A simple logical matrix and sequent calculus for Parry's logic of Analytic Implication

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Abstract

We provide a logical matrix semantics and a Gentzen-style sequent calculus for the first-degree entailments valid in W. T. Parry's logic of Analytic Implication. We achieve the former by introducing a logical matrix closely related to that inducing paracomplete weak Kleene logic, and the latter by presenting a calculus where the initial sequents and the left and right rules for negation are subject to linguistic constraints.

1.   Background and Aim

The logic of Analytic Implication was developed during the early 1930s by W. T. Parry, at the time a doctoral student in Philosophy at Harvard University, under the supervision of A. N. Whitehead.[1] His system was one of the early respondents to the so-called paradoxes of strict implication, exhibited by the logics developed by C. I. Lewis.[2] A few decades later, the term *relevance logics* would be coined to refer to those systems lacking the aforementioned paradoxes as reported, e.g., in the very first paragraphs of [29]. In this vein, because these alleged paradoxes do not hold of the notion of implication characteristic of Parry's logic (henceforth, PAI), it is appropriate to classify his system as a relevance logic *avant la lettre*.

The main and distinctive feature of PAI (and of the many systems of analytic implication belonging to its ilk) is the rejection of the classically valid principle of *Addition*, sometimes also referred to as Disjunction Introduction. In other words, the principle leading from a formula φ to a disjunction of the form $\varphi \vee \psi$, where ψ is an arbitrary formula. Parry blamed on this principle the derivability of the paradoxes of strict implication—given that it is famously featured in Lewis' derivation of an arbitrary formula ψ from a contradiction of the form $\varphi \wedge \neg\varphi$. His rejection of Addition was not due to a particular fixation with that principle, but rather a consequence of a more general diagnosis. Parry

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[1] The main publications which featured presentations of this system were his Doctoral Thesis [34], and his articles [35], [36], [37], [38], and the posthumously published [39].

[2] These paradoxes pertain to the fact that, on the one hand, necessarily true propositions (e.g., logical truths) are implied by any proposition whatsoever and, on the other hand, necessarily false propositions (e.g., logical contradictions) imply any proposition whatsoever.

thought that genuine entailments should be subject to a substantive constraint, christened by him the "Proscriptive Principle". In the literature, it is commonly accepted that when working with a propositional language and focusing on an implication of the form $\varphi \to \psi$ this constraint translates into requiring the inclusion of the set of propositional variables appearing in ψ in the set of propositional variables appearing in φ—as extensively discussed, e.g., in [20]. In other words, where $Var(\chi)$ refers to the set of propositional variables appearing in a formula χ, that an implication of the form $\varphi \to \psi$ complains with the Proscriptive Principle amounts to:

$$Var(\psi) \subseteq Var(\varphi)$$

Following the usual terminology, let us denote by "first-degree entailments" those entailments of the form $\varphi \to \psi$ where φ and ψ do not contain occurrences of the implication connective. Thus, as proved by K. Fine in [21], F. Johnson in [27], and later by others (e.g., in [31] and [20]) the first-degree fragment of Parry's PAI can be characterized in terms of the set of inferences that are valid in Classical Logic (CL, hereafter) that respect the Proscriptive Principle—thereby corroborating a hypothesis discussed by Gödel in [25]. That is to say, when $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{PAI}} \varphi \to \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{CL}} \psi, \text{ and} \\ Var(\psi) \subseteq Var(\varphi) \end{cases}$$

Our aim in this article is to develop a simple semantics and a simple sequent calculus for those first-degree entailments valid in Parry's system. To accomplish these tasks, it will be easier to consider such a set of first-degree entailments as a logical system in its own right. This can be easily done by considering the "first-degree fragment" $\mathsf{L}_{\mathsf{fde}}$ of a logic L formulated in a language with an implication connective $\to$, along the following lines, where $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{L}} \varphi \to \psi \quad \Longleftrightarrow \quad \varphi \vdash_{\mathsf{L}_{\mathsf{fde}}} \psi$$

Thus, it will make things clearer in what follows to notice that we will be providing a semantics and a calculus for the first-degree entailment fragment of PAI, that is to say, for the logic $\mathsf{PAI}_{\mathsf{fde}}$. In this respect, then, our aims here are twofold. Our first goal is to develop a simple matrix semantics for $\mathsf{PAI}_{\mathsf{fde}}$, following the standard techniques of Abstract Algebraic Logic discussed, e.g., in [23]. Our second goal is to develop a simple sequent calculus for $\mathsf{PAI}_{\mathsf{fde}}$, following the techniques applied to obtain linguistically constrained sequent calculi presented, e.g., in [11], [12], [14], [24], and [45].

To this extent, the article is structured as follows. In Section 2 we provide a single logical matrix for $\mathsf{PAI}_{\mathsf{fde}}$, building on previous work with so-called infectious logics and paraconsistent subsystems of 3-valued weak Kleene logic. In Section 3 we analyze further aspects of the semantics, establishing its connections with certain structures called Płonka sums, and proving its potential to be

generalized in order to interpret other systems respecting Parry's Proscriptive Principle. In Section 4 we provide a Gentzen-style sequent calculus for $\mathsf{PAI_{fde}}$, taking a great deal of inspiration from work in calculi of this sort for infectious logics and for subsystems of PAI, like Angell's logic of Analytic Containment. In Section 5 we analyze some further aspects of the calculus, related to its so-called external consequence relation, and also to the potential to generalize its features so as to provide calculi for other logics satisfying the Proscriptive Principle. In Section 6 we finish up with some concluding remarks and directions of future work.

This being said, before delving into the proper contents of the article, let us briefly make explicit that we will be working with a propositional language $\mathcal{L}$ counting with a denumerable set Var of propositional variables $p, q, r, \ldots$ and with logical connectives $\neg, \wedge, \vee$—intended to represent negation, conjunction and disjunction, respectively. Thus, $\mathbf{FOR}(\mathcal{L})$ will be the algebra of well-formed formulae, standardly defined, whose carrier set is the set of well-formed formulae $FOR(\mathcal{L})$. In this respect, lower case Greek letters $\varphi, \psi, \chi, \ldots$ will be considered as schematic formulae, whereas upper case Greek letters $\Gamma, \Delta, \Theta, \ldots$ will be considered as schematic sets of formulae.

2. A simple logical matrix for Parry's logic

In this section, our goal is to provide matrix semantics for $\mathsf{PAI_{fde}}$, that is to say, the first-degree entailment fragment of Parry's logic of Analytic Implication. In this respect, it should be duly noted that semantics for Parry's full system have been previously given by K. Fine in [21], in the vein of Kripke frames with a subject-matter assignment function—thus employing intensional semantics. With respect to the first-degree fragment of PAI, extensional semantics have been given by T. Smiley in [43] and F. Paoli in [31], in the form of algebraic semantics—particularly, semantics where logical consequence is defined in terms of certain order-theoretic relations holding between the elements of the carrier set of a given algebra as, e.g., in L. Humberstone's [26, p. 246].

In what follows, we will endow $\mathsf{PAI_{fde}}$ with extensional semantics of a different and more common kind, that of the so-called *logical matrices*—extensively studied in the literature on Abstract Algebraic Logic, e.g., in [23]. In a nutshell, matrix semantics are given by a set of truth-values and an associated set of truth-tables describing the operations these values partake on, and the results thereof. Logical consequence is then defined as necessary preservation from premises to conclusion, of a certain subset of truth-values, called designated values. Whence, if the premise is designated, so must the conclusion.

More formally, for a given propositional language $\mathcal{L}$ a logical $\mathcal{L}$-matrix $\mathcal{M}$ is a pair $\langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D is a subset of A, the universe or carrier set of $\mathbf{A}$. Letting an $\mathcal{M}$-valuation v be an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, a logical matrix $\mathcal{M}$ induces a consequence relation $\vDash_{\mathcal{M}}$ in the following, standard manner, where $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \iff \text{ for every } \mathcal{M}\text{-valuation } v: \text{ if } v(\Gamma) \subseteq D, \text{ then } v(\varphi) \in D$$

Hereafter, when the consequence relation that is characteristic of a logic L is induced by a single matrix $\mathcal{M}$, we will take the liberty of referring to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$. For example, let $\mathbf{B}$ be the 2-element Boolean Algebra whose carrier set is $\{\mathbf{t}, \mathbf{f}\}$, and whose operations are depicted by the usual 2-valued truth-tables. Thus, $\langle \mathbf{B}, \{\mathbf{t}\}\rangle$ is the usual logical matrix associated with CL, inducing the well-known consequence relation of Classical Logic—which, in what follows, we will interchangeably refer to as $\vdash_{\mathsf{CL}}$ and $\vDash_{\mathsf{CL}}$. In the rest of this section, we examine one way of arriving at a simple matrix semantics for $\mathsf{PAI_{fde}}$, that is to say, at a logical matrix that will properly induce the target logic.

As remarked before and as noted elsewhere—e.g., by T. Ferguson in [18]—the satisfaction of the Proscriptive Principle by a consequence relation or a logic has as an immediate consequence the failure of the inference schema commonly referred to as *Addition*, that is, of the inference schema whose premise is φ and whose conclusion is $\varphi \vee \psi$. Thus, in the quest for appropriate matrix semantics for $\mathsf{PAI_{fde}}$ we may naturally look for logical matrices inducing consequence relations where Addition, as an inference schema, turns out to be invalid. The (paracomplete) weak Kleene logic K_3^w—also referred sometimes as the classical, internal, or $\{\neg, \wedge, \vee\}$-fragment of D. Bochvar's logic of nonsense from [3]—is famously well-known for being a system of this sort. This logic can be seen as induced by a single logical matrix built using the 3-element weak Kleene algebra, whose carrier set is $\{\mathbf{t}, \mathbf{e}, \mathbf{f}\}$ and whose operations $\neg, \wedge, \vee$ can be depicted by the following "truth-tables":

	$\neg$		$\wedge$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{f}$		$\vee$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$		$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{f}$		$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{e}$	$\mathbf{e}$		$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$		$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{t}$		$\mathbf{f}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{f}$		$\mathbf{f}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{f}$

Definition 2.1. K_3^w is the logic induced by the logical matrix $\langle \mathbf{WK}, \{\mathbf{t}\}\rangle$

As such, it is easily observed that Addition is invalid in K_3^w, i.e., $\varphi \nvDash_{\mathsf{K}_3^w} \varphi \vee \psi$. Just let φ be p and let ψ be q. Consider a weak Kleene valuation v such that $v(p) = \mathbf{t}$ and $v(q) = \mathbf{e}$. In such a valuation $v(p \vee q) = \mathbf{e}$. But then the premise is designated and the conclusion is not, whence the inference in question is invalid and, therefore, the schema is invalid too.

Thus, K_3^w being a subclassical system that invalidates Addition may lead one to wonder whether or not it coincides with $\mathsf{PAI_{fde}}$. It is straightforward to observe that it is not. This can be easily shown by noticing that said weak Kleene logic does not satisfy the Proscriptive Principle. To wit, K_3^w does not satisfy this principle because, among others, the inference schema usually referred to as *Explosion* is valid in it. In other words, that $\varphi \wedge \neg\varphi \vDash_{\mathsf{K}_3^w} \psi$. However, in what pertains to this inference schema it might well happen that $Var(\psi) \nsubseteq Var(\varphi \wedge \neg\varphi)$. Whence, K_3^w does not satisfy the Proscriptive Principle.

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Indeed, a more general fact concerning this 3-valued logic and the non-satisfaction of said principle holds. Let a formula φ be an *anti-theorem* if and only if φ entails every formula χ. As shown by A. Urquhart in [48], and detailed by us down below, the presence of anti-theorems is the only thing that differentiates K_3^w from our target logic, $\mathsf{PAI}_{\mathtt{fde}}$.

Observation 2.2 ([48]). *For all $\varphi, \psi \in FOR(\mathcal{L})$:*

$$\varphi \vDash_{\mathsf{K}_3^w} \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vDash_{\mathsf{CL}} \psi \ and \ Var(\psi) \subseteq Var(\varphi), \ or \\ \varphi \vDash_{\mathsf{CL}} \chi \ for \ all \ \chi \in FOR(\mathcal{L}) \end{cases}$$

Interestingly, this already points towards a way of satisfying the Proscriptive Principle in the context of certain subsystems of weak Kleene logic—as highlighted in [18]. To understand such a link, it is informative to observe that K_3^w belongs to a broader collection of so-called *infectious logics*, that have been the focus of a growing collection of recent works.[3] Intuitively, infectious logics can be seen as induced by logical matrices whose underlying algebras count with a contaminating, absorptive or—in the words of N. Rescher in [41, p. 29]— "infectious" element. That is to say, an element which is the output of every operation that it is an input of. We can give a more formal understanding of these ideas, as follows.

Definition 2.3. An algebra $\mathbf{A}$ has an *infectious* element u if and only if for every n-ary operation $\P$ of $\mathbf{A}$, and every $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } u \in \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = u$$

More formally, then, a logical matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ has an infectious truth-value if and only if $\mathbf{A}$ has an infectious element. Furthermore, if a logic L is induced by a single logical matrix $\mathcal{M}$ which has an infectious value, then we may say that L is an *infectious logic*. With these definitions in mind, the following result due to T. Ferguson, linking infectious sublogics of K_3^w and systems satisfying the Proscriptive Principle can now be understood.

