Revenue Guarantees in Generalized Second Price Auctions

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Sponsored search auctions are the main source of revenue for search engines. In such an auction, a set of utility-maximizing advertisers compete for a set of ad slots. The assignment of advertisers to slots depends on bids they submit; these bids may be different than the true valuations of the advertisers for the slots. Variants of the celebrated VCG auction mechanism guarantee that advertisers act truthfully and, under mild assumptions, lead to revenue or social welfare maximization. Still, the sponsored search industry mostly uses generalized second price (GSP) auctions; these auctions are known to be non-truthful and suboptimal in terms of social welfare and revenue. In an attempt to explain this tradition, we study a Bayesian setting where the valuations of advertisers are drawn independently from a regular probability distribution. In this setting, it is well known by the work of Myerson [1981] that the optimal revenue is obtained by the VCG mechanism with a particular reserve price that depends on the probability distribution. We show that by appropriately setting the reserve price, the revenue over any Bayes-Nash equilibrium of the game induced by the GSP auction is at most a small constant factor away from the optimal revenue, improving previous results of Lucier et al. [2012]. Our analysis is based on the Bayes-Nash equilibrium conditions and on the properties of regular probability distributions.

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1. INTRODUCTION

The sale of advertising space is the main source of income for information providers on the Internet. For example, a query to a search engine creates advertising space that is sold to potential advertisers through auctions that are known as *sponsored search auctions* (or ad auctions). In their influential papers, Edelman et al. [2007] and

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Varian [2007] have proposed a (now standard) model for this process. According to this model, a set of utility-maximizing advertisers compete for a set of ad slots with non-increasing click-through rates. The auctioneer collects bids from the advertisers and assigns them to slots (usually, in non-increasing order of their bids). In addition, the auctioneer assigns a payment per click to each advertiser. Depending on the way the payments are computed, different auctions can be defined. Typical examples are the Vickrey-Clark-Groves (VCG), the generalized second price (GSP), and the generalized first price (GFP) auction. Naturally, the advertisers are engaged as players in a strategic game defined by the auction; the bid submitted by each player is such that it maximizes her utility (i.e., the total difference of her valuation minus her payment over all clicks) given the bids of the other players. This behavior leads to equilibria, i.e., states of the induced game from which no player has an incentive to unilaterally deviate.

Traditionally, truthfulness has been recognized as an important desideratum in the Economics literature on auctions [Krishna 2002]. In truthful auctions, truth-telling is an equilibrium according to specific equilibrium notions (e.g., dominant strategy, Nash, or Bayes-Nash equilibrium). Such a mechanism guarantees that the social welfare (i.e., the total value of the players) is maximized. VCG is a typical example of a truthful auction [Vickrey 1961; Clarke 1971; Groves 1973]. In contrast, GSP auctions are not truthful [Edelman et al. 2007; Varian 2007]; still, they are the main auction mechanisms used in the sponsored search industry adopted by leaders such as Google, Microsoft, and Yahoo!

In an attempt to explain this prevalence, several papers have provided bounds on the social welfare of GSP auctions [Lahaie 2006; Paes Leme and Tardos 2010; Caragiannis et al. 2011; Lucier and Paes Leme 2011] over different classes of equilibria (pure Nash, coarse-correlated, Bayes-Nash). The main message from these studies is that the social welfare is always a constant fraction of the optimal one. However, one would expect that revenue (as opposed to social welfare) maximization is the major concern from the point of view of the sponsored search industry. In this paper, following previous work by Lucier et al. [2012], we aim to provide a theoretical justification for the wide adoption of the GSP mechanism by focusing on the revenue generated by (variants of) these auctions.

In order to model the inherent uncertainty in advertisers' beliefs, we consider a Bayesian (or incomplete information) setting [Harsanyi 1967] where the advertisers have random valuations drawn independently from a common probability distribution. This is the classical setting that has been studied extensively since the seminal work of Myerson [1981] for single-item auctions (which is a special case of ad auctions). The results of [Myerson 1981] carry over to our model as follows. Under mild assumptions, the revenue generated by a player in a Bayes-Nash equilibrium depends only on the distribution of the click-through rate of the ad slot the player is assigned to for her different valuations. Hence, two Bayes-Nash equilibria that correspond to the same allocation yield the same revenue even if they are induced by different auction mechanisms; this statement is known as *revenue equivalence*. The allocation that optimizes the expected revenue is one in which low-bidding advertisers are excluded and the remaining ones are assigned to ad slots in non-increasing order of their valuations. Such an allocation is a Bayes-Nash equilibrium of the variation of the VCG mechanism where an appropriate reserve price (henceforth called Myerson reserve) is set in order to exclude the low-bidding advertisers. Revenue maximization in Bayesian auctions is extensively covered in the recent survey article of Hartline [2013].

GSP auctions may lead to different Bayes-Nash equilibria [Gomes and Sweeney 2013] in which a player with a higher valuation is assigned with positive probability to a slot with lower click-through rate than another player with lower valuation.

This implies that the revenue is suboptimal. Our purpose is to quantify the revenue suboptimality over all Bayes-Nash equilibria of GSP auctions by proving worst-case *revenue guarantees*. A revenue guarantee of ρ for an auction mechanism implies that, at any Bayes-Nash equilibrium, the revenue generated is at least a $1/\rho$ fraction of the optimal one.

Following Myerson's revenue equivalence statement (see also [Lucier et al. 2012] for a concrete example), the use of reserve prices together with GSP is absolutely necessary in order to obtain non-trivial revenue guarantees. Furthermore, it is not clear whether the Myerson reserve is the choice that minimizes the revenue guarantee in GSP auctions. This issue is the subject of experimental work by Ostrovsky and Schwarz [2011]. Revenue maximization in variants of GSP auctions with reserve prices is studied (analytically and experimentally) in recent papers by Roberts et al. [2013] and Thompson and Leyton-Brown [2013].

Lucier et al. [2012] provide revenue guarantees for GSP auctions. Among other results for full information settings, they consider two different Bayesian models. When the advertisers' valuations are drawn independently from a common probability distribution with monotone hazard rate (MHR), they show that GSP auctions with Myerson reserve have a revenue guarantee of at most 6. This bound is obtained by comparing the utility of players at the Bayes-Nash equilibrium with the utility they would have by deviating to a single alternative bid (and by exploiting the special properties of MHR distributions). The class of MHR distributions is wide enough and includes many common distributions (such as uniform, normal, and exponential). In the more general case where the valuations are regular, the same bound is obtained using a different reserve price. This reserve is computed using a prophet inequality [Krengel and Sucheston 1977]. Prophet inequalities have been proved useful in several Bayesian auction settings in the past [Hajiaghayi et al. 2007; Chawla et al. 2010].

In this work, we consider the same Bayesian settings with [Lucier et al. 2012], significantly extend their analysis and improve their results. We show that when the players have i.i.d. valuations drawn from a regular distribution, there is a reserve price so that the revenue guarantee is at most 4.72. For MHR valuations, we present a bound of 3.46. In both cases, the reserve price is either Myerson's or another one that maximizes the revenue obtained by the player allocated to the first slot. The latter is computed by developing new prophet-like inequalities that exploit the particular characteristics of the valuations. Furthermore, we show that the revenue guarantee of GSP auctions with Myerson reserve is at most 3.90 for MHR valuations. In order to analyze GSP auctions with Myerson reserve, we extend the techniques developed in [Caragiannis et al. 2011; Lucier and Paes Leme 2011] (see also [Caragiannis et al. 2012]) for bounding the inefficiency of GSP auctions (without reserves) in terms of the social welfare. In particular, the Bayes-Nash equilibrium condition implies that the utility of each player does not improve when she deviates to any other bid. This yields a series of inequalities which we take into account with different weights. These weights are given by families of functions that are defined in such a way that a relation between the revenue at a Bayes-Nash equilibrium and the optimal revenue is revealed; we refer to them as *deviation weight function families*.

The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. Our prophet-type bounds are presented in Section 3. The role of deviation weight function families in the analysis is explored in Section 4 with two technical proofs appearing in appendix. Then, Section 5 is devoted to the proofs of our main statements. We conclude with open problems and a discussion in Section 6.

2. PRELIMINARIES

We consider a Bayesian setting with n players and n slots[1] where slot $j \in [n]$ has a click-through rate α_j that corresponds to the frequency of clicking an ad in slot j. We add an artificial $(n+1)$-th slot with click-through rate 0 and index the slots so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \alpha_{n+1} = 0$. Each player's valuation (per click) is non-negative and is drawn from a publicly known probability distribution.

