Metainferential Duality

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Abstract

The aim of this article is to discuss the extent to which certain substructural logics are related through the phenomenon of *duality*. Roughly speaking, metainferences are inferences between collections of inferences, and thus *substructural* logics can be regarded as those logics which have fewer valid metainferences that Classical Logic. In order to investigate duality in substructural logics, we will focus on the case study of the logics **ST** and **TS**, the former lacking Cut, the latter Reflexivity. The sense in which these logics, and these metainferences, are dual has yet to be explained in the context of a thorough and detailed exposition of duality for frameworks of this sort. Thus, our intent here is to try to elucidate whether or not this way of talking holds some ground—specially generalizing one notion of duality available in the specialized literature, the so-called notion of *negation duality*. In doing so, we hope to hint at broader points that might need to be addressed when studying duality in relation to substructural logics.

1 Introduction

The aim of this article is to discuss the extent to which certain substructural logics are related through the phenomenon of *duality*. Roughly speaking, metainferences are inferences between collections of inferences, and thus *substructural* logics can be regarded as those logics which have fewer valid metainferences that Classical Logic—the logician and philosopher's usual reference point. In particular, there are some special metainferences, the so-called "structural rules", which often fail in substructural logics and can be informally understood as those features or properties of logical consequence itself, not of some specific logical vocabulary. Usually, the list of such features includes Reflexivity, Monotonicity, Contraction, and Cut—although others may deserve to be included as well.

In order to investigate duality in substructural logics, we will focus on the case study of the logics **ST** and **TS**, the former defended by Cobreros, Égré, Ripley, and van Rooij in works like [8], [7], [9], [10], and [23] and [24], the latter advocated for by French in works like [15]. These logics are substructural in the aforementioned understanding, with the distinctive feature that **ST** is non-transitive whereas **TS** is non-reflexive. Although in the seminal article [7] both **ST** and **TS** were regarded as inferentially self-dual, in formal and in informal contexts, scholars have pointed out to that these systems offer, metainferentially speaking, dual approaches and dual solutions to the same logical and philosophical issues—highlighting that as metainferences, Cut and Reflexivity are also dual to each other. For example, in [15] French claims that there is a strong duality between these approaches, and cites Ripley [24], Hösli and Jäger [17], and Frankowski in [14] as implicitly or explicitly supporting this point of view. However, the sense in which these logics and these metainferences are dual has not been discussed in any of these works, and has thus yet to be explained in the context of a thorough and detailed exposition of duality for frameworks of this sort.[1]

Thus, our intent here is to try to elucidate whether or not this way of talking holds some ground—specially generalizing one notion of duality available in the specialized literature, the so-called *negation duality*. In doing so, we hope to provide an account of duality for metainferences, such as Cut and Reflexivity, and to hint at broader points that might need to be addressed when studying duality in relation to substructural logics.

To this end, we embark on the task of providing a fully general, philosophically motivated, and technically sound way of accounting for the duality of metainferences and for substructural logics in general. In order to do so, we first present one notion of duality for inferential logics, the so-called *negation duality*, and show that applied to **ST** and **TS** as inferential logics describes them as self-dual, and thus as unrelated to each other. Next, we generalize this notion to metainferences, and show that, under this generalization, **ST** and **TS** are metainferentially dual.

For this purpose, the article is structured in the following way. In Section 2, we introduce some formal definitions that will be used throughout the whole article. Next, in Section 3, we present the notion of *negation duality* (which can be traced back to Kleene's [19] and show that it's not adequate for characterizing metainferential duality. In Section 4, we provide a general definition of duality for metainferences and for substructural logics, which allows to prove that **ST** and **TS** are dual. In Section 4, we also show that in this new framework Cut and Reflexivity can be taken as dual under equivalence. In Section 5, we discuss in which sense all these positive results are based on the notion of local validity and, appealing to [28], we show that all the results provided can be obtained using instead a global notion. In Section 6 we explore the possibility of establishing

[1] It should be noted that a recently published article by Cobreros, La Rosa and Tranchini [11] independently arrived at conclusions similar to ours, also arguing in favor of the duality between **ST** and **TS** on different grounds that we do. Although in private conversation we have discussed some of the similarities and differences of the notions implemented in the two approaches, a more comprehensive study in this respect is in order. Unfortunately, carrying this out here would lead us too far afield, and thus we hope to do it on in future work.

the duality between **ST** and **TS**, and between Cut and Reflexivity, through translations from metainferential logics to inferential logics, and argue against this approach. In Section 7, we conclude with some final remarks and directions of future work.