Observation 2.4 ([18]). *Let L be a logic induced by the single matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ such that (i) $\mathbf{A}$ has an infectious element u, (ii) $u \notin D$, and (iii) L has no anti-theorems. Then, L satisfies the Proscriptive Principle.*

Thus, in terms of contemporary literature, one way to arrive at logics satisfying Parry's principle is to look at *paraconsistent infectious subsystems of weak Kleene logic*.

This result is of great significance in the quest for appropriate semantics for the first-degree entailment fragment of Parry's logic of Analytic Implication, as it certainly suggests a plausible path that will lead us to our goal. We will not claim that said path is the only, the best, or even the most efficient

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[3]The collection of recent works focused on these systems is growing by the hour, but just to name a few outside of those already cited elsewhere in this article: [6], [7], [9], [10], [11], [12], [13], [15], [17], [18], [19], [30], [33], [44], [45], [46], among many others.

way to arrive at such semantics, but only that it is the one suggested by the previous discussion.[4] In a few words, the idea would be to exploit the fact that anti-theorems are the only thing coming in between weak Kleene logic and $\mathsf{PAI_{fde}}$, by expanding the matrix inducing $\mathsf{K_3^w}$ with an additional truth-value, whose presence would prevent there being anti-theorems at all—without thereby having any other logical costs.

Special care should be put in exploring this alternative, though. Paraconsistent infectious sublogics of $\mathsf{K_3^w}$ have been studied in the literature that do, indeed, satisfy the Proscriptive Principle. However, all of them fail to pin down exactly the set of first-degree entailments that are valid in PAI. For instance, G. Priest's system $\mathsf{S_{fde}}$ discussed in [40] is a logic of this sort which, nevertheless, invalidates certain inferences that are valid in our target system $\mathsf{PAI_{fde}}$—prime among them *Disjunctive Syllogism* which, if the material conditional $\varphi \supset \psi$ is defined as $\neg\varphi \vee \psi$, amounts to *Modus Ponens*.

Therefore, if we intend to conceive $\mathsf{PAI_{fde}}$ as a paraconsistent infectious sublogic of $\mathsf{K_3^w}$, the logical matrix inducing it would need to *diverge as little and as seamlessly as possible* from that inducing weak Kleene logic. In fact, if such a matrix would count with an additional truth-value, its inclusion should have the effect of making all formulae satisfiable—thereby precluding the existence of anti-theorems—without any other problematic logical side-effect.

Below we show that this strategy can be happily carried out, with the help of a logical matrix built using a 4-element algebra found first (to the best of our knowledge) in F. Paoli's work [31].[5] The algebra in question will be, for obvious reasons, referred to us as the **FP** algebra. It's carrier set is $\{\mathbf{t}, \mathbf{o}, \mathbf{e}, \mathbf{f}\}$ and its operations $\neg, \wedge, \vee$ can be, as usual, described using the "truth-tables" appearing below:

	$\neg$		$\wedge$	$\mathbf{t}$	$\mathbf{o}$	$\mathbf{e}$	$\mathbf{f}$		$\vee$	$\mathbf{t}$	$\mathbf{o}$	$\mathbf{e}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$		$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$		$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{o}$	$\mathbf{o}$		$\mathbf{o}$	$\mathbf{e}$	$\mathbf{o}$	$\mathbf{e}$	$\mathbf{e}$		$\mathbf{o}$	$\mathbf{e}$	$\mathbf{o}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{e}$	$\mathbf{e}$		$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$		$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{t}$		$\mathbf{f}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$		$\mathbf{f}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$

For the purpose of showing that this algebra can be the basis of a logical matrix that will characterize $\mathsf{PAI_{fde}}$, we provide the following definition of a logic we will provisionally call $\mathsf{CL}[\mathbf{eo}]$, later proving that its formula-to-formula

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[4]To wit, there are other "purely semantic" ways of characterizing $\mathsf{PAI_{fde}}$ by appealing, in a broader sense, to logical matrices. Just to name one, consider the intersection of CL and the logic characterized by—using the terminology of E. Chemla, P. Égré and B. Spector in [8]—the "definedness" consequence relation over the **WK** algebra, induced by focusing on the single matrix $\langle\mathbf{WK}, \{\mathbf{t}, \mathbf{f}\}\rangle$. The latter consequence relation is such that φ entails ψ if and only if $Var(\psi) \subseteq Var(\varphi)$, whence the characterization already mentioned. Whether it is possible to obtain a single logical matrix out of the 2-valued semantics for CL and the 3-valued semantics for this logic—in the same vein in which K. Fine obtains a 16-valued logical matrix for Angell's $\mathsf{AC_{fde}}$ out of two 4-valued matrices in [22]—is an open question for us.

[5]It should be mentioned that F. Paoli employs this algebra to provide algebraic semantics, not matrix semantics, for the first-degree entailment fragment of R. Epstein's Relatedness logic from [16].

valid inferences are identical to those first-degree entailments that are valid in PAI. In other words, borrowing the terminology of L. Humberstone's [26], that the FMLA-FMLA fragment of $\mathsf{CL[eo]} = \mathsf{PAI_{fde}}$.[6]

Definition 2.5. $\mathsf{CL[eo]}$ is the logic induced by the logical matrix $\langle \mathbf{FP}, \{\mathbf{t}, \mathbf{o}\} \rangle$

Before we get into the proof that $\mathsf{CL[eo]}$ characterizes the first-degree entailments valid in PAI, let us note not only that it is a paraconsistent infectious sublogic of K_3^w, but also that its underlying logical matrix conforms to the requirements outlined above. First, that it is a paraconsistent logic can be noticed by observing that all classically-unsatisfiable formulae φ are satisfiable in $\mathsf{CL[eo]}$, simply by considering the valuation v such that $v(p) = \mathbf{o}$, for all $p \in Var(\varphi)$. Whence, $\mathsf{CL[eo]}$ has no anti-theorems. Secondly, that it is an infectious sublogic of K_3^w can be easily realized by noting that the truth-value $\mathbf{e}$ is an infectious element in the algebra $\mathbf{FP}$, and that it is undesignated in the corresponding logical matrix. Third, and finally, it can be observed that the logical matrix underlying $\mathsf{CL[eo]}$ diverges as little as possible from the logical matrix inducing K_3^w. In fact, this can be read off the "truth-tables" for $\mathbf{FP}$. While this algebra has an additional value compared to the 3-element weak Kleene algebra, said extra truth-value $\mathbf{o}$ follows as closely as possible the behavior of the infectious value $\mathbf{e}$—"mimicking" it in almost all contexts, as it were. Indeed, when the truth-value $\mathbf{o}$ is an input of an operation the result of such an operation is almost always the result that we would have if $\mathbf{o}$ were replaced by $\mathbf{e}$—except when all the inputs are $\mathbf{o}$, in which case the result is $\mathbf{o}$, too. We take these to be the technical incarnation of the idea that understanding $\mathsf{PAI_{fde}}$ as a paraconsistent infectious sublogic of K_3^w requires working with a logical matrix that would differ from that underlying weak Kleene logic as seamlessly as possible.

Technicalities aside, we should flag out that matters of interpretation regarding the truth-values underlying the algebra in question are indeed unclear. Thus, while in the context of the $\mathbf{WK}$ algebra it is common to interpret the truth-value $\mathbf{e}$ being assigned to a formula as the fact that said formula is meaningless or nonsensical, it is difficult to transpose this reading to the present context. In particular, it is difficult to understand what sort of phenomena is represented by the assignment of $\mathbf{o}$ to a given formula—an interpretative inconvenience already acknowledged by F. Paoli in [31]. Nevertheless, our goal in this section is not to provide a philosophically convincing matrix semantics for $\mathsf{PAI_{fde}}$, but rather to simply provide a semantics of this sort for it. Therefore, to proving that this can be done with the help of the matrix outlined before we now turn.

Theorem 2.6. *The* FMLA-FMLA *fragment of* $\mathsf{CL[eo]} = \mathsf{PAI_{fde}}$

Proof. Assume $\varphi \vDash_{\mathsf{CL}} \psi$ and $Var(\psi) \subseteq Var(\varphi)$, and suppose for *reductio* that $\varphi \nvDash_{\mathsf{CL[eo]}} \psi$. Then, there is a $\mathsf{CL[eo]}$-valuation v such that $v(\varphi) \in \{\mathbf{t}, \mathbf{o}\}$ and

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[6]The FMLA-FMLA fragment of a logic L is the restriction of L to what is called, e.g., in [26, p. 108] the FMLA-FMLA framework. That is to say, set of inferences that are valid in such a logic which have only one formula as a premise and only one formula as a conclusion.

$v(\psi) \in \{\mathbf{e}, \mathbf{f}\}$. In case $v(\varphi) = \mathbf{t}$ this would allow us to infer, given the operations in **FP**, that for all $p \in Var(\varphi)$, $v(p) \in \{\mathbf{t}, \mathbf{f}\}$. From that and our assumptions, we could moreover infer that for all $q \in Var(\psi)$, $v(q) \in \{\mathbf{t}, \mathbf{f}\}$. In other words, v must be a valuation such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$, whence $\varphi \nvDash_{\mathsf{CL}} \psi$. But this contradicts our initial assumption, from which we infer that $v(\varphi) \neq \mathbf{t}$. Then, in case $v(\varphi) = \mathbf{o}$, given the operations in **FP**, we could infer that for all $p \in Var(\varphi)$, $v(p) = \mathbf{o}$. Moreover, our supposition that $v(\psi) \in \{\mathbf{e}, \mathbf{f}\}$ implies there is some $q \in Var(\psi)$ such that $v(q) \neq \mathbf{o}$. However, this contradicts our initial assumption that $Var(\psi) \subseteq Var(\varphi)$, whence we should infer that $v(\varphi) \neq \mathbf{o}$. Therefore, if $\varphi \vDash_{\mathsf{CL}} \psi$ and $Var(\psi) \subseteq Var(\varphi)$, then there cannot be a $\mathsf{CL}[\mathbf{eo}]$-valuation v such that $v(\varphi) \in \{\mathbf{t}, \mathbf{o}\}$ and $v(\psi) \in \{\mathbf{e}, \mathbf{f}\}$. This establishes that $\varphi \vDash_{\mathsf{CL}[\mathbf{eo}]} \psi$.

Assume either $\varphi \nvDash_{\mathsf{CL}} \psi$, or $Var(\psi) \nsubseteq Var(\varphi)$. If the former is the case, then there is a CL-valuation v such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$. However, given CL-valuations are a subset of $\mathsf{CL}[\mathbf{eo}]$-valuations, this establishes that there is a $\mathsf{CL}[\mathbf{eo}]$-valuation v' such that $v'(\varphi) = \mathbf{t}$ and $v'(\psi) = \mathbf{f}$. From this it follows that $\varphi \nvDash_{\mathsf{CL}[\mathbf{eo}]} \psi$. If the latter is the case, it is possible to construct a $\mathsf{CL}[\mathbf{eo}]$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{e} & \text{if } p \in Var(\psi) \setminus Var(\varphi) \\ \mathbf{o} & \text{otherwise} \end{cases}$$

For such a valuation it is possible to show, through a simple induction on the complexity of formulae, that $v(\varphi) = \mathbf{o}$ while $v(\psi) = \mathbf{e}$. From this it follows that $\varphi \nvDash_{\mathsf{CL}[\mathbf{eo}]} \psi$. □

Thus, knowing that the Fmla-Fmla fragment of $\mathsf{CL}[\mathbf{eo}] = \mathsf{PAI_{fde}}$ in what follows we will allow ourselves to refer to the Fmla-Fmla fragment of $\vDash_{\mathsf{CL}[\mathbf{eo}]}$ as $\vDash_{\mathsf{PAI_{fde}}}$, and to similarly refer to the 4-valued $\mathsf{CL}[\mathbf{eo}]$-valuations as $\mathsf{PAI_{fde}}$-valuations, without loss of generality.

Having provided a semantic characterization of $\mathsf{PAI_{fde}}$ in terms of a 4-valued logical matrix, in the next section we discuss with more detail some aspects of the employed algebraic structure, connecting it with the so-called Płonka sums and offering potential generalizations for other similar logical systems.

3.   Further remarks on the semantics

The aim of this section is to delve a little bit deeper into some aspects of the semantics just offered. To this end, we divide it into two subsections. In the first, we discuss the extent to which some constructions called Płonka sums can be used to better understand the matrix semantics presented above. In the second, we show that some features of said semantics can be generalized in order to provide matrices for similar fragments of other logics—fragments that also respect Parry's Proscriptive Principle.[7]

3.1. Płonka sums

Płonka sums are operations on algebras that allow to define new structures from a previously given collection thereof. As stated in [5], they are useful tools to study the so-called regular varieties, that is to say, equational classes axiomatized by equations that contain the same variables on both sides. But this is something we will not discuss in-depth here. Instead, in what follows, we will show the extent to which the aforementioned algebra **FP** featured in our matrix semantics for $\mathsf{PAI_{fde}}$ can be understood as the Płonka sum of other algebras—more precisely, of Boolean algebras. For this purpose, we will detail some pieces of terminology present in some works in Abstract Algebraic Logic first.