The GSP auction mechanism we consider uses a reserve price t and assigns slots to players according to the bids they submit. Player i submits a bid $b_i(v_i)$ that depends on her valuation v_i; the bidding function b_i is the strategy of player i. Given a realization of valuations, let $\mathbf{b} = (b_1, \ldots, b_n)$ denote a bid vector and define the permutation π so that $\pi(j)$ is the player with the j-th highest bid (breaking ties arbitrarily). The mechanism assigns slot j to player $\pi(j)$ whenever $b_{\pi(j)} \geq t$; if $b_{\pi(j)} < t$, the player is not allocated any slot. In such an allocation, let $\sigma(i)$ denote the slot that is allocated to player i. This is well-defined when player i is assigned a slot; if this is not the case, we follow the convention that $\sigma(i) = n+1$. Given $\mathbf{b}$, the mechanism also defines a payment p_i for each player i that is allocated a slot $\sigma(i) \leq n$. This payment is defined as the maximum between the reserve price t and the next highest bid $b_{\pi(\sigma(i)+1)}$ (following the convention that $b_{\pi(n+1)} = 0$). Then, the *utility* of player i is $u_i(\mathbf{b}) = \alpha_{\sigma(i)}(v_i - p_i)$. A set of players' strategies is a Bayes-Nash equilibrium if no player has an incentive to deviate from her strategy in order to increase her expected utility. This means that a bid vector $\mathbf{b}$ is a Bayes-Nash equilibrium if for every player i and every possible valuation x,

$$\mathbb{E}[u_i(\mathbf{b})|v_i = x] \geq \mathbb{E}[u_i(b'_i, \mathbf{b}_{-i})|v_i = x]$$

for every alternative bid b'_i. Note that the expectation is taken over the randomness of the valuations of the other players and the notation $(b'_i, \mathbf{b}_{-i})$ is used for the bid vector where player i has deviated to b'_i and the remaining players bid as in $\mathbf{b}$. The revenue generated by the mechanism at a Bayes-Nash equilibrium $\mathbf{b}$ is

$$\mathcal{R}_t(\mathbf{b}) = \mathbb{E}[\sum_i \alpha_{\sigma(i)} p_i].$$

We focus on the case where the valuations of players are drawn independently from a common probability distribution $\mathcal{D}$ with probability density function f and cumulative distribution function F. Given a distribution $\mathcal{D}$ over players' valuations, the *virtual valuation* function is

$$\phi(x) = x - \frac{1 - F(x)}{f(x)}.$$

We consider *regular* probability distributions where $\phi(x)$ is non-decreasing. The work of Myerson [1981] implies that the expected revenue from player i at a Bayes-Nash equilibrium $\mathbf{b}$ of any auction mechanism is $\mathbb{E}[\alpha_{\sigma(i)}\phi(v_i)]$, i.e., it depends only on the allocation of player i and her virtual valuation. Hence, the total expected revenue is maximized when the players with non-negative virtual valuations are assigned to slots in non-increasing order of their virtual valuations and players with negative virtual valuations are not assigned any slot. A mechanism that imposes this allocation as a Bayes-Nash equilibrium (and, hence, is revenue-maximizing) is the celebrated VCG mechanism with reserve price t such that $\phi(t) = 0$. We refer to this as Myerson reserve and denote it by r in the following. We use the notation μ to denote such an allocation. Note that, in μ, players with zero virtual valuation can be either allocated slots or not;

[1]Our model can simulate cases where the number of slots is smaller than the number of players by adding fictitious slots with zero click-through rate.

such players do not contribute to the optimal revenue which we denote by

$$\mathcal{R}_{OPT} = \mathbb{E}[\sum_i \alpha_{\mu(i)} \phi(v_i)].$$

As it has been observed in [Gomes and Sweeney 2013], GSP may not admit the allocation μ as a Bayes-Nash equilibrium. This immediately implies that the revenue over Bayes-Nash equilibria would be suboptimal. In order to capture the revenue loss due to the selfish behavior of the players, we use the notion of *revenue guarantee*.

Definition. The revenue guarantee of an auction game with reserve price t is $\max_{\mathbf{b}} \frac{\mathcal{R}_{OPT}}{\mathcal{R}_t(\mathbf{b})}$, where $\mathbf{b}$ runs over all Bayes-Nash equilibria of the game.

A particular subclass of regular probability distributions are those with *monotone hazard rate* (MHR). A regular distribution $\mathcal{D}$ is MHR if its hazard rate function $h(x) = f(x)/(1-F(x))$ is non-decreasing. These distributions have some nice properties (see [Barlow and Marshall 1964]) such as $F(r) \leq 1 - 1/e$ and $\phi(x) \geq x - r$ for every $x \geq r$; we will use these properties in our analysis.

In our proofs, we use the notation σ to refer to the random allocation that corresponds to a Bayes-Nash equilibrium. Note that, a player with valuation strictly higher than the reserve always has an incentive to bid at least the reserve and be allocated a slot. When her valuation equals the reserve, she is indifferent between bidding the reserve or not participating in the auction. For auctions with Myerson reserve, when comparing a Bayes-Nash equilibrium to the revenue-maximizing allocation μ, we assume that a player with valuation equal to the reserve has the same behavior in both σ and μ (this implies that $\mathbb{E}[\sum_i \alpha_{\sigma(i)}] = \mathbb{E}[\sum_i \alpha_{\mu(i)}]$). This assumption is without loss of generality since such a player contributes zero to the optimal revenue anyway. In our proofs, we also use the random variable $o(j)$ to denote the player with the j-th highest valuation (breaking ties arbitrarily). Hence, $\mu(i) = o^{-1}(i)$ if the virtual valuation of player i is positive and $\mu(i) = n+1$ if it is negative. When the virtual valuation of player i is zero, it can be either $\mu(i) = o^{-1}(i)$ or $\mu(i) = n+1$.

When considering GSP auctions, we make the assumption that players are *conservative*: whenever the valuation of player i is v_i, she only selects a bid $b_i(v_i) \in [0, v_i]$ at Bayes-Nash equilibria. This is a rather natural assumption since any bid $b_i(v_i) > v_i$ is weakly dominated by bidding $b_i(v_i) = v_i$ [Paes Leme and Tardos 2010].

In the following, we use the notation x^+ to denote $\max\{x, 0\}$ while the expression $x\mathbb{1}\{E\}$ equals x when the event E is true and 0 otherwise.

3. ACHIEVING MINIMUM REVENUE GUARANTEES

Our purpose in this section is to show that by appropriately setting the reserve price, we can guarantee a high revenue from the advertiser that occupies the first slot at any Bayes-Nash equilibrium. Even though this approach will not give us a "standalone" result, it will be very useful later when we combine it with the analysis of GSP auctions with Myerson reserve. These high revenue guarantees from the advertiser that occupies the first slot are similar in spirit to prophet inequalities in optimal stopping theory [Krengel and Sucheston 1977] as well as revenue guarantees from single-item auction mechanisms that use posted prices (see [Chawla et al. 2010] for more general results in this direction).

The analysis in this section will make extensive use of the following lemma.

LEMMA 3.1. *Consider n random valuations v_1, ..., v_n that are drawn i.i.d. from a regular distribution $\mathcal{D}$. Then, for every $t \geq r$, it holds that*

$$\mathbb{E}[\max_i \phi(v_i)^+] \leq \phi(t) + \frac{n(1-F(t))^2}{f(t)}.$$

PROOF. We will first prove that $\mathbb{E}[(\phi(v_i) - \phi(t))^+] = \frac{(1-F(t))^2}{f(t)}$. Indeed, we can easily verify that

$$\int_t^\infty (1-F(x))\,\mathrm{d}x = \int_t^\infty (x-t)f(x)\,\mathrm{d}x,$$

and, hence,

$$\begin{aligned}\mathbb{E}[(\phi(v_i) - \phi(t))^+] &= \int_t^\infty \left(x - \frac{1-F(x)}{f(x)} - t + \frac{1-F(t)}{f(t)}\right) f(x)\,\mathrm{d}x \\ &= \int_t^\infty (x-t)\,f(x)\,\mathrm{d}x - \int_t^\infty (1-F(x))\ \mathrm{d}x + \int_t^\infty \frac{1-F(t)}{f(t)} f(x)\,\mathrm{d}x \\ &= \frac{(1-F(t))^2}{f(t)}\end{aligned}$$

for every player i. Now, using this last equality, we obtain that

$$\begin{aligned}\mathbb{E}[\max_i \phi(v_i)^+] &\leq \phi(t) + \mathbb{E}[\max_i(\phi(v_i) - \phi(t))^+] \\ &\leq \phi(t) + \sum_i \mathbb{E}[(\phi(v_i) - \phi(t))^+] \\ &= \phi(t) + n\mathbb{E}[(\phi(v_i) - \phi(t))^+] \\ &= \phi(t) + \frac{n(1-F(t))^2}{f(t)},\end{aligned}$$

and the proof of the lemma is complete. Note that the second inequality follows since $(\phi(v_i) - \phi(t))^+$ is non-negative and by linearity of expectation. □

We are ready to present a first application of Lemma 3.1. In particular, we will show (in Lemma 3.2) that, by appropriately setting the reserve price, the revenue obtained from the first slot is at least a constant fraction of the optimal revenue $\alpha_1\mathbb{E}[\max_i \phi(v_i)^+]$ among single-item auctions in which the first slot is auctioned off.