2 Technicalities

In this section, we provide some technical definitions concerning non-classical logics and metainferences, that will be of use later. In the course of this article, for $\mathcal{L}$ a propositional language, and $\mathbf{FOR}(\mathcal{L})$, the (algebra of) well-formed formulae—standardly defined—we will use lowercase Greek letters for (schematic) formulae, and uppercase Greek letters for (schematic) collections[2] of formulae of $\mathcal{L}$.

In this context, an *inference token* $\Gamma \Rightarrow \Delta$ of a language $\mathcal{L}$ is a pair $\langle \Gamma, \Delta \rangle$, where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$. Furthermore, a *simple inference schema* of a language $\mathcal{L}$ is the set of all and only the inference tokens that can be obtained from one of its members—its "basic instance"—by uniformly substituting some propositional variable p in it by some formula φ. Lastly, an *inference schema with contexts* is the union of a simple inference schema ρ with a subset of $\{\langle \Gamma \cup \Sigma, \Delta \cup \Pi \rangle \mid \langle \Gamma, \Delta \rangle \in \rho\}$.[3] When it generates no ambiguity, we will indistinctly refer as "schemata" to both simple schemata and to those with contexts.

To begin with, let us have a look at the rather well-known system of Classical Logic. For this purpose, let us employ the usual notion of a (matrix) logic $\mathbf{L}$, as induced by a logical matrix $\mathcal{M}$. Thus, for $\mathcal{L}$ a propositional language, an $\mathcal{L}$-logical matrix is a structure $\mathcal{M} = \langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D is a subset of the universe or carrier set of $\mathbf{A}$. With regard to such a structure, an $\mathcal{M}$-valuation v is an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$. Through valuation functions, logical matrices can be used to define logical consequence relations in the following way.

Definition 1. *For an $\mathcal{L}$-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies an inference token $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if, if $v(\gamma) \in \mathcal{D}$, for all $\gamma \in \Gamma$ then $v(\delta) \in \mathcal{D}$, for some $\delta \in \Delta$. An inference token $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, for all $\mathcal{M}$-valuations v.*

Naturally, inference schemata are $\mathcal{M}$-valid if and only if all their inference tokens are valid. In addition, when some logic $\mathbf{L}$ is induced by a single matrix $\mathcal{M}$, we may interchangeably refer to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathbf{L}}$. This being said, we can identify Classical Logic (**CL**, for short) with the (matrix) logic induced by the logical matrix $\langle \mathbf{B}, \{\mathbf{t}\} \rangle$, where $\mathbf{B}$ is the usual two-element Boolean algebra counting with elements $\mathbf{t}$ and $\mathbf{f}$ to represent truth and falsity, respectively. With respect to Classical Logic, then, a system $\mathbf{L}$ will be said to be (inferentially) *subclassical*

[2]Given that our main concern are the logics **ST** and **TS**, for the most part of the article, these collections will simply be taken to be sets of formulae. The need to consider multisets will only briefly come into play when considering the failure of the structural rule of Contraction.

[3]We thank an anonymous referee for pointing out the need to clarify this generalization.

if and only if some inference schema that is valid in Classical Logic does not hold in it.

Now, our main topic of interest in this article is non-classical logics, especially substructural logics—i.e., logics which invalidate some metainference schema that is valid in Classical Logic, on which more will be said below. However, before presenting these sorts of non-classical frameworks, let us introduce two more subclassical matrix logics that will be of interest when discussing the negative or partial results on the duality of some substructural logics. These systems are the three-valued logics **LP** and $\mathbf{K_3}$, which can be respectively seen as the matrix logics induced by the matrices $\langle \mathbf{SK}, \{\mathbf{t}, \mathbf{i}\}\rangle$ and $\langle \mathbf{SK}, \{\mathbf{t}\}\rangle$—where **SK** is the three-element (strong) Kleene algebra defined by the so-called "truth-tables" appearing below.

	$\neg$
t	**f**
i	**i**
f	**t**

$\wedge$	**t**	**i**	**f**
t	**t**	**i**	**f**
i	**i**	**i**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**i**	**f**
t	**t**	**t**	**t**
i	**t**	**i**	**i**
f	**t**	**i**	**f**

Figure 1: Strong Kleene truth-tables

In passing, and because it will be of some relevance when discussing our target substructural systems, let us highlight some of the inferences that are valid in **CL** and that are invalid in these two non-classical systems. Saliently, **LP** is usually referred to as a paraconsistent logic because it invalidates the classically-valid inference schema called *Ex Falso Sequitur Quodlibet*—i.e., $\varphi \wedge \neg\varphi \Rightarrow$—whereas $\mathbf{K_3}$ is usually referred to as a paracomplete logic because it invalidates classically-valid inference schema called *Tertium Non Datur*—i.e., $\Rightarrow \varphi \vee \neg\varphi$.