Before moving on, it may as well be noted that **FP** is a special kind of algebra. It is what F. Paoli and M. Pra Baldi call in [32] a *generalized involutive bisemilattice*. A structure of this sort is an algebra counting with two partial orders (in our case, where $x, y \in \{\mathbf{t}, \mathbf{o}, \mathbf{e}, \mathbf{f}\}$, the orders $\leq_\wedge$ and $\leq_\vee$ respectively defined by letting $x \leq_\wedge y$ if and only if $x \wedge y = x$, and $x \leq_\vee y$ if and only if $x \vee y = y$), with an involution (in our case, the operation $\neg$), with no constants for the infimum and the supremum elements. Interestingly enough, these authors show that every generalized involutive bisemilattice is decomposable as a Płonka sum over a semilattice direct system of Boolean algebras [32, Theorem 4]. Whence, our comments below oriented at representing **FP** in this way provide an exemplification of their general result. In this vein, then, let us proceed in detailing some crucial definitions.

Definition 3.1 ([4, 5, 6, 32]). A *semilattice direct system* of algebras of language $\mathcal{L}$ is a triple $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{f_{ij} \mid i \leq_{\mathbf{I}} j\} \rangle$, where:

(i) $\mathbf{I} = \langle I, \leq_{\mathbf{I}} \rangle$ is a join-semilattice,

(ii) $\{\mathbf{A}_i\}_{i \in I}$ is a family of similar algebras of language $\mathcal{L}$ with parwise disjoint universes,

(iii) for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, f_{ij} is an homomorphism from $\mathbf{A}_i$ to $\mathbf{A}_j$, and moreover f_{ii} is the identity map for every $i \in I$, and whenever $i \leq_{\mathbf{I}} j \leq_{\mathbf{I}} k$, we have that $f_{ik} = f_{jk} \circ f_{ij}$.

Definition 3.2 ([4, 5, 6, 32]). The *Płonka sum* $\mathrm{Pl}(\mathbb{A})$ of a semilattice direct system of algebras $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{f_{ij} \mid i \leq_{\mathbf{I}} j\} \rangle$ is the algebra of language $\mathcal{L}$ whose universe is $\bigcup_{i \in \mathbf{I}} A_i$, and for which every basic n-ary operation $\P$ of $\mathbf{A}$ (with $n \geq 1$) is defined as follows, where $a_1, \ldots, a_n \in \bigcup_{i \in \mathbf{I}} A_i$:

$$\P^{\mathrm{Pl}(\mathbb{A})}(a_1, \ldots, a_n) = \P^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n))$$

where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_m$.

With these tools at hand, it is routine to show that the algebra **FP** can be obtained as a Płonka sum of Boolean algebras. To see this, let $\mathbf{A}_i$ denote the 2-element Boolean algebra with universe $\{\mathbf{t}, \mathbf{f}\}$, and let $\mathbf{A}_j$ and $\mathbf{A}_{i \vee j}$ denote

two disjoint but isomorphic 1-element trivial Boolean algebras with universes, respectively, $\{\mathbf{o}\}$ and $\{\mathbf{e}\}$. Then, consider the semilattice direct system $\mathbb{A}' = \langle \{\mathbf{A}_i,\}_{i \in I'}, \mathbf{I}', \{f_{ij} \mid i \leq_{\mathbf{I}'} j\}\rangle$, where $\mathbf{I}'$ is the join-semilattice depicted by the following Hasse diagram:

$$
\begin{array}{c}
i \vee j \\
\diagup \qquad \diagdown \\
i \qquad\qquad\qquad j
\end{array}
$$
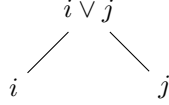
Given these elements, it is straightforward to observe that **FP** can be conceived as $\mathrm{Pl}(\mathbb{A}')$, the Płonka sum of the aforementioned semilattice direct system of Boolean algebras. Notice, furthermore, that the mappings in $\{f_{ij} \mid i \leq_{\mathbf{I}'} j\}$ are unique, whence such a sum is uniquely determined.

Interestingly, many scholars have shown that Płonka sums can not only be applied to certain collections of algebras, but also to collections of logical matrices. Thus, one may wonder whether our own matrix $\langle \mathbf{FP}, \{\mathbf{t}, \mathbf{o}\}\rangle$ for $\mathsf{PAI_{fde}}$ can be obtained as the Płonka sum of a previously given system of matrices. Considering this question already requires noting that Płonka sums for matrices are not uniquely defined, but rather defined for different kinds of systems of logical matrices. Up to now, these comprise the so-called direct systems of logical matrices and the r-direct systems of logical matrices, which we define below.

Definition 3.3 ([4, 5])**.** A *direct* (alternatively, *r-direct*) *system* of logical matrices of language $\mathcal{L}$ is a triple $\mathbb{M} = \langle \{\langle \mathbf{A}_i, D_i\rangle\}_{i \in I}, \mathbf{I}, \{f_{ij} \mid i \leq_{\mathbf{I}} j\}\rangle$, where:

(i) $\mathbf{I} = \langle I, \leq_{\mathbf{I}}\rangle$ is a join-semilattice,

(ii) $\{\langle \mathbf{A}_i, D_i\rangle\}_{i \in I}$ is a family of matrices with parwise disjoint universes (alternatively, where also $\mathbf{I}^+ = \{i \in I \mid D_i \neq \emptyset\}$ is a sub-semilattice of $\mathbf{I}$),

(iii) for every $i, j \in I$ where $i \leq_{\mathbf{I}} j$, f_{ij} is an homomorphism from $\mathbf{A}_i$ to $\mathbf{A}_j$, such that $f_{ij}(D_i) \subseteq D_j$ (alternatively, such that $f_{ij}^{-1}(D_j) = D_i$ whenever $D_j \neq \emptyset$), and moreover f_{ii} is the identity map for every $i \in I$, and whenever $i \leq_{\mathbf{I}} j \leq_{\mathbf{I}} k$, we have that $f_{ik} = f_{jk} \circ f_{ij}$.

Definition 3.4 ([4, 5])**.** The *Płonka sum* $\mathrm{Pl}(\mathbb{M})$ of a direct (alternatively, r-direct) system of logical matrices $\mathbb{M} = \langle \{\langle \mathbf{A}_i, D_i\rangle\}_{i \in I}, \mathbf{I}, \{f_{ij} \mid i \leq_{\mathbf{I}} j\}\rangle$ is the logical matrix $\langle \mathrm{Pl}(\mathbb{A}), \bigcup_{i \in I} D_i\rangle$ of language $\mathcal{L}$, where $\mathrm{Pl}(\mathbb{A})$ is the Płonka sum of the semilattice direct system of algebras formed by the algebra-reducts of the logical matrices in $\mathbb{M}$.

In this regard, it is important to notice that given that **FP** can be uniquely determined as the Płonka sum of the aforementioned semilattice direct system of algebras, this limits our choices of direct (r-direct) systems of matrices that would induce the matrix $\langle \mathbf{FP}, \{\mathbf{t}, \mathbf{o}\}\rangle$. In particular, the only reasonable choice

given our previous remarks would be the system $\mathbb{M}' = \langle \{\langle \mathbf{A}_i, D_i\rangle\}_{i \in I}, \mathbf{I}', \{f_{ij} \mid i \leq_\mathbf{I} j\}\rangle$, where $\mathcal{M}_i = \langle \mathbf{A}_i, \{\mathbf{t}\}\rangle$, $\mathcal{M}_j = \langle \mathbf{A}_j, \{\mathbf{o}\}\rangle$, $\mathcal{M}_{i \vee j} = \langle \mathbf{A}_{i \vee j}, \emptyset \rangle$—where all these elements are as in the system $\mathbb{A}'$ above.

However, such a family of logical matrices *cannot constitute a direct system* of logical matrices because it does not fulfill clause (iii) of Definition 3.3. To see this, notice that the requirements that $f_{i,i \vee j}(D_i) \subseteq D_{i \vee j}$ and $f_{j,i \vee j}(D_j) \subseteq D_{i \vee j}$ would translate both into the condition $\{\mathbf{e}\} \subseteq \emptyset$, which cannot be true. Similarly, the family of matrices $\mathbb{M}'$ *cannot constitute a r-direct system* of logical matrices because it does not fulfill clause (ii) of Definition 3.3. To see this, notice that the parenthetical requirement asks that the set of indices of logical matrices in the system counting with non-empty sets of designated values should form a sub-semilattice of the original semilattice of indices, which is not the case here, since the set $\{i, j\}$ does not satisfy this condition. Whence, our own matrix semantics for $\mathsf{PAI_{fde}}$ cannot—under the currently available definitions—be seen as the Płonka sum of a direct or r-direct system of logical matrices.

This, of course, does not prevent the exploration of further accounts of systems of logical matrices that would accommodate this issue, allowing us to look at our matrix as the Płonka sum of an appropriately conceived system of logical matrices. To wit, the notion of r-direct system of logical matrices was only recently developed by S. Bonzio and M. Pra Baldi in [4], in order to express certain logical matrices as Płonka sums—something that was not possible before their introduction of such a novel definition. It would be extremely interesting to explore more possibilities along this path in the near future.

In line with these considerations connecting our semantics with Płonka sums and some other elements of Abstract Algebraic Logic, one may wonder what the answer to a plethora of other questions are. For instance, regarding our matrix semantics for $\mathsf{PAI_{fde}}$ we could name the following: what does a full description of its matrix models look like, and which are its reduced models? What are its quasivariety semantics, and is it possible to provide algebraic versions of the completeness and Cut Elimination results for it? All these questions are deeply intriguing, and it would be immensely interesting to explore them. However, discussing them here would take us a little bit too far afield, and would require more space than we have. This being said, we hope to tackle them in future work.

3.2. Some generalizations

The matrix semantics for $\mathsf{PAI_{fde}}$ that we presented above provides a way to approach an interesting fragment of Classical Logic—i.e., those Fmla-Fmla valid inferences of Classical Logic that also respect Parry's Proscriptive Principle. In other words, according to the definition below, it provides a matrix semantics for the Fmla-Fmla fragment of $\mathsf{CL_{PP}}$.

Definition 3.5 ([18])**.** Given a logic L, the Fmla-Fmla fragment of L that satisfies Parry's Proscriptive Principle is the Fmla-Fmla *fragment of* $\mathsf{L_{PP}}$, which can be defined as follows:

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$$\varphi \vdash_{\mathsf{L_{PP}}} \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{L}} \psi, \text{ and} \\ Var(\psi) \subseteq Var(\varphi) \end{cases}$$

In what follows we show that the elements of our semantics can be useful in offering—given an arbitrary matrix logic L—a matrix semantics for $\mathsf{L_{PP}}$. To show this, let us first fix some terminology.

Definition 3.6. An algebra $\mathbf{A}$ has distinct elements $j, k \in A$ such that j "mimics" k if and only if for all n-ary operations $\P$ of $\mathbf{A}$ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{j\} \subsetneq \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = \P^{\mathbf{A}}((a_1, \ldots, a_n)[j/k])$$

where $(a_1, \ldots, a_n)[j/k]$ is the result of replacing each occurrence of j for an occurrence of k in $a_1, \ldots, a_n$.

Definition 3.7. An algebra $\mathbf{A}$ has a universally idempotent element j if and only if for all n-ary operations $\P$ of $\mathbf{A}$ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{j\} = \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = j$$

We now move on to consider extensions of algebras with infectious or universally idempotent and mimicking elements.

Definition 3.8 ([12]). Given an algebra $\mathbf{A}$, the algebra $\mathbf{A}[u]$ is its extension with an infectious element $u \notin A$, such that for all n-ary operations $\P$ of $\mathbf{A}[u]$ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{u\}$:

$$\P^{\mathbf{A}[u]}(a_1, \ldots, a_n) = \begin{cases} u & \text{if } u \in \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

Definition 3.9. Given an algebra $\mathbf{A}$, the algebra $\mathbf{A}[j]$ is its extension with a universally idempotent element $j \notin A$ that "mimics" an element $k \in A$, such that for all n-ary operations $\P$ of $\mathbf{A}[j]$ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{j\}$:

$$\P^{\mathbf{A}[j]}(a_1, \ldots, a_n) = \begin{cases} j & \text{if } \{j\} = \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}((a_1, \ldots, a_n)[j/k]) & \text{if } \{j\} \subsetneq \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

where $(a_1, \ldots, a_n)[j/k]$ is the result of replacing each occurrence of j for an occurrence of k in $a_1, \ldots, a_n$.

Below, we will show that given a matrix logic L induced by a single logical matrix, extending it in the way in which our semantics for $\mathsf{PAI_{fde}}$ extend the semantics for Classical Logic, gives a semantics for $\mathsf{L_{PP}}$—regardless of whether L counts with anti-theorems or not. This result is, in fact, a generalization of Theorem 2.6, as a closer look into the proof will reveal.