LEMMA 3.2. *Let $\mathbf{b}$ be a Bayes-Nash equilibrium for a GSP auction game with n players with random valuations v_1, ..., v_n drawn i.i.d. from a regular distribution $\mathcal{D}$. Then, there exists $r' \geq r$ such that*

$$\mathcal{R}_{r'}(\mathbf{b}) \geq (1-1/e)\alpha_1\mathbb{E}[\max_i \phi(v_i)^+].$$

PROOF. Let $t \geq r$ and observe that $\phi(t) \geq 0$. By the definition of the virtual valuation we have $t = \phi(t) + \frac{1-F(t)}{f(t)}$. By multiplying both sides with $1 - F^n(t)$, which denotes the probability that at least one player has valuation at least t and is therefore allocated a slot, we get

$$\begin{aligned}t(1-F^n(t)) &= \left(\phi(t) + \frac{1-F(t)}{f(t)}\right)(1-F^n(t)) \\ &= \left(\phi(t) + \frac{n(1-F(t))^2}{n(1-F(t))f(t)}\right)(1-F^n(t)). \qquad (1)\end{aligned}$$

Note that the left-hand side of the above equality multiplied with α_1 is a lower bound on the revenue of GSP with reserve t. Let t^* be such that $F(t^*) = 1 - 1/n$ (and, equivalently, $n(1 - F(t^*)) = 1$). If $t^* \geq r$, using (1), Lemma 3.1 (with $t = t^*$), and the fact that $(1 - 1/n)^n \leq 1/e$, we obtain

$$\begin{aligned}\mathcal{R}_{t^*}(\mathbf{b}) &\geq \alpha_1 t^*(1 - F^n(t^*)) \\ &= \alpha_1 \left(\phi(t^*) + \frac{n(1 - F(t^*))^2}{f(t^*)}\right)(1 - F^n(t^*)) \\ &\geq \alpha_1 \mathbb{E}[\max_i \phi(v_i)^+](1 - (1 - 1/n)^n) \\ &\geq (1 - 1/e)\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+].\end{aligned}$$

Otherwise (if $t^* < r$), using (1), Lemma 3.1 (with $t = r$), as well as the facts that the function $g(y) = \frac{1-y^n}{n(1-y)}$ is non-decreasing in $[0, 1]$ and $(1 - 1/n)^n \leq 1/e$, we obtain that

$$\begin{aligned}\mathcal{R}_r(\mathbf{b}) &\geq \alpha_1 r(1 - F^n(r)) \\ &= \alpha_1 \frac{n(1 - F(r))^2}{n(1 - F(r))f(r)}(1 - F^n(r)) \\ &\geq \frac{1 - F^n(r)}{n(1 - F(r))}\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+] \\ &\geq \frac{1 - F^n(t^*)}{n(1 - F(t^*))}\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+] \\ &\geq (1 - 1/e)\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+].\end{aligned}$$

The lemma follows by setting $r' = t^*$ when $t^* \geq r$ and $r' = r$ otherwise. □

An alternative (and simpler) proof of Lemma 3.2 could first argue that the revenue obtained by the first slot is lower-bounded by the revenue of a posted price mechanism that auctions off the first slot and, then, use the much more general result of Chawla et al. [2010] to reach the $1 - 1/e$ bound. We have selected to present this alternative proof in order to introduce the use of Lemma 3.1 that we also exploit for the particular case of MHR valuations.

LEMMA 3.3. *Let* b *be a Bayes-Nash equilibrium for a GSP auction game with* n *players with random valuations* v_1, ..., v_n *drawn i.i.d. from an MHR distribution* $\mathcal{D}$. *Then, there exists* $r' \geq r$ *such that*

$$\mathcal{R}_{r'}(\mathbf{b}) \geq (1 - e^{-2})\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+] - (1 - e^{-2})\alpha_1 r(1 - F^n(r)).$$

PROOF. We assume that $\mathbb{E}[\max_i \phi(v_i)^+] \geq r(1 - F^n(r))$ since the lemma holds trivially otherwise[2]. Let t^* be such that $F(t^*) = 1 - \eta/n$ where $\eta = 2 - (1 - 1/e)^n$. We distinguish between two cases depending on whether $t^* \geq r$ or not.

We first consider the case $t^* \geq r$. We will use the definition of the virtual valuation, the fact that the hazard rate function satisfies $h(t^*) \geq h(r) = 1/r$, the definition of t^*, Lemma 3.1 (with $t = t^*$), and the fact that $F(r) \leq 1 - 1/e$ which implies that

[2]Actually, $\mathbb{E}[\max_i \phi(v_i)^+]$ is never smaller than $r(1 - F^n(r))$. To see why this is true, consider single-item auctions of the first slot. Then, $\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+]$ is the optimal revenue, while $\alpha_1 r(1-F^n(r))$ is the revenue obtained with a posted price equal to Myerson reserve.

$1 - F^n(r) \geq \eta - 1$. We have

$$\begin{aligned} t^*(1-F^n(t^*)) &= \left(\phi(t^*) + \frac{1}{h(t^*)}\right)(1-F^n(t^*)) \\ &= \left(\phi(t^*) + \frac{\eta}{h(t^*)} - \frac{\eta-1}{h(t^*)}\right)(1-F^n(t^*)) \\ &\geq \left(\phi(t^*) + \frac{n(1-F(t^*))^2}{f(t^*)} \cdot \frac{\eta}{n(1-F(t^*))} - (\eta-1)r\right)(1-F^n(t^*)) \\ &= (1-F^n(t^*))\left(\phi(t^*) + \frac{n(1-F(t^*))^2}{f(t^*)} - (\eta-1)r\right) \\ &\geq \left(1 - \left(1 - \frac{2-(1-1/e)^n}{n}\right)^n\right)\left(\mathbb{E}[\max_i \phi(v_i)^+] - r(1-F^n(r))\right). \end{aligned}$$

Note that the left side of the above equality multiplied with α_1 is a lower bound on the revenue of GSP with reserve t^*. Also, $\left(1 - \frac{2-(1-1/e)^n}{n}\right)^n$ is non-decreasing in n and approaches e^{-2} from below as n tends to infinity. Furthermore, the right-hand side of the above inequality in non-negative. Hence,

$$\mathcal{R}_{t^*}(\mathbf{b}) \geq (1-e^{-2})\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+] - (1-e^{-2})\alpha_1 r(1-F^n(r))$$

as desired.

We now consider the case $t^* < r$. We have

$$1 - \eta/n = F(t^*) \leq F(r) \leq 1 - 1/e,$$

which implies that $n \leq 5$. Tedious calculations yield

$$\frac{1-F^n(r)}{n(1-F(r))} = \frac{1 + F(r) + ... + F^{n-1}(r)}{n} \geq \frac{1-e^{-2}}{2-e^{-2}}$$

for $n \in \{2, 3, 4, 5\}$ since $F(r) \geq 1 - \eta/n$. Hence,

$$\begin{aligned} \mathcal{R}_r(\mathbf{b}) &\geq \alpha_1 r(1-F^n(r)) \\ &\geq (1-e^{-2})\alpha_1 nr(1-F(r)) - (1-e^{-2})\alpha_1 r(1-F^n(r)) \\ &\geq (1-e^{-2})\alpha_1 \mathbb{E}[\max_i \phi(v_i)^+] - (1-e^{-2})\alpha_1 r(1-F^n(r)), \end{aligned}$$

where the last inequality follows by applying Lemma 3.1 with $t = r$. The lemma follows by setting $r' = t^*$ when $t^* \geq r$ and $r' = r$ otherwise. □

4. DEVIATION WEIGHT FUNCTION FAMILIES

The main idea we use for the analysis of Bayes-Nash equilibria of auction games with reserve price t is that the utility of player i with valuation $v_i = x \geq t$ does not increase when this player deviates to any other bid in $[t, x]$. This provides us with infinitely many inequalities on the utility of player i that are expressed in terms of her valuation, the bids of the other players, and the reserve price. Our technique combines these infinite lower bounds by considering their weighted average. The specific weights with which we consider the different inequalities are given by families of functions with particular properties that we call deviation weight function families. These are defined in the following.

Definition 4.1. Let $\beta, \gamma, \delta \geq 0$ and consider the family of functions $\mathcal{G} = \{g_\xi : \xi \in [0, 1)\}$ where g_ξ is a non-negative function defined in $[\xi, 1]$. $\mathcal{G}$ is a (β, γ, δ)–DWFF (deviation weight function family) if the following two properties hold for every $\xi \in [0,1)$:

$$i) \quad \int_{\xi}^{1} g_\xi(y)\,\mathrm{d}y = 1,$$
$$ii) \quad \int_{z}^{1} (1-y) g_\xi(y)\,\mathrm{d}y \geq \beta - \gamma z + \delta \xi, \quad \forall z \in [\xi, 1].$$

The next technical lemma (proof in Appendix A) presents such a family of deviation weight functions.

LEMMA 4.2. *Consider the family of functions $\mathcal{G}_1$ consisting of the functions $g_\xi : [\xi, 1] \to \mathbb{R}_+$ defined as follows for every $\xi \in [0,1)$:*

$$g_\xi(y) = \begin{cases} \frac{\kappa}{1-y}, & y \in [\xi, \xi + (1-\xi)\lambda), \\ 0, & \textit{otherwise}, \end{cases}$$

where $\lambda \in (0,1)$ and $\kappa = -\frac{1}{\ln(1-\lambda)}$. Then, $\mathcal{G}_1$ is a $(\kappa\lambda, \kappa, \kappa(1-\lambda))$–DWFF.

The following lemma is used in order to prove two of our three bounds together with the deviation weight function family presented in Lemma 4.2.

LEMMA 4.3. *Consider a Bayes-Nash equilibrium $\mathbf{b}$ for a GSP auction game with n players and reserve price $t \geq r$. Then, the following two inequalities hold for every player i.*

$$\mathbb{E}[u_i(\mathbf{b})] \geq \sum_{j \geq c} \mathbb{E}[\alpha_j(\beta v_i - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i) = j\}], \tag{2}$$
$$\mathbb{E}[\alpha_{\sigma(i)}\phi(v_i)] \geq \sum_{j \geq c} \mathbb{E}[\alpha_j(\beta\phi(v_i) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}], \tag{3}$$

where c is any integer in $[n]$, β, γ, and δ are such that a (β,γ,δ)–DWFF exists, and μ is any revenue-maximizing allocation[3].

PROOF. Consider a Bayes-Nash equilibrium $\mathbf{b}$. Since $\mathbf{b}$ is a Bayes-Nash equilibrium, we have

$$\begin{aligned} \mathbb{E}[u_i(\mathbf{b})|v_i = x] &\geq \mathbb{E}[u_i(yx, \mathbf{b}_{-i})|v_i = x] \\ &\geq \sum_{j \geq c} \mathbb{E}[u_i(yx, \mathbf{b}_{-i})\mathbb{1}\{\mu(i) = j\}|v_i = x], \end{aligned} \tag{4}$$

for every $y \in [t/x, 1]$ and every integer $c \in [n]$.

Furthermore, when $\mu(i) = j$ and $v_i = x$, player i occupies slot j (or a higher one) when deviating to a bid $b'_i \in [t, x]$ provided that $b'_i > b_{\pi_i(j)}$, where $\pi_i(j)$ denotes the advertiser with the j-th highest bid excluding advertiser i. In this case, the payment of player i per click is at most b'_i. Hence,

$$\begin{aligned} u_i(yx, \mathbf{b}_{-i}) &\geq \alpha_j(x - yx)\mathbb{1}\{y > b_{\pi_i(j)}/x\} \\ &\geq \alpha_j x(1-y)\mathbb{1}\{y > b_{\pi(j)}/x\}, \end{aligned} \tag{5}$$

for every $y \in [t/x, 1]$.

[3]We remark that we have stated inequalities (2) and (3) in a more general form than what is required in the proofs of Theorems 5.2 and 5.3 later. In particular, we will always invoke them with $c = 2$ and $t = r$.

Now, let $\mathcal{G}$ be a (β,γ,δ)–DWFF and let $g = g_{t/x} \in \mathcal{G}$. Using its properties, as well as the above two inequalities, we have

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b})|v_i = x] &= \int_{t/x}^{1} \mathbb{E}[u_i(\mathbf{b})|v_i = x]g(y)\,\mathrm{d}y \\
&\geq \int_{t/x}^{1} \sum_{j\geq c} \mathbb{E}[u_i(yx,\mathbf{b}_{-i})\mathbb{1}\{\mu(i)=j\}|v_i = x]g(y)\,\mathrm{d}y \\
&= \sum_{j\geq c} \mathbb{E}[\int_{t/x}^{1} u_i(yx,\mathbf{b}_{-i})g(y)\,\mathrm{d}y\mathbb{1}\{\mu(i)=j\}|v_i = x] \\
&\geq \sum_{j\geq c} \mathbb{E}[\alpha_j x \int_{b_{\pi(j)}/x}^{1} (1-y)g(y)\,\mathrm{d}y\mathbb{1}\{\mu(i)=j\}|v_i = x] \\
&\geq \sum_{j\geq c} \mathbb{E}[\alpha_j x(\beta - \gamma b_{\pi(j)}/x + \delta t/x)\mathbb{1}\{\mu(i)=j\}|v_i = x] \\
&= \sum_{j\geq c} \mathbb{E}[\alpha_j (\beta x - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i)=j\}|v_i = x]. \qquad (6)
\end{aligned}$$

The first equality follows by the first property in Definition 4.1 for function g, the first inequality follows by inequality (4), the second equality follows by linearity of expectation, the second inequality follows by inequality (5) and since $b_{\pi(j)} \geq t$ when $\mu(i) = j$, and the third inequality follows by the second property in Definition 4.1 for function g. Note that we have silently assumed that $x > 0$. In the extreme case where this is not true (i.e., $x = 0$), inequality (6) clearly holds since $\mu(i) = n+1$ (i.e., advertiser i is not assigned to any slot in the revenue maximizing allocation) and the right-hand side of inequality (6) is zero.

In order to prove inequality (2), we will bound the unconditional utility of player i using inequality (6) and by integrating over her range of valuations that allow her to participate in the auction.

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b})] &\geq \int_{t}^{\infty} \mathbb{E}[u_i(\mathbf{b})|v_i = x]f(x)\,\mathrm{d}x \\
&\geq \int_{t}^{\infty} \sum_{j\geq c} \mathbb{E}[\alpha_j(\beta x - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i)=j\}|v_i = x]f(x)\,\mathrm{d}x \\
&= \sum_{j\geq c} \int_{t}^{\infty} \mathbb{E}[\alpha_j(\beta x - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i)=j\}|v_i = x]f(x)\,\mathrm{d}x \\
&= \sum_{j\geq c} \mathbb{E}[\alpha_j(\beta v_i - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i)=j\}]
\end{aligned}$$

as desired.

In order to prove inequality (3), we will first use the observation that the utility of player i with valuation $x \geq t \geq r$ is not higher than $\alpha_{\sigma(i)}x$, the fact that $\phi(x) \leq x$, and inequality (6) to obtain

$$\begin{aligned}\mathbb{E}[\alpha_{\sigma(i)}\phi(x)|v_i = x] &= \frac{\phi(x)}{x}\mathbb{E}[\alpha_{\sigma(i)}x|v_i = x]\\ &\geq \frac{\phi(x)}{x}\mathbb{E}[u_i(\mathbf{b})|v_i = x]\\ &\geq \frac{\phi(x)}{x}\sum_{j\geq c}\mathbb{E}[\alpha_j(\beta x - \gamma b_{\pi(j)} + \delta t)\mathbb{1}\{\mu(i) = j\}|v_i = x]\\ &\geq \sum_{j\geq c}\mathbb{E}[\alpha_j(\beta\phi(x) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}|v_i = x].\end{aligned}$$

Again, we have assumed that $x > 0$. If this is not the case, the inequality clearly holds. Using this inequality, we bound the (unconditional) expected revenue from player i by integrating over her range of valuations that allow her to participate in the auction.

$$\begin{aligned}\mathbb{E}[\alpha_{\sigma(i)}\phi(v_i)] &\geq \int_t^\infty \sum_{j\geq c}\mathbb{E}[\alpha_j(\beta\phi(x) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}|v_i = x]f(x)\,\mathrm{d}x\\ &= \sum_{j\geq c}\int_t^\infty \mathbb{E}[\alpha_j(\beta\phi(x) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}|v_i = x]f(x)\,\mathrm{d}x\\ &= \sum_{j\geq c}\mathbb{E}[\alpha_j(\beta\phi(v_i) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}].\end{aligned}$$

This completes the proof of the lemma. □

4.1. A slightly different deviation weight function family

We now introduce a more complex family of deviation weight functions that we will exploit for the case of MHR valuations and GSP auction games with Myerson reserve.

Definition 4.4. Let $\beta, \gamma > 0$ and consider the family of functions $\mathcal{G} = \{g_\xi : \xi \in [0, 1)\}$ where g_ξ is a non-negative function defined in $[\xi, 1]$. $\mathcal{G}$ is a (β, γ)–DWFF (deviation weight function family) if the following three properties hold for every $\xi \in [0, 1)$:

$$\begin{aligned}&i) \quad \int_\xi^1 g_\xi(y)\,\mathrm{d}y = 1,\\ &ii) \quad \int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y \geq \beta - \gamma z + (\gamma - \beta)\xi, \quad \forall z \in [\xi, 1],\\ &iii) \quad (1-z)\int_z^1 g_\xi(y)\,\mathrm{d}y \geq \beta - (\gamma - \beta - 1)z + (\gamma - 2\beta - 1)\xi, \quad \forall z \in [\xi, 1].\end{aligned}$$

The next technical lemma (proof in Appendix B) presents such a family of deviation weight functions.

LEMMA 4.5. *Consider the family of functions $\mathcal{G}_2$ consisting of the functions g_ξ : $[\xi, 1] \to \mathbb{R}_+$ defined as follows for every $\xi \in [0, 1)$:*

$$g_\xi(y) = \begin{cases} \frac{\gamma}{1-y}, & y \in [\xi, \xi + (1-\xi)\lambda),\\ \frac{\kappa(1-\xi)}{(1-y)^2}, & y \in [\xi + (1-\xi)\lambda, \xi + (1-\xi)\mu),\\ 0, & \textit{otherwise},\end{cases}$$

where $\kappa = \gamma - 2\beta - 1 \geq 0$, $\mu = \frac{\beta}{\gamma-\beta-1}$, $\beta = \frac{\gamma(1-\lambda)\ln(1-\lambda)+1+(\gamma-2)\lambda}{1+\lambda}$, *and* $0 < \lambda \leq \mu < 1$, *such that* $\ln\left(\frac{1-\lambda}{1-\mu}\right) = \frac{\beta-\gamma\lambda}{\kappa}$. *Then,* $\mathcal{G}_2$ *is a* (β,γ)*–DWFF.*

We now provide a lower bound on the utility of each player at a Bayes-Nash equilibrium that depends on the existence of a (β,γ)–DWFF.

LEMMA 4.6. *Consider a Bayes-Nash equilibrium* $\mathbf{b}$ *for a GSP auction game with* n *players and reserve price* t*. Then, for every player* i*, it holds that*

$$\begin{aligned}\mathbb{E}[u_i(\mathbf{b})] \geq \mathbb{E}[\beta\alpha_{\mu(i)}v_i - (\gamma-\beta-1)\alpha_{\mu(i)}b_{\pi(\mu(i))} + (\gamma-2\beta-1)\alpha_{\mu(i)}t] \\ + (\beta+1)\sum_{j\geq 2}\mathbb{E}[(\alpha_j t - \alpha_j b_{\pi(j)})\mathbb{1}\{\mu(i)=j\}],\end{aligned}$$

where β *and* γ *are such that a* (β,γ)*–DWFF exists and* μ *is any revenue-maximizing allocation.*

PROOF. Consider a Bayes-Nash equilibrium $\mathbf{b}$. Since $\mathbf{b}$ is a Bayes-Nash equilibrium, we have

$$\begin{aligned}\mathbb{E}[u_i(\mathbf{b})|v_i=x] &\geq \mathbb{E}[u_i(yx,\mathbf{b}_{-i})|v_i=x] \\ &= \sum_j \mathbb{E}[u_i(yx,\mathbf{b}_{-i})\mathbb{1}\{\mu(i)=j\}|v_i=x],\end{aligned}$$

for every $y \in [t/x, 1]$.

Furthermore, when $\mu(i) = j$ and $v_i = x$, player i occupies slot j (or a higher one) when deviating to a bid $b'_i \in [t,x]$ provided that $b'_i > b_{\pi_i(j)}$, where $\pi_i(j)$ denotes the advertiser with the j-th highest bid excluding advertiser i. In this case, the payment of player i per click is at most b'_i. Hence,

$$\begin{aligned}u_i(yx,\mathbf{b}_{-i}) &\geq \alpha_j(x-yx)\mathbb{1}\{y > b_{\pi_i(j)}/x\} \\ &\geq \alpha_j x(1-y)\mathbb{1}\{y > b_{\pi(j)}/x\},\end{aligned} \tag{7}$$

for every $y \in [t/x, 1]$.

A slightly stronger bound holds if we consider deviating to the first slot. In particular, when $\mu(i) = 1$ and $v_i = x$, player i occupies slot 1 when deviating to a bid $b'_i \in [t,x]$ provided that $b'_i > b_{\pi_i(1)}$. In this case, the payment of player i per click is at most $b_{\pi_i(1)}$. Hence,

$$\begin{aligned}u_i(yx,\mathbf{b}_{-i}) &\geq \alpha_1(x-b_{\pi_i(1)})\mathbb{1}\{y > b_{\pi_i(1)}/x\} \\ &\geq \alpha_1 x(1-b_{\pi(1)}/x)\mathbb{1}\{y > b_{\pi(1)}/x\},\end{aligned} \tag{8}$$

for every $y \in [t/x, 1]$.

Now, let $\mathcal{G}$ be a (β,γ)–DWFF and let $g = g_{t/x} \in \mathcal{G}$. Using its properties, as well as the above three inequalities, we have

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b})|v_i = x] &= \int_{t/x}^{1} \mathbb{E}[u_i(\mathbf{b})|v_i = x] g(y)\,\mathrm{d}y \\
&\geq \int_{t/x}^{1} \sum_j \mathbb{E}[u_i(yx, \mathbf{b}_{-i})\mathbb{1}\{\mu(i) = j\}|v_i = x] g(y)\,\mathrm{d}y \\
&= \sum_j \mathbb{E}[\int_{t/x}^{1} u_i(yx, \mathbf{b}_{-i}) g(y)\,\mathrm{d}y \mathbb{1}\{\mu(i) = j\}|v_i = x] \\
&= \mathbb{E}[\int_{t/x}^{1} u_i(yx, \mathbf{b}_{-i}) g(y)\,\mathrm{d}y \mathbb{1}\{\mu(i) = 1\}|v_i = x] \\
&\quad + \sum_{j\geq 2} \mathbb{E}[\int_{t/x}^{1} u_i(yx, \mathbf{b}_{-i}) g(y)\,\mathrm{d}y \mathbb{1}\{\mu(i) = j\}|v_i = x] \\
&\geq \mathbb{E}[\alpha_1 x(1 - b_{\pi(1)}/x) \int_{b_{\pi(1)}/x}^{1} g(y)\,\mathrm{d}y \mathbb{1}\{\mu(i) = 1\}|v_i = x] \\
&\quad + \sum_{j\geq 2} \mathbb{E}[\alpha_j x \int_{b_{\pi(j)}/x}^{1} (1-y) g(y)\,\mathrm{d}y \mathbb{1}\{\mu(i) = j\}|v_i = x],
\end{aligned}$$

where the first equality follows by the first property in Definition 4.4 for function g, the first inequality follows by the equilibrium inequality, the second equality follows by linearity of expectation, the second inequality follows by inequalities (7) and (8) and since $b_{\pi(j)} \geq t$ when $\mu(i) = j$, for all j. Now, from the second and third property in Definition 4.4 for function g, we get

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b})|v_i = x] &\geq \mathbb{E}[(\beta\alpha_1 x - (\gamma-\beta-1)\alpha_1 b_{\pi(1)} + (\gamma - 2\beta - 1)\alpha_1 t)\mathbb{1}\{\mu(i) = 1\}|v_i = x] \\
&\quad + \sum_{j\geq 2} \mathbb{E}[(\beta\alpha_j x - \gamma\alpha_j b_{\pi(j)} + (\gamma-\beta)\alpha_j t)\mathbb{1}\{\mu(i) = j\}|v_i = x] \\
&= \sum_j \mathbb{E}[(\beta\alpha_j x - (\gamma-\beta-1)\alpha_j b_{\pi(j)} + (\gamma - 2\beta - 1)\alpha_j t)\mathbb{1}\{\mu(i) = j\}|v_i = x] \\
&\quad + (\beta+1)\sum_{j\geq 2} \mathbb{E}[(\alpha_j t - \alpha_j b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}|v_i = x] \\
&= \mathbb{E}[\beta\alpha_{\mu(i)} x - (\gamma-\beta-1)\alpha_{\mu(i)} b_{\pi(\mu(i))} + (\gamma - 2\beta - 1)\alpha_{\mu(i)} t|v_i = x] \\
&\quad + (\beta+1)\sum_{j\geq 2} \mathbb{E}[(\alpha_j t - \alpha_j b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}|v_i = x]. \qquad (9)
\end{aligned}$$

Note that we have silently assumed that $x > 0$; again, if this is not the case, inequality (9) clearly holds. We now bound the unconditional expected utility of player i by integrating over the range of valuations that allow her to participate in the auction.