Moving on to our main target, let us first clarify now what substructural logics are. Given the definitions that we will provide shortly, it will become apparent why the logics of our case study—the systems **ST** and **TS**—are logics of this kind. Let us begin by discussing what a metainference is. Intuitively speaking, by this we mean an inference holding between inferences—instead of the usual inferences, which relate (collections of) formulae. In this vein, a metainference establishes that from such and such inferences, such and such follows. That is, either another inference or inferences (in plural, as we will discuss later in the positive sections) follow.

Formally speaking, a *metainference token* of a language $\mathcal{L}$ is a pair $\langle S, s\rangle$, where S is a set of inference tokens and s is an inference token of $\mathcal{L}$. Furthermore, a *metainference schema* of a language $\mathcal{L}$ is the set of all and only the metainference tokens that can be obtained from one of its members—its "basic instance"—by uniformly substituting some propositional variable p in it by some formula φ.

With the help of these definitions, the usual *structural features* of logical consequence can be formally reinterpreted as *metainferential schemata* of the following form—in line with their usual depiction, e.g., in the context of sequent calculi.

Reflexivity: $$\frac{}{\varphi \Rightarrow \varphi}$$

Contraction: $$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

Cut: $$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Of vital importance for our project is the question: when does a certain metainference token or metainference schema hold in the context of a given logic? For the purpose of answering this, it is crucial to observe the following definition of metainferential validity.[4]

Definition 2. *For an $\mathcal{L}$(-p, -q)-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies a metainference token $\langle S, s\rangle$ if and only if, if v satisfies all the inference tokens in S, then v satisfies the inference token s. A metainference token $\langle S, s\rangle$ is (locally) $\mathcal{M}$-valid if and only if all $\mathcal{M}$-valuations satisfy it.*

Naturally, again, a metainference schema is valid if and only if all its instances are. Therefore, let us define a *substructural logic* as a logic where some metainference that is valid in Classical Logic does not hold.

With all of these elements in place, let us present our target logics **ST** and **TS**. Proof-theoretically, these systems can be—respectively—seen as Cut-free and Reflexivity-free versions of Classical Logic, where Cut and Reflexivity are not derivable. More interesting to us, though, is the semantic introduction of these logics. Since they are substructural, they will not be definable in terms of standard logical matrices—i.e., it will not be possible to induce them with the help of such structures. However, Malinowski [20] first and Frankowski [13] later defined algebraic structures that generalize the notion of a logical matrix and that are capable of rendering substructural logics of the desired kind. In particular, Malinowski defined q-matrices which appropriately induce non-reflexive logics, whereas Frankowski defined p-matrices which accordingly induce non-transitive logics.

More concretely, for $\mathcal{L}$ a propositional language, an $\mathcal{L}$-p-matrix is a structure $\langle \mathbf{A}, \mathcal{D}^+, \mathcal{D}^-\rangle$, where $\mathbf{A}$ is an algebra of the same similarity type as $\mathcal{L}$, and $\mathcal{D}^+, \mathcal{D}^- \subseteq \mathcal{V}$ and $\mathcal{D}^+ \subseteq \mathcal{D}^-$. With regard to such structures, valuations (as standardly defined, previously) can be used to define logical consequence relations

[4] The definitions of an $\mathcal{L}$-p-matrix and $\mathcal{L}$-q-matrix are provided below.

in the following way.[5]

Definition 3. *For an $\mathcal{L}$-p-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies an inference token $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if, if $v(\gamma) \in \mathcal{D}^+$, for all $\gamma \in \Gamma$ then $v(\delta) \in \mathcal{D}^-$, for some $\delta \in \Delta$. An inference token $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, for all $\mathcal{M}$-valuations v.*

Moreover, for $\mathcal{L}$ a propositional language, an $\mathcal{L}$-q-matrix is a structure $\langle \mathbf{A}, \mathcal{D}^+, \mathcal{D}^- \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type as $\mathcal{L}$, and $\mathcal{D}^+, \mathcal{D}^- \subseteq \mathcal{V}$ and $\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset$. Once again, with regard to such structures, valuations (as standardly defined, previously) can be used to define logical consequence relations in the following way.

Definition 4. *For an $\mathcal{L}$-q-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies an inference token $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if, if $v(\gamma) \notin \mathcal{D}^-$, for all $\gamma \in \Gamma$ then $v(\delta) \in \mathcal{D}^+$, for some $\delta \in \Delta$. An inference token $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, for all $\mathcal{M}$-valuations v.*

Naturally, inference schemata are $\mathcal{M}$-valid in either p- or q-matrices if and only if all their inference tokens are valid.