Theorem 3.10. *Let* L *be a logic (possibly counting with anti-theorems) induced by the single matrix* $\mathcal{M} = \langle \mathbf{A}, D \rangle$*, and let* L$[uj]$ *be the logic induced by the single matrix* $\mathcal{M}[uj] = \langle \mathbf{A}[uj], D \cup \{j\} \rangle$*, where* $\mathbf{A}[u]$ *is the algebra resulting from extending* $\mathbf{A}$ *with an infectious element* u*, and* $\mathbf{A}[uj]$ *is the algebra resulting from extending* $\mathbf{A}[u]$ *with a universally idempotent element* j *that "mimics" the infectious element* u*. Then, the* FMLA-FMLA *fragment of* L$[uj]$ = L$_{\mathsf{PP}}$*.*

Proof. Assume $\varphi \vDash_{\mathsf{L}} \psi$ and $Var(\psi) \subseteq Var(\varphi)$, and suppose for *reductio* that $\varphi \nvDash_{\mathsf{L}[uj]} \psi$. Then, there is a L$[uj]$-valuation v such that $v(\varphi) \in D \cup \{j\}$ and $v(\psi) \notin D \cup \{j\}$. In case $v(\varphi) \in D$ this would allow us to infer that $v(\varphi) \notin \{u, j\}$ and, given the operations in $\mathbf{A}[uj]$, that for all $p \in Var(\varphi)$, $v(p) \notin \{u, j\}$. From that and our assumptions, we could moreover infer that for all $q \in Var(\psi)$, $v(q) \notin \{u, j\}$. In other words, v must be a valuation such that $v(\varphi) \in D$ and $v(\psi) \notin D$, whence $\varphi \nvDash_{\mathsf{L}} \psi$. But this contradicts our initial assumption, from which we infer that $v(\varphi) \notin D$. Then, in case $v(\varphi) = j$, given the operations in $\mathbf{A}[uj]$, we could infer that for all $p \in Var(\varphi)$, $v(p) = j$. Moreover, our supposition that $v(\psi) \notin D \cup \{j\}$ implies there is some $q \in Var(\psi)$ such that $v(q) \neq j$. However, this contradicts our initial assumption that $Var(\psi) \subseteq Var(\varphi)$, whence we should infer that $v(\varphi) \neq j$. Therefore, if $\varphi \vDash_{\mathsf{L}} \psi$ and $Var(\psi) \subseteq Var(\varphi)$, then there cannot be a L$[uj]$-valuation v such that $v(\varphi) \in D \cup \{j\}$ and $v(\psi) \notin D \cup \{j\}$. This establishes that $\varphi \vDash_{\mathsf{L}[uj]} \psi$.

Assume either $\varphi \nvDash_{\mathsf{L}} \psi$, or $Var(\psi) \nsubseteq Var(\varphi)$. If the former is the case, then there is a L-valuation v such that $v(\varphi) \in D$ and $v(\psi) \notin D$. However, given L-valuations are a subset of L$[uj]$-valuations, this establishes that there is a L$[uj]$-valuation v' such that $v'(\varphi) \in D$ and $v'(\psi) \notin D$. From this it follows that $\varphi \nvDash_{\mathsf{L}[uj]} \psi$. If the latter is the case, it is possible to construct a L$[uj]$-valuation v such that:

$$v(p) = \begin{cases} u & \text{if } p \in Var(\psi) \setminus Var(\varphi) \\ j & \text{otherwise} \end{cases}$$

For such a valuation it is possible to show, through a simple induction on the complexity of formulae, that $v(\varphi) = j$ while $v(\psi) = u$. From this it follows that $\varphi \nvDash_{\mathsf{L}[uj]} \psi$. $\qquad \square$

A number of things can be said as closing remarks to this section. First, the previous results suggest semantics for L$_{\mathsf{PP}}$ where L is a sublogic of Classical Logic counting with anti-theorems, like strong Kleene logic K$_3$. Semantics of this sort were already available for anti-theoremless sublogics of Classical Logic, like G. Priest's LP, but were not found anywhere for sublogics of this kind that have anti-theorems. Secondly, and more importantly, the above definitions serves as a technical vindication of the intuitions laid out in the previous section, when reflecting upon the necessary ingredients to arrive at a matrix semantics for PAI$_{\mathtt{fde}}$ starting from matrix semantics for K$_3^w$. More particularly, this allows to understand the **FP** algebra as the algebra $\mathbf{B}[eo]$, where $\mathbf{B}$ is the 2-element Boolean algebra, $\mathbf{B}[e]$ is the extension of $\mathbf{B}$ with an infectious element e, and $\mathbf{B}[eo]$ is the extension of $\mathbf{B}[e]$ with a universally idempotent element $\mathbf{o}$ that

mimics the infectious element **e**. Furthermore, these remarks explain the notation for CL[**eo**], whose FMLA-FMLA fragment coincides—as we proved in the previous section—with $\mathsf{PAI_{fde}}$.

Having said all this, we now proceed to the task of endowing our target logic with a simple sequent calculus. In doing so, we will make use of the semantic developments above, by employing them both to prove the soundness and completeness of the target proof theory.

4. A simple sequent calculus for Parry's logic

In this section we provide a sound and complete sequent calculus for $\mathsf{PAI_{fde}}$, that is to say, for the set of first-degree entailments valid in Parry's logic of Analytic Implication. Proof-systems for Parry's logic as a whole have been given by W. T. Parry himself in [35] and, as regards $\mathsf{PAI_{fde}}$, although there is no axiom system that characterizes it, a natural deduction calculus has been presented by F. Johnson in [28].

Here, with the purpose of endowing our target logic with a Gentzen-style sequent calculus we will follow the ideas and techniques discussed by R. French in [24], where a calculus of this sort is given for K. Fine's axiomatization of the first-degree entailments that are valid in R. Angell's logic of Analytic Containment AC—in other words, for $\mathsf{AC_{fde}}$. For this task, we will work with sequents of the form $\Gamma \succ \Delta$ defined as pairs $\langle \Gamma, \Delta \rangle$ where Γ and Δ are finite multisets of formulae. In this context, sequents will receive a concrete interpretation, as we will establish that $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in the target calculus if and only if the first-degree entailment $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi_1 \vee \cdots \vee \psi_m$ is valid in PAI—in other words, if and only if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$. It will be important to bear this in mind when conducting the soundness and completeness proofs.

The main idea behind the calculus that we introduce below—borrowed from R. French's [24] and from M. I. Corbalán and M. Coniglio's [14]—is to have sequent rules (be it initial sequents, operational or structural rules), that are bound to linguistic restrictions. That is to say, rules that can be applied only if certain constraints regarding the parametric or active formulae are met. These restrictions guarantee that the rules preserve the satisfaction of the Proscriptive Principle or, put differently, that the rules are subject-matter preserving. This is completely analogous to what is done in [24], although the conception of subject-matter underlying R. Angell's logic (and thus the linguistic restrictions needed to secure subject-matter preservation) are different in that case.[8]

We will now proceed to present the set of rules that define our calculus $\mathcal{G}_{\mathsf{PAI_{fde}}}$, later showing the adequacy of the formalism. In what follows it will be important

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[8]As witnessed by the remarks below, R. French states in [24] that given Angell's conception of the subject-matter of a given formula φ, such a subject-matter can be formalized as $t(\varphi) = \{p \in Var \mid p$ appears in φ under the scope of an even number of negations$\} \cup \{p \in Var \mid p$ appears in φ under the scope of an odd number of negations$\}$—whereas we should remember that, as discussed above, given the conception of subject-matter underlying Parry's logic, such a subject-matter can be formalized as $Var(\varphi)$.

to state explicitly that *Lit* will denote the set of *literals* of our language $\mathcal{L}$, that is, the collection of all the propositional variables $p, q, r, \ldots$ and their respective negations $\neg p, \neg q, \neg r, \ldots$ —letting $p^{\pm}, q^{\pm}, r^{\pm}, \ldots$ denote schematic literals. Let us note, in passing, that for $\Theta \subseteq FOR(\mathcal{L})$, $Var(\Theta) = \bigcup_{\theta \in \Theta} Var(\theta)$.

Definition 4.1. The calculus $\mathcal{G}_{\mathsf{PAI_{fde}}}$ is constituted by the following rules:

Initial Sequents:

$$[Initial] \quad \Gamma, p \succ p, \Delta \quad \text{where } Var(\Delta, p) \subseteq Var(\Gamma, p)$$

Structural Rules:

$$\frac{\Gamma, \varphi, \varphi \succ \Delta}{\Gamma, \varphi \succ \Delta} \ [WL] \qquad\qquad \frac{\Gamma \succ \varphi, \varphi, \Delta}{\Gamma \succ \varphi, \Delta} \ [WR]$$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma' \succ \varphi, \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'} \ [Cut]$$

Operational Rules:

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta} \ [\neg L]^{\dagger} \qquad\qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta} \ [\neg R]^{\ddagger}$$

$$\dagger: \ \text{where } \Delta \neq \emptyset \qquad\qquad \ddagger: \ \text{where } Var(\varphi) \subseteq Var(\Gamma)$$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma, \psi \succ \Delta}{\Gamma, \varphi \vee \psi \succ \Delta} \ [\vee L] \qquad\qquad \frac{\Gamma \succ \varphi, \psi, \Delta}{\Gamma \succ \varphi \vee \psi, \Delta} \ [\vee R]$$

$$\frac{\Gamma, \varphi, \psi \succ \Delta}{\Gamma, \varphi \wedge \psi \succ \Delta} \ [\wedge L] \qquad\qquad \frac{\Gamma \succ \varphi, \Delta \quad \Gamma \succ \psi, \Delta}{\Gamma \succ \varphi \wedge \psi, \Delta} \ [\wedge R]$$

"De Morgan" Rules:

$$\frac{\Gamma, \varphi \succ \Delta}{\Gamma, \neg\neg\varphi \succ \Delta} \ [\neg\neg L] \qquad\qquad \frac{\Gamma \succ \varphi, \Delta}{\Gamma \succ \neg\neg\varphi, \Delta} \ [\neg\neg R]$$

$$\frac{\Gamma, \neg\varphi \succ \Delta \quad \Gamma, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \wedge \psi) \succ \Delta} \ [\neg\wedge L] \qquad\qquad \frac{\Gamma \succ \neg\varphi, \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \wedge \psi), \Delta} \ [\neg\wedge R]$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \vee \psi) \succ \Delta} \ [\neg\vee L] \qquad\qquad \frac{\Gamma \succ \neg\varphi, \Delta \quad \Gamma \succ \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \vee \psi), \Delta} \ [\neg\vee R]$$

Let us make a few remarks about the calculus we just presented. As we shall see later using a simple argument stemming from the way in which we prove

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our completeness result below, the rule of Cut is eliminable from $\mathcal{G}_{\mathsf{PAI_{fde}}}$. Also regarding the structural rules, it shall be noted that the $[Initial]$ is a form of the structural rule of Identity or Reflexivity, with Left and Right Weakening "absorbed"—to some extent. Indeed, as we will show below, adopting these initial sequents allows for the rule of Left Weakening to be admissible in its unrestricted form, while allowing for the admissibility of a constrained rule of Right Weakening, guaranteeing the satisfaction of the Proscriptive Principle.

Observation 4.2. *The following forms of the Weakening rules are admissible in $\mathcal{G}_{\mathsf{PAI_{fde}}}$:*

$$\frac{\Gamma \succ \Delta}{\Gamma, \varphi \succ \Delta} \; [KL] \qquad\qquad\qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \varphi, \Delta} \; [KR]^{\ddagger}$$

$$\ddagger : \; where \; Var(\varphi) \subseteq Var(\Gamma)$$

Proof. The proof is a straightforward modification of R. French's proof of his Theorem 3.6 in [24], for an analogous fact regarding the admissibility of another restricted form of the Weakening rules in his calculus for Angell's logic of Analytic Containment.[9] Moving on to the proper proof, regarding $[KL]$, suppose we have a derivation of $\Gamma \succ \Delta$. We can turn this into a derivation of $\Gamma, \varphi \succ \Delta$ by adding φ to the left-hand side of each of the nodes of the derivation, as the uppermost node will still constitute a rightful instance of $[Initial]$. Similarly, regarding $[KR]^{\ddagger}$, suppose we have a derivation of $\Gamma \succ \Delta$ and that $Var(\varphi) \subseteq Var(\Gamma)$. We can turn this into a derivation of $\Gamma \succ \varphi, \Delta$ by, first, adding Γ to the left-hand side of each of the nodes of the derivation, and φ to the right-hand side of each of the nodes of the derivation. In such a case, the uppermost node will still constitute a rightful instance of $[Initial]$, although we will end up with an endsequent of the form $\Gamma, \Gamma \succ \Delta, \varphi$. Secondly, to turn this into a derivation of $\Gamma \succ \varphi, \Delta$, we apply $|\Gamma|$-many times the rule $[WL]$ in order to arrive at our target sequent $\Gamma \succ \varphi, \Delta$. $\qquad\qquad\square$

An additional point of interest noticed by an anonymous reviewer, resides in comparing our calculus with those proposed for the weak Kleene logics, in our case, the calculus for paracomplete weak Kleene logic K_3^w presented in [14] being the relevant one. In this respect, it is important to highlight four items.

First, as mentioned earlier, our calculus has axioms with contexts or side-formulae which "absorb" the Weakening rules—in their constrained admissible forms stated above. In the case of the calculus for K_3^w, axioms have no contexts or side-formulae and the Weakening rules are primitive. But nothing really important depends on this, for axioms could have been formulated with such contexts, thereby "absorbing" the Weakening rules in their unconstrained forms. This brings us to a *second*, already visible issue, regarding the Weakening rules.

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[9]To be more precise: the only modification needed for his argument to apply to our case lies in replacing the mention of $t(A)$ and $t(\Gamma)$ in the proof of the aforementioned theorem, respectively, for $Var(\varphi)$ and $Var(\Gamma)$.

Whereas in our calculus the only admissible form of such rules is the one where the Right Weakening rule is constrained, in the calculus for K_3^w both Weakening rules are unconstrained. A *third* focal point can be placed over the Disjunction rules, especially the $[\vee R]$ rule. Thus, while it has no restrictions in our calculus, in the calculus for K_3^w it does have a restriction, requiring that $Var(\varphi, \psi) \subseteq Var(\Gamma)$. The reason for this divergence resides in the different strengths that the Weakening rules, especially the $[KR]$ rule, have in both systems. Thus, in our case there is no instance where the premise of the $[\vee R]$ rule that can violate the Proscriptive Principle, because the admissible forms of the $[KR]$ rule would not allow for such a thing. In the case of the calculus for K_3^w, however, there could be a case where the premise of the $[\vee R]$ rule satisfies the Proscriptive Principle without its conclusion doing so, precisely because one of the disjuncts could be obtained by an application of the unrestricted (and, therefore, not necessarily Proscriptive Principle-complying) $[KR]$ rule. A *fourth* and final comparison point can be considered around the $[\neg L]$ rule. Traditionally, having this rule in its unrestricted form corresponds to the validity of Explosion in the logic in question. However, given the remarks made in Section 2 it is clear that such an inference is not generally valid in $\mathsf{PAI}_{\mathtt{fde}}$, even though it is valid in K_3^w. In this vein, the only forms of Explosion that are valid in our logic are those respecting the Proscriptive Principle—that is to say, those inferences $\varphi \wedge \neg\varphi \vDash \psi$ where $Var(\psi) \subseteq Var(\varphi)$. Concomitantly, this explains the presence of the unrestricted $[\neg L]$ rule in the calculus for K_3^w, and the presence of a rule in our calculus which is restricted precisely because of the constraints imposed to the $[KR]$ rule, which guarantee the satisfaction of the Proscriptive Principle.[10]

This being said, let us deal with the proof of the soundness result for our calculus. Thus, the next result we discuss shows that every provable sequent of $\mathcal{G}_{\mathsf{PAI}_{\mathtt{fde}}}$ encodes a corresponding first-degree entailment that is valid in PAI—or, what is the same, a valid inference of $\mathsf{PAI}_{\mathtt{fde}}$. For the purpose of proving this, we will appeal to the characterization of said set of valid entailments in the paragraphs above.

Lemma 4.3. *All the rules of $\mathcal{G}_{\mathsf{PAI}_{\mathtt{fde}}}$ preserve $\mathsf{PAI}_{\mathtt{fde}}$-validity. In other words, for each of the rules of the calculus, if the premise sequents are valid in $\mathsf{PAI}_{\mathtt{fde}}$, so is the conclusion sequent of that rule.*

Proof. We show this by cases—focusing on the restricted rules and leaving the rest as exercises to the reader—assuming the premise sequents of a rule are valid in $\mathsf{PAI}_{\mathtt{fde}}$, and later proving that its conclusion sequent is also valid in said logic. In all cases below, we will assume that Γ can be redescribed as $\gamma_1, \ldots, \gamma_n$, and that Δ can be redescribed as $\delta_1, \ldots, \delta_m$.

> [*Initial*]: Suppose v is a $\mathsf{PAI}_{\mathtt{fde}}$-valuation such that $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge p) \subseteq \{\mathbf{t}, \mathbf{o}\}$ and that $Var(\Delta, p) \subseteq Var(\Gamma, p)$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge p) = \mathbf{t}$, then

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[10]A further, though inessential, divergence from our calculus and the calculus for K_3^w resides in the presence of the so-called De Morgan rules in the former and their absence in the latter. This difference is inessential, because although they make things more manageable for the completeness proof of our own calculus, they are derivable in it as well as in the calculus for K_3^w.

all $q \in Var(\Gamma, p)$ are such that $v(q) \in \{\mathbf{t}, \mathbf{f}\}$ and in particular $v(p) = \mathbf{t}$. Whence, $v(p \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{t}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge p) = \mathbf{o}$, then all $q \in Var(\Gamma, p)$ are such that $v(q) = \mathbf{o}$. Given $Var(\Delta, p) \subseteq Var(\Gamma, p)$, we know that all $q' \in Var(\Delta, p)$ are such that $v(q') = \mathbf{o}$. Whence, $v(p \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{o}$. Therefore, $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge p \vDash_{\mathsf{PAI_{fde}}} p \vee \delta_1 \vee \cdots \vee \delta_m$.

$[\neg L]^\dagger$: Assume $\Delta \neq \emptyset$, such that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{PAI_{fde}}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$. Suppose v is a $\mathsf{PAI_{fde}}$-valuation such that $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi) \subseteq \{\mathbf{t}, \mathbf{o}\}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi) = \mathbf{t}$, then $v(\varphi) = \mathbf{f}$, whence by hypothesis it is guaranteed that $v(\delta_1 \vee \cdots \vee \delta_m) \subseteq \{\mathbf{t}, \mathbf{o}\}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi) = \mathbf{o}$, then all $q \in Var(\Gamma, \neg\varphi)$ are such that $v(q) = \mathbf{o}$, whence by hypothesis all $q' \in Var(\Delta)$ are such that $v(q') = \mathbf{o}$, which entails that $v(\delta_1 \vee \cdots \vee \delta_m) = \mathbf{o}$. Therefore, $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi \vDash_{\mathsf{PAI_{fde}}} \delta_1 \vee \cdots \vee \delta_m$.

$[\neg R]^\ddagger$: Assume $Var(\varphi) \subseteq Var(\Gamma)$ and $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{PAI_{fde}}} \delta_1 \vee \cdots \vee \delta_m$. Notice that these two assumptions imply that $Var(\Delta) \subseteq Var(\Gamma)$. Suppose v is a $\mathsf{PAI_{fde}}$-valuation such that $v(\gamma_1 \wedge \cdots \wedge \gamma_n) \subseteq \{\mathbf{t}, \mathbf{o}\}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n) = \mathbf{t}$, then all $q \in Var(\Gamma)$ are such that $v(q) \in \{\mathbf{t}, \mathbf{f}\}$. If, additionally, $v(\varphi) = \mathbf{t}$, then by hypothesis $v(\delta_1 \vee \cdots \vee \delta_m) \subseteq \{\mathbf{t}, \mathbf{o}\}$. Otherwise, if $v(\varphi) = \mathbf{f}$, then $v(\neg\varphi) = \mathbf{t}$. This, given the fact that $Var(\Delta) \subseteq Var(\Gamma)$ leads to the fact that all $q' \in Var(\Delta)$ are such that $v(q') \in \{\mathbf{t}, \mathbf{f}\}$. All of this implies that $v(\neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{t}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n) = \mathbf{o}$, then all $q \in Var(\Gamma)$ are such that $v(q) = \mathbf{o}$. Given $Var(\Delta) \subseteq Var(\Gamma)$ and $Var(\varphi) \subseteq Var(\Gamma)$, we know that all $q' \in Var(\Delta, \neg\varphi)$ are such that $v(q') = \mathbf{o}$. Whence, $v(\neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{o}$. Therefore, $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{PAI_{fde}}} \neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m$.

The case of $[Cut]$ as well as the case of the $[W]$, $[\wedge]$, $[\vee]$, and the "De Morgan" rules are straightforward and thus are left to the reader as an exercise. $\square$

Theorem 4.4 (Soundness)**.** *If the sequent* $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ *is provable in* $\mathcal{G}_{\mathsf{PAI_{fde}}}$, *then* $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Proof. We know that the initial sequents are valid in $\mathsf{PAI_{fde}}$ and that all rules preserve $\mathsf{PAI_{fde}}$ validity. A straightforward induction on the height of the derivation shows (using Lemma 4.3 in the inductive step) that all provable sequents encode inferences that are valid in $\mathsf{PAI_{fde}}$. Thus, if $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, then the corresponding inference is valid in $\mathsf{PAI_{fde}}$—in other words, $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$. $\square$

Having discussed soundness, we will now turn to the more laborious task of providing a completeness proof for $\mathcal{G}_{\mathsf{PAI_{fde}}}$ with regard to $\mathsf{PAI_{fde}}$. To achieve this goal, and because it will make things fairly more manageable in the proofs below, we will now present a further restricted calculus that is nonetheless equivalent to the one we already introduced.

Definition 4.5. The calculus $\mathcal{G}^-_{\mathsf{PAI_{fde}}}$ is the result of replacing $[Initial]$ with the following rule:

$[LiteralInitial]$ $\Gamma, p \succ p, \Delta$ where $\Gamma, \Delta \subseteq Lit$ and $Var(\Delta, p) \subseteq Var(\Gamma, p)$

Lemma 4.6. *If a sequent is derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$ it is derivable in $\mathcal{G}^{-}_{\mathsf{PAI_{fde}}}$, and vice-versa.*

Proof. In the Appendix. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

With these tools in hand we now move on to the proper completeness result.

Theorem 4.7 (Completeness)**.** *If $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$.*

Proof. We start by assuming that the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is unprovable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. We, then, consider two cases:

(i) $Var(\psi_1, \ldots, \psi_m) \nsubseteq Var(\varphi_1, \ldots, \varphi_n)$

(ii) $Var(\psi_1, \ldots, \psi_m) \subseteq Var(\varphi_1, \ldots, \varphi_n)$

showing that in both cases we can design valuations that witness $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (i): if $Var(\psi_1, \ldots, \psi_m) \nsubseteq Var(\varphi_1, \ldots, \varphi_n)$ is the case, consider a $\mathsf{PAI_{fde}}$-valuation v such that:

$$
v(p) = \begin{cases} \mathbf{e} & \text{if } p \in Var(\psi_1, \ldots, \psi_m) \setminus Var(\varphi_1, \ldots, \varphi_n) \\ \mathbf{o} & \text{otherwise} \end{cases}
$$

It is then straightforward to notice, by a simple induction on the complexity of formulae, that there will be a $\psi_i \in \{\psi_1, \ldots, \psi_m\}$ such that $v(\psi_i) = \mathbf{e}$, whereas all $\varphi_j \in \{\varphi_1, \ldots, \varphi_n\}$ will be such that $v(\varphi_j) = \mathbf{o}$. A quick inspections of the **FP** algebra allows to notice that this renders $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{o}$, while at the same time giving $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{e}$. Whence, $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (ii): if $Var(\psi_1, \ldots, \psi_m) \subseteq Var(\varphi_1, \ldots, \varphi_n)$ is the case, in order to show that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$ we will apply the method of reduction trees as explored, e.g., in [47] by G. Takeuti and in [42] by D. Ripley.

The idea is to start with a sequent that we assume to be unprovable later extending it in a finite series of steps with the help of *reduction rules* that will finally render a *reduction tree*. Thus, we start with an unprovable sequent and build a tree above it, with each node consisting of a sequent that results from an application of the reduction rules to the sequent below it. As we extend the tree, we will sometimes find that the tip of a branch is an instance of one of [$LiteralInitial$]—in such a case we will consider this branch *closed* and will stop performing reductions on it. Contrary to that, if a branch is not closed after applying all the possible reduction rules, we will consider this branch *open*. Let us now detail the rules that we apply to the sequents at the top of each branch of the tree, at each stage of the reduction process. Let us note, in passing, that this technique requires an enumeration of the formulae of our language.

- To reduce a sequent of the form $\Gamma, \varphi \wedge \psi \succ \Delta$, extend the branch with the sequent $\Gamma, \varphi, \psi \succ \Delta$.

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- To reduce a sequent of the form $\Gamma \succ \varphi \wedge \psi, \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma \succ \varphi, \Delta$; to the other, add the sequent $\Gamma \succ \psi, \Delta$

- To reduce a sequent of the form $\Gamma \succ \varphi \vee \psi, \Delta$, extend the branch with the sequent $\Gamma \succ \varphi, \psi, \Delta$.

- To reduce a sequent of the form $\Gamma, \varphi \vee \psi \succ \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma, \varphi \succ \Delta$; to the other, add the sequent $\Gamma, \psi \succ \Delta$.

- To reduce a sequent of the form $\Gamma, \neg\neg\varphi \succ \Delta$, extend the branch with the sequent $\Gamma, \varphi \succ \Delta$.

- To reduce a sequent of the form $\Gamma \succ \neg\neg\varphi, \Delta$, extend the branch with the sequent $\Gamma \succ \varphi, \Delta$.

- To reduce a sequent of the form $\Gamma, \neg(\varphi \wedge \psi) \succ \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma, \neg\varphi \succ \Delta$; to the other, add the sequent $\Gamma, \neg\psi \succ \Delta$.

- To reduce a sequent of the form $\Gamma \succ \neg(\varphi \wedge \psi), \Delta$, extend the branch with the sequent $\Gamma \succ \neg\varphi, \neg\psi, \Delta$.

- To reduce a sequent of the form $\Gamma, \neg(\varphi \vee \psi) \succ \Delta$, extend the branch with the sequent $\Gamma, \neg\varphi, \neg\psi \succ \Delta$.

- To reduce a sequent of the form $\Gamma \succ \neg(\varphi \vee \psi), \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma \succ \neg\varphi, \Delta$; to the other, add the sequent $\Gamma \succ \neg\psi, \Delta$.

- To reduce a sequent of the form $\Gamma, \neg\varphi \succ \Delta$, consider whether $\neg\varphi$ is of the form $\neg(\psi \wedge \chi)$ or $\neg(\psi \vee \chi)$, or $\neg\neg\psi$, in which case proceed to apply the corresponding reduction rules detailed above.[11] If this is not the case, then if $\Delta \neq \emptyset$, extend the branch with the sequent $\Gamma \succ \varphi, \Delta$; otherwise, if $\Delta = \emptyset$, do nothing and proceed to reduce the next formula, if there is one.

- To reduce a sequent of the form $\Gamma \succ \neg\varphi, \Delta$, consider whether $\neg\varphi$ is of the form $\neg(\psi \wedge \chi)$ or $\neg(\psi \vee \chi)$, or $\neg\neg\psi$, in which case proceed to apply the corresponding reduction rules detailed above. If this is not the case, then if $Var(\varphi) \subseteq Var(\Gamma)$, extend the branch with the sequent $\Gamma, \varphi \succ \Delta$; otherwise, if $Var(\varphi) \nsubseteq Var(\Gamma)$, do nothing and proceed to reduce the next formula, if there is one.

Suppose we star with a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ where $Var(\psi_1, \ldots, \psi_m) \subseteq Var(\varphi_1, \ldots, \varphi_n)$ and follow this process as many times as necessary for there to be no more legal applications of the reduction rules. Then, either all branches of the tree will be closed (whence, we have a proof of the sequent that was assumed to be unprovable, contradicting our initial hypothesis), or some branch will be open. Suppose the latter is the case, and that there is an open branch. The next step in our proof is to show that it is possible to find a $\mathsf{PAI_{fde}}$-valuation that witnesses $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

For this purpose, let us temporarily relabel the sequents in the open branch as $\Gamma_1 \succ \Delta_1, \ldots, \Gamma_k \succ \Delta_k$, letting $\Gamma_1 \succ \Delta_1$ be $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ and letting

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[11] Notice that, if this is not the case, then $\neg\varphi$ is a negated literal—that is, $\varphi = \neg p$, for some propositional variable p.

$\Gamma_k \succ \Delta_k$ be the sequent at the tip of the open branch. Furthermore, let the sequent $\Gamma \succ \Delta$—where $\Gamma = \cup\{\Gamma_i \mid 1 \leq i \leq k\}$ and $\Delta = \cup\{\Delta_i \mid 1 \leq i \leq k\}$—be the sequent that "collects" all the sequents in the open branch. Consider the $\mathsf{PAI_{fde}}$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{t} & \text{if } p \in \Gamma \text{ or } \neg p \in \Delta \\ \mathbf{f} & \text{otherwise} \end{cases}$$

We now prove by induction on the complexity of φ that v is a $\mathsf{PAI_{fde}}$-valuation such that $v(\varphi) = \mathbf{t}$ if and only if $\varphi \in \Gamma$ and $v(\varphi) = \mathbf{f}$ if and only if $\varphi \in \Delta$.

Before going into the base cases though, lets us highlight a number of facts regarding literals and our newly defined Γ and Δ. As is easy to observe, all these remarks can be derived from the definition of Γ and Δ, the fact that none of the $\Gamma_i \succ \Delta_i$ $(1 \leq i \leq k)$ is an instance of $[LiteralInitial]$, the fact that the reduction rules preserve the satisfaction of the Proscriptive Principle, and the further pair of facts that propositional variables cannot be reduced by any of the reduction rules (and so are "preserved" up in the reduction process), and that if literals of the form $\neg p$ cannot be reduced by any of the reduction rules, they are "preserved" up in the reduction process. So, these remarks are: (i) for all propositional variables p, $p \notin \Gamma \cap \Delta$; (ii) for all negated literals $\neg p$, $\neg p \notin \Gamma \cap \Delta$; (iii) for all propositional variables p, $\{p, \neg p\} \nsubseteq \Gamma$; (iv) for all propositional variables p, $\{p, \neg p\} \nsubseteq \Delta$.[12]

Base cases:

- If $\varphi = p$, then if $p \in \Gamma$, $v(p) = \mathbf{t}$ by definition of v. Otherwise, if $p \in \Delta$, for example, $v(p) = \mathbf{f}$ by definition. Notice that, by the remarks above, we know that either $p \notin \Gamma$, or $p \notin \Delta$—granting the well-definedness of v.

- If $\varphi = \neg p$, then if $\neg p \in \Gamma$, we know that $p \notin \Gamma$ by the remarks above and, thus, $v(p) = \mathbf{f}$ by definition of v. Whence, $v(\neg p) = \mathbf{t}$. Otherwise, if $\neg p \in \Delta$, we know that $p \notin \Delta$ by the remarks above and, thus, $v(p) = \mathbf{t}$ by definition. Whence, $v(\neg p) = \mathbf{f}$. Notice that, by the remarks above, we know that either $\neg p \notin \Gamma$, or $\neg p \notin \Delta$—granting the well-definedness of v.

Inductive step: we assume that for all formulae of lesser complexity than φ, the hypothesis holds and show that it also holds for φ.

- If $\varphi = \psi \wedge \chi$, then if $\psi \wedge \chi \in \Gamma$ we know that $\psi, \chi \in \Gamma$. By the IH we know that $v(\psi) = v(\chi) = \mathbf{t}$. Thus, $v(\psi \wedge \chi) = \mathbf{t}$. Otherwise, if $\psi \wedge \chi \in \Delta$, then either $\psi \in \Delta$ or $\chi \in \Delta$. By the IH we know that either $v(\psi) = \mathbf{f}$ or $v(\chi) = \mathbf{f}$. Whence, $v(\psi \wedge \chi) = \mathbf{f}$.

- If $\varphi = \psi \vee \chi$, then if $\psi \vee \chi \in \Gamma$ we know that either $\psi \in \Gamma$ or $\chi \in \Gamma$. By the IH we knoww that either $v(\psi) = \mathbf{t}$ or $v(\chi) = \mathbf{t}$. Whence, $v(\psi \vee \chi) = \mathbf{t}$. Otherwise, if $\psi \vee \chi \in \Delta$, we know that $\psi, \chi \in \Delta$. By the IH this implies that $v(\psi) = v(\chi) = \mathbf{f}$. Whence, $v(\psi \vee \chi) = \mathbf{f}$.

- If $\varphi = \neg\neg\psi$, then if $\neg\neg\psi \in \Gamma$ we know that $\psi \in \Gamma$. By the IH we know that $v(\psi) = \mathbf{t}$, whence $v(\neg\neg\psi) = \mathbf{t}$. Otherwise, if $\neg\neg\psi \in \Delta$ we know that $\psi \in \Delta$. By the IH we know that $v(\psi) = \mathbf{f}$, whence $v(\neg\neg\psi) = \mathbf{f}$.

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[12]Notice, furthermore, that by simple reasoning these facts can be generalized to apply to arbitrary formulae φ and not only to propositional variables and negated literals.

- If $\varphi = \neg(\psi \wedge \chi)$, then if $\neg(\psi \wedge \chi) \in \Gamma$ we know that either $\neg\psi \in \Gamma$ or $\neg\chi \in \Gamma$. By the IH we know that either $v(\neg\psi) = \mathbf{t}$ or $v(\neg\chi) = \mathbf{t}$, which implies that either $v(\psi) = \mathbf{f}$ or $v(\chi) = \mathbf{f}$. Whence, $v(\psi \wedge \chi) = \mathbf{f}$ and furthermore $v(\neg(\psi \wedge \chi)) = \mathbf{t}$. Otherwise, if $\neg(\psi \wedge \chi) \in \Delta$, we know that $\neg\psi, \neg\chi \in \Delta$. By the IH this implies that $v(\neg\psi) = v(\neg\chi) = \mathbf{f}$, which further implies that $v(\psi) = v(\chi) = \mathbf{t}$. Whence, $v(\psi \wedge \chi) = \mathbf{t}$ and furthermore $v(\neg(\psi \wedge \chi)) = \mathbf{f}$.

- If $\varphi = \neg(\psi \vee \chi)$, then if $\neg(\psi \vee \chi) \in \Gamma$ we know that $\neg\psi, \neg\chi \in \Gamma$. By the IH we know that $v(\neg\psi) = v(\neg\chi) = \mathbf{t}$, which implies that $v(\psi) = v(\chi) = \mathbf{f}$. Thus, $v(\psi \vee \chi) = \mathbf{f}$, whence $v(\neg(\psi \vee \chi)) = \mathbf{t}$. Otherwise, if $\neg(\psi \vee \chi) \in \Delta$, then either $\neg\psi \in \Delta$ or $\neg\chi \in \Delta$. By the IH we know that either $v(\neg\psi) = \mathbf{f}$ or $v(\neg\chi) = \mathbf{f}$, which implies that either $v(\psi) = \mathbf{t}$, or $v(\chi) = \mathbf{t}$. Whence, $v(\psi \vee \chi) = \mathbf{t}$ and furthermore $v(\neg(\psi \vee \chi)) = \mathbf{f}$.

Given this, and since $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ and $\{\psi_1, \ldots, \psi_m\} \subseteq \Delta$, we know that for all i such that $1 \leq i \leq n$, $v(\varphi_i) = \mathbf{t}$, and for all j such that $1 \leq j \leq m$, $v(\psi_j) = \mathbf{f}$. Whence, by looking at the Boolean reduct of the $\mathbf{FP}$ algebra it is easy to notice that $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{t}$ and $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{f}$. Therefore, v is a $\mathsf{PAI_{fde}}$-valuation witnessing the fact that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$. This establishes that if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}^-_{\mathsf{PAI_{fde}}}$.

Finally, since $\mathcal{G}^-_{\mathsf{PAI_{fde}}}$ and $\mathcal{G}_{\mathsf{PAI_{fde}}}$ are equivalent due to Lemma 4.6 (proven in the Appendix) the above establishes that if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. $\square$

Corollary 4.8 (Cut-elimination)**.** *The Cut rule is eliminable from* $\mathcal{G}_{\mathsf{PAI_{fde}}}$

Proof. By Theorem 4.4, if there is a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, then $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$. Furthermore, by Theorem 4.7, if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{PAI_{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then applying the method of reduction trees gives a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. However, notice that this proof does not feature any instance of the Cut rule and is, thus, a Cut-free proof. Whence, for any sequent provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, there is a proof of it that does not use the Cut rule. $\square$

To close this section, we provide an answer to an inquiry by an anonymous reviewer concerning the existence of a Deduction Theorem for our calculus. In order to consider this issue, it should be taken into account beforehand that the only available conditional connective that we have in our system is the material conditional, definable as $\varphi \supset \psi =_{def} \neg\varphi \vee \psi$. For this connective, it can be easily shown that the unrestricted or fully general Deduction Theorem does not hold. To wit, consider the fact that the sequent $q, p \succ p$ is provable in our calculus $\mathcal{G}_{\mathsf{PAI_{fde}}}$ although the sequent $q \succ p \supset p$ (alternatively read as $q \succ \neg p \vee p$) is not provable. However, this in itself suggests that a restricted form of this result can be provided, as we prove below.

Observation 4.9. *The following form of the Deduction Theorem holds for* $\mathcal{G}_{\mathsf{PAI_{fde}}}$:

$$\Gamma, \varphi \succ \psi, \Delta \ \textit{is provable} \quad \Longleftrightarrow \quad \Gamma \succ \varphi \supset \psi, \Delta \ \textit{is provable}$$

where $Var(\varphi) \subseteq Var(\Gamma)$.

Proof. For the left to right direction, assume that $\Gamma, \varphi \succ \psi, \Delta$ is provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. Now, extend the proof of $\Gamma, \varphi \succ \psi, \Delta$ with an application of the $[\neg R]$ rule to arrive at the sequent $\Gamma \succ \neg\varphi, \psi, \Delta$, noting that such an application is legitimate because $Var(\varphi) \subseteq Var(\Gamma)$ is guaranteed by hypothesis. Then, extend such a proof with an application of the $[\vee R]$ rule to arrive at the sequent $\Gamma \succ \neg\varphi \vee \psi, \Delta$—which given our remarks above is a definitional variant of the sequent $\Gamma \succ \varphi \supset \psi, \Delta$.

For the right to left direction, assume that $\Gamma, \varphi \succ \psi, \Delta$ is not provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. As above, let us assume that Γ can be redescribed as $\gamma_1, \ldots, \gamma_n$, and that Δ can be redescribed as $\delta_1, \ldots, \delta_m$. Then, by Theorem 4.7 (that is, by completeness), we know that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \nvDash_{\mathsf{PAI_{fde}}} \psi \vee \delta_1 \vee \cdots \vee \delta_m$. This implies that there is a $\mathsf{PAI_{fde}}$-valuation v such that $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi) \in \{\mathbf{t}, \mathbf{o}\}$ while $v(\psi \vee \delta_1 \vee \cdots \vee \delta_m) \in \{\mathbf{e}, \mathbf{f}\}$. If $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi) = \mathbf{o}$, then by the operation in the **FP** algebra we know that $v(\varphi) = v(\neg\varphi) = \mathbf{o}$, which given the previous information would then imply that $v(\neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{e}$. Thus, v would be a $\mathsf{PAI_{fde}}$-valuation witnessing the fact that $\gamma_1 \wedge \cdots \wedge \gamma_n \nvDash_{\mathsf{PAI_{fde}}} \neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m$. Similarly, if $v(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi) = \mathbf{t}$, then by the operation in the **FP** algebra we know that $v(\gamma_1 \wedge \cdots \wedge \gamma_n) = v(\varphi) = \mathbf{t}$. Then, if $v(\psi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{e}$, we are guaranteed that $v(\neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{e}$. Thus, v would be a $\mathsf{PAI_{fde}}$-valuation witnessing the fact that $\gamma_1 \wedge \cdots \wedge \gamma_n \nvDash_{\mathsf{PAI_{fde}}} \neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m$. If, on the other hand, $v(\neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{f}$, this would imply that $v(\psi) = \mathbf{f}$. Since by our assumption $v(\varphi) = \mathbf{t}$, this implies that $v(\neg\varphi) = \mathbf{f}$. From this it follows that $v(\neg\varphi \vee \psi) = \mathbf{f}$, which furthermore results in $v(\neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m) = \mathbf{f}$. Thus, v would be a $\mathsf{PAI_{fde}}$-valuation witnessing the fact that $\gamma_1 \wedge \cdots \wedge \gamma_n \nvDash_{\mathsf{PAI_{fde}}} \neg\varphi \vee \psi \vee \delta_1 \vee \cdots \vee \delta_m$. In all of these cases, by Theorem 4.4 (that is, by soundness) this implies that $\Gamma \succ \neg\varphi \vee \psi, \Delta$ is not provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, which by definition is the same as saying that $\Gamma \succ \varphi \supset \psi, \Delta$ is not provable in our calculus. $\square$

Having provided a proof theory for $\mathsf{PAI_{fde}}$, in the next section we discuss with more detail some aspects of the employed calculus, exploring its so-called external consequence relation and offering some pointers towards potential generalizations for other similar logical systems.

5. Further remarks on the sequent calculus

The aim of this section is to comment on some aspects of the sequent calculus for $\mathsf{PAI_{fde}}$ presented above. To this extent, we divide it in two subsections. In the first, we discuss the so-called external consequence relation of our calculus $\mathcal{G}_{\mathsf{PAI_{fde}}}$. In the second, we reflect on the possibility of generalizing the techniques applied in the previous section, in order to provide calculi for the Fmla-Fmla

fragment that respects Parry's Proscriptive Principle of other non-classical logics.[13]

5.1. External Consequence

In [1, 2] and other works, A. Avron proposes to observe that it is possible to associate to any given Gentzen-style sequent calculus $\mathcal{G}_{\mathsf{L}}$ at least a pair of consequence relations. These are the so-called *internal* and *external* consequence relations. In a nutshell, the internal consequence relation $\vdash_{\mathrm{I}(\mathcal{G}_{\mathsf{L}})}$ consists in the relation holding between formulae within provable sequents. This is the usual consequence relation associated with sequent calculi.

Definition 5.1. Given a Gentzen-style sequent calculus $\mathcal{G}_{\mathsf{L}}$, its *internal consequence* relation $\mathrm{I}(\mathcal{G}_{\mathsf{L}})$ is defined so that:

$$\Gamma \vdash_{\mathrm{I}(\mathcal{G}_{\mathsf{L}})} \psi \quad \Longleftrightarrow \quad \Gamma \succ \psi \text{ is provable in } \mathcal{G}_{\mathsf{L}}$$

However, there is yet another option which is constituted by the external consequence relation $\vdash_{\mathrm{E}(\mathcal{G}_{\mathsf{L}})}$. Briefly, this relation consists in adding formulae as axioms to the sequent calculus, and determining which formulae we obtain as theorems thereby. Thus, if we can prove the sequent $\succ\psi$, when adding the sequents in $\{\succ\varphi \mid \varphi \in \Gamma\}$ as axioms to $\mathcal{G}_{\mathsf{L}}$, then we say that Γ entails ψ according to $\vdash_{\mathrm{E}(\mathcal{G}_{\mathsf{L}})}$.

Definition 5.2. Given a Gentzen-style sequent calculus $\mathcal{G}_{\mathsf{L}}$, its *external consequence* relation $\mathrm{E}(\mathcal{G}_{\mathsf{L}})$ is defined so that:

$$\Gamma \vdash_{\mathrm{E}(\mathcal{G}_{\mathsf{L}})} \psi \quad \Longleftrightarrow \quad \succ\psi \text{ is provable in } \mathcal{G}_{\mathsf{L}} \cup \{\succ\varphi \mid \varphi \in \Gamma\}$$

Thus, one may wonder what the external consequence relation of our own calculus $\mathcal{G}_{\mathsf{PAl_{fde}}}$ is. In this respect, it shall be noted that although the notion of external consequence is formulated in the SET-FMLA framework, we might as well be interested in its restriction to the FMLA-FMLA framework. This may be of special interest concerning sequent calculi $\mathcal{G}_{\mathsf{L}}$ for which provable sequents of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ are interpreted as encoding validity claims of the form $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \psi_1 \vee \cdots \vee \psi_m$ in a logic L—such as in our own calculus presented in the previous section. For cases of this sort, it may be interesting to determine which is the logic corresponding to the FMLA-FMLA fragment of the $\mathrm{E}(\mathcal{G}_{\mathsf{PAl_{fde}}})$. In what follows, we devote to answering this question in what pertains to our calculus.

As shown below, it can be straightforwardly noted that the FMLA-FMLA fragment of $\mathrm{E}(\mathcal{G}_{\mathsf{PAl_{fde}}})$—the external consequence of our target calculus—is exactly the FMLA-FMLA fragment of the consequence relation of $\mathsf{CL}[\mathbf{eo}]$, that is to say, the consequence relation associated to $\mathsf{PAl_{fde}}$.

Theorem 5.3. *The* FMLA-FMLA *fragment of* $\mathrm{E}(\mathcal{G}_{\mathsf{PAl_{fde}}}) = \mathsf{PAl_{fde}}$.

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[13]We would like to thank an anonymous reviewer for suggesting me to consider the issues in this section.

Proof. Assume that φ entails ψ according to $\mathsf{PAI_{fde}}$, i.e., $\varphi \models_{\mathsf{PAI_{fde}}} \psi$. Suppose, now that $\succ\varphi$ is added as an axiom to $\mathcal{G}_{\mathsf{PAI_{fde}}}$. By Theorem 4.7 (that is, by completeness), the former implies that $\varphi \succ \psi$ is provable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. By an application of the $[Cut]$ rule to the sequents $\succ\varphi$ and $\varphi \succ \psi$, we arrive at the sequent $\succ\psi$, thereby showing that φ entails ψ in $\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})$, i.e., $\varphi \vdash_{\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})} \psi$.

Assume that φ does not entail ψ according to $\mathsf{PAI_{fde}}$, i.e., $\varphi \nvDash_{\mathsf{PAI_{fde}}} \psi$. Suppose, for *reductio*, that φ entails ψ in $\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})$, i.e., $\varphi \vdash_{\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})} \psi$. Now, this implies that there is a proof in $\mathcal{G}_{\mathsf{PAI_{fde}}}$ from $\succ\varphi$ to $\succ\psi$, that is to say, a proof whose topmost node is $\succ\varphi$ and whose last node is $\succ\psi$. But, given this, we can transform this proof into a proof of $\varphi \succ \psi$. First, we add φ to the left-hand side of each node of the proof, in compliance with the $[KL]$ rule. Then, we extend up the topmost node $\varphi \succ \varphi$ by routinely reducing it to one or several instances of $[Initial]$, in accordance with the reduction rules discussed in Section 4. Thus, we end with a proof whose topmost nodes are instances of $[Initial]$, and whose last node is $\varphi \succ \psi$. This will constitute a proof in $\mathcal{G}_{\mathsf{PAI_{fde}}}$ of this last sequent. However, by our assumption that $\varphi \nvDash_{\mathsf{PAI_{fde}}} \psi$ in conjunction with Theorem 4.4 (that is, soundness), we know that there is no proof of $\varphi \succ \psi$ in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. This contradiction allows us to infer that φ does not entail ψ in $\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})$, i.e., $\varphi \nvdash_{\mathrm{E}(\mathcal{G}_{\mathsf{PAI_{fde}}})} \psi$. $\square$

5.2. Some generalizations

An anonymous reviewer wonders whether it is possible to generalize the techniques implemented to obtain $\mathcal{G}_{\mathsf{PAI_{fde}}}$ in order to arrive at appropriate calculi for the Fmla-Fmla fragment respecting Parry's Proscriptive Principle of other logics like, e.g., Intuitionistic Logic (IL, for short). We would like to point towards an answer to this question, providing as many details as possible. To understand what this would amount to, it is important to make a few remarks first.

We present a calculus for $\mathsf{PAI_{fde}}$, and thus providing an intuitionistic analog of our calculus would require entertaining an intuitionistic variant of $\mathsf{PAI_{fde}}$— let us refer, in what follows, to such a logic as $\mathsf{iPAI_{fde}}$. Basically, this is the restriction to the Fmla-Fmla framework of the fragment of Intuitionistic Logic that respects the Proscriptive Principle, in our previous terminology, of $\mathsf{IL_{PP}}$. In fewer words, then, the Fmla-Fmla fragment of $\mathsf{IL_{PP}} = \mathsf{iPAI_{fde}}$.

In order to present a calculus for $\mathsf{iPAI_{fde}}$, we need to consider that the usual calculi for IL is usually presented in a very idiosyncratic fashion. To begin with, it is usually presented as operating over sequents of the form $\Gamma \succ \Delta$, understood as pairs $\langle \Gamma, \Delta \rangle$ where Γ and Δ are finite multisets of formulae, and Δ can have at most one occurrence of one formula—thus, in contrast with our own calculus, Δ can be either empty or a singleton of the form $\{\chi\}$. Moreover, it is usually presented using purely additive, i.e., context-sharing rules—whereas our calculus is presented through some rules which are additive and some which are multiplicative, i.e., context-independent. Furthermore, precisely because of the cardinality restrictions over the right-hand side of the sequents, the admissible form of the Weakening Right rule cannot be absorbed in the axioms in any way and has therefore to be separately stated—contrary to the case of our calculus.

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Thus, all these changes need to be made in transitioning from our calculus $\mathcal{G}_{\mathsf{PAI_{fde}}}$ for $\mathsf{PAI_{fde}}$, to a calculus which we may call $\mathcal{G}_{\mathsf{iPAI_{fde}}}$ for the intuitionistic variant $\mathsf{iPAI_{fde}}$.

In this regard, we conjecture that the set of rules appearing below provides an adequate calculus for this logic, although for matters of space we cannot go through the details of the corresponding soundness and completeness proofs, for they would take us too far afield. As regards the system $\mathsf{iPAI_{fde}}$, let us notice in passing that no semantics have been offered for it, and that it has not been studied in complete detail, apart from the discussion of its implication fragment by T. Ferguson in [19]. In connection to this, below we will represent intuitionistic implication with the binary connective $\rightarrow$.[14]

Definition 5.4. The calculus $\mathcal{G}_{\mathsf{iPAI_{fde}}}$ is constituted by the following rules:

Initial Sequents:

$$[Initial] \quad \Gamma, p \succ p$$

Structural Rules:

$$\frac{\Gamma, \varphi, \varphi \succ \chi}{\Gamma, \varphi \succ \chi} \ [WL] \quad \frac{\Gamma \succ}{\Gamma \succ \psi} \ [KR]^{\dagger} \quad \frac{\Gamma, \varphi \succ \chi \quad \Gamma \succ \varphi}{\Gamma \succ \chi} \ [Cut]$$

$$\dagger: \text{ where } Var(\psi) \subseteq Var(\Gamma)$$

Operational Rules:

$$\frac{\Gamma \succ \varphi}{\Gamma, \neg\varphi \succ} \ [\neg L] \quad \frac{\Gamma, \varphi \succ}{\Gamma \succ \neg\varphi} \ [\neg R]^{\ddagger}$$

$$\ddagger: \text{ where } Var(\varphi) \subseteq Var(\Gamma)$$

$$\frac{\Gamma, \varphi \succ \chi \quad \Gamma, \psi \succ \chi}{\Gamma, \varphi \vee \psi \succ \chi} \ [\vee L] \quad \frac{\Gamma \succ \varphi}{\Gamma \succ \varphi \vee \psi} \ [\vee R]^{\dagger} \quad \frac{\Gamma \succ \psi}{\Gamma \succ \varphi \vee \psi} \ [\vee R]^{\ddagger}$$

$$\dagger: \text{ where } Var(\psi) \subseteq Var(\Gamma) \quad \ddagger: \text{ where } Var(\varphi) \subseteq Var(\Gamma)$$

$$\frac{\Gamma, \varphi \succ \chi}{\Gamma, \varphi \wedge \psi \succ \chi} \ [\wedge L] \quad \frac{\Gamma, \psi \succ \chi}{\Gamma, \varphi \wedge \psi \succ \chi} \ [\wedge L] \quad \frac{\Gamma \succ \varphi \quad \Gamma \succ \psi}{\Gamma \succ \varphi \wedge \psi} \ [\wedge R]$$

$$\frac{\Gamma \succ \varphi \quad \Gamma, \psi \succ \chi}{\Gamma, \varphi \rightarrow \psi \succ \chi} \ [\rightarrow L] \quad \frac{\Gamma, \varphi \succ \psi}{\Gamma \succ \varphi \rightarrow \psi} \ [\rightarrow R]^{\ddagger}$$

$$\ddagger: \text{ where } Var(\varphi) \subseteq Var(\Gamma)$$

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[14]For matters of space, we do not discuss the De Morgan rules, although some of them are derivable in the present calculus. This being said, not all of them are, as some—like the $[\neg\neg L]$, $[\neg\neg R]$, and $[\neg\wedge L]$ rules—are even inadmissible in Intuitionistic Logic.

Notice that, just like in our calculus, some of the rules of $\mathcal{G}_{\mathsf{iPAI_{fde}}}$ have linguistic restrictions that guarantee the satisfaction of Parry's Proscriptive Principle. In the case of $[\neg R]$, this is apparent just like in our calculus, as it is in the case of the $[KR]$ rule, which cannot be absorbed in the axioms (like in our calculus), due to the cardinality constraints proper of a calculus for IL. In fact, in the case of the pair of $[\vee R]$, restrictions are present precisely because of such cardinality constraints, and the fact that rules are presented in an additive manner. Similar remarks apply to other rules.

This being said, we believe that similar strategies could be applied to obtain appropriate fragments, of other logics, where these fragments also satisfy the Proscriptive Principle. For instance, substructural logics could be explored in this respect, thereby studying fragments of this sort of, e.g., Linear, Affine Logic, and more. We hope to study these systems in future research soon.

6. Conclusion

In this article, we presented semantics for $\mathsf{PAI_{fde}}$ of a type unlike that of previous discussions concerning this logic. In fact, while previous characterizations of this logic have been done using intensional semantics, or extensional semantics of an algebraic kind, we have shown that it is possible to analyze the target notion of entailment using the tools of logical matrices—understanding logical consequence as necessary preservation of designated values. In the paragraphs above, we presented a calculus for $\mathsf{PAI_{fde}}$ that follows the path of working with rules (either structural or operational) that are subject to linguistic restrictions, guaranteeing the preservation of the relevant property characteristic of Parry's logic: the Proscriptive Principle.

A lot of avenues of further research open up exactly where this article lefts off—especially regarding the semantic techniques developed here. One of these concerns the potential to generalize the techniques implemented to work with $\mathsf{PAI_{fde}}$, in order to characterize different relevant logics complying with principles other than Parry's. In this vein, one may look at what is called the dual of Parry's Proscriptive Principle in, e.g., [12] and [45], which requires that for a first-degree entailment of the form $\varphi \rightarrow \psi$, it holds that $Var(\varphi) \subseteq Var(\psi)$. A system whose valid implications comply with this principle is R. Epstein's Dual Dependence logic (hereafter, DD), regarding which it can be said—per the analysis of, e.g., F. Paoli in [31] that:

$$\vdash_{\mathsf{DD}} \varphi \rightarrow \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{CL}} \psi, \text{ and} \\ Var(\varphi) \subseteq Var(\psi) \end{cases}$$

Then, letting $\mathsf{L_{dPP}}$ be defined, for any given logic, in an analogous way to how $\mathsf{L_{PP}}$ is defined above—noting that $\mathsf{DD_{fde}} =$ the FMLA-FMLA fragment of $\mathsf{CL_{dPP}}$.

Whence, one may wonder whether it is possible to apply the techniques above to provide matrix semantics and Gentzen-style sequent calculi not only for $\mathsf{DD_{fde}}$, but for (the FMLA-FMLA fragment of) $\mathsf{L_{dPP}}$, for any given logic L.

Following the previous remarks, we conjecture that the answer is affirmative. In fact, regarding the general semantic result in Theorem 3.10, all that would be needed is to change, respectively, the requirement that the infectious value u is undesignated for the requirement that it be designated, and vice-versa with the "mimicking" value j. Concerning a calculus for $\mathsf{DD_{fde}}$, then, it would suffice to consider a proper dualization of the linguistic constraints imposed on $[Initial]$ and the left and right rules for negation.[15]

Finally, alongside these issues, there is the question of how may all these discussions generalize to the first-order case. We briefly point out that, in this regard, we agree with K. Fine in [21] and R. French in [24] that an appropriate generalization of the notion of subject-matter of a formula φ—formalized in an ideal way in the context of propositional languages as $Var(\varphi)$—should render the subject-matter of $\forall x \varphi(x)$ as being equal to that of the sum or collection of all instances of $\varphi(x)$. We leave the matter of how to best deal with these issues for future investigations, hoping to discuss them in the near future.

Appendix: Equivalence of $\mathcal{G}^{-}_{\mathsf{PAI_{fde}}}$ and $\mathcal{G}_{\mathsf{PAI_{fde}}}$

The following proof of our Lemma 4.6 is a straightforward adaptation of R. French's proof of his Lemma 3.9 in [24], for an analogous fact regarding a further restricted form of his calculus $\mathcal{G}_{AC}$ for Angell's logic of Analytic Containment— where the Γ and Δ featured in the initial sequents can also only be sets of literals.[16] Finally, for the proof below to go through it is necessary to consider the following definition of complexity.

Definition 6.1 ([24])**.** Given a formula φ, let the De Morgan Complexity $dmc(\varphi)$ be defined as follows:

- $dmc(p) = 0$

- $dmc(\neg p) = 0$

- $dmc(\varphi \wedge \psi) = dmc(\neg(\varphi \wedge \psi)) = 1 + dmc(\varphi) + dmc(\psi)$

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[15]Therefore, letting these rules be of the following form:

$$[Initial] \quad \Gamma, p \succ p, \Delta \quad \text{where } Var(\Gamma, p) \subseteq Var(\Delta, p)$$

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta} \ [\neg L]^{\ddagger} \qquad\qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta} \ [\neg R]^{\dagger}$$

$$\ddagger: \text{ where } Var(\varphi) \subseteq Var(\Delta) \qquad\qquad \dagger: \text{ where } \Gamma \neq \emptyset$$

[16]The only modification needed for his argument to apply to our case lies in replacing the mention of $t(A)$ and $t(\Gamma)$ in the proof of the aforementioned theorem, respectively, for $Var(\varphi)$ and $Var(\Gamma)$.

- $dmc(\varphi \vee \psi) = dmc(\neg(\varphi \vee \psi)) = 1 + dmc(\varphi) + dmc(\psi)$

- $dmc(\neg\neg\varphi) = 1 + dmc(\varphi)$

Proof of Lemma 4.6. First, since all instances of $[LiteralInitial]$ are instances of $[Initial]$, it readily follows that if a sequent is derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}^{-}$, it is also derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$. Secondly, to show that if a sequent is derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, it is also derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}^{-}$ we suppose we have a proof starting with an instance of $[Initial]$ of the following form, where $Var(\Delta, p) \subseteq Var(\Gamma, p)$:

$$\Gamma, p \succ p, \Delta$$

which is not an instance of $[LiteralInitial]$, meaning that $\Gamma, \Delta \nsubseteq Lit$. We then show how to transform said proof in a $\mathcal{G}_{\mathsf{PAI_{fde}}}^{-}$ proof. For this purpose, we show how Γ and Δ can each be legitimately replaced by a set of literals. We do this first for Δ, and then for Γ.

To show that Δ can be replaced by a set of literals, we perform an induction on the sum of the De Morgan complexities of the formulae in Δ. If such a sum is 0, then every formula in Δ is a literal, as needed. If such a sum is positive and less or equal than n, then we assume that the instance of $[Initial]$ that our $\mathcal{G}_{\mathsf{PAI_{fde}}}$ proof starts off with is of one of the following forms (with the rest of the cases being strictly analogous to these):

$$(i) \; \Gamma, p \succ p, \Delta, \varphi \vee \psi \quad (ii) \; \Gamma, p \succ p, \Delta, \varphi \wedge \psi \quad (iii) \; \Gamma, p \succ p, \Delta, \neg\neg\varphi$$

noticing that, in all the cases above, the sum of De Morgan complexities of the right-hand side of each sequent is $n + 1$. We proceed to show how, in each case, we can derive each of these instances of $[Initial]$ from instances whose right-hand side is of lower De Morgan complexity, as follows:

$$(i) \; \frac{\Gamma, p \succ p, \Delta, \varphi, \psi}{\Gamma, p \succ p, \Delta, \varphi \vee \psi} \quad (ii) \; \frac{\Gamma, p \succ p, \Delta, \varphi \quad \Gamma, p \succ p, \Delta, \psi}{\Gamma, p \succ p, \Delta, \varphi \wedge \psi} \quad (iii) \; \frac{\Gamma, p \succ p, \Delta, \varphi}{\Gamma, p \succ p, \Delta, \neg\neg\varphi}$$

Thus, given that in each of these cases the premise-sequents have right-hand sides of at least one lower De Morgan complexity, by the IH we know that they can be derived from instances of $[Initial]$ where Δ contains only literals. From this it follows that we can replace all instances of $[Initial]$ with instances of the following sequent schema, letting $p_1^{\pm}, \ldots, p_m^{\pm}$ be literals:

$$\Gamma, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm} \quad \text{where } Var(p_1^{\pm}, \ldots, p_m^{\pm}, p) \subseteq Var(\Gamma, p)$$

To show that Γ can be replaced by a set of literals, we perform once again an induction on the sum of the De Morgan complexities of the formulae in Γ. If such a sum is 0, then every formula in Γ is a literal, as needed. If such a sum is positive and less or equal than n, then we assume that the instance of $[Initial]$ that our $\mathcal{G}_{\mathsf{PAI_{fde}}}$ proof starts off with is of one of the following forms (with the rest of the cases being strictly analogous to these):

$$(i')\ \Gamma, \varphi \wedge \psi, p \succ p, \Delta \quad (ii')\ \Gamma, \varphi \vee \psi, p \succ p, \Delta \quad (iii')\ \Gamma, \neg\neg\varphi, p \succ p, \Delta$$

noticing that, in all the cases above, the sum of De Morgan complexities of the left-hand side of each sequent is $n+1$. We proceed to show how, in each case, we can derive each of these instances of $[Initial]$ from instances whose left-hand side is of lower De Morgan complexity, as follows:

$$(i')\ \frac{\Gamma, \varphi, \psi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}{\Gamma, \varphi \wedge \psi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}$$

$$(ii')\ \frac{\dfrac{\dfrac{\Gamma, \varphi, p \succ p, r_1^{\pm}, \ldots, r_k^{\pm}}{\Gamma, \varphi \vee \psi, \varphi, p \succ p, r_1^{\pm}, \ldots, r_k^{\pm}}\ [KL]}{\Gamma, \varphi \vee \psi, \varphi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}\ [KR]\quad \dfrac{\dfrac{\Gamma, \varphi, p \succ p, s_1^{\pm}, \ldots, s_j^{\pm}}{\Gamma, \varphi \vee \psi, \psi, p \succ p, s_1^{\pm}, \ldots, s_j^{\pm}}\ [KL]}{\Gamma, \varphi \vee \psi, \psi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}\ [KR]}{\dfrac{\dfrac{\Gamma, \varphi \vee \psi, \varphi \vee \psi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}{\Gamma, \varphi \vee \psi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}\ [WL]}{}}\ [\vee L]$$

where $\{r_1^{\pm}, \ldots, r_k^{\pm}\} = \{p_i^{\pm} \in \{p_1^{\pm}, \ldots, p_m^{\pm}\} \mid Var(p_i^{\pm}) \in Var(\Gamma, \varphi)\}$ and similarly $\{s_1^{\pm}, \ldots, s_j^{\pm}\} = \{p_i^{\pm} \in \{p_1^{\pm}, \ldots, p_m^{\pm}\} \mid Var(p_i^{\pm}) \in Var(\Gamma, \psi)\}$.

$$(iii')\ \frac{\Gamma, \varphi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}{\Gamma, \neg\neg\varphi, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm}}$$

Thus, given that in each of these cases the premise-sequents have left-hand sides of at least one lower De Morgan complexity, by the IH we know that they can be derived from instances of $[Initial]$ where Γ contains only literals. From this it follows that we can replace all instances of $[Initial]$ with instances of the following sequent schema, letting $q_1^{\pm}, \ldots, q_n^{\pm}, p_1^{\pm}, \ldots, p_m^{\pm}$ be literals:

$$q_1^{\pm}, \ldots, q_n^{\pm}, p \succ p, p_1^{\pm}, \ldots, p_m^{\pm} \quad \text{where } Var(p_1^{\pm}, \ldots, p_m^{\pm}, p) \subseteq Var(q_1^{\pm}, \ldots, q_n^{\pm}, p)$$

This concludes showing how a $\mathcal{G}_{\mathsf{PAI_{fde}}}$ proof can be turned into a $\mathcal{G}_{\mathsf{PAI_{fde}}}^{-}$ proof. Therefore, if a sequent is derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}$, then it is derivable in $\mathcal{G}_{\mathsf{PAI_{fde}}}^{-}$. $\qquad\square$

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