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b})] &= \int_{t}^{\infty} \mathbb{E}[u_i(\mathbf{b})|v_i = x] \cdot f(x)\,\mathrm{d}x \\
&\geq \beta\mathbb{E}[\alpha_{\mu(i)} v_i] - (\gamma-\beta-1)\mathbb{E}[\alpha_{\mu(i)} b_{\pi(\mu(i))}] + (\gamma-2\beta-1)\mathbb{E}[\alpha_{\mu(i)} t] \\
&\quad + (\beta+1)\sum_{j\geq 2} \mathbb{E}[(\alpha_j t - \alpha_j b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}].
\end{aligned}$$

This completes the proof of the lemma. □

5. REVENUE GUARANTEES IN GSP AUCTIONS

We will now exploit the techniques developed in the previous sections in order to prove our bounds for GSP auctions. Throughout this section, we denote by O_j the event that slot j is occupied in the revenue-maximizing allocation considered. The next lemma provides a lower bound on the revenue of GSP auctions with Myerson reserve in terms of the click-through rates, the bids and r.

LEMMA 5.1. *Consider a Bayes-Nash equilibrium* $\mathbf{b}$ *for a GSP auction game with Myerson reserve price* r *and* n *players. It holds that*

$$\sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] \leq \mathcal{R}_r(\mathbf{b}) - \alpha_1 r\cdot \Pr[\mathrm{O}_1].$$

PROOF. Consider a Bayes-Nash equilibrium $\mathbf{b}$ for a GSP auction game with Myerson reserve price r. Define $\Pr[\mathrm{O}_{n+1}] = 0$. Consider some player whose valuation exceeds r and is thus allocated some slot. Note that the player's payment per click is determined by the bid of the player allocated just below her, if there is one, otherwise, the player's (per click) payment is set to r. It holds that

$$\begin{aligned}
\mathcal{R}_r(\mathbf{b}) &= \sum_j \alpha_j r(\Pr[\mathrm{O}_j]-\Pr[\mathrm{O}_{j+1}]) + \sum_j \mathbb{E}[\alpha_j b_{\pi(j+1)}\mathbb{1}\{\mathrm{O}_{j+1}\}]\\
&= \sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] + \sum_j \alpha_j r(\Pr[\mathrm{O}_j]-\Pr[\mathrm{O}_{j+1}]) + \sum_j \mathbb{E}[(\alpha_j-\alpha_{j+1}) b_{\pi(j+1)}\mathbb{1}\{\mathrm{O}_{j+1}\}]\\
&\geq \sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] + \sum_j \alpha_j r(\Pr[\mathrm{O}_j]-\Pr[\mathrm{O}_{j+1}]) + \sum_j (\alpha_j-\alpha_{j+1}) r\cdot\Pr[\mathrm{O}_{j+1}]\\
&= \sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] + \sum_j \alpha_j r\Pr[\mathrm{O}_j] - \sum_j \alpha_{j+1} r\cdot\Pr[\mathrm{O}_{j+1}]\\
&= \sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] + \alpha_1 r\cdot\Pr[\mathrm{O}_1].
\end{aligned}$$

The proof follows by rearranging the terms in the last inequality. □

5.1. Regular probability distributions

We are ready to present the formal statement of our first main result (Theorem 5.2). It states that the GSP auction with either the Myerson reserve or the reserve in the statement of Lemma 3.2 yields a 4.72-approximation to the optimal revenue when valuations are drawn from a regular distribution. In the proof, we combine several of the lemmas presented above, namely Lemma 3.2, inequality (3) from Lemma 4.3, Lemma 5.1, and the DWFF from Lemma 4.2.

THEOREM 5.2. *Consider a regular distribution* $\mathcal{D}$*. There exists some* r^**, such that the revenue guarantee over Bayes-Nash equilibria of GSP auction games with reserve price* r^* *is* 4.72*, when valuations are drawn i.i.d. from* $\mathcal{D}$*.*

PROOF. By Lemma 3.2, we have that there exists $r' \geq r$ such that the expected revenue over any Bayes-Nash equilibrium $\mathbf{b}'$ of the GSP auction game with reserve price r' satisfies

$$\mathcal{R}_{r'}(\mathbf{b}') \geq (1-1/e)\mathbb{E}[\alpha_1\phi(v_{o(1)})^+]. \tag{10}$$

Now, let $\mathbf{b}''$ be any Bayes-Nash equilibrium of the GSP auction game with Myerson reserve and let β, γ, and δ be parameters so that a (β, γ, δ)–DWFF exists. Using inequality (3) from Lemma 4.3 with $c = 2$ and Lemma 5.1 we obtain

$$\begin{aligned}\mathcal{R}_r(\mathbf{b}'') &= \sum_i \mathbb{E}[\alpha_{\sigma(i)}\phi(v_i)] \\ &\geq \sum_i \sum_{j\geq 2} \mathbb{E}[\alpha_j(\beta\phi(v_i) - \gamma b_{\pi(j)})\mathbb{1}\{\mu(i) = j\}] \\ &= \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] - \gamma \sum_{j\geq 2} \mathbb{E}[\alpha_j b_{\pi(j)} \mathbb{1}\{\mathbf{O}_j\}] \\ &\geq \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] - \gamma \mathcal{R}_r(\mathbf{b}'').\end{aligned}$$

In other words,

$$(1+\gamma)\mathcal{R}_r(\mathbf{b}'') \geq \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+].$$

Using this last inequality together with inequality (10), we obtain

$$\begin{aligned}\left(1+\gamma+\frac{e\beta}{e-1}\right)\max\{\mathcal{R}_r(\mathbf{b}''), \mathcal{R}_{r'}(\mathbf{b}')\} &\geq (1+\gamma)\mathcal{R}_r(\mathbf{b}'') + \frac{e\beta}{e-1}\mathcal{R}_{r'}(\mathbf{b}') \\ &\geq \beta \sum_j \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] \\ &= \beta \mathcal{R}_{OPT}.\end{aligned}$$

We conclude that there exists some reserve price r^* (either r or r') such that for any Bayes-Nash equilibrium $\mathbf{b}$ it holds that

$$\frac{\mathcal{R}_{OPT}}{\mathcal{R}_{r^*}(\mathbf{b})} \leq \frac{1+\gamma}{\beta} + \frac{e}{e-1}.$$

By Lemma 4.2, the family $\mathcal{G}_1$ is a $(\beta, \gamma, 0)$–DWFF with $\beta = \kappa\lambda$ and $\gamma = \kappa$, where $\lambda \in (0, 1)$ and $\kappa = -\frac{1}{\ln(1-\lambda)}$. By substituting β and γ with these values and using $\lambda \approx 0.682$, the right-hand side of our last inequality is upper-bounded by 4.72. □

5.2. MHR probability distributions

Our next result applies specifically to MHR valuations and provides our strongest revenue guarantee. Its proof combines inequality (2) from Lemma 4.3, Lemma 5.1, Lemma 3.3, and the DWFF defined in Lemma 4.2.

THEOREM 5.3. *Consider an MHR distribution $\mathcal{D}$. There exists some r^*, such that the revenue guarantee over Bayes-Nash equilibria of GSP auction games with reserve price r^* is 3.46, when valuations are drawn i.i.d. from $\mathcal{D}$.*

PROOF. Let $\mathbf{b}'$ be any Bayes-Nash equilibrium of the GSP auction game with Myerson reserve and let β, γ, and δ be parameters so that a (β, γ, δ)–DWFF exists. Since $\mathcal{D}$ is an MHR probability distribution, we have

$$\mathbb{E}[\alpha_{\sigma(i)} r] \geq \mathbb{E}[\alpha_{\sigma(i)}(v_i - \phi(v_i))] = \mathbb{E}[u_i(\mathbf{b}')]$$

for every player i. By summing over all players and using inequality (2) from Lemma 4.3 with $c = 2$, we obtain

$$\begin{aligned}\sum_i \mathbb{E}[\alpha_{\sigma(i)} r] &\geq \sum_i \mathbb{E}[u_i(\mathbf{b}')] \\ &\geq \sum_i \sum_{j\geq 2} \mathbb{E}[\alpha_j(\beta v_i - \gamma b_{\pi(j)} + \delta r)\mathbb{1}\{\mu(i) = j\}] \\ &\geq \sum_{j\geq 2} \mathbb{E}[\alpha_j(\beta\phi(v_{o(j)})^+ - \gamma b_{\pi(j)} + \delta r)\mathbb{1}\{\mathsf{O}_j\}] \\ &= \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] - \gamma \sum_{j\geq 2} \mathbb{E}[\alpha_j b_{\pi(j)} \mathbb{1}\{\mathsf{O}_j\}] + \delta \sum_{j\geq 2} \mathbb{E}[\alpha_j r \mathbb{1}\{\mathsf{O}_j\}] \\ &\geq \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] - \gamma \mathcal{R}_r(\mathbf{b}') + (\gamma - \delta)\mathbb{E}[\alpha_1 r \mathbb{1}\{O_1\}] + \delta \sum_j \mathbb{E}[\alpha_j r \mathbb{1}\{\mathsf{O}_j\}] \\ &= \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] - \gamma \mathcal{R}_r(\mathbf{b}') + (\gamma - \delta)\mathbb{E}[\alpha_1 r \mathbb{1}\{O_1\}] + \delta \sum_i \mathbb{E}[\alpha_{\mu(i)} r].\end{aligned}$$