In this vein, we can simply present our target substructural logics **ST** and **TS** as induced, respectively, by the p-matrix $\langle \mathbf{SK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{i}\} \rangle$ and the q-matrix $\langle \mathbf{SK}, \{\mathbf{t}\}, \{\mathbf{f}\} \rangle$—where **SK** is the three-element (strong) Kleene algebra. Given theses definitions, it is straightforward to observe that **ST** has the same valid inferences that Classical Logic, whereas **TS** has no valid inferences at all (if the language does not count with truth-constants for the truth-values **t** or **f**).

More importantly—recalling our definition of metainferential validity from a few paragraphs back—it can be easily observed that both systems are substructural. The special and distinctive feature of **ST** being that it is non-transitive (meaning that the metainference called Cut does not hold in it), while **TS** is non-reflexive (meaning that the metainference called Reflexivity does not hold in it).

With all these pieces in place, let us now proceed to the discussion of duality, and to the assessment of how it resonates in the context of the non-classical logics that we just introduced.

[5] A word on how q- and p-matrices generalize the usual notion of a logical matrix is in order. In a usual logical matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ the truth-values of the matrix, i.e. the elements of $\mathcal{V}$, are presented in a *dichotomized* way. By this we mean that they either belong to $\mathcal{D}$—and, hence, are designated—or they belong to $\mathcal{V} \setminus \mathcal{D}$—and, hence, are anti-designated. Contrary to this, q- and p-matrices start from a non-dichotomized classification of the truth-values of the given matrix—i.e. the members of $\mathcal{V}$—letting them belong to two sets, which we here call $\mathcal{D}^+$ and $\mathcal{D}^-$ (we adopt this terminology—i.e. talk of $\mathcal{D}^+$ and $\mathcal{D}^-$—introduced by [27], emphasizing that we will take q- and p-logics to be induced by different type of structures, i.e. respectively q- and p-matrices. In this vein, what will be distinctive of these type of structures will be the properties of the sets $\mathcal{D}^+$ and $\mathcal{D}^-$, as detailed previously.) We will, then, allow these sets to be *jointly non-exhaustive* and *mutually non-exclusive*. Paradigmatically, the first note of this generalization is associated with q-matrices, where it is allowed that $\mathcal{D}^+ \cup \mathcal{D}^- \neq \mathcal{V}$ (see e.g. [20, p. 12]). Analogously, the second note of this generalization is associated with p-matrices, where it is allowed that $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$ (see e.g. [13, p. 45]).

3 Duality in the logico-philosophical literature

In this section, we introduce the notion of duality we will be focusing on throughout this article, namely the notion of *negation duality*. This will allow us to evaluate whether employing it leads to establishing the desired type of connection between the inferences of **ST** and of **TS**, and to prepare the ground for its generalization to metainferences such as Cut and Reflexivity.[6]

Before going to this notion, a few words on the general concept of duality are in order. The notion of duality is very widespread in many fields of thought. In this respect, in [1, p.1] Atiyah says that fundamentally "duality gives two different points of view of looking at the same object", attributing its origin in mathematics and physics to the invention of projective geometry. Projective planes can be defined by points, lines and an incidence relation, but also by lines, points and the inverse of this relation.[7] With the process of algebraization of logic which began in the XIX century, the interest in these sorts of structures was transposed to logic. Its first appearance is attributed to Schröder's [26], and also Kleene refers to Hilbert and Ackermann, and Church in [16, p.15-16] and [6, pp.106-107], respectively.

Kleene himself took a stab at this phenomenon in [19, p.23], with the analysis of the alleged duality between the logical operations of conjunction and disjunction in the context of Classical Logic. Dear to Kleene's reflections, it appears, is the idea that the opposition between truth and falsity is at the core of the notion of duality in logic and that negation is the tool we may use to express implicit relations of duality of this sort. To wit, observe the following story that he uses to motivate the duality between disjunction and conjunction:

> Suppose a visitor from Mars is confused by what he observes upon his arrival on Earth and mistakes our true "t" for false "F"(...). Then our table for & would for him be read as our table for $\vee$ for us, and vice versa.

He then, goes on to analyze what such a Martian would be doing, in confusing things in that peculiar way. For this purpose he appeals to an elucidation of the duality of two truth-functions f and f' saying that they are duals whenever $f(x_1, \ldots, x_m) = y$ if and only if $f'(n(x_1), \ldots, n(x_m)) = n(y)$—where the function $n(x) = 1 - x$ represents the negation of each input. If we have this in mind, then, it is easy to check that conjunction and disjunction are duals in Classical

[6] There are, indeed, other notions of duality defined in the literature, e.g., those defined by Humberstone in [18, p. 93], which we may call *connective-duality* and *converse-duality*. As an anonymous reviewer has suggested, presenting these notions in full detail would deviate us too much from our main point, which is the generalization of the notion of negation duality to substructural logics. Therefore, we leave a deeper exploration of these definitions and their related applications to the target systems for future work.

[7] Thus, as is well known, truths about a plane can be obtained from those about its dual plane, interchanging the words "point" and "line" and inverting the relation—thus, preserving theoremhood. For instance: *Two points can be joined by a unique line* and *Two lines meet in one point.*

Logic, and that the Martian is only dualizing their truth-functions, through the application of negation.[8]

In what follows, we will see that as regards consequence relations or validity claims, negation can also play an important role in establishing their duality. In fact, negation lays at the core of a widespread conception of duality that we call *negation duality* below. This account exploits the idea that negation may be the lens or medium through which we could be able to project the duality between truth and falsity, using it to surface the further duality between consequence relations or logical systems.

The core of the idea of negation duality is, given an inference, to move all of the formulas from premises to conclusions, and vice versa, and replace them by their negations. The motivation for proceeding in this way can be drawn from a close inspection of the usual understanding of logical consequence. Indeed, if logical consequence is understood as either truth-preservation (from premises to conclusions) or as falsity-preservation (from conclusion to premises), then an interesting question may arise. Given some logic **L** endowed with a certain semantics, there might be some inferences which according to the usual interpretation of such semantics are backward falsity-preserving, but which are nevertheless deemed as invalid by the corresponding apparatus. If this is the case, then, one may ask if it is possible to find a system where all the missed falsity-preserving inferences are accounted for. We may call this system the negation dual of **L**.[9]

This being said, let $^{\neg} : FOR(\mathcal{L}) \longrightarrow FOR(\mathcal{L})$ be a function that maps each formula φ to its negation $\neg\varphi$, letting $\neg(\Gamma) = \{\neg\gamma \mid \gamma \in \Gamma\}$, for $\Gamma \subseteq FOR(\mathcal{L})$.

Definition 5. *The negation dual of an inference token* $\Gamma \Rightarrow \Delta$ *is the inference token* $\neg(\Delta) \Rightarrow \neg(\Gamma)$.

Once again, an inference schema is the negation dual of another if and only if each of the inference tokens of one of them has a negation dual that is an inference token of the other. Furthermore, a logic **L** is negation dual to a logic $\mathbf{L}^{\neg}$ if and only if all inference schemata valid in **L** have negation dual inference schemata that hold in $\mathbf{L}^{\neg}$ and vice versa.

It should be highlighted that, as is easy to notice, this notion renders $\mathbf{K_3}$ and **LP** negation duals, and that the relation of duality holding between them is involutive—that is, the negation dual of the negation dual of $\mathbf{K_3}$ (that is,

[8]Notice that in the case where f is the truth-function associated with Boolean negation, its dual f' is also the truth-function associated with Boolean negation. In other words, we could say that negation is *self dual*. We would like to thank an anonymous reviewer for providing us with these historical references and this abstract idea of duality.

[9]For example, according to the usual paraconsistent understanding of **LP**, the inference $\varphi \wedge \neg\varphi \Rightarrow \psi$ is invalid because $\varphi \wedge \neg\varphi$ can be true (because φ can be both true and false) without ψ being true. However, the same understanding would indicate that $\varphi \wedge \neg\varphi$ is always false, thereby implying that the inference $\varphi \wedge \neg\varphi \Rightarrow \psi$ preserves falsity from conclusion to premises—which, for some, may be sufficient for its validity. In such a case, since its validity is not recognized within **LP** one may wonder which is the system that validates all the backward falsity-preserving inferences of **LP**. The answer, as advertised previously, is $\mathbf{K_3}$. Similar considerations apply, in the case of $\psi \Rightarrow \varphi \vee \neg\varphi$, for $\mathbf{K_3}$ and **LP**.

LP) is $\mathbf{K_3}$, and similarly for **LP**. However, under this notion of duality Ex Falso Sequitur Quodlibet and Tertium Non Datur are *not* negation duals to each other. Instead, the negation dual of the former is $\neg(\varphi \vee \neg\varphi) \Rightarrow$, whereas the negation dual of the latter is $\Rightarrow \neg(\varphi \wedge \neg\varphi)$. It is true, though, that in the context of **LP** the inference schema $\neg(\varphi \vee \neg\varphi) \Rightarrow$ is logically equivalent to Ex Falso Sequitur Quodlibet, whereas in the context of $\mathbf{K_3}$ the inference schema $\Rightarrow \neg(\varphi \wedge \neg\varphi)$ is logically equivalent to Tertium Non Datur. So, in a way, these inference schemata can be regarded as negation duals *modulo logical equivalence*.[10]

Nevertheless, with regard to our target substructural logics this notion is useless, since it links every inference valid in a logic to a corresponding negation dual inference that is also valid in the same logic:

Fact 6. **ST** *is self-negation dual, and* **TS** *is self-negation dual.*

Proof. Let us see the case of **ST**. Let us take any valid inference $\Gamma \Rightarrow \Delta$ valid in **ST**. Thus, for every valuation, v, either $v(\gamma) \in \{\mathbf{f}, \mathbf{i}\}$ for some $\gamma \in \Gamma$, or $v(\delta) \in \{\mathbf{i}, \mathbf{t}\}$, for some $\delta \in \Delta$. Thus, using the tables for negation, for every valuation either $v(\neg\gamma) \in \{\mathbf{t}, \mathbf{i}\}$ for some $\gamma \in \Gamma$, or $v(\neg\delta) \in \{\mathbf{i}, \mathbf{f}\}$, for some $\delta \in \Delta$. Therefore, $\neg(\Delta) \Rightarrow \neg(\Gamma)$ is a valid inference in **ST**. The case for **TS** is analogous. □

This result coincides with the original idea of Cobreros et al. in [7], where the authors explicitly state the self-duality of these logics. In the same line, there is another interesting way of thinking about the self-duality of the inferences of **ST** and **TS**, which is a more abstract way of taking the idea of negation duality as applied to the semantics and may also be related to Kleene's ideas. As suggested by an anonymous reviewer, let us say (as in [5]) that a set of truth-values A is in a p-relation R to some other set B provided either some $a_i \in A$ does not belong to D^+ or some $b_j \in B$ belongs to D^-—and similarly for a q-relation.[11] In this vein, we could define the dual relation R' to R as determined by the fact that either some $a_i \in A$ is such that $n(a_i)$ does not belong to the set $n(D^+)$ made of negations of values in D^+, or some $b_j \in B$ is such that $n(b_j)$ belongs to the set $n(D^-)$ of negations of values in D^-.

Now, applying this general idea to the case of R being the **ST**-consequence relation (or similarly to **TS**), it's easy to check that it is in fact self-dual. By definition, $D^+ = \{\mathbf{t}\}$ and $D^- = \{\mathbf{t}, \mathbf{i}\}$, so if we take an **ST**-valid inference $\Gamma \Rightarrow \Delta$ then for every valuation v, there is some $\gamma \in \Gamma$ such that $v(\gamma) \notin D^+$ or there is some $\delta \in \Delta$ such that $v(\delta) \in D^-$. So, defining the function $n(x)$ as the Strong Kleene negation, the dual of the **ST**-consequence relation is such that for every valuation v, there is some $\gamma \in \Gamma$ such that $v(\neg\gamma) \notin n(D^+) = \{\mathbf{f}\}$ or there is some $\delta \in \Delta$ such that $v(\neg\delta) \in n(D^-) = \{\mathbf{f}, \mathbf{i}\}$. Thus, there is some $\delta \in \Delta$ such that $v(\neg\delta) \notin D^+$ or there is some $\gamma \in \Gamma$ such that $v(\neg\gamma) \in D^-$, and

[10] We will go back to this point in the next Section.

[11] This is another way of stating our Definition 3 of the previous Section.

therefore $\neg\Delta$ and $\neg\Gamma$ are in the **ST**-consequence relation. Similarly, for **TS**.[12]

Finally, there is not an obvious way of extending this concept to metainferences, since negation is not involved in either Cut or Reflexivity, rendering both self-negation dual. Then, one may ask if there is a way of generalizing the notion of negation duality that we can come up with that serves this purpose. Indeed, we believe there is.

4 Duality for metainferences

In the previous section, we introduced the notion of negation duality, and showed that it is not entirely satisfactory to provide the desired duality results between the substructural logics **ST** and **TS**, and between the structural schematic metainferences Cut and Reflexivity. In this section we embrace the task of providing philosophical and technical means to flesh out the intuitions claiming that these are in fact dual.

In order to do this, we will draw inspiration from the notion of negation duality. This will lead us to a necessary modification of the way we look at metainferences or, equivalently, to an extension of the framework in which we study metainferences. These changes will be of two sorts. On the one hand, we will allow metainferences to have multiple conclusions—that is, sets (including the empty set) of inferences as conclusions. On the other hand, we will allow for inferences to be of two sorts: positive and negative. Negative inferences will represent the negation of regular (positive) inferences—in a sense to be made clear, shortly. With the help of these elements, we will introduce a novel notion of metainferential duality. In light of this definition, we will properly show that, according to our intuitions, **ST** and **TS** are metainferentially dual. Interestingly enough, we will also show that similarly to what happens to the link between Ex Falso Sequitur Quodlibet and Tertium Non Datur, Cut and Reflexivity are not properly speaking metainferentially dual—although they can be regarded as duals modulo equivalence.

First, notice that in order to find the negation dual of an inference token the notion of negation duality requires us to flip around premises and conclusions. Thus, the negation dual of an inference token with multiple (i.e., with a set of) premises will be an inference token with multiple conclusions. Similarly, the kind of dual that we will be studying hereafter is such that the metainferential dual of a metainference token with multiple premises will necessarily be a metainference token with a set of conclusions. This is why we will need to change our framework.

Let us remark that, besides the purpose of this particular investigation, there seem to be good reasons to admit this extension. After all, we permit multiple conclusions in the context of inferences holding between (collections of) formulae. So, if metainferences (as argued in [4]) are inferences holding between different kinds of relata—in this case, inferences themselves—then there appears

[12] We owe thanks to an anonymous reviewer for providing us with this more abstract way of showing the self duality of these logics at the inferential level.

to be no reason to disallow multiple conclusions in this case but not in the other. Is there anything in particular about inferences as relata of inferences themselves that prevents us from having a unified picture, where both premises and conclusions can be sets? We cannot think of a reason thereof and so, in absence of grounds to refrain from this generalization, we consider it to be acceptable. Furthermore, since premises and meta premises are read conjunctively, conclusions and meta conclusions should be read disjunctively. Intuitively speaking, the separation between the premises (conclusions) of a metainference plays the same role that the commas in a premise (conclusion) of an inference—something that will be salient in the definition of satisfaction for metainferences that we detail below.

Secondly, notice that in order to find the negation dual of an inference token the notion of negation duality requires that we map each formula belonging to the premise set to its negation, and analogously for each formula of the conclusion set. Thus, we need some sort of surrogate of negation that will appropriately apply to the relata of metainferences—that is, to inferences themselves. In other words, we need to find a cogent and perspicuous way to negate inferences. In the context of our discussion, this requires providing some syntactic and semantic details.

Regarding the former, we will start by calling "positive inferences" the regular inference tokens of the form $\Gamma \Rightarrow \Delta$, now alternatively denoting them by $\Gamma \Rightarrow^{+} \Delta$. In this vein, we will denote by $\Gamma \Rightarrow^{-} \Delta$ the negative inference that can be considered the "negation" of the positive or regular inference $\Gamma \Rightarrow^{+} \Delta$. Conversely, we may also call $\Gamma \Rightarrow^{+} \Delta$ the "negation" of $\Gamma \Rightarrow^{-} \Delta$, taking negation for inferences to be involutive—something that will be salient when we discuss the semantic reading of negative inferences.

Regarding the latter, that is, the way in which satisfaction should be understood for negative inferences, we should say the following. Just like formula-negation operates at the formula-level, toggling between truth and falsity, we will let inference-negation operate at the meta-level, affecting the inference satisfaction conditions. While the satisfaction conditions for a positive inference in a given valuation consist in the fact that *if* all premises receive a designated value, *then* so do some of the conclusions, the satisfaction conditions for the corresponding negative inference invert that to a relevant extent. Thus, a valuation satisfies its corresponding negative inference if and only if it assigns a designated value to each premise *and* it assigns a *non*-designated value to each conclusion. We can visualize this more formally in the definition below.[13]

Definition 7. *For an $\mathcal{L}$-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies a negative inference token $\Gamma \Rightarrow^{-} \Delta$, in symbols $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^{-} \Delta$, if and only if $v(\gamma) \in \mathcal{D}$, for every $\gamma \in \Gamma$, and $v(\delta) \notin \mathcal{D}$, for every $\delta \in \Delta$. A negative inference token $\Gamma \Rightarrow^{-} \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow^{-} \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^{-} \Delta$, for all $\mathcal{M}$-valuations v.*

Notice that by a simple inspection of the satisfaction conditions, a valuation

[13] We would like to thank an anonymous referee for asking us to clarify this point.

v satisfies a negated inference $\Gamma \Rightarrow^- \Delta$ if and only if it satisfies all of the members of the following set of positive inferences $\{\Rightarrow^+ \gamma, \delta \Rightarrow^+\}$, for every $\gamma \in \Gamma$ and $\delta \in \Delta$. This is so, because for a valuation v, $v(\varphi) \in \mathcal{D}$ if and only if $v \vDash_{\mathcal{M}} \Rightarrow^+ \varphi$, and $v(\varphi) \notin \mathcal{D}$ if and only if $v \vDash_{\mathcal{M}} \varphi \Rightarrow^+$.

We would like to highlight that in a classical setting, the satisfaction conditions for a negative inference correspond to the counterexamplification conditions of the corresponding positive inference. This equivalence can be reflected by the fact that our newly introduced negation for inferences satisfies certain principles that we may analogously refer to as Exclusion and Exhaustion—to parallel what happens at the level of negation for formulae. Thus, in Classical Logic the following holds, for every Boolean valuation v.

$$\text{Exclusion: not } (v \vDash \Gamma \Rightarrow^+ \Delta \text{ and } v \vDash \Gamma \Rightarrow^- \Delta)$$

$$\text{Exhaustion: } v \vDash \Gamma \Rightarrow^+ \Delta \text{ or } v \vDash \Gamma \Rightarrow^- \Delta$$

In this spirit, it is interesting to observe that negation for inferences can be as non-classical as negation for formulae can be. Indeed, just like some non-classical systems allow for some formula and its negation to be both true and false, while other systems allow for some formula and its negation to be neither true nor false, we may have non-classical logics that treat inferences and their negations in a similar way. In other words, although every Boolean valuation satisfies either the positive-inference or its corresponding negative-inference, but not both, there may be some logics allowing for positive inferences of the form $\Gamma \Rightarrow^+ \Delta$ and negative inferences of the form $\Gamma \Rightarrow^- \Delta$ to be both satisfied at the same time (violating Exclusion) whereas some other systems may allow for neither of them to be satisfied (violating Exhaustion).

Actually, as we will see, this is what happens once we take into account p-logics and q-logics, especially **ST** and **TS**. However, in order to see this, we need to define the conditions of satisfaction of a positive inference and of its corresponding negative inference, generalizing the Definition 7. To carry out this generalization, we have to pay attention to the fact that in p- and q-logics we have two different sets of designated values—thus, the generalizations should be sensible to those particularities. For this purpose, let us first adapt the Definition 7 for $p-$logics and $q-$logics, as we did in Definitions 3 and 4 for positive inferences:

Definition 8. *For an $\mathcal{L}$-p-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies a negative inference token $\Gamma \Rightarrow^- \Delta$, in symbols $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$, if and only if $v(\gamma) \in \mathcal{D}^-$, for all $\gamma \in \Gamma$, and $v(\delta) \notin \mathcal{D}^+$ for all $\delta \in \Delta$. As before, a negative inference token $\Gamma \Rightarrow^- \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$, for all $\mathcal{M}$-valuations v.*

Definition 9. *For an $\mathcal{L}$-q-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies a negative inference token $\Gamma \Rightarrow^- \Delta$, in symbols $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$, if and only if $v(\gamma) \in \mathcal{D}^+$, for all $\gamma \in \Gamma$, and $v(\delta) \in \mathcal{D}^-$ for all $\delta \in \Delta$. As before, a negative inference token $\Gamma \Rightarrow^- \Delta$ is $\mathcal{M}$-valid (written $\vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$) if and only if $v \vDash_{\mathcal{M}} \Gamma \Rightarrow^- \Delta$, for all $\mathcal{M}$-valuations v.*

Notice that in this case too, the effect is the same as in the Definition 7, a valuation v satisfies a negated inference $\Gamma \Rightarrow^- \Delta$ if and only if it satisfies the set of positive inferences $\{\Rightarrow^+ \gamma, \delta \Rightarrow^+\}$, for every $\gamma \in \Gamma$ and $\delta \in \Delta$. Let us see this for p-logics. For any valuation, $v \vDash \Rightarrow^+ \gamma$ if and only if $v(\gamma) \in \mathcal{D}^-$, and $v \vDash \delta \Rightarrow^+$ if and only if $v(\delta) \notin \mathcal{D}^+$. The case of q-logics is similar.

As advertised, observing negative inferences in the context of p- and q-logics like **ST** and **TS** allows to surface cases where negation for inferences displays a certain non-classical behavior—thus, not complying with the Exclusion or Exhaustion clauses depicted above. To observe this, consider a (positive or negative) inference $\Gamma \Rightarrow^{+/-} \Delta$ and take a strong Kleene valuation v, where for all propositional variables p, $v(p) = \mathbf{i}$. Then, $v(\Gamma) = v(\Delta) = \mathbf{i}$. Such a valuation satisfies both $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$ in **ST**, but does not satisfy neither $\Gamma \Rightarrow^+ \Delta$ nor $\Gamma \Rightarrow^- \Delta$ in **TS**. Some could, following the path initiated by [3], see this as a sort of revelation of the paraconsistent nