

and the optimum control over the interval is

$$\underline{X} = -\tfrac{1}{2}\ \underline{H}^{-1}\underline{B}^T(\underline{Q}^{-1})^T\underline{J}^{-1}(T)\underline{C} \tag{137}$$

When this $\underline{X}$ is used as the input to the plant which has the input-output relationships of Eq. (105), minimum energy will be used and the system will be optimized. All the designer need do now is to find a regulator to place in front of the plant, as G in Fig. 10, which has outputs of Eq. (137) for any m inputs the designer wishes to choose.

Minimum Time Response

For many systems the most critical problem is for the output to reach a given state in the least possible time. Thus the optimal problem becomes that of finding the best control function which minimizes the response time of the system subject to the

constraint that the input vectors can be saturated and cannot be greater than some given value, $X_i \leq k_i$. It can be formulated as to minimize

$$\int_0^{t_r} dt \tag{138}$$

with the final value constraint

$$\underline{Y}(t_r) = \underline{D}(t_r) = \underline{Q}(t_r)\underline{C} + \int_0^{t_r} \underline{Q}(t_r)\underline{Q}^{-1}(t')\underline{B}\underline{X}(t')dt' \tag{139}$$

or

$$\underline{Q}^{-1}(0)\underline{D}(0) - \underline{C} = \int_0^{t_r} \left\{ \frac{d\left[-\underline{Q}^{-1}(t')\underline{D}(t')\right]}{dt'} + \left[\underline{Q}^{-1}(t')\underline{B}\right]\underline{X}(t') \right\}dt'$$

Using the Lagrange multiplier, the problem becomes the minimization of

$$\int_0^{t_r} \left\{ 1 + \frac{\lambda d\left[-\underline{Q}^{-1}(t')\underline{D}(t')\right]}{dt'} + \left[\underline{Q}(t')\underline{B}\right]^T \underline{\lambda}^T \underline{X}(t') \right\}dt' \tag{141}$$

The Euler-Lagrange formula leads to an inconclusive result so physical intuition must be used. The minimization of the integrand will minimize any given integral, thus causing the integrand of Eq. (141) to assume its least possible value, and will yield the minimization of the response time. The only variable in the integrand is in the last term so if the last term can be made to be always negative, the least possible integrand will be obtained. This can be done by requiring

$$X_i = -k_i \, \mathrm{sgn}\left\{ \left[\underline{Q}^{-1}(t)\underline{B}\right]^T \right\}_i \tag{142}$$

where

$$\text{sgn } \theta = \begin{cases} 1 & \theta \quad 0 \\ 0 & \theta = 0 \\ -1 & \theta \quad 0 \end{cases}$$

Thus placing the input variables in the form of Eq. (142) results in minimal response time. This problem is carried further by Lee (103) where he proposes a method of finding just what the minimum response time will be. Other authors[1] discuss this problem in detail.

The problem of optimization can be extended to many other areas in multivariable systems. One major question which will arise is whether an optimal solution really exists. Markus and Lee (115) present an excellent article which discusses this problem. Many other recent papers[2] have been written on this subject and cover the many forms of optimization. The alert control engineer and system designer will wish to read them.

CONCLUSIONS

The field of multivariable control systems has really just begun to grow. From its beginnings in the 1930's and 1940's it has progressed as most new areas by going from a specific case, through more general problems, and on to more inclusive ideas. The first case was that of obtaining noninteraction among the inputs and was described in detail in this paper. The synthesis problem was presented and its method of solution shown. An example 3x2 system

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[1]References 6, 75, 156, 175, 184, and 198.
[2]References 5, 11, 23, 28, 36, 38, 42, 53, 54, 55, 69, 74, 76, 83, 93, 101, 104, 116, 121, 127, 138, 148, 150, and 151.

wes synthesized in Appendix A. The problem of finding the con-
troller elements yielding the leest mean squere output error with
correlated inputs wes presented end the solution cerried to the
obtaining of e Wiener-Hopf equetion. Plants heving peremeter
veriations, but with an appropriete controller preceding the plent,
could give es smell as desired deviations from e specified overell
transfer function. The idees of system optimizetion were pre-
sented and two specific problems, one of minimum energy and the
other of minimum response time, were explained.

There ere an increesing number of papers being written con-
cerning the designing of multivarieble systems through use of Bode
plots and Nyquist diegrams. Their applicetion is very difficult
in lerge dimension systems but is finding more acceptence when
used with smell and simple plents.

The most repidly expending eree is that of overell system
optimizetion. The increesing emount of reseerch concerning this
still rether broed aree will mean that control engineers end de-
signers will find it even more difficult to keep ebreast of the
latest developments. In this aree the future challenge is pre-
sented.

ACKNOWLEDGMENT

APPENDICES

APPENDIX A

Example Noninteraction Problem

As an example, assume that the multivariable plant given has three inputs and two outputs, the plant parameters are time invariant, and the plant elements expressed in the complex frequency domain are

$$p_{11} = \frac{6}{40s + 1} \qquad p_{12} = \frac{3}{30s + 1} \qquad p_{13} = \frac{5}{45s + 1}$$

$$p_{21} = \frac{1}{25s + 1} \qquad p_{22} = \frac{1}{15s + 1} \qquad p_{23} = \frac{6}{35s + 1}$$

$$(143)$$

Let it be assumed that the desired overall transfer matrix is to have the principle of noninteraction and further to have the elements

$$w_{11} = \frac{1}{10s + 1} \qquad\qquad w_{22} = \frac{1}{20s + 1}$$

$$w_{33} = \frac{1}{30s + 1} \qquad\qquad w_{ij} = 0 \qquad i \neq j$$

$$(144)$$

The general system block diagram is shown in Fig. 11. For convenience assume the measuring instruments in the feedback loop are such that the feedback matrix is the unit matrix. Thus the problem is to find the elements of the controller which will transform the input-output relationship from that of the plant alone given through use of Eq. (143) into an overall relationship given through use of Eq. (144).
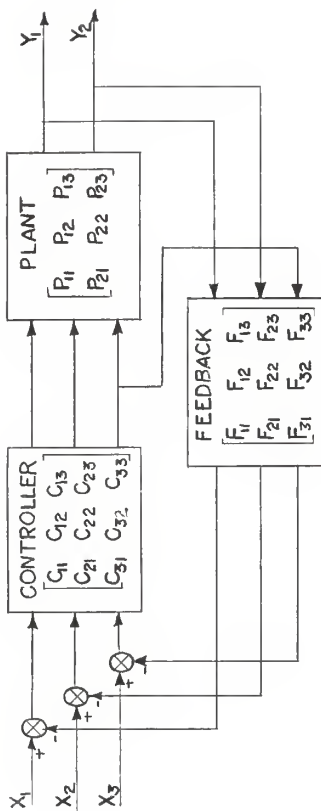
FIG. II. THE 3×2 MULTIVARIABLE SYSTEM BLOCK DIAGRAM.

From Eqs. (22), (24), (25), and (26), the following submatrices were defined.

$$\underline{P}_a = \begin{bmatrix} p_{11} & p_{12} \\ \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{6}{40s + 1} & \dfrac{3}{30s + 1} \\ \\ \dfrac{1}{25s + 1} & \dfrac{1}{15s + 1} \end{bmatrix} \tag{145}$$

$$\underline{P}_b = P_{33} = 1 \tag{146}$$

$$\underline{P}_c = \begin{bmatrix} p_{13} \\ \\ p_{23} \end{bmatrix} = \begin{bmatrix} \dfrac{5}{45s + 1} \\ \\ \dfrac{6}{35s + 1} \end{bmatrix} \tag{147}$$

$$\underline{P}_d = \begin{matrix} p_{31} & p_{32} \end{matrix} = 0 \tag{148}$$

$$\underline{W}_a = \begin{bmatrix} \dfrac{1}{10s + 1} & 0 \\ \\ 0 & \dfrac{1}{20s + 1} \end{bmatrix} \tag{149}$$

$$\underline{W}_b = \dfrac{1}{30s + 1} \tag{150}$$

$$\underline{W}_c = \underline{W}_d = 0 \tag{151}$$

$$\underline{F}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{152}$$

$$\underline{F}_b = 1 \tag{153}$$

$$\underline{F}_c = \underline{F}_d = 0 \tag{154}$$

From Eq. (51)

$$\underline{C}_a = (\underline{P}_a)^{-1} \underline{W}_a (\underline{I} - \underline{F}_a \underline{W}_a)^{-1}$$

$$= \begin{bmatrix} \dfrac{6}{40s+1} & \dfrac{1}{30s+1} \\[3mm] \dfrac{1}{25s+1} & \dfrac{1}{15s+1} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{10s+1} & 0 \\[3mm] 0 & \dfrac{1}{20s+1} \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{10s+1} & 0 \\[3mm] 0 & \dfrac{1}{20s+1} \end{bmatrix} \right\}^{-1} \tag{155}$$

$$= \left[\dfrac{1}{(s+.0306+j.0133)(s+.0306-j.0133)} \right]$$

$$\begin{bmatrix} \dfrac{(25s+1)(30s+1)(40s+1)}{30s} & -\dfrac{(15s+1)(25s+1)(40s+1)}{20s} \\[4mm] -\dfrac{(15s+1)(30s+1)(40s+1)}{30s} & \dfrac{(15s+1)(25s+1)(30s+1)}{10s} \end{bmatrix} \tag{156}$$

From Eq. (52)

$$\underline{C}_b = \underline{W}_b(\underline{I} - \underline{F}_b\underline{W}_b)^{-1} = \dfrac{1}{30s+1}(1 - 1 \cdot \dfrac{1}{30s+1})^{-1} \tag{157}$$

$$= \dfrac{1}{30s} \tag{158}$$

From Eq. (53)

$$\underline{C}_c = -\underline{P}_a^{-1}\underline{P}_c\underline{W}_b(\underline{I} - \underline{F}_b\underline{W}_b)^{-1} = -\underline{P}_a^{-1}\underline{P}_c\underline{C}_b$$

$$= -\begin{bmatrix} \dfrac{6}{40s+1} & \dfrac{3}{30s+1} \\[3mm] \dfrac{1}{25s+1} & \dfrac{1}{15s+1} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{5}{45s+1} \\[3mm] \dfrac{6}{35s+1} \end{bmatrix} \dfrac{1}{30s} \tag{159}$$

$$= \begin{bmatrix} \dfrac{13(25s+1)(40s+1)(s+.0880)(s+.0314)}{30s(45s+1)(35s+1)(s+.0306+j.0133)(s+.0306-j.0133)} \\[2ex] \dfrac{-31(15s+1)(30s+1)(s+.0420)(s+.0220)}{30s(35s+1)(45s+1)(s+.0306+j.0133)(s+.0306-j.0133)} \end{bmatrix} \qquad (160)$$

From Eq. (54)

$$\underline{C}_d = 0 \qquad (161)$$

Thus from Eqs. (156), (158), (160), and (161) the elements of the controller which cause overall system noninteraction are readily seen to be

$$c_{11} = \frac{(25s+1)(30s+1)(40s+1)}{30s(s+.0306 + j.0133)(s+.0306-j.0133)} \qquad (162)$$

$$c_{12} = - \frac{(15s+1)(25s+1)(40s+1)}{20s(s+.0306+j.0133)(s+.0306-j.0133)} \qquad (163)$$

$$c_{13} = \frac{13(25s+1)(40s+1)(s+.0880)(s+.0314)}{30s(35s+1)(45s+1)(s+.0306+j.0133)(s+.0306-j.0133)} \qquad (164)$$

$$c_{21} = - \frac{(15s+1)(30s+1)(40s+1)}{30s(s+.0306+j.0133)(s+.0306-j.0133)} \qquad (165)$$

$$c_{22} = \frac{(15s+1)(25s+1)(30s+1)}{10s(s+.0306+j.0133)(s+.0306-j.0133)} \qquad (166)$$

$$c_{23} = \frac{-31(15s+1)(30s+1)(s+.0420)(s+.0220)}{30s(35s+1)(45s+1)(s+.0306+j.0133)(s+.0306-j.0133)} \qquad (167)$$

$$c_{31} = 0 \qquad (168)$$

$$c_{32} = 0 \qquad (169)$$

$$c_{33} = \frac{1}{30s} \qquad (170)$$

APPENDIX B

Forming Equations (96), (97), and (98)

Cramer's rule states[1]

$$X_{ij} = \frac{D_{ij}}{D} \tag{171}$$

when $\underline{Y} = \underline{A}\underline{X}$ (172)

where D is the determinant of $\underline{A}$

D_{ij} is the determinant of $\underline{A}$ with the i'th column

of $\underline{A}$ replaced by the j'th column of $\underline{Y}$

So from Eq. (95) the first column of the matrix $\Delta_w, \underline{S}$ is

$$\Delta_w, s_{\ell 1} = \sum_{i=1}^{n} W'_{i\ell} \tau_{i\ell} \qquad \ell = 1, 2, \ldots, n \tag{173}$$

or

$$
\begin{bmatrix}
\Delta_w, s_{11} \\
\Delta_w, s_{21} \\
\cdot \\
\cdot \\
\cdot \\
\Delta_w, s_{n1}
\end{bmatrix}
=
\begin{bmatrix}
W'_{11} & W'_{21} & \cdots & W'_{n1} \\
W'_{12} & W'_{22} & \cdots & W'_{n2} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
W'_{1n} & W'_{2n} & \cdots & W'_{nn}
\end{bmatrix}
\begin{bmatrix}
\tau_{11} \\
\tau_{12} \\
\cdot \\
\cdot \\
\cdot \\
\tau_{1n}
\end{bmatrix}
\tag{174}
$$

(Remember the interchanging of the indices when forming the adjoint, so that W_{ij} is at the j'th row and i'th column of the above nxn matrix.) Thus using Cramer's rule

---

[1]Turnbull, The Theory of Determinants, Matrices, and Invariants, 2nd ed. London: Blakie and Sons, Ltd., 1945, p. 11. C. R. Wylie, Advanced Engineering Mathematics, 2nd ed. New York: McGraw-Hill Book Co., 1960, p. 32.

$$\mathcal{T}_{11} = \begin{vmatrix} \Delta_{w'}S_{11} & W'_{21} & \cdots & W'_{n1} \\ \Delta_{w'}S_{21} & W'_{22} & \cdots & W'_{n2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Delta_{w'}S_{n1} & W'_{2n} & \cdots & W'_{nn} \end{vmatrix} \frac{1}{\Delta_{w'}} \qquad (175)$$

where as previously defined $\Delta_{w'}$ is the determinant of the ad-joint of $\underline{W}'$. So

$$\mathcal{T}_{11} = \frac{1}{\Delta_{w'}} \left[\Delta_{w'}S_{11} \text{ (cofactor of } W'_{11}) \right.$$

$$+ \Delta_{w'}S_{21} \text{ (cofactor of } W'_{12}) + \cdots$$

$$\left. + \Delta_{w'}S_{n1} \text{ (cofactor of } W'_{1n}) \right] \qquad (176)$$

Noting that

$$\frac{\text{adjoint of } \underline{W}'}{\Delta_{w'}} = (\underline{W}')^{-1} \qquad (177)$$

or $\qquad$ adjoint of $\underline{W}' = \Delta_{w'}(\underline{W}')^{-1} \qquad (178)$

and that the

cofactor of the ij'th element of the adjoint of W'

$$= \text{cofactor of } W'_{ji} = \text{ij'th cofactor of } \Delta_{w'}(\underline{W}')^{-1}$$

$$\overset{\Delta}{=} M_{ij} \qquad (179)$$

So from (176)

$$\mathcal{T}_{11} = \frac{\Delta_{w'}}{\Delta_{w'}} \left[S_{11}M_{11} + S_{21}M_{21} + \cdots + S_{n1}M_{n1} \right] \qquad (180)$$

which is Eq. (96).

———————————

Jacobi's theorem states,[1] "Each minor of Δ is proportional to the corresponding complementary cofactor of Δ^{-1}, the ratio being Δ."

Let
$$\Delta = \Delta_{W'} = \left| \underline{W} \right| \tag{181}$$

$$\Delta^{-1} = \left| \underline{W}' \right|^{-1} = \begin{vmatrix} \dfrac{W'_{11}}{\Delta_{W'}} & \dfrac{W'_{21}}{\Delta_{W'}} & \cdots & \dfrac{W'_{j1}}{\Delta_{W'}} & \cdots & \dfrac{W'_{n1}}{\Delta_{W'}} \\[2ex] \dfrac{W'_{12}}{\Delta_{W'}} & \dfrac{W'_{22}}{\Delta_{W'}} & \cdots & \dfrac{W'_{j2}}{\Delta_{W'}} & \cdots & \dfrac{W'_{n2}}{\Delta_{W'}} \\[2ex] \vdots & \vdots & & \vdots & & \vdots \\[2ex] \dfrac{W'_{1i}}{\Delta_{W'}} & \dfrac{W'_{2i}}{\Delta_{W'}} & \cdots & \dfrac{W'_{ji}}{\Delta_{W'}} & \cdots & \dfrac{W'_{ni}}{\Delta_{W'}} \\[2ex] \vdots & \vdots & & \vdots & & \vdots \\[2ex] \dfrac{W'_{1n}}{\Delta_{W'}} & \dfrac{W'_{2n}}{\Delta_{W'}} & \cdots & \dfrac{W'_{jn}}{\Delta_{W'}} & \cdots & \dfrac{W'_{nn}}{\Delta_{W'}} \end{vmatrix} \tag{182}$$

jth column

$\leftarrow$ i'th row

[1] Turnbull, op. cit., p. 77.

$$\text{ij'th cofactor of } |\underline{W}'| = (-1)^{i+j}
\begin{vmatrix}
\dfrac{W'_{11}}{\Delta_{w'}} & \dfrac{W'_{21}}{\Delta_{w'}} & \cdots & \dfrac{W'_{j-1,1}}{\Delta_{w'}} & \dfrac{W'_{j+1,1}}{\Delta_{w'}} & \cdots & \dfrac{W'_{n1}}{\Delta_{w'}} \\[2ex]
\dfrac{W'_{12}}{\Delta_{w'}} & \dfrac{W'_{22}}{\Delta_{w'}} & \cdots & \dfrac{W'_{j-1,2}}{\Delta_{w'}} & \dfrac{W'_{j+1,2}}{\Delta_{w'}} & \cdots & \dfrac{W'_{n2}}{\Delta_{w'}} \\[1ex]
\vdots & \vdots & & \vdots & \vdots & & \vdots \\[1ex]
\dfrac{W'_{1,i-1}}{\Delta_{w'}} & \dfrac{W'_{2,i-1}}{\Delta_{w'}} & \cdots & \dfrac{W'_{j-1,i-1}}{\Delta_{w'}} & \dfrac{W'_{j+1,i-1}}{\Delta_{w'}} & \cdots & \dfrac{W'_{n,i-1}}{\Delta_{w'}} \\[2ex]
\dfrac{W'_{1,i+1}}{\Delta_{w'}} & \dfrac{W'_{2,i+1}}{\Delta_{w'}} & \cdots & \dfrac{W'_{j-1,i+1}}{\Delta_{w'}} & \dfrac{W'_{j+1,i+1}}{\Delta_{w'}} & \cdots & \dfrac{W'_{n,i+1}}{\Delta_{w'}} \\[1ex]
\vdots & \vdots & & \vdots & \vdots & & \vdots \\[1ex]
\dfrac{W'_{1n}}{\Delta_{w'}} & \dfrac{W'_{2n}}{\Delta_{w'}} & \cdots & \dfrac{W'_{j-1,n}}{\Delta_{w'}} & \dfrac{W'_{j+1,n}}{\Delta_{w'}} & \cdots & \dfrac{W'_{nn}}{\Delta_{w'}}
\end{vmatrix}$$

$$(183)$$

$$= \frac{(-1)^{i+j}}{\Delta_{w'}^{n-1}}
\begin{vmatrix}
W'_{11} & W'_{21} & \cdots & W'_{j-1,1} & W'_{j+1,1} & \cdots & W'_{n1} \\
W'_{12} & W'_{22} & \cdots & W'_{j-1,2} & W'_{j+1,2} & \cdots & W'_{n2} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
W'_{1,i-1} & W'_{2,i-1} & & W'_{j-1,i-1} & W'_{j+1,i-1} & & W'_{n,i-1} \\
W'_{1,i+1} & W'_{2,i+1} & & W'_{j-1,i+1} & W'_{j+1,i+1} & & W'_{n,i+1} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
W'_{1n} & W'_{2n} & & W'_{j-1,n} & W'_{j+1,n} & & W'_{nn}
\end{vmatrix}$$

$$(184)$$

$$= \frac{1}{\Delta_{w'}^{n-1}} \quad \text{(ij'th cofactor of the adjoint of } \underline{W}')$$

$$= \frac{1}{\Delta_{w'}^{n-1}} \quad (ij\text{'th cofactor of } \Delta_{w'}(w')\}$$

$$= \frac{1}{\Delta_{w'}^{n-1}} \quad M_{ij} \tag{185}$$

The minor of the w_{ji}'th element of $\underline{W}$ is w_{ji} and its complementary cofactor will be W_{ij}. So from the theorem one has

$$\frac{w'_{ji}}{\dfrac{M_{ij}}{\Delta_{w'}^{n-1}}} = \Delta_{w'} \tag{186}$$

or

$$M_{ij} = \frac{w'_{ji} \Delta_{w'}^{n-1}}{\Delta_{w'}} = w'_{ji} \Delta_{w'}^{n-2} \tag{187}$$

which is Eq. (97) identically.

To arrive at Eq. (98), form the product of the determinant of $\underline{W}'$ and the determinant of its adjoint, $\underline{W}^A$.[1]

$$\left| \underline{W}' \right| = \Delta_{w'} = \begin{vmatrix} w'_{11} & w'_{12} & \cdots & w'_{1n} \\ w'_{21} & w'_{22} & \cdots & w'_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w'_{n1} & w'_{n2} & \cdots & w'_{nn} \end{vmatrix}$$

$$\left| \underline{W}'^A \right| = \Delta_{W'} = \begin{vmatrix} W'_{11} & W'_{21} & \cdots & W'_{n1} \\ W'_{12} & W'_{22} & \cdots & W'_{n2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ W'_{1n} & W'_{2n} & \cdots & W'_{nn} \end{vmatrix}$$

---

[1] Turnbull, op. cit., p. 67.

$$
\Delta_{w'}\,\Delta_{W'} = \begin{vmatrix}
\displaystyle\sum_{i=1}^{n} w'_{1i}W'_{1i} & \displaystyle\sum_{i=1}^{n} w'_{1i}W'_{2i} & \cdots & \displaystyle\sum_{i=1}^{n} w'_{1i}W'_{ni} \\[2ex]
\displaystyle\sum_{i=1}^{n} w'_{2i}W'_{1i} & \displaystyle\sum_{i=1}^{n} w'_{2i}W'_{2i} & \cdots & \displaystyle\sum_{i=1}^{n} w'_{2i}W'_{ni} \\[2ex]
\vdots & \vdots & & \vdots \\[2ex]
\displaystyle\sum_{i=1}^{n} w'_{ni}W'_{1i} & \displaystyle\sum_{i=1}^{n} w'_{ni}W'_{2i} & \cdots & \displaystyle\sum_{i=1}^{n} w'_{ni}W'_{ni}
\end{vmatrix}
\tag{188}
$$

Now

$$
\sum_{i=1}^{n} w'_{ji}W'_{ji} = \Delta_{w'}
\tag{189}
$$

and

$$
\sum_{i=1}^{n} w'_{ji}W'_{ki} = 0 \qquad\qquad j \neq k
$$

(The magnitude of W'_{ki} is the determinant $\underline{W}'$ with the k'th row and

the i'th column deleted, so $\displaystyle\sum_{i=1}^{n} w'_{ji}W'_{ki}$ would be the determinant

of the matrix $\underline{W}$ with the k'th row replaced by w_{ji}, $i = 1, 2, \ldots,$

n, the j'th row of $\underline{W}$. But this would mean that two rows would be

the same, so $\displaystyle\sum_{i=1}^{n} w'_{ji}W'_{ik} = 0$.) So

$$
\Delta_{w'}\,\Delta_{W'} = \begin{vmatrix}
\Delta_{w'} & 0 & \cdots & 0 \\
0 & \Delta_{w'} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & & \Delta_{w'}
\end{vmatrix}
$$

$$= \Delta_{w'}^{n} \tag{190}$$

and if
$$\Delta_{w'} \neq 0$$

then
$$\Delta_{W'} = \Delta_{w'}^{n-1} \tag{191}$$

which is seen to be Eq. (98).

APPENDIX C

Recent Papers

Since the writing of this review a symposium on Multivariable Linear Control Systems has been held by the Society for Industrial and Applied Mathematics. The following papers were delivered and are included in this work as a supplement for the sake of completeness. The annotations are those supplied by the bulletin announcing the meeting. Copies of the papers are expected to be included in the second issue of the new SIAM Series on Control which will be published early in 1963.

1. "Controllability and observability in multivariable control systems", Prof. E. G. Gilbert.

The representation of linear multivariable feedback systems by transfer-function matrices is essential to most known synthesis procedures. The purpose of this paper is to investigate carefully the limitations of such representations, primarily the possible neglect of some modes of response. Controllability and observability of interconnected multivariable systems forms the basis of this study.

Some of the topics treated are: methods for determining the controllability and observability of feedback systems in terms of open-loop controllability and observability; the minimum dynamic order of a transfer-function matrix; stability constraints in compensation, their realization without undue complexity.

2. "Mathematical description of linear dynamical systems",
Dr. R. E. Kalman.

There are two different ways of describing dynamical systems: (i) by means of state variables and (ii) by input/output relations. The first method may be regarded as an axiomatization of Newton's laws of causality and taken to be the basic definition of a system.

It is then shown (in the linear case) that the input/output relations determine only one part of a system, that which is completely observable and completely controllable. Using the theory of controllability and observability, methods are given for calculating irreducible realization of a given impulse-response matrix. In particular, an explicit procedure is given to determine the minimal number of state variables necessary to realize a given transfer-function matrix. Difficulties arising from the use of reducible realizations are discussed briefly.

3. "Current status of multivariable linear control system theory", Dr. K. J. Astrom, Dr. J. E. Bertram, Dr. J. J. Florentin, Dr. P. D. Joseph, Dr. R. E. Kalman.

The purpose of this paper is to present the main theoretical results in the field, without proofs but with careful control-theoretic, mathematical and physical interpretation. Included is an extensive bibliography.

The main topics are:
    (1) Origin of the equations of motion; definition of state variables.
    (2) Statement of the general control problem.
    (3) Solution of deterministic control problems via the Hamilton-Jacobi equation.
    (4) Solution of deterministic control problems via the Euler equations.
    (5) Optimal linear filters.
    (6) Stability, controllability, and observability.
    (7) Jointly optimal control and estimation systems.
    (8) Modifications of the theory for sampled-data systems.

4. "Neighboring-optimum terminal control for multivariable nonlinear systems", Dr. J. V. Breakwell, and Prof. A. E. Bryson.

A feedback control scheme is described that optimizes a terminal quantity and satisfies certain terminal constraints in the presence of small disturbances. The scheme is based on a linear perturbation from a nominal optimum path and involves essentially the second variation in the calculus of variations. Measurements of deviations of the state variables from their nominal values constitute an error vector which is multiplied by a matrix of time-varying gains (precalculated and stored in a memory) to yield a control variable deviation vector; this

deviation vector is added to the nominal control variable vector to provide a control signal. A simple example of the use of the scheme is given for thrust direction control to place a vehicle in orbit.

5. "On computing optimal control with inequality constraints", Prof. Y. C. Ho, and Mr. P. Brentani.

This paper describes the recently developed successive approximation techniques for solving general problems of optimal control and programming with inequality constraints on the control or state variables. The exposition is self-contained and proceeds from first principles. Examples of application are given.

6. "Air vehicle trajectory optimization", by Dr. H. J. Kelley, Mr. M. Falco, and Mr. D. J. Ball.

The method of gradients is applied to the optimization of flight paths and throttle programs for high performance air-breathing vehicles. Problems of maximum range, minimum fuel and minimum time subject to a variety of constraints are treated. The inequality constraint on thrust $0 \leqq T \leqq T_{max}$ is treated by introduction of an appropriate throttle variable having limited range and by a corresponding modification of the gradient process.

Some digital computations of flight performance for a hypothetical Mach 3 turbojet vehicle are presented: minimum-fuel and minimum-time climbs and maximum range mission profiles. The successive approximation process is illustrated and the relative merits of two versions of the gradient method are examined from the viewpoint of speed of convergence. The use of the Gel'fand-Tsetlin method of ravines to accelerate convergence in the most obdurate cases is discussed and illustrated.

7. "Optimization and control using perturbation theory to find neighboring optimum paths", Mr. J. L. Speyer.

This paper is concerned with the generation of the control program for a nonlinear system, over a finite time interval, which optimizes a terminal quantity. A rapidly converging procedure is presented for finding an exact solution to the two-point boundary-value problem of the calculus of variations. This procedure is closely related to the neighboring-optimum feedback control scheme proposed by Breakwell and Bryson and by Kelley. Application of both the optimization procedure and the control scheme is demonstrated for the trajectory of a vehicle entering the earth's atmosphere at parabolic speed.

The procedure starts with an exact optimum solution of a set of differential equations which, however, does not satisfy the desired boundary conditions. Then a neighboring-optimum solution is determined, using the second variation as an intermediate step, which comes closer to satisfying the desired boundary conditions.

This process is repeated until the boundary conditions are satisfied as closely as desired. Unlike previous schemes, convergence to the exact optimum is assured and rapid.

8. "Multivariable linear filter theory applied to space vehicle guidance", Dr. G. L. Smith, and Dr. S. F. Schmidt.

Midcourse guidance of a spacecraft consists of (1) estimating the vehicle's position and velocity (i.e., the state) from noisy observations obtained by instruments on board the vehicle or by a ground tracking network, (2) using a guidance law to compute from the estimated state a velocity correction which if applied would result in satisfying desired end-point conditions, and (3) applying the computed correction if a decision logic so indicates. This paper shows how the equations for processing the observational data to obtain the estimated state may be derived using linear filter theory concepts suggested by R. E. Kalman.

A method is given for computing optimal observation and velocity correction schedules, using filter decision theory. This method amounts to utilizing the statistics always available from the guidance computations to decide at every point in time which (if any) of a number of possible observations should be made and whether or not the indicated velocity correction should be implemented.

The calculations necessary (performed on a digital computer) are described and some representative results are presented. These include a circumlunar guidance study and an analysis of the expected improvement in knowledge of the astrodynamic constants from observations of a lunar probe.

9. "The use of variational parameters in multivariable linear prediction and filtering theory", Dr. S. Pines and Mr. Alphonse Sistino.

The choice of the state variables in the application of Kalman's method in multivariable prediction and filtering theory affects the rate of convergence and the numerical accuracy of the various decision functions arising out of the theory. This phenomenon is closely related to the use of perturbation variables to obtain slowly varying parameters to describe a system in dynamics and celestial mechanics. To replace the usual state variables, a new set of parameters are developed which have the property that only one of these variables contain a secular time term. The transformed state variables (except for the one describing the energy) remain almost constant along the trajectory, and the numerical procedures required may be accurately and simply carried out.

Another difficulty is the need for integrating the $4n^2$ linear perturbation differential equations of partial derivatives in addition to the 2n first order nonlinear differential equations

of motion. A method is described using a perturbation technique based on the Encke method which computes the transition matrix in closed form as a function of the time and the instantaneous state variables without the need for integrating the $4n^2$ differential equations. Only the 2n equations of motion need be integrated and the transition matrix may be computed directly in terms of the known times and the instantaneous values of the state variables.

The method is applied to a system of equations associated with the motion of a space vehicle in a central force field under the action of perturbations. A set of variational parameters is described, closed form expressions for the transition matrix are presented, and the procedure for updating the covariance matrix is given in detail.

10. "Satellite attitude control; an application of automatic optimal design techniques", Mr. T. S. Englar and Dr. R. E. Kalman.

The object of this paper is to illustrate the application of the general linear optimal design technique to satellite atti- tude control. The technique is based on the Hamilton-Jacobi formulation, and is implemented with the aid of a digital computer.

Several examples of optimal linear design are calculated and discussed. Particular attention is paid to the selection of quadratic performance criteria in the optimization program. These criteria are successfully modified according to certain heuristic principles until the engineering design requirements are set.

Problems of numerical computation are also discussed. The speed of convergence and accuracy of the computations is analyzed.

11. "Some control problems in inertial guidance", Dr. K. J. Astrom.

This paper applies the computational techniques of linear optimal control theory to certain standard problems in aerospace control, for instance, gyro-stabilization of inertial platforms. Particular attention is paid to the effects of widely separated time constants within the same overall control loop. The details of the computations are described and their accuracy analyzed.

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1. Amara, R. C.
   "The Linear Least Squares Synthesis of Continuous and Sampled-Data Multivariable Systems," Technical Report No. 40, Stanford Electronics Laboratory, Stanford University, Stanford, California, July, 1958.
   A detailed derivation is presented of the equations which enable the determination of the controller elements which cause a minimization of the sum of the mean square errors between the set of actual outputs and a set of desired outputs subject to arbitrary auto- and cross-correlations. Both filter and control system cases are discussed with continuous, discrete, and hybrid (continuous-discrete) signals.

2. _____.
   "The Linear Least Squares Synthesis of Multivariable Control Systems," Transactions, AIEE, vol. 78, pt. II, May, 1959, pp. 115-119.
   A practical synthesis procedure is given for obtaining the controller elements of a multivariable control system with the minimization of the sum of the mean square output error. Continuous signals are used. This covers a small part of the author's Technical Report No. 40 listed as reference 1.

3. _____.
   "Application of Matrix Methods to the Linear Least Square Synthesis of Multivariable Systems," Journal of the Franklin Institute, vol. 268, no. 1, July, 1959, pp. 1-16.
   Multivariable filter and control systems are discussed with a criterion of minimization of the sum of the mean square output errors. Continuous time signals are used in the determination of the controller elements. This is part of the material covered in the author's Technical Report No. 40 listed as reference 1.

4. _____.
   "Hybrid Multivariable Control Systems," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 349-354.
   The hybrid (continuous-discrete) signal multivariable control system is analyzed so that the controller elements are obtained allowing a minimization of the sum of the mean square error. The optimization of the sampling period is also discussed. This is also a part of the author's Technical Report No. 40 listed as reference 1.

5. Aoki, Masanao.
   "Optimal and Suboptimal Control," Transactions, PGAC, IRE, vol. AC-5, no. 3, Aug., 1960, pp. 171-178.

Using dynamic programming techniques, the recurrence relation of the sequence of control forces required to cause optimization of a given performance index for a system is found. For various systems it is noted that the first control force in the sequence may cause near optimal conditions. Thus a suboptimal process which closely approximates the desired optimal policy is defined by using only the first control force.

6. Athanassiades, M. and D. Falb.
"Time Optimal Control for Plants with Numerator Dynamics," Transactions, PGAC, IRE, vol. AC-7, no. 4, July, 1962, pp. 47-50.
A time optimal control system whose transfer function contains zeros as well ae poles is discussed. The optimal control concept is extended (not for the first time in this paper) to require that the output reach a closed, bounded, convex, nonempty set in minimum time rather than one particular point in the set.

7. Auricoste, J. G.
"Applications of Digital Computers to Process Control," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 282-287.
The digital computer is shown to be a powerful tool in the optimization and control of multivariable systems. A description is given of the application of a digital computer in an ammonia generating plant. Mathematical equations and the computer flow diagrams for control are given.

8. Axelby, G. S., and R. H. Plath.
"Reduction of Control Loop Errors with a Prediction Computer," Proceedings, Computers in Control Systems Conference, Atlantic City, Oct., 1957, publ. by AIEE, May, 1958, pp. 100-109.
The authors discuss the simple two input-two output syetem with the criterion of maintaining an rms rate error less than a specified value. The system is described as a tracking servo with both signal and noise inputs. The use of a prediction computer is proposed and its effect upon the realizability, noise reduction, and tracking bandwidth ie given.

9. Becker, H.
"The Theory of Multiple Control," Regelungstechnik, vol. 7, no. 4, Jan., 1959, pp. 133-134.
A specific case using the approach developed by Kavanagh is inveetigated. Two multipole circuits are connected in tandem with the output of the second connected back to the input of the first. Realizability is the main problem and a formal solution is obtained.

10. Bellman, Richard E., I. Glicksberg, and O. A. Gross.
"Some Aspects of the Mathematical Theory of Control Processes," Rand Corporation Report R-313, 1958, pp. 168-170.
In the optimal inventory and stock control of a multidimensional system it is found that the distribution of the demand of the items has no cross-correlation.

11. Bertram, J. E., and P. E. Sarachik.
"On Optimal Computer Control," Automatic and Remote Control,
Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR,
1960, Butterworth's, London, 1961, vol. 1, pp. 419-422.
This paper presents the use of a digital computer to cause
a plant to go from a certain output which is expressed as an
arbitrary point in an m (the number of outputs) dimensional
space to a desired point in minimum time. The inputs are
expressed by an n dimensional input vector. Also included
is the optimal control by minimizing the control cost but
with the restriction of having a maximum input value.

12. Birkel, George, Jr.
"Hybrid Computers for Process Control," Transactions, AIEE,
vol. 79, no. 52, pt. I, Jan., 1961, pp. 726-734.
The high speeds and accuracies, over large numerical
ranges, available from the combined analog and digital or
hybrid computer cause it to be an excellent tool in solving
simple process control equations in industry.

13. Bishop, A. B., and H. R. Chope.
"Regression Techniques in Multivariate Adaptive Control
Systems," Transactions, PGAC, IRE, vol. AC-7, no. 2, Mar.,
1962, pp. 107-116.
The authors discuss an n input-one output multivariable
system based upon the drying of paper fiber. Here the
method of determination of which input will be varied when
a change in the output occurs is based upon the cost of making a unit change in each input. The input which causes the
minimum cost is the one which is varied.

14. Bohn, E. V.
"Stabilization of Linear Multivariable Feedback Control
Systems," Transactions, PGAC, IRE, vol. AC-5, no. 4, Sept.,
1960, pp. 321-327.
A new definition of noninteraction is proposed which states
that noninteraction is obtained when the error vector becomes
zero. A particular type of multivariable is discussed where
the compensating or controller matrix is the inverse or adjoint of the plant matrix. It is shown that the results obtained by using this method are the same as would have been
obtained had the methods proposed by Kavanagh been used.

15. _____.
"Design and Synthesis Methods for a Class of Multivariable

Feedback Control Systems Based on Single-variable Methods,"
Transactions, AIEE, vol. 81, pt. II, July, 1962, pp. 109-116.
    The author continues the discussion of the material presented in his 1960 article (reference 14) by giving a series of examples of the proposed stability criterion.

16. Boksenbom, Aaron S., and Richard Hood.
    "General Algebraic Method Applied to Control Analysis of Complex Engine Types," *NACA Technical Report 980*, Lewis Flight Propulsion Lab., Cleveland, Ohio, April, 1949.
    By using very complicated algebraic methods rather than more sophisticated matrix methods, the noninteraction requirements of a general nxn system are attained. A specific system concerning a turbojet engine is discussed and equations expressing the elements of the controller are found.

17. Brockett, R. W.
    "The Control of Linear Multivariable Systems," *SRC 7-A-62-7*, Systems Research Center, Case Institute of Technology, Cleveland, Ohio, Jan., 1962.
    The author presents an entirely new approach to the solving of Wiener-Hopf equations. Also presented is a definition of cohesion which describes the extent to which the system depends upon interaction for optimization. Several examples are given.

18. Brockett, R. W., and Mihajlo D. Mesarović.
    "Synthesis of Linear Multivariable Systems," Conference Paper, AIEE Winter General Meeting, New York, N.Y., Jan. 28-Feb. 2, 1962, AIEE Paper No. 62-74.
    The material presented is a step-by-step method for the solution to the Wiener-Hopf equation. A quick derivation of the procedure is given along with some applications. It is based upon the material presented in Brockett's paper, reference 17.

19. Brown, Gordon S., and Donald P. Campbell.
    Principles of Servomechanisms, Dynamics, and Synthesis of Closed-loop Control Systems. New York: John Wiley and Sons, Inc., 1948, pp. 295-306.
    The indicated pages contain a general discussion of the single input-single output system with noise or disturbances added. This is the simplest of the multivariable control systems but a very important special case.

20. Bryson, Arthur E., and Walter F. Denham.
    "Multivariable Terminal Control for Minimum Mean Square Deviation from a Nominal Path," *BR-1333*, Raytheon Company, Missile and Space Division, Bedford, Mass., Sept., 1961.
    The authors propose a new scheme where a feedback control matrix is calculated which when multiplied by the error vector between the desired and actual output state and then

given. The method proposed is best applicable to small nxn systems.

26. Chestnut, Harold, and Robert W. Mayer.
    Servomechanisms and Regulating System Design, vol. 1, 2nd ed. New York: John Wiley and Sons, Inc., 1959, pp. 452-486.
    The basic two input-two output system, one of the inputs being a disturbance or noise, is presented. Different multiloop and specific motor-generator problems are discussed.

27. Chien, K. L., E. I. Ergin, C. Ling, and A. Lee.
    "The Noninteracting Controller for a Steam-generating System," Control Engineering, vol. 5, no. 10, Oct., 1958, pp. 95-101.
    The authors present an excellent practical application of attaining noninteracting control of a multivariable plant. The paper deals with the interaction of steam-generating boilers with steam, feed water, and fuel rates as inputs and drum pressure and water level as outputs. By using an analog computer to simulate the plant, the attainment of noninteraction was verified when the calculated parameters were used in the controller.

28. Chinaev, P. I.
    "On the Principles of Synthesizing Automatic Systems with Many Controlled Quantities," Avtomatika i Telemekhanika, vol. 21, no. 6, June, 1960, pp. 761-771; Automation and Remote Control, vol. 21, no. 6, June, 1960, pp. 529-537.
    The basic mathematical relationships of an interrelated automatic system are derived using matrices. The controller elements are found through use of physical realizability, invariance, and optimality conditions.

29. Chiu, P. C., and C. R. Webb.
    "A Generalized Synthesis Method for Noninteracting Control of Multivariable Systems," Conference Paper, AIEE Summer General Meeting, Denver, Colo., June 17-22, 1962, AIEE Paper No. CP 62-1057.
    The authors present a new design procedure for obtaining noninteraction control from a general nxm plant. Consideration is made for the additional action of disturbances such as supply disturbances, interplant disturbances, and demand disturbances.

30. Courcoulas, J. H., and J. M. Ham.
    "Incremental Control Equations for Tandem Rolling Mills," Transactions, AIEE, vol. 75, pt. II, 1956, (January, 1957 Section), pp. 363-374.
    This paper gives an application of multivariable systems in a strip-rolling mill where five tandem stand rollers control many variables of the output steel. Inputs are thickness of input strip, roll force in the stand, roll torque in the stand, input strip tension, and output strip tension. Outputs are output strip thickness, thickness between stands,

linear speed of strip between stands, and speed of output strip.

31. Cruickshank, A. J. O.
"Matrix Formulation of Control System Equations," Matrix and Tensor Quarterly, Tensor Club of Great Britain, vol. 5, no. 3, 1955, pp. 76-83.
   The author presents one of the first papers using matrix operations in the mathematical description of 2x2 and 3x3 multidimensional systems.

32. Dickinson, M. M.
"Comparison of DDA and GP Equipment," Transactions, AIEE, vol. 79, no. 52, pt. I, Jan., 1961, pp. 706-708.
   The Digital Differential Analyzer and the General Purpose computer are compared by the solutions obtained from a particular multidimensional missile control problem. The results showed that the two systems are competitive in most respects.

33. Draper, Charles S., Walter Wrigley, and John Hovorka.
Inertial Guidance. New York: Pergamon Press, 1960.
   This book contains an excellent discussion of various systems in an inertial guidance complex. The equations and corresponding control system for some coordinate transformations are given.

34. Druzhinin, N. N.
"Equations for Control and Interlinkage of Multi-motor Electric Drive and Technology in Continuous Rolling Mills," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 132-137.
   The basic equations of performance of a set of stands in a steel rolling mill are given. The effect of the variation of certain parameters is described and an analog computer block diagram is presented for representation of a two-stand mill and control system.

35. Dudnkov, E. G.
"Determination of the Optimal Setting-up of Industrial Automatic Control Systems from Experimental Data," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 360-364.
   The methods of obtaining the elements of the plant matrix are presented. The conditions for autonomous control (similar to the discussion in the section on Noninteraction in this report) and for an unlinked system (one where a controller is connected to each input of an interrelated system and noninteraction obtained) are briefly given.

36. Eaton, J. H., and L. A. Zadeh.
"Optimal Pursuit Strategies in Discrete-State Probabilistic Systems," Transactions, ASME, Series D, Journal of Basic Engineering, vol. 84, no. 1, March, 1962, pp. 23-29.
 A cost function is established and equations derived for the arriving at a fixed state from a given initial state at minimum cost. The system is then extended to the solution of the moving target problem.

37. Eckmann, D. P. and I. Lefsowitz.
"Optimizing Control of a Chemical Process," Control Engineering, vol. 4, no. 9, Sept., 1957, pp. 197-202.
 The authors present a practical application of the optimization with respect to minimization of a cost function of a multivariable system. The paper describes the control of the batch hydrogenation of cottonseed oil with autoclave level, temperature, pressure, composition measurements, and product specifications as inputs. An analog computer was used to simulate the system and the path followed to obtain optimization for various states was found.

38. _____.
"Principles of Model Techniques in Optimizing Control," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 2, pp. 970-976.
 The basic principles of the use of optimization via a direct experimental method and via a model method in automatic control systems are presented. The paper is written in a general nature and no details of any specific form of optimization are given.

39. Egan, J. F., and G. J. Murphy.
"Optimum Synthesis of Multiport Systems Containing Modulators with Periodic Carriers," Transactions, AIEE, vol. 80, pt. II, Sept., 1961, pp. 203-208.
 A minimization synthesis of the sum of the squares of the output errors in a filter system is presented and is in a broader sense than that given by Hsieh and Leondes. It is a straightforward and easily followed approach where the synthesis of two cases is considered. These two cases are: (1) before the signal and carrier are joined, and (2) after the carrier and signal have been joined.

40. Ergin, E. I., and C. Ling.
"Development of a Non-interacting Controller for Boilers," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 347-353.
 This paper approaches the multivariable system directly by determining the plant parameters of a specific system, and then by using previously developed formulas determines the

controller elements needed for attaining noninteraction. A
simplification of the resultant controller elements is made
for ease of synthesis and the resulting responses showed
only a slight sacrifice resulting in some noninteraction.

41. Feder, Melvin S., and Richard Hood.
    "Analysis for Control Applications of Dynamic Characteris-
    tics of Turbojet Engine with Tail-pipe Burning," NACA Tech-
    nical Note 2183, Lewis Flight Propulsion Lab., Cleveland,
    Ohio, Sept., 1950.
       By use of thermodynamics the transfer functions for a run-
    ning turbojet engine were found and the results applied to
    the formulas developed by Boksenbom and Hood for the at-
    tainment of an overall noninteracting system. This paper
    describes one of the first practical applications of non-
    interaction control.

42. Fel'baum, A. A.
    "Automatic Optimalizer," Avtomatika i Telemekhanika, vol.
    19, no. 8, August, 1958, pp. 731-743; Automation and Remote
    Control, vol. 19, no. 8, August, 1958, pp. 718-728.
       The paper presents the problem of minimization by some
    machine of a function of several variables, $P(x_1, x_2, \ldots$
    $x_n)$ with the restriction $J_i(x_1, x_2, \ldots x_n) = 0, i = 1, 2,$
    $\ldots$ m. Several approaches to the solution are shown and
    circuits of the proposed machine models are given.

43. Fialko, G. M.
    "Use of Electrothermal Isodrome Devices for the Intercon-
    nected Control of Production Processes," Avtomatika i Tele-
    mekhanika, vol. 21, no. 6, June, 1960, pp. 812-820; Automa-
    tion and Remote Control, vol. 21, no. 6, June, 1960,
    pp. 568-574.
       The author proposes the use of electrothermal devices tied
    into the system and connected into the control loop, enabling
    autonomy conditions to be more easily obtained.

44. Fogel, L. J.
    "New Instrumentation Concepts for Manned Flight," Proceed-
    ings, IRE, vol. 47, no. 11, Nov., 1959, pp. 1978-1992.
       The paper presents a broad application of many multivari-
    able systems which are combined to give a visual aid in
    pilot control of high-speed vehicles.

45. Frazer, R. A., W. J. Duncan, and A. R. Collar.
    Elementary Matrices and Some Applications to Dynamics and
    Differential Equations. New York: The Macmillan Co., 1947.
       The authors give an excellent review of the theory of ma-
    trices and some examples of their use in describing multi-
    variable systems in airplanes, vibrating structures, and
    others.

46. Freeman, Herbert
    "Multiple Sampled-data Control Systems," Technical Report
    T-12/B, Electronics Research Lab., Columbia University,

New York, N. Y., Sept., 1955.
   The development of the equations for attainment of non-
interaction for sampled-data systems is given. An exten-
sion is made for the determination of the stability and
realizability criteria.

47. _____.
   "The Synthesis of Multipole Control Systems," Technical Re-
port T-15/B, Electronics Research Lab., Columbia University,
New York, N. Y., April, 1956.
   The attainment of any desired overall transfer function
from a given multivariable plant is described. The ground
work given by Boksenbom and Hood is covered and extended in
a step-by-step synthesis procedure of the controller which
gives "independent output restoration" or noninteraction.
The stability and realizability conditions for a general
sampled-data system are found and discussed in detail.

48. _____.
   "A Synthesis Method for Multipole Control Systems," Trans-
actions, AIEE, vol. 76, pt. II, March, 1957, pp. 28-31.
   The article presents a method for synthesis of a control-
ler which with a given multivariable plant and knowledge of
the desired overall transfer function will cause system
noninteraction or any other specified performance.

49. _____.
   "Stability and Physical Realizability Considerations in the
Synthesis of Multipole Control Systems," Transactions, AIEE,
vol. 77, pt. II, March, 1958, pp. 1-15.
   The synthesis procedure for obtaining a desired overall
transfer function of a system with a given stable multivar-
iable plant is outlined. A technique is then proposed which
adds constraints on the overall response and enables a
stable design to be achieved when certain of the plant's
elements are unstable.

50. Friedland, Bernard.
   "A Technique for the Analysis of Time-Varying Sampled-data
Systems," Transactions, AIEE, vol. 75, pt. II, Jan., 1957,
pp. 407-414.
   This paper gives the basic ideas behind sampled-data sys-
tems and how by matrix methods the control systems with feed-
back can be analyzed. The application of other components
to cause an unstable system to become stable is discussed.
The relationship of this system to multivariable systems is
that the input is sampled and each sample can be considered
as a different input.

51. _____.
   "Theory of Time-Varying Sampled-Data Systems," Technical Re-
port T-19/B, Electronics Research Lab., Columbia University,
New York, N. Y., April, 1957.
   The author presents the analysis of linear time-varying

sampled-data systems using infinite, triangular transmission matrices $\underline{H}$ and infinite signal vectors $\underline{x}$ such that the input-output relationship has the form $\underline{y} = \underline{Hx}$. Some transmission matrices of common components are given. With the use of infinite matrix theory the stability and steady-state behavior are described.

52. _____.
"Time-Varying Analysis of a Guidance System," _Transactions_, AIEE, vol. 77, pt. II, May, 1958, pp. 75-81.
This paper contains an application of the author's _Technical Report_ T-19/B to the closed loop control system problem of steering a missile toward a target, both of which travel in a common plane.

53. _____.
"The Structure of Optimum Control Systems," _Transactions_, ASME, Series D, Journal of Basic Engineering, vol. 84, no. 1, March, 1962, pp. 1-12.
This article presents an excellent explanation of the optimal control concept and the various ways minimization can occur.

54. Gibson, J. E.
"Self-optimizing or Adaptive Control Systems," _Automatic and Remote Control_, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 2, pp. 586-595.
The multidimensional self-optimalizing system contains some type of _F_igure of _M_erit which is expressed so that a computer can calculate its value and vary the system so that the optimal (minimum or maximum generally) value is obtained. In some systems the extremum cannot be obtained so additional restrictions must be given.

55. Gibson, John E., and Eugene S. McVey.
"Multidimensional Adaptive Control," _Proceedings_, National Electronics Conference, 228 N. LaSalle St., Chicago 1, Illinois, vol. 15, 1959, pp. 17-26.
The adaptive multidimensional control system is defined and the optimization by the method of steepest descent to reach the desired operating point explained. Laboratory model tests of a specific system are compared to the results from an idealized digital computer model.

56. Gille, J. C., M. J. Pelegrin, and P. Decaulne.
Feedback Control Systems, Analysis, Synthesis, and Design. New York: McGraw-Hill Book Co., 1959, pp. 364-373.
The authors describe the general multivariable system and present possible methods of expressing it mathematically. The reader is cautioned about several printing errors in the book.

57. Glushkov, V. M., L. N. Dashevskii, and A. I. Nikitin.
"Use of Electronic Computers for Bessemer Process Automa-
tion," Avtomatika i Telemekhanika, vol. 21, no. 6, June,
1960, pp. 877-883; Automation and Remote Control, vol. 21,
no. 6, June, 1960, pp. 617-621.
   The determination of the time for ending the "blow" of a
Bessemer converter is dependent upon the quick analysis of
a large number of variables. This particular multivariable
system is discussed in detail.

58. Goldberg, Edwin M., and Harry D. Felsenthal, Jr.
"Estimating the Dynamic Characteristics of Multi-coupled
Physical System," Report T-138, Instrumentation Lab., Massa-
chusetts Institute of Technology, Cambridge, Mass., May,
1957.
   Using the Cauchy Mapping Theorem rapid estimations of the
limiting values of the parameters of a multivariable system
are obtained. Two examples are given, one of which is the
fourth-order characteristic determinant of a helicopter in
hovering flight.

59. Golomb, Michael, and Eugene Usdin, Jr.
"A Theory of Multidimensional Servo Systems," Journal of the
Franklin Institute, vol. 253, no. 1, Jan., 1952, pp. 29-57.
   The paper is concerned with n servo systems which have
their error signals coupled. General performance criteria
are established by use of a defined coefficient matrix de-
pendent upon the quality of the outputs. Through the use of
stability criteria, conditions for combining stable and un-
stable servos into an overall stable system are listed.
Optimization of the output error in an rms sense is
discussed.

60. Goodman, T. P.
"Determination of the Characteristics of Multi-input and
Nonlinear Systems from Normal Operating Records," Transac-
tions, ASME, vol. 79, no. 3, April, 1957, pp. 567-575.
   The author proposes the use of statistical auto-correla-
tion and cross-correlation functions of the multivariable
plant's inputs to find the impulse responses or weighting
functions. Deconvolution by means of an electronic delay-
line synthesizer enables both linear and nonlinear systems
to be described.

61. Gordon, M. M.
"Complex Automation of an Open-Hearth Furnace with Gas Heat-
ing," Avtomatika i Telemekhanika, vol. 21, no. 6, June, 1960,
pp. 849-856; Automation and Remote Control, vol. 21, no. 6,
June, 1960, pp. 596-601.
   The paper contains a discussion of the automatic control
of an open-hearth furnace thermal system with eight control-
ling loops. Methods of measuring the several controlling
parameters are considered.

62. Gould, L. A., and W. Kipiniak.
"Dynamic Optimization and Control of a Stirred-Tank Chem-
ical Reactor," Transactions, AIEE, vol. 79, no. 52, pt. II,
Jan., 1961, pp. 734-746.
    The average rate of return on investment is used as the
function to be optimized (maximized in this case) for a
chemical plant with certain restrictions, such as equipment
limitations and safety considerations.

63. Grabbe, Eugene M.
"Flight Control and the Digital Computer," Control Engi-
neering, vol. 2, no. 10, Oct., 1955, pp. 64-68.
    The use of Digitac, Digital Tactical Automatic Control,
for automatic aircraft navigation and weapons control was
developed at Hughes Aircraft. The system is based upon
position determination in a hyperbolic coordinate system.

64. _____.
"Digital Computer Control Systems--An Annotated Bibliog-
raphy," Automatic and Remote Control, Proceedings of the
First International Congress of the International Federation
of Automatic Control, Moscow, USSR, 1960, Butterworth's,
London, 1961, vol. 2, pp. 1074-1087.
    A list and comments on over 200 books and articles con-
cerned with control systems incorporating the use of a
digital computer are given and listed by year of publication.

65. Grabbe, Eugene M., Simon Ramo, and Dean E. Wooldridge.
Handbook of Automation, Computation, and Control, vol. 3,
Chapters 10 and 13. New York: John Wiley and Sons, Inc.,
1961.
    Chapter 10 gives the basic ideas of multivariable systems
and discusses the 2x2 system. Chapter 13 presents the over-
all picture of multivariable systems and how computers can
be used in the controlling and optimization of such systems.

66. Halijak, Charles A.
"Linear Operator Analysis of Carrier-Frequency Servomechan-
isms," PhD Thesis, University of Wisconsin, 1956.
    A general servomechanism which has three inputs and one out-
put is discussed. The analysis, the shortcomings, and a new
theory of calculating the performance of such systems are
given in detail.

67. _____.
"A Two-Dimensional Servo Model for Seeker Systems," unpub-
lished report, Bendix Aviation Corporation, Systems Division,
Ann Arbor, Mich., Oct., 1960.
    A special multivariable system is considered wherein a
servo model is developed for the purpose of moving one co-
ordinate system into another with special control equations.
Stationary and constant velocity target systems are con-
sidered with implications into three-dimensional type systems.

68. Halijak, Charles A., and Thomas J. Higgins.
    "Some Aspects of A-C Control Systems Analysis," District
    Paper, AIEE Great Lakes District Meeting, Milwaukee, Wisc.,
    April 27-29, 1960, DP 60-686.
      The steady-state analysis of a-c control systems using
    trigonometry forms and the derivation of the data spectrum
    are given. The harmonic analysis of a three-input, one-
    output system is presented.

69. Han, K. W., and G. J. Thaler.
    "Multi-parameter Self Adaptation Using Auxiliary Models,"
    Conference Paper, AIEE Winter General Meeting, New York,
    N.Y., Jan. 28-Feb. 2, 1962, CP 62-76.
      The design techniques for obtaining the optimum response
    of a multiparameter system based upon minimizing the devia-
    tion of the system response from a model response are con-
    sidered. The proposed methods do not tend to home on false
    minima.

70. Harris, H.
    "Theory of Mutually Interacting Servomechanisms," unpub-
    lished report, Sperry Gyroscope Co., April, 1945.
      One of the first papers on multivariable systems.

71. Hayes, W. S., and D. L. Pitman.
    "Guidance and Control of Mobile-launched Ballistic Missiles,"
    Paper No. 1219 (Revised), Missiles and Space Systems Engi-
    neering, Douglas Aircraft Company, Inc., Santa Monica Divi-
    sion, Santa Monica, Calif., Sept. 28, 1961, pp. 54-58.
      These lecture notes contain a brief discussion on a 2x2
    attitude error computer for the programming of azimuth
    trajectories for ballistic missiles.

72. Heising, C. R.
    "Evaluation of a Turbojet Control on an Engine Simulator,"
    Proceedings, Computers in Control Systems Conference, At-
    lantic City, Oct., 1957; publ. by AIEE, New York, May,
    1958, pp. 54-57.
      A simulator has been developed which imitates a turbojet
    engine at all altitudes and flight conditions for use in
    control development work.

73. Helson, Henry, and David B. Lowdenslager.
    "Prediction Theory and Fourier Series in Several Variables,"
    Acta Mathematica, vol. 99, 1958, pp. 165-202.
      An extensive mathematical development of Fourier series
    and prediction theory for functions of several variables is
    presented.

74. Ho, Yu-Chi
    "Solution Space Approach to Optimal Control Problems,"
    Transactions, ASME, Series D, Journal of Basic Engineering,
    vol. 83, no. 1, March, 1961, pp. 53-58.

The paper mentions the complexity of and gives some basic approaches for optimal control. The time-varying system is introduced.

75. _____.
"A Successive Approximation Technique for Optimal Control Systems Subject to Input Saturation," *Transactions*, ASME, Series D, Journal of Basic Engineering, vol. 84, no. 1, March, 1962, pp. 33-40.
    The author considers the basic optimal control for minimizing the response time of a system when each input is limited to be equal to or less than a certain value.

76. _____.
"A Computational Technique for Optimal Control Problems with State Variable Constraint," *Rand* *Corporation* *Report* *RM-3047-NASA*, *NASA* *Publication* *N-107, 854*.

77. Horowitz, Isaac M.
"Synthesis of Linear, Multivariable Feedback Control Systems," *Transactions*, PGAC, IRE, vol. AC-5, no. 2, June, 1960, pp. 94-105.
    An excellent discussion on the reduction of the system sensitivity to parameter variation is given. The paper then continues with a discussion of the design of the controller in the higher than significant system response frequency region.

78. Hsieh, H. C.
"On the Optimum Synthesis of Random Sampling Multipole Filters with Stationary Inputs," *Transactions*, AIEE, vol. 80, pt. II, Nov., 1961, pp. 239-247.
    The optimum filters for systems with nonperiodic input samples are given. The detail procedures for obtaining this optimum case are discussed and two simple examples described.

79. Hsieh, H. C., and C. T. Leondes.
"On the Optimum Synthesis of Multipole Control Systems in the Wiener Sense," *Transactions*, PGAC, IRE, vol. AC-4, no. 2, Nov., 1959, pp. 16-29.
    The authors present the synthesis solution of the multivariable system with the minimization of the sum of the mean square errors between the actual output and the desired output.

80. _____.
"Techniques for the Optimum Synthesis of Multipole Control Systems with Random Processes as Inputs," *Transactions*, PGAC, IRE, vol. AC-4, no. 3, Dec., 1959, pp. 212-231.
    An approach different from Amara's (references 1-4) is used in the synthesis of the control for minimization of the sum of the mean square output errors with the inputs containing noise.

81. _____.
   "On the Optimum Synthesis of Sampled-Data Multipole Filters
   with Random and Nonrandom Inputs," Transactions, PGAC, IRE,
   vol. AC-5, no. 3, Aug., 1960, pp. 193-208.
   The earlier work presented in the authors' two previous
   papers is expanded to the sampled-data case. The synthesis
   procedures are developed so as to specify the weight func-
   tions of the filters "so that the system error has a zero
   ensemble mean and the system ensemble mean square error is
   minimum".

82. Jeffrey, A.
   "The Stability of Interacting Control Systems," Automatic
   and Remote Control, Proceedings of the First International
   Congress of the International Federation of Automatic Con-
   trol, Moscow, USSR, 1960, Butterworth's, London, 1961,
   vol. 1, pp. 126-131.
   A control system with interrelated error signals is con-
   sidered with regard to stability. The problem is discussed
   and expressed by means of exponential polynomials.

83. Joseph, Peter D., and Julius T. Tou.
   "On Linear Control Theory," Transactions, AIEE, vol. 80,
   pt. II, Sept., 1961, pp. 193-196.
   A sampled-data system containing both a controller and an
   estimator is described using a vector difference equation.
   The authors prove that optimizing the controller and the
   estimator separately leads to the optimal overall system
   performance.

84. Kalman, R. E.
   "A New Approach to Linear Filtering and Prediction Problems,"
   Transactions, ASME, Series D, Journal of Basic Engineering,
   vol. 82, no. 1, March, 1960, pp. 35-45.
   Use is made in this article of the Bode-Shannon repre-
   sentation and the "state-transition" method of analysis of
   the classical filtering and prediction problem.

85. _____.
   "On the General Theory of Control Systems," Automatic and
   Remote Control, Proceedings of the First International Con-
   gress of the International Federation of Automatic Control,
   Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1,
   pp. 481-492.
   The author presents a purely theoretical discussion of
   control which is comparable to Shannon's work in information
   theory. Concepts of controllability and observability are
   introduced to describe and classify certain systems. From
   the principle of duality the Weiner filtering problem is
   shown to be the natural dual of the optimal regulation
   problem.

86. Kashmar, C. M., E. L. Peterson, and F. X. Remond.
    "A General Approach to the Numerical Solution of Multi-
    dimensional Nonlinear Boundary-valued Variational Problem,"
    Report No. R 60TMP-27, Technical Military Planning Operation,
    General Electric Co., Santa Barbara, Calif., April 15, 1960.
       The application and limitations of using dynamic program-
    ming in the solving of multidimensional nonlinear boundary-
    valued variational problems are considered. Solutions are
    presented via numerical methods applicable on available
    computers.

87. Kavanagh, R. J.
    "Linear Multivariable Control Systems," PhD Thesis, Uni-
    versity of Toronto, Toronto, Ontario, Canada, 1956.
       Through the application of matrix methods the analysis,
    synthesis and realizability of multivariable systems is
    discussed in basic terms.

88. _____.
    "Application of Matrix Methods to Multivariable Control
    Systems," Journal of the Franklin Institute, vol. 262,
    no. 5, Nov., 1956, pp. 349-367.
       The author presents the analysis and synthesis of the
    general multivariable system. The realizability criterion
    is developed through the use of independency of the rows,
    columns, or ranks of the entire control system matrix.

89. _____.
    "Noninteracting Controls in Linear Multivariable Systems,"
    Transactions, AIEE, vol. 76, pt. II, May, 1957, pp. 95-99.
       The paper gives the basic matrix operations of obtaining
    the synthesis of the controller elements of a multivariable
    system which is designed for a specific overall transfer
    function and for the special case of noninteraction.

90. _____.
    "A Control System Interpretation of Certain Properties of
    Matrices," Matrix and Tensor Quarterly, Tensor Club of Great
    Britain, vol. 7, no. 4, Dec., 1957, pp. 32-38.
       The controller is synthesized by matrix methods and the
    results of using various types of matrices and operations
    are investigated.

91. _____.
    "Multivariable Control Systems Synthesis," Transactions,
    AIEE, vol. 77, pt. II, Nov., 1958, pp. 425-429.
       This paper contains possible modifications of the multi-
    variable system by specifying the forms of the controller
    and/or the feedback matrix. These matrices are shown to be
    in terms of the plant and the desired overall transfer
    matrices.

92. Kinnen, E., and D. S. Liu.
"Linear Multivariable Control System Design with Root Loci,"
Transactions, AIEE, vol. 81, pt. II, May, 1962, pp. 41-44.
   A general approach procedure on the use of root loci in a
multivariable system is given and is followed by a 2x2 ex-
ample. The proposed method is quite detailed and seems best
applied to small nxn systems.

93. Kipiniak, Walerian
Dynamic Optimization and Control - A Variational Approach.
New York: John Wiley and Sons, Inc., 1961.
   The optimization problems considered are basically that
of finding the control law and thus the synthesis of a con-
troller which makes the system tend toward an expected value
at some future finite or infinite time.

94. Kirchmayer, L. K.
"Differential Analyzer Aids Design of Electronic Utility
Automatic Dispatching System," Proceedings, Computers in
Control Systems Conference, Atlantic City, Oct., 1957;
publ. by AIEE, New York, May, 1958, pp. 60-67.
   A power distribution system is discussed whose turbine
generators are controlled by an analog computer so as to
minimize the fuel expenditures. The paper then continues by
describing the use of a DDA in the design of an automatic
dispatching system.

95. Klein, Georges.
"Oscillations de Couplage dans un Servomecanisme de Recopie
a Deux Variables," Annales des Telecommunications, vol. 8,
no. 7, July, 1953, pp. 254-259.
   Through the stability study of two coupled servos, the root
position needs are found and the idea of a global phase mar-
gin is proposed.

96. Kraus, Gunther.
"Ein Beitrag zur Bemessung von Automatischen Pilotreglern,"
Archiv der Elektrischen Übertragung, vol. 19, March, 1956,
pp. 175-187.
   The paper presents the effect of the relationships of the
number of pilots and the number and grouping of the equaliz-
ers in automatic level control systems.

97. Kron, G.
"Tensorial Analysis of Control Systems," Journal of Applied
Mechanics, vol. 15, June, 1948, pp. 107-124.
   The author considers a set of linear algebraic equations
which could represent the resultant equations of intercon-
nected servo systems. The effect of the interrelations is
described mathematically.

98. Kukhtenko, A. I.
"The Dynamics of Devices Which Imitate Living Organisms,"

Automatic and Remote Control, Proceedings of the First In-
ternational Congress of the International Federation of
Automatic Control, Moscow, USSR, 1960, Butterworth's, Lon-
don, 1961, vol. 2, pp. 658-663.
   The mathematical equations describing various living
organisms are given and possible control system simulations
based upon these equations are presented.

99. Kulebakin, V. S.
   "The Theory of Invariance of Regulating and Control Sys-
tems," _Automatic and Remote Control_, Proceedings of the
First International Congress of the International Federa-
tion of Automatic Control, Moscow, USSR, 1960, Butter-
worth's, London, 1961, vol. 1, pp. 106-116.
   The conditions for invariance of a specific three-variable
problem with two inputs and one system parameter capable of
being varied are presented and discussed.

100. Kuroda, Yoshiteru.
   "On the Optimization of Multi-Variable Control Systems,"
Proceedings, 9th Japan National Congress for Applied
Mechanics, 1959, pp. 327-330.
   A minimization of the sum of the mean square output errors
is carried out for three sizes of multivariable systems.

101. Kurzweil, Fred, Jr.
   "Dynamic Synthesis of Higher-Order, Optimum Saturating Sys-
tems," _Transactions_, ASME, Series D, Journal of Basic Engi-
neering, vol. 83, no. 1, March, 1961, pp. 45-52.
   The author presents a new technique in the time optimiza-
tion of higher-order saturating systems. It recalculates
the optimum forcing function point by point and thus
reaches the optimum trajectory for the system. The fact
that this type of synthesis of the forcing function obtains
the optimum trajectory was proven experimentally on a digi-
tal computer.

102. Lange, A. S.
   "Automatic Control of Three-dimensional Quantities," _Trans-
actions_, PGAC, IRE, vol. AC-4, no. 1, May, 1959, pp. 21-30;
vol. AC-5, no. 1, Jan., 1960, pp. 38-57; vol. AC-5, no. 2,
June, 1960, pp. 106-117.
   The mathematical relations for various subsystems of space
guidance and control systems are given. Basic gyroscope
and accelerometer systems are described in detail.

103. Lee, E. Bruce.
   "Mathematical Aspects of the Synthesis of Linear Minimum
Response Time Controllers," _Transactions_, PGAC, IRE, vol.
AC-5, no. 4, Sept., 1960, pp. 283-289.
   A systematic procedure is presented for finding the input
forcing function needed for time optimal, i.e., minimum re-
sponse time, control of a multivariable system.

104. _____.
"Design of Optimum Multivariable Control Systems," Trans-
actions, ASME, Series D, Journal of Basic Engineering,
vol. 83, no. 1, March, 1961, pp. 85-90.
   A difference equation is used to express a multivariable
system and its solution is formulated. A loss function is
defined which is to be minimized subject to a boundary con-
dition which is entered into the equation through use of
the Lagrange multiplier. Optimization of the system and the
synthesis of the controller for minimum energy and minimum
time are discussed. The minimization of the response time
subject to an energy constraint is also given.

105. Lees, Sidney.
"Interpreting Dynamic Measurements of Physical Systems,"
Transactions, ASME, vol. 80, no. 5, May, 1958, pp. 833-857.
   This is an excellent paper which discusses many means of
obtaining the characteristics of multivariable systems.
Both linear and nonlinear physical systems have been classi-
fied into six categories relating possible methods of ap-
proach: derivatives method, steady-state frequency method,
standard curves method, random input method, indirect fre-
quency method, and combination method.

106. Lees, S., Harry D. Felsenthal, Jr., and Edwin M. Goldberg.
"Estimating the Roots of the Characteristic Determinant for
Multicoupled Systems," Transactions, ASME, Series D, Jour-
nal of Basic Engineering, vol. 81, no. 1, March, 1960,
pp. 85-90.
   A procedure is developed which determines the effect on
the entire system of the variation of certain elements.
Using Cauchy's residue theorem the undamped natural fre-
quency, damping ratio, and the real parts of the roots of
the characteristic determinant are bounded.

107. Lefkowitz, Irving, and Donald P. Eckman.
"Application and Analysis of a Computer Control System,"
Transactions, ASME, Series D, Journal of Basic Engineering,
vol. 81, no. 4, Dec., 1959, pp. 569-577.
   A digital computer is used to control a multivariable
system for minimizing the cost of operation for a specified
performance.

108. Leondes, C. T., and M. Nothman.
"Real-Time Hybrid Computers for Control Systems," Proceed-
ings, Computers in Control Systems Conference, Atlantic
City, Oct., 1957, publ. by AIEE, May, 1958, pp. 88-94.
   This paper presents the main advantages of the use of
hybrid, both continuous and sampled-data signal, computers
in the control of multivariable systems.

109. Lerner, A. Ya.
"The Use of Self-adjusting Automatic Control Systems,"

Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 226-230.
   The general idea behind the self-adjusting system is discussed and several broad examples given. Nothing specific is shown.

110. Letov, A. M.
   "A Conditionally Stable Control System," Proceedings, Computers in Control Systems Conference, Atlantic City, Oct., 1957; publ. by AIEE, May, 1958, pp. 29-34.
   The author uses a computer to synthesize the controller elements of a regulator system obtaining overall system stability.

111. Locke, Arthur S.
   Guidance. Princeton, N.J.: D. Van Nostrand Co., 1955.
   This book contains a basic yet complete coverage of many different forms of missile guidance systems.

112. Long, M. V., and E. G. Holzman.
   "Approaching the Control Problem of the Automatic Chemical or Petroleum Plant," Transactions, ASME, vol. 75, no. 10, Oct., 1953, pp. 1373-1381.
   A comparison is made between two automatic control systems which were derived from different points of view. The advantages and disadvantages of the required computer elements needed in both systems are discussed.

113. Lur'e, A. I.
   "Some Non-Linear Problems in the Theory of Automatic Control," Gos. Isdat. Tekh. Teor. Lit., 1951, USSR; translated from the Russian, Her Majesty's Stationery Office, Code No. 70-760, 1957.
   The author develops a canonical form of a possible multivariable system and then discusses the solutions of the problems of stability, oscillations, and threshold stability for this system. Lyapunov functions are constructed and many examples are given of nonlinear problems.

114. Margolis, M., and C. T. Leondes.
   "On the Theory of Adaptive Control Systems; the Learning Model Approach," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 2, pp. 556-563.
   A specific adaptive multivariable system is discussed which contains unspecified elements which are varied until a defined error function is minimized.

115. Markus, L., and E. B. Lee.
   "On the Existence of Optimal Controls," Transactions, ASME,

Series D, Journal of Basic Engineering, vol. 84, no. 1,
March, 1962, pp. 13-22.
  The results given in this paper will enable the system
designer to test if optimal control of certain types of
systems exists.

116. McCausland, Ian.
     "Optimal Control of Multiactuator Systems," Transactions,
     AIEE, vol. 78, pt. II, no. 42, May, 1959, pp. 67-73.
       Some relevant portions of economic theory are applied to
     the multivariable problem. A penalty function is defined
     and criteria are established to place the operating point
     in a position which minimizes the penalty.

117. McCracken, L. G.
     "An Extension of Wiener Theory to Multivariable Controls,"
     Proceedings, IRE International Convention, 1961.
       The minimization of the sum of the squared output errors
     of a given multivariable system is performed. The result-
     ant expression is a Wiener-Hopf equation which is solved by
     means of Fourier transforms. Thus the values of the con-
     troller elements are obtained.

118. Meerov, M. V.
     "Some Peculiarities of the Structure of Multipole Control
     Systems," Automatic and Remote Control, Proceedings of the
     First International Congress of the International Federa-
     tion of Automatic Control, Moscow, USSR, 1960, Butter-
     worth's, London, 1961, vol. 1, pp. 85-89.
       This paper is concerned with a cascaded multivariable
     system where different values of gain and different multi-
     loop configurations are shown to affect the amount of out-
     put interaction.

119. Meierov, P.
     "On the Control of Multivariable Systems," Second Symposium
     on Automatic Control, Academy of Science, USSR.

120. Merchav, S. J.
     "Compatibility of a Two-Degree-of-Freedom System with a Set
     of Independent Specifications," Transactions, PGAC, IRE,
     vol. AC-7, no. 1, Jan., 1962, pp. 67-72.
       A two input-one output system is discussed where the
     transfer function is related to a one input-one output sys-
     tem with noise entering at the input to the plant. A dis-
     cussion is given to show how to obtain a prescribed set of
     transfer functions for compatibility. Two examples are
     given.

121. Merriam, C. W., III.
     "An Optimization Theory for Feedback Control System Design,"
     Information and Control, vol. 3, no. 1, March, 1960,
     pp. 32-59.

The author presents a general approach to the minimization of the error between the actual outputs and the desired outputs. The theory presented is applicable to certain nonlinear systems and also to systems that experience saturation of the input signals.

122. Mesarovic, Mihajlo D.
The Control of Multivariable Systems. New York: John Wiley and Sons, Inc., 1960.
Two canonical forms are introduced to describe the many different multivariable systems. A definition and measure of interaction are given, enabling a comparison of different systems. The possibility of the uniqueness of the synthesis of a system is found to be related to two defined characteristic numbers.

123. _____.
"Dynamic Response of Large Complex Systems," Journal of the Franklin Institute, vol. 269, no. 4, April, 1960, pp. 274-298.
Two canonical forms are presented and the multivariable systems are classified by these forms. Any possible subsystem is noted and its interrelations observed so as to simplify the overall system. The precise behavior would still be described deterministically.

124. _____.
"Control of Multivariable Systems," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 100-105.
The author stresses the importance of the multivariable system as a basis for a new control theory and not just as an extension or generalization of single variable control theory.

125. _____.
"On the Existence and Uniqueness of the Optimal Multivariable System Synthesis," Transactions, PGAC, IRE, vol. AC-5, no. 3, August, 1960, pp. 166-170.
The binary structure of the multivariable system with environmental and behavioral inputs is described. When the inputs and outputs are specified, a unique synthesis of the system can be obtained if the conditions listed in the paper are met.

126. _____.
"Black-Box Outputs Interaction," unpublished report, Case Institute of Technology, Cleveland, Ohio, 1960.
The author discusses the various methods of obtaining a measure of possible interactions inside a black box.

127. Mesarovic, M. D., and G. J. Coviello.
"Optimal Control of Multivariable Systems," Conference

Paper, AIEE Summer General Meeting, Denver, Colo., June 21, 1962, AIEE Paper No. CP 62-1280.
    The concept of the necessity of interaction if optimal control is to be achieved is brought out. A novel approach in the control of large systems is suggested which is based upon the theory of independence of the system's dimensions.

128. Milsum, J. H.
    "Statistical Optimization of Regulators Employing a Binary Error Criterion," Transactions, ASME, Series D, Journal of Basic Engineering, vol. 81, no. 2, June, 1959, pp. 254-262.
    The use of an analog computer enables proper instrumentation of a regulator which is subject to both reference and statistical load disturbances. The error in the output is expressed as a penalty and a criterion is established so that controller minimizes the penalty.

129. _____.
    "Adaptive Control of Processes by an Economic Criterion," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 231-240.
    A system penalty function based upon the cost of operation is formed in the controller and an adaptive circuit continuously computes the derivative of this function. The controller then drives the system so as to have a minimum penalty--when the derivative is zero.

130. Mironov, V. D.
    "Development of Apparatus for the Automatic Control of Boiler Aggregates," Avtomatika i Telemekhanika, vol. 21, no. 6, June, 1960, pp. 833-839; Automation and Remote Control, vol. 21, no. 6, June, 1960, pp. 584-588.
    The methods of using electronic sampled-data systems in control systems are discussed.

131. Mishkin, Eli, and Ludwig Braun.
    Adaptive Control Systems. New York: McGraw-Hill Book Co., 1961, pp. 104-105.
    Very cursory comments on multivariable systems.

132. Mitchell, D. S., and C. R. Webb.
    "A Study of Interaction in a Multi-Loop Control System," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 142-152.
    The authors present the requirements for noninteraction of a 2x2 system and verify the results experimentally with the use of an analog computer.

133. Miyakawa, Hiroshi.
    "Sampling Theorem of Stationary Stochastic Variables in

Multimensional Space," *Journal*, Institution of Electrical Communication Engineers of Japan (in Japanese), vol. 42, no. 4, 1959, pp. 421-427 (abstract in English UDC 519.2).
   The author presents a new approach to the control process by using the method of generalized harmonic analysis in the n dimensional space of an nxn system. An extended sampling theorem is given for this approach and the effectiveness of various sampling lattices are given.

134. Moore, Roger A., and Dennis F. Meronek.
   "A Digital Computer Program for a Generalized Inertial Guidance System Error Analysis," Report TR-59-0000-00647, Space Technology Laboratories, Inc., Los Angeles, California, March 30, 1959.
   The equations of the errors in the guidance instruments causing a missile to miss its target are considered. A digital computer is used in the control to determine the required forcing functions to reduce the errors detected.

135. Murrish, Charles H.
   "Optimization in a Mean-Square-Error Sense of a Two-Dimensional Sampled-Data System," PhD Thesis, University of Wisconsin, Madison, Wisconsin, 1960.
   This thesis presents the problem and analysis of the multivariable sampled-data system with noninteracting outputs to obtain an explicit solution for the nonconstrained optimizing transfer function in the minimum mean square error sense.

136. Murrish, Charles H., and Thomas J. Higgins.
   "Optimization in a Mean Square Sense of a Two-Dimensional Sampled-Data System," District Paper, AIEE Great Lakes District Meeting, Minneapolis, Minn., April 19-21, 1961, AIEE Paper No. DP 61-592.
   The basic analysis and synthesis of the noninteracting system and the minimum mean square output error system are presented.

137. Narendra, Kumpati S., and Roger M. Goldwyn.
   "Application of the Matrix Methods to the Optimum Synthesis of Multivariable Systems Subject to Constraints," Conference Paper, AIEE Winter General Meeting, New York, N. Y., Jan. 28-Feb. 2, 1962, AIEE Paper No. 62-87.
   A new and concise notation is proposed and with this notation the inputs are convolved with the system weighting function. The resulting function is then correlated with the outputs and this equation is transformed into the complex frequency domain. The final equation leads to a form of the Wiener-Hopf equation.

138. Nelson, W. L.
   "Optimal Control Methods for On-Off Sampling Systems," *Transactions*, ASME, Series D, Journal of Basic Engineering,

vol. 84, no. 1, March, 1962, pp. 91-100.
This paper considers the optimal on-off regulation (minimum response time) of a plant whose state is known only at periodic instants of time.

139. Newman, David B.
"The Analysis of Cross-Coupling Effects on the Stability of Two-Dimensional, Orthogonal Feedback Control Systems," Transactions, PGAC, IRE, vol. AC-5, no. 4, Sept., 1960, pp. 314-320.
A detailed account of a symmetric two-dimensional cross-coupled system is given. Formulas are developed and Nyquist stability conditions are discussed. Two examples are given, one of which shows how two unstable systems can be combined to obtain a stable overall system.

140. Nishida, Fujio.
"Synthesis of Sampled-Data Controllers in Multivariable Control Systems," Journal, Institute of Electrical Engineers of Japan, vol. 80, no. 856, Jan., 1960, pp. 75-82 (in Japanese).
The author gives the necessary and sufficient conditions for the noninteracting and finite settling specifications for an open-loop multivariable system. Several examples are given, one of which shows that with an auxiliary controller the minimum settling time can be reduced by one sampling period.

141. _____.
"Synthesis of Multi-Variable Control Systems by Means of Sampled-Data Compensations," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 328-332.
A discussion is given on the noninteraction and the minimum mean square output error criteria for a sampled-data multivariable system. The concept of reducing by one sampling period of the settling time through use of an auxiliary controller is presented in detail.

142. Novik, David.
"Some Linear Dynamics of Two-Spool Turbojet Engines," NACA TN 3274, National Advisory Committee for Aeronautics, Washington, D.C., June, 1956.
The author analytically derives and experimentally corroborates the transfer functions of changes of inlet temperature and exhaust nozzle area of a turbojet engine.

143. Novoselov, V. S.
"Investigation of Probability Stability Using Example of Automatic Control of Aircraft Course," Avtomatika i Telemekhanika, vol. 21, no. 6, June, 1960, pp. 666-673; Automation and Remote Control, vol. 21, no. 6, June, 1960,

pp. 462-466.
    By defining a varying degree of stability, the author presents an investigation concerning the probabilistic nature of a constant-speed servo system.

144. Orcutt, J. C., and D. E. Lamb.
    "Stability of a Fixed Bed Catalytic Reactor System with Feed-effluent Heat Exchange," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 274-281.
    The basic equations of a chemical reactor heat exchanger with small perturbations about the steady-state are developed and the stability criterion established. An analog computer simulation was used to verify the results.

145. Pack, George J., and W. E. Phillips, Jr.
    "Analog Study of Interacting and Noninteracting Multiple-Loop Control Systems for Turbojet Engines," NACA Technical Note 3112, National Advisory Committee for Aeronautics, Washington, D.C., March, 1954.
    This paper goes into complete detail of a two input-two output linear interconnected multivariable system. It gives the system transfer function, related graphs of the variables showing stable and unstable regions, and additional elements needed to obtain noninteraction. Additional compensation terms are shown to allow an increased gain and an increased stability region.

146. Patha, John T.
    "Six Degree Dynamics," Document No. 5D-5639, Boeing Airplane Co., Seattle, Washington, Oct., 1959.
    This paper presents the equations for attitude and reaction to controls of a body essentially in space.

147. Peschon, J., and W. H. Horton.
    "On Timesharing of Control System Components," Transactions, PGAC, IRE, vol. AC-7, no. 4, July, 1962, pp. 33-38.
    A discussion is given concerning the equations of performance of a system which has one control element which acts in turn on each input of n independent servo units. While the element is connected to one input, the other inputs are not connected to anything. Cross coupling or interaction between the control loops occurs when this proposed more economical system is used. The cost of construction is lower, because only one control element is used rather than n. This saving, even with some interaction, may more than offset the cost of an additional controller to obtain noninteraction.

148. Pessen, D. W.
    "Optimum Three-mode Controller Settings for Automatic Startup," Transactions, ASME, vol. 75, no. 7, July, 1953, pp. 843-849.

An electronic process-analog verified formula found empirically for the settings on a specific class of three-mode controller. The settings produce automatic start-up without overshoot and produce close to optimum response to load disturbances.

149. Peterson, Edwin L.
"Cross-Coupling in Two-Channel Servos," Conference Paper, AIEE Winter General Meeting, New York, Jan. 22, 1954.
A discussion is given concerning the stability problem when two servos have some type of interrelation.

150. _____.
"Optimization of Multi-output Linear Time-Varying Systems Subject to Multiple or Redundant Non-stationary Inputs," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 2, pp. 745-752.
A specific multivariable system concerning radio-inertial velocity sensing is considered where the radio component determines the position and the inertial component senses the vehicle acceleration. Optimization is applied to the minimization of the mean square output error.

151. _____.
Statistical Analysis and Optimization of Systems. New York: John Wiley and Sons, Inc., 1961.
The synthesis procedures for optimal control with constraints for the general linear multivariable system are discussed and examples given. A numerical process is presented for obtaining the optimal control of certain non-linear systems.

152. Petrov, B. N.
"The Invariance Principle and the Condition for its Application during the Calculation of Linear and Non-Linear Systems," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 117-125.
The basic method of obtaining invariance in dynamic systems is presented. The criterion given is applied to both general linear and nonlinear systems.

153. Pitman, George R., Jr.
Inertial Guidance. New York: John Wiley and Sons, Inc., 1962.
A series of articles written by various authors has been compiled to present both the general overview and the finer points of many aspects of inertial guidance. A selected bibliography is given to aid further research in any of the aspects.

154. Piven, V. D.
"The Problem of Autonomy in the Technique of Automatic Control," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 153-156.
    The attainment of noninteraction through use of element-by-element calculation and without the use of matrix methods is presented.

155. Povejsil, D. J., and A. M. Fuchs.
"A Method for the Preliminary Synthesis of a Complex Multiple-loop Control System," Transactions, AIEE, vol. 74, pt. II, July, 1955, pp. 129-134.
    A general multivariable plant is considered where control elements are continually being added to the system and the resultant characteristic equation after each addition compared to the specified desired overall characteristic equation. This process continues until the desired equation is obtained. It is a slow and tedious method but one which must be used in many instances.

156. Rang, E. R.
"A Switching Criterion for Certain Time-Optimal Regulating Systems," Transactions, ASME, Series D, Journal of Basic Engineering, vol. 84, no. 1, March, 1962, pp. 30-32.
    A rule is presented which allows for the computing of the initial relay position for time-optimal regulation. Any set of initial conditions may be used. The method given does not require the solution of any transcendental switching equations.

157. Raymond, François.
"Sur la Stabilite d'un Asservissement Lineaire Multiple," Comptes Rendus Académie des Sciences Paris, vol. 235, 1952, pp. 508-510.
    A multiple servo system is described mathematically and with certain hypothesis a theorem concerning the stability requirements is proven for two special cases.

158. _____.
"Introduction À L'étude des Asservissements Multiples Simultanés (Servoméchanismes à Plusieurs Variables)," Bulletin de la Societe Francaise des Mécaniciens, Editions Science et Indestrie, no. 7, 1953, pp. 18-25.
    The paper contains a review of a multiple input-multiple output servomechanism system using matrix methods. The stability criterion is established.

159. Rechberger, H., and H. Sequenz.
"Simulation of a Nuclear Reactor by a Digital Model," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London,

1961, vol. 2, pp. 955-961.
   The equations expressing neutron density in a nuclear reactor containing more than one source are obtained and the system's necessary control features found through application of a digital simulator.

160. Roberson, R. E.
   "Attitude Control of a Satellite Vehicle - An Outline of the Problems," Proceedings, VIII International Astronautical Federation Congress, Springer-Verlag, Wien, 1958, pp. 317-339.

161. Robinson, Arthur S.
   "The Synthesis of Computer-Limited Sampled-Data Control Systems," Proceedings, Computers in Control Systems Conference, Atlantic City, Oct., 1957; publ. by AIEE, New York, May, 1958, pp. 77-87.
   A discussion is presented concerning the use of digital computers in multivariable control systems where the time of computing the desired functions in the computer establishes a limit to the number of inputs allowed.

162. _____.
   "The Synthesis of Computer-Limited Sampled-Data Simulation and Filtering Systems," Proceedings, Eastern Joint Computer Conference, Washington, D.C., Dec., 1957.
   The use of a digital computer to simulate dynamic transfer characteristics is described and applied in an array of output "holds" or filters.

163. Rosenbrock, H. H.
   "The Distinctive Problems of Process Control," R&D/235, Constructors John Brown Ltd., Research and Development Station, Kingston Road, Leatherhead, Surrey, U. K., August, 1961.
   The extension of control systems into the process control area is progressing at a slow pace because of the lack of basic theory which can describe the plant. A proposed general theory is given and applications are made to distillation systems.

164. _____.
   "The Control of Distillation Columns," Transactions, Institution of Chemical Engineers, Printed by Adlard and Sons Ltd., Bartholomew Press, Dorking, Surrey, U. K., vol. 40, no. 1, 1962, pp. 35-53.
   A detailed study is given on a 2x2 chemical distillation column. It is found that many effects secondary to the main purpose of the column cause knowledge of the detailed behavior of the system to be difficult to obtain. By use of a "disturbance function", however, more extended results as to the mathematical description of the system can be obtained.

165. Ruschinski, V. M.
"Dynamics of the Automatic Control of a Boiler-Turbine
Unit," Automatic and Remote Control, Proceedings of the
First International Congress of the International Federa-
tion of Automatic Control, Moscow, USSR, 1960, Butter-
worth's, London, 1961, vol. 4, pp. 365-372.
    This paper gives a graphical presentation of the effect
of the variation of a few parameters upon derived equations
of performance of a specific system.

166. Sakrison, David J.
"Application of Stochastic Approximation Methods to System
Optimization," Technical Report 391, Massachusetts Insti-
tute of Technology, Research Laboratory of Electronics,
Cambridge, Mass., July 10, 1962.
    The author considers the optimization of a system with
respect to the minimization of the error between the actual
output and the desired output. The assumption is made that
there are a certain number of variable parameters in the
system which can be changed so as to enable optimization to
occur. An iterative type solution is used.

167. Sarachik, Phillip, and J. R. Ragazzini.
"A Two Dimensional Feedback Control System," Technical
Report T-16/B, Electronics Research Lab., Columbia Uni-
versity, New York, N. Y., August 15, 1956.
    The class of feedback control systems where an accurate
relationship between the two outputs of a particular linear
or nonlinear system is discussed. Application is made to
a cutting lathe tool where tool depth and bed speed are
the controlled output variables.

168. Sarkisyan, E. P., M. M. Agababyan, and P. S. Saakyan.
"Adaptive Automatic Control System for Obtaining Aluminum
by the Electrolytic Process Using a Computing Device,"
Avtomatika i Telemekhanika, vol. 21, no. 6, June, 1960,
pp. 806-811; Automation and Remote Control, vol. 21, no. 6,
June, 1960, pp. 563-567.
    The authors present a description of an adaptive computer
controller of an aluminum producing system where the com-
ponent of cost is minimized by the variation of the anode
position.

169. Savant, C. J., R. C. Howard, C. B. Solloway, and C. A. Savant.
Principles of Inertial Navigation. New York: McGraw-Hill
Book Co., 1961.
    A basic discussion is given on the fundamentals of in-
ertial navigation, the autonavigator components, the mathe-
matics involved, and the elements of ballistic guidance.

170. Sawaragi, Y., N. Sugai, and Y. Sunahara.
"A Statistical Study on the Analysis and Synthesis of Non-
Linear Automatic Control Systems Subject to Random Inputs,"

Automatic and *Remote* *Control*, Proceedings of the First In-
ternational Congress of the International Federation of
Automatic Control, Moscow, USSR, 1960, Butterworth's,
London, 1961, vol. 2, pp. 774-778.
   The authors give a graphical technique for the analysis
and synthesis of a single input-single output system with
disturbances.

171. Seifert, Howard.
    Space *Technology*. New York: John Wiley and Sons, Inc.,
    1959, Chapter 25, written by Donald P. Ling.
       A specific radio-inertial system is controlled to op-
    timize with respect to a minimum mean square output error.

172. Shinbrot, Marvin.
    "Optimization of Time-Varying Linear Systems with Non-
    stationary Inputs," *Transactions*, ASME, vol. 80, no. 2,
    Feb., 1958, pp. 457-462.
       In the optimization of certain systems with nonstationary
    inputs integral equations arise which can be solved if the
    correlation functions of the inputs are of a certain type.
    Examples of the particular problem which can be analyzed
    are presented.

173. Shumilovskii, N. N., L. V. Melttser, Men Chin Chzhun,
    A. A. Naumov, and L. A. Pivovarov.
    "Automatic Control of Composition of a Mixture Containing
    Many Components," *Automatic* and *Remote* *Control*, Proceedings
    of the First International Congress of the International
    Federation of Automatic Control, Moscow, USSR, 1960, But-
    terworth's, London, 1961, vol. 3, pp. 381-385.
       The equations are found to describe the effect of emis-
    sion from radioactive isotopes when adjacent to different
    materials in varying states. These equations are then ap-
    plied to possible control configurations of multivariable
    systems.

174. Sinaiko, Wallace H.
    Selected *Papers* *on* *Human* *Factors* *in* *the* *Design* *and* Use *of*
    Control *Systems*. New York: Dover Publications, Inc., 1961.
       The basic human operator is indeed a definite multivari-
    able controller and the effect of various environmental
    factors upon the person is observed.

175. Smith, Fred B.
    "Time-Optimal Control of Higher-Order Systems," *Transac-*
    tions, PGAC, IRE, vol. AC-6, no. 1, Feb., 1961, pp. 16-21.
       A method for obtaining the forcing functions in terms of
    the state variables and not using the phase space concept
    is discussed. The transcendental equations obtained re-
    quire an on-line solution.

176. Smith, Gerald L., Stanley F. Schmidt, and Leonard A. McGee.
"Application of Statistical Filter Theory to the Optimal
Estimation of Position and Velocity on Board a Circumlunar
Vehicle," NASA Technical Report R-135, Ames Research Center,
Moffett Field, Calif., 1962.
      The report contains a specific example of a multivariable
system which deals with the position and velocity measure-
ments of a space vehicle and ways to correct them, and thus
obtain the desired reentry into the earth's atmosphere.
Minimization of the effects of noise or errors in measure-
ments, and therefore minimization of the amount of fuel
consumed for these corrections, is discussed.

177. Stakhovskii, R. I.
"Twin-Channel Automatic Optimalizer," Avtomatika i Tele-
mekhanika, vol. 19, no. 8, August, 1958, pp. 744-756;
Automation and Remote Control, vol. 19, no. 8, August,
1958, pp. 729-740.
      The author presents in detail the circuits and some units
of a twin-channel electronic automatic optimalizer for the
location of minima with restrictions. Experimental results
for different models are given.

178. Stear, Edwin B.
"On the Synthesis of Optimum Multipole Filters," Technical
Note 60-27, for Air Force Office of Scientific Research,
University of Southern California, Department of Engineer-
ing, Los Angeles, Calif., May, 1960, AFOSR TN 60-617.
      This paper gives the general quadratic error criteria for
synthesis of linear "least squares" multipole filters. It
is shown that the "optimum multipole filter for any element
of the class of criteria is also the optimum multipole
filter for any other element of the class".

179. Stear, E. B., and C. T. Leondes.
"Extended Synthesis Techniques for Multipole Sampled-Data
Control Systems," Technical Note 59-67, for Air Force Of-
fice of Scientific Research, University of Southern Cali-
fornia, Department of Engineering, Los Angeles, Calif.,
May, 1960, AFOSR TN 59-1223.
      The use of a digital computer as the controller of a
multivariable system is discussed where the specifications
are known only at arbitrary instants of time. A lower
bound on the sampling rate is established based upon system
requirements.

180. Steeg, C. W., and R. V. Morris.
"The Solution of Differential Equations in the Time Domain,"
Proceedings, Computers in Control Systems Conference, At-
lantic City, Oct., 1957; publ. by AIEE, May, 1958, pp.
139-148.
      Some of the many classical principles of the solution of
differential equations arising in some multivariable systems
are given.

181. Stewart, Robert M.
"A Simple Graphical Method for Constructing Families of Nyquist Diagrams," *Journal of Aeronautical Sciences*, vol. 18, no. 11, Nov., 1951, pp. 767-768.
    The author proposes sets of Nyquist plots for stability study of multivariable systems. An example of a two-input and one-output system is given. This is one of the first discussions of this method of approach.

182. _____.
"Statistical Design and Analysis of Multiply-Instrumented Control Systems," *Transactions*, PGAC, IRE, vol. AC-3, Nov., 1957, pp. 29-31.
    Wiener's linear least-square filter theory is extended to some common types of multivariable control systems.

183. Stewart, R. M., and R. J. Parks.
"Degenerate Solutions and Algebraic Approach to the Multiple-Input Linear Filter Design Problem," *Transactions*, PGCT, IRE, vol. CT-4, no. 1, March, 1957, pp. 10-15.
    The basic minimization of the output error criterion is established and applied to some multivariable systems.

184. Stone, C. R., Charles W. Johnson, F. B. Smith, E. B. Lee, and C. A. Harvey.
"Time Optimal Control of Linear Systems," *Aero Document R-ED 6134*, Aeronautical Division, Minneapolis-Honeywell Regulator Co., Minneapolis, Minn., Sept. 1, 1959.
    The time-optimal controller is generally described and the system configuration discussed. Closed-form solutions of a set of simultaneous equations are given for second- and third-order systems. It is an excellent paper on time-optimal control.

185. Stout, T. M.
"System Considerations in Computer Control of a Semicontinuous Chemical Process," *Proceedings*, Computers in Control Systems Conference, Atlantic City, Oct., 1957; publ. by AIEE, May, 1958, pp. 111-121.
    The use of a digital computer in multivariable systems is analyzed concerning evaluation of process objectives, economics, and evaluation of the system. A specific example is cited concerning the isomerization process for converting normal butane into isobutane.

186. _____.
"Mathematical Models for Computer Control Systems," *Automatic and Remote Control*, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 2, pp. 991-997.
    The optimization is regarded in three steps: an objective to be minimized or maximized, a certain number of various constraints, and as a mathematical model.

187. Strejc, Vladimir.
"The General Theory of Autonimity and Invariance of Linear Systems of Control," _Acta Technica_, no. 3, March, 1960, pp. 235-258.
The generalized theory of autonomous and invariant multi-variable control systems with different parts being inter-related is presented. This follows the same method as used by Freeman (references 46-49).

188. Svoboda, F., and M. Martinek.
"Digital Computer for Generation of Data for Automatic Machine Control," _Automatic and Remote Control_, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 4, pp. 9-20.
A system is given for the control of a milling machine in a one plane enabling the feed for cutting straight lines and circles.

189. Syuzyumova, E. M.
"Some Questions Concerning Coordinated Control Systems," _Avtomatika i Telemekhanika_, vol. 21, no. 6, June, 1960, pp. 749-760; _Automation and Remote Control_, vol. 21, no. 6, June, 1960, pp. 521-528.
A theorem is considered concerning the inverse of matrices used in the formulation of the static independence conditions of control variables. Dynamic independence of nonlinear systems is also considered.

190. Taft, V. A.
"On the Analysis of the Stability of the Periodic Modes of Operation in Nonlinear Control Systems with Many Degrees of Freedom," _Avtomatika i Telemekhanika_, vol. 20, no. 9, Sept., 1959, pp. 1163-1170; _Automation and Remote Control_, vol. 20, no. 9, Sept., 1959, pp. 1132-1140.
By use of a generalized corollary of Hill's equation a derivation is given of the characteristic equation of a multivariable system with time variant elements. The results are further examined by Mikhailov criterion for analyzing the stability of the operation.

191. Tou, Julius T.
"A Simplified Technique for the Determination of Output Transforms of Multiloop, Multisampler, Variable-rate, Discrete-Data Systems," _Proceedings_, IRE, vol. 49, no. 2, March, 1961, pp. 646-647.
For multiloop, multisampled control systems a simple and general approach for finding the output transforms is given.

192. Truxal, J. G.
Automatic Feedback Control Systems. New York: McGraw-Hill Book Co., 1955, pp. 318-336.
The basic multiloop system with two inputs, one of which is generally described as a disturbance, is discussed.

193. Truxal, J. G. and E. Mishkin.
"On the Evaluation of an Attitude Control System," vol. X
of Proceedings of the Symposium on Active Networks and
Feedback Systems, Polytechnic Press of the Polytechnic
Institute of Brooklyn, Brooklyn, N. Y., 1961, pp. 81-91.
   This paper describes the analytical approach in three di-
mensions to the attitude control of an orbiting body.
Euler's equations of motion are developed and specialized
to spherical, oblong ellipsoidal, and oblate ellipsoidal
bodies. The stability of these simplified cases is shown.

194. Tsien, H. S.
Engineering Cybernetics. New York: McGraw-Hill Book Co.,
1954, pp. 53-69.
   The author repeats the work of Boksenbom and Hood
(reference 16).

195. Tsypkin, J. Z.
"Some Problems of the Theory of Discrete Automatic Systems,"
Proceedings, Computers in Control Systems Conference, At-
lantic City, Oct., 1957; publ. by AIEE, May, 1958, pp. 11-
21.
   Impulse automatic and digital automatic systems are dis-
cussed for time optimal and minimum mean square output
error conditions.

196. Vaeth, James E.
"Flywheel Control of Space Vehicles," Transactions, PGAC,
IRE, vol. AC-5, no. 3, August, 1960, pp. 247-253.
   A three-dimensional analog computer study of a multivari-
able system is presented.

197. Waite, D. P., and E. E. Lynch.
"Computer Verification of Steam Generator Instrumentation
for a Nuclear Power Plant," Proceedings, Computers in Con-
trol Systems Conference, Atlantic City, Oct., 1957; publ.
by AIEE, May, 1958, pp. 42-47.
   The analog computer may be employed in many ways in the
instrumentation and control of multivariable systems as
illustrated in a nuclear power plant.

198. Wang, P. K. C.
"The Design of Time-Optimal Electro-Hydraulic Servomechan-
isms," Conference Paper, AIEE Winter General Meeting, New
York, N. Y., Jan. 29-Feb. 3, 1961, AIEE Paper No. CP 61-175.
   A method in designing a near time-optimal control of
large-signal dynamic systems is discussed and illustrated
in an electro-hydraulic servomechanism with step inputs.

199. Waymeyer, W. K., and T. H. Young.
"Coupling in Cruciform Missile Control Systems," Conference
Paper, AIEE Winter General Meeting, New York, N. Y., Jan.
29-Feb. 2, 1962, AIEE Paper No. 62-490.

The equations of motion of a missile in flight are given and the necessary control elements found. It is noted in application that interactions occur in the control systems due to effects caused when the missile is placed near its limits of required performance.

200. Westcott, J. H.
"Design of Multivariable Optimum Filters," Transactions, ASME, vol. 80, no. 2, Feb., 1958, pp. 463-467.
The filter problem is presented which requires the resemblance in a mean square error sense of a message from different sources where the statistical characteristics of the input message and disturbances are known.

201. Wiener, N. L., and P. Masani.
"Multivariate Stochastic Processes," Acta Mathematica, Part I in vol. 98: 1-2, 1957, pp. 111-150; Part II in vol. 99:1-2, 1958, pp. 93-137.
This is the basic reference for the mathematical procedures of the "least-squares" method involving minimization of the sum of mean square output errors for multipole filters.

202. Wong, E., and J. E. Thomas.
"Multidimensional Prediction and Filtering Problem and the Fractorization of Special Matrices," Journal of the Franklin Institute, vol. 272, no. 2, August, 1961.
A method of solving the Wiener-Hopf equation is presented.

203. Xu-Yen, Tu.
"Theory of an Harmonically Acting Control System with a Large Number of Controlled Variables," Automatic and Remote Control, Proceedings of the First International Congress of the International Federation of Automatic Control, Moscow, USSR, 1960, Butterworth's, London, 1961, vol. 1, pp. 90-99.
A more advanced method than used by Sarachik and Ragazzini (reference 167) is proposed where the outputs are some functionally or harmonically related. The control process thus is governed by a variation in the prescribed functional relationship of the outputs.

204. Youla, D. C.
"On the Factorization of Rational Matrices," Transactions, PGIT, IRE, vol. IT-7, no. 3, July, 1961, pp. 172-189.
The author gives an extensive treatment of the analysis of factoring matrices by citing lemmas, theorems, and proofs. The matrices are from equations one of which is the Wiener-Hopf equation.

205. Younkin, George W.
"Numeripoint Position System with Linear Inductosyn Scales," unidentified report, Giddings and Lewis Machine Tool Co., Fond du Lac, Wisconsin, March 26, 1959.

The paper presents a detailed description of a two-axis numerical position control system.

206. Zadeh, L. A.
"Nonlinear Multipoles," Proceedings, National Academy of Science, vol. 39, no. 4, April, 1953, pp. 274-280.
The basic definitions and mathematical operations of matrices in multivariable systems is treated. This is one of the first papers which describes the use of matrices in this field.

MULTIVARIABLE SYSTEMS

by

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B. S. (E. E.) and B. S. (Math.),
Kansas State University, 1962

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The control of systems having more than one input and one or more outputs has evolved into the study of multivariable systems. This new control system has many basic aims. Some of them are the attainment of noninteraction control, minimization of the mean square output error, minimization of the effect of plant parameter variation, and optimization of some function generally related to the cost of operation. These goals have been fulfilled as research progressed from meager beginnings in the 1930's to the present time, as evidenced by over 200 publications. The effects permeate the control industry from chemical mixing plants to guidance of space vehicles.