The last inequality follows by Lemma 5.1. Since $\sum_i \mathbb{E}[\alpha_{\mu(i)} r] = \sum_i \mathbb{E}[\alpha_{\sigma(i)} r]$, we obtain that

$$\gamma \mathcal{R}_r(\mathbf{b}') \geq \beta \sum_{j\geq 2} \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] + (\gamma - \delta)\alpha_1 r \cdot \Pr[O_1] + (\delta - 1)\sum_i \mathbb{E}[\alpha_{\sigma(i)} r]. \tag{11}$$

By Lemma 3.3, we have that there exists $r' \geq r$ such that the expected revenue over any Bayes-Nash equilibrium $\mathbf{b}''$ of the GSP auction game with reserve price r' satisfies

$$\mathcal{R}_{r'}(\mathbf{b}'') \geq (1 - e^{-2})\mathbb{E}[\alpha_1 \phi(v_{o(1)})^+] - (1 - e^{-2})\mathbb{E}[\alpha_1 r \mathbb{1}\{O_1\}].$$

Using this last inequality together with inequality (11), we obtain

$$\begin{aligned}&\left(\gamma + \frac{e^2\beta}{e^2 - 1}\right)\max\{\mathcal{R}_r(\mathbf{b}'), \mathcal{R}_{r'}(\mathbf{b}'')\} \\ &\geq \gamma \mathcal{R}_r(\mathbf{b}') + \frac{e^2\beta}{e^2 - 1}\mathcal{R}_{r'}(\mathbf{b}'') \\ &\geq \beta \sum_j \mathbb{E}[\alpha_j \phi(v_{o(j)})^+] + (\gamma - \delta - \beta)\mathbb{E}[\alpha_1 r \mathbb{1}\{O_1\}] + (\delta - 1)\sum_i \mathbb{E}[\alpha_{\sigma(i)} r] \\ &\geq \beta \mathcal{R}_{OPT} + (\gamma - \delta - \beta)\mathbb{E}[\alpha_1 r \mathbb{1}\{O_1\}] + (\delta - 1)\sum_i \mathbb{E}[\alpha_{\sigma(i)} r].\end{aligned}$$

By Lemma 4.2, the family $\mathcal{G}_1$ is a (β, γ, δ)–DWFF with $\beta = \gamma - \delta = \kappa\lambda$, $\gamma = \kappa$, and $\delta = \kappa(1-\lambda)$, where $\lambda \in (0, 1)$ and $\kappa = -\frac{1}{\ln(1-\lambda)}$. By setting $\lambda \approx 0.432$ so that $\delta = \kappa(1-\lambda) = 1$, the above inequality implies that there exists some reserve price r^* (either r or r') such that for any Bayes-Nash equilibrium $\mathbf{b}$ of the corresponding GSP auction game, it holds that

$$\frac{\mathcal{R}_{OPT}}{\mathcal{R}_{r^*}(\mathbf{b})} \leq \frac{1}{\lambda} + \frac{e^2}{e^2 - 1} \approx 3.46,$$

as desired. □

5.3. MHR probability distributions and GSP auctions with Myerson reserve

We now specifically focus on the important GSP auction with Myerson reserve and present a revenue guarantee of 3.90 for MHR distributions. This bound follows using the slightly more involved deviation weight function family from Lemma 4.5 together with Lemmas 4.6 and 5.1.

THEOREM 5.4. *Consider an MHR distribution $\mathcal{D}$. The revenue guarantee over Bayes-Nash equilibria of GSP auction games with Myerson reserve price r is 3.90, when valuations are drawn i.i.d. from $\mathcal{D}$.*

PROOF. Let $\mathbf{b}$ be any Bayes-Nash equilibrium of the GSP auction game with Myerson reserve and let β and γ be parameters so that a (β,γ)–DWFF $\mathcal{G}$ exists. We first derive a lower bound on the expected utility of each player using Lemma 4.6. We have

$$\mathbb{E}[u_i(\mathbf{b})] \geq \mathbb{E}[\beta\alpha_{\mu(i)}v_i - (\gamma-\beta-1)\alpha_{\mu(i)}b_{\pi(\mu(i))} + (\gamma-2\beta-1)\alpha_{\mu(i)}r] + (\beta+1)\sum_{j\geq 2}^{n}\mathbb{E}[\alpha_j r - \alpha_j b_{\pi(j)}\mathbb{1}\{\mu(i)=j\}].$$

By summing over all players, we get

$$\sum_i \mathbb{E}[u_i(\mathbf{b})] \geq \beta\sum_i \mathbb{E}[\alpha_{\mu(i)}v_i] - (\gamma-\beta-1)\sum_i \mathbb{E}[\alpha_{\mu(i)}b_{\pi(\mu(i))}] + (\gamma-2\beta-1)\sum_i \mathbb{E}[\alpha_{\mu(i)}r] - (\beta+1)\sum_i\sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mu(i)=j\}] + (\beta+1)\sum_i\sum_{j\geq 2}\mathbb{E}[\alpha_j r\mathbb{1}\{\mu(i)=j\}]. \quad (12)$$

We will argue about some of the terms of Inequality (12) separately. First note that

$$\sum_i \mathbb{E}[u_i(\mathbf{b})] = \sum_i \mathbb{E}[\alpha_{\sigma(i)}(v_i - \phi(v_i))] \leq \sum_i \mathbb{E}[\alpha_{\sigma(i)}r]. \quad (13)$$

Using O_j to denote the event that slot j is occupied in μ (or σ) when reserve price r is set, we have

$$\begin{aligned}\sum_i \mathbb{E}[\alpha_{\mu(i)}b_{\pi(\mu(i))}] &= \sum_j \mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}]\\ &\leq \sum_j \mathbb{E}[\alpha_j v_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}]\\ &= \sum_i \mathbb{E}[\alpha_{\sigma(i)}v_i]\\ &\leq \sum_i \mathbb{E}[\alpha_{\sigma(i)}(\phi(v_i)+r)],\end{aligned}$$

which implies that

$$\sum_j \mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] \leq \mathcal{R}_r(\mathbf{b}) + \sum_i \mathbb{E}[\alpha_{\sigma(i)}r]. \quad (14)$$

Furthermore,

$$\sum_i\sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mu(i)=j\}] = \sum_{j\geq 2}\mathbb{E}[\alpha_j b_{\pi(j)}\mathbb{1}\{\mathrm{O}_j\}] \quad (15)$$

and

$$\sum_i\sum_{j\geq 2}\mathbb{E}[\alpha_j r\mathbb{1}\{\mu(i)=j\}] = \sum_{j\geq 2}\alpha_j r\cdot\Pr[\mathrm{O}_j]. \quad (16)$$

Using inequalities (12)-(16) and Lemma 5.1, we obtain

$$\begin{aligned}\sum_i \mathbb{E}[\alpha_{\sigma(i)} r] &\geq \beta \sum_i \mathbb{E}[\alpha_{\mu(i)} v_i] - \gamma \mathcal{R}_r(\mathbf{b}) + (\gamma - 2\beta - 1) \sum_i \mathbb{E}[\alpha_{\mu(i)} r] \\ &\quad - (\gamma - \beta - 1) \sum_i \mathbb{E}[\alpha_{\sigma(i)} r] + (\beta + 1) \sum_{j \geq 2} \alpha_j r \cdot \Pr[\mathbf{O}_j] \\ &\quad + (\beta + 1) \alpha_1 r \cdot \Pr[\mathbf{O}_1] \\ &\geq \beta \mathcal{R}_{OPT} - \gamma \mathcal{R}_r(\mathbf{b}) + \sum_i \mathbb{E}[\alpha_{\sigma(i)} r],\end{aligned}$$

which implies that

$$\frac{\mathcal{R}_{OPT}}{\mathcal{R}_r(\mathbf{b})} \leq \frac{\gamma}{\beta}.$$

We use the family $\mathcal{G}_2$ from Lemma 4.5 with parameters $\kappa = 0.158$, $\lambda = 0.163$, $\mu \approx 0.794$, $\beta \approx 0.608$, and $\gamma = 2.374$ to obtain a revenue guarantee of 3.90. The proof of the lemma is now complete. □

6. CONCLUSIONS

Even though we have considerably improved and extended the results of Lucier et al. [2012], we conjecture that our revenue bounds are not tight. The work of Gomes and Sweeney [2013] implies that the revenue guarantee of GSP auctions with Myerson reserve is in general higher than 1; however, no explicit lower bound is known. Due to the difficulty in computing Bayes-Nash equilibria analytically, coming up with a specific lower bound construction is interesting and would reveal the gap of our revenue guarantees. A concrete question here is whether Myerson reserve is the best choice as a reserve price.

Furthermore, note that the analysis in Section 4 considers infinitely many deviations in order to bound the utility of each advertiser. We believe that this argument could be improved by taking into account the fact that advertisers are utility-maximizers. Such a detailed reasoning about the structure of equilibria was recently applied by Chawla and Hartline [2013] on GFP auctions; their results imply that GFP auctions with Myerson reserve are revenue-optimal. Interestingly, even though GFP auctions were used by Yahoo! until 2004, they are no longer popular and, instead, GSP is the de facto standard of the sponsored search industry today. So, we believe that further exploring the revenue guarantees of variants of GSP auctions is important.

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Online Appendix to: Revenue Guarantees in Generalized Second Price Auctions

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A. PROOF OF LEMMA 4.2

Let $\xi \in [0,1)$. We first compute $\int_\xi^1 g_\xi(y)\,\mathrm{d}y$. It holds that

$$\int_\xi^1 g_\xi(y)\,\mathrm{d}y = \int_\xi^{\xi+(1-\xi)\lambda} \frac{\kappa}{1-y}\,\mathrm{d}y = -\kappa\ln(1-\lambda) = 1.$$

In order to prove the second property of Definition 4.1 we need to show that

$$\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \kappa\lambda + \kappa z - \kappa(1-\lambda)\xi \geq 0, \quad \forall z \in [\xi, 1].$$

We consider two cases depending on z. First, we consider $z \in [\xi+(1-\xi)\lambda, 1]$, in which case we have

$$\begin{aligned}\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \kappa\lambda + \kappa z - \kappa(1-\lambda)\xi &= \kappa(z-\lambda-(1-\lambda)\xi)\\ &\geq \kappa(\xi+(1-\xi)\lambda-\lambda-(1-\lambda)\xi)\\ &= 0.\end{aligned}$$

For $z \in [\xi, \xi+(1-\xi)\lambda)$, we have

$$\begin{aligned}\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \kappa\lambda + \kappa z - \kappa(1-\lambda)\xi &= \int_z^{\xi+(1-\xi)\lambda} \kappa\,\mathrm{d}y - \kappa\lambda + \kappa z - \kappa(1-\lambda)\xi\\ &= 0,\end{aligned}$$

and the proof of the lemma is complete. □

B. PROOF OF LEMMA 4.5

Let $\xi \in [0,1)$. We begin by computing $\int_\xi^1 g_\xi(y)\,\mathrm{d}y$. We have

$$\begin{aligned}\int_\xi^1 g_\xi(y)\,\mathrm{d}y &= \int_\xi^{\xi+(1-\xi)\lambda} \frac{\gamma}{1-y}\,\mathrm{d}y + \int_{\xi+(1-\xi)\lambda}^{\xi+(1-\xi)\mu} \frac{\kappa(1-\xi)}{(1-y)^2}\,\mathrm{d}y\\ &-\gamma\ln(1-\lambda) + \frac{\kappa(\mu-\lambda)}{(1-\mu)(1-\lambda)}.\end{aligned}$$

It holds that

$$\int_\xi^1 g_\xi(y)\,\mathrm{d}y = -\gamma\ln(1-\lambda) + \frac{\beta-\lambda(\gamma-\beta-1)}{1-\lambda} = 1,$$

where we get the first equality by substituting κ and μ, and the second equality by substituting β.

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Regarding the second property of Definition 4.4, we need to prove that

$$\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi \geq 0, \quad \forall z \in [\xi, 1].$$

We consider three cases depending on z. If $z \in [\xi + (1-\xi)\mu, 1)$, we have

$$\begin{aligned}\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi &= -\beta + \gamma z - (\gamma-\beta)\xi \\ &\geq (1-\xi)\left(\gamma\mu - \beta\right) \\ &\geq 0.\end{aligned}$$

If $z \in [\xi + (1-\xi)\lambda, \xi + (1-\xi)\mu)$, we have

$$\begin{aligned}\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi &= \int_z^{\xi+(1-\xi)\mu} \frac{\kappa(1-\xi)}{1-y}\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi \\ &= \kappa(1-\xi)\ln\left(\frac{1-z}{(1-\xi)(1-\mu)}\right) - \beta + \gamma z - (\gamma-\beta)\xi \\ &\geq \kappa(1-\xi)\ln\left(\frac{1-\lambda}{1-\mu}\right) - (\beta - \gamma\lambda)(1-\xi) \\ &= 0,\end{aligned}$$

where the inequality holds since the function is non-decreasing for $z \in [\xi + (1-\xi)\lambda, \xi + (1-\xi)\mu)$, and the last equality holds by definition. For $z \in [\xi, \xi + (1-\xi)\lambda)$, we have

$$\begin{aligned}&\int_z^1 (1-y)g_\xi(y)\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi \\ &= \int_z^{\xi+(1-\xi)\lambda} \gamma\,\mathrm{d}y + \int_{\xi+(1-\xi)\lambda}^{\xi+(1-\xi)\mu} \frac{\kappa(1-\xi)}{1-y}\,\mathrm{d}y - \beta + \gamma z - (\gamma-\beta)\xi \\ &= \gamma(\xi + (1-\xi)\lambda - z) + \kappa(1-\xi)\ln\left(\frac{1-\lambda}{1-\mu}\right) - \beta + \gamma z - (\gamma-\beta)\xi \\ &= \gamma\lambda(1-\xi) + \kappa(1-\xi)\ln\left(\frac{1-\lambda}{1-\mu}\right) - \beta(1-\xi) \\ &= 0,\end{aligned}$$

where the last equality holds by definition.

For the third property of Definition 4.4, we need to prove that

$$(1-z)\int_z^1 g_\xi(y)\,\mathrm{d}y - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \geq 0, \quad \forall z \in [\xi, 1].$$

Again, we distinguish between three cases depending on z. First we consider $z \in [\xi + (1-\xi)\mu, 1)$. We have

$$\begin{aligned}&(1-z)\int_z^1 g_\xi(y)\,\mathrm{d}y - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\ &= -\beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\ &\geq -\beta(1-\xi) + (\gamma-\beta-1)(1-\xi)\mu \\ &= 0.\end{aligned}$$

For $z \in [\xi + (1-\xi)\lambda, \xi + (1-\xi)\mu)$, we have

$$\begin{aligned}
&(1-z)\int_z^1 g_\xi(y)\,\mathrm{d}y - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\
&= (1-z)\int_z^{\xi+(1-\xi)\mu} \frac{\kappa(1-\xi)}{(1-y)^2}\,\mathrm{d}y - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\
&= \frac{\kappa(1-z)}{1-\mu} - \kappa(1-\xi) - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\
&= \kappa\left(\frac{1}{1-\mu} - 1\right) - \beta \\
&= 0,
\end{aligned}$$

where the second to last equality holds since the coefficients of z cancel out. Finally, if $z \in [\xi, \xi + (1-\xi)\lambda)$, we have

$$\begin{aligned}
&(1-z)\int_z^1 g_\xi(y)\,\mathrm{d}y - \beta + (\gamma-\beta-1)z - (\gamma-2\beta-1)\xi \\
&= (1-z)\int_z^{\xi+(1-\xi)\lambda} \frac{\gamma}{1-y}\,\mathrm{d}y + (1-z)\int_{\xi+(1-\xi)\lambda}^{\xi+(1-\xi)\mu} \frac{\kappa(1-\xi)}{(1-y)^2}\,\mathrm{d}y - \beta + (\gamma-\beta-1)z \\
&\quad - (\gamma-2\beta-1)\xi \\
&= \gamma(1-z)\ln\left(\frac{1-z}{(1-\xi)(1-\lambda)}\right) + \kappa(1-z)\frac{\mu-\lambda}{(1-\mu)(1-\lambda)} - \beta + (\gamma-\beta-1)z \\
&\quad - (\gamma-2\beta-1)\xi \\
&\geq \kappa(1-\xi)\frac{\mu-\lambda}{1-\mu} - \beta(1-\xi) + (\gamma-\beta-1)(1-\xi)\lambda \\
&= 0,
\end{aligned}$$

where the inequality follows since the derivative with respect to z is negative. The proof of the lemma is complete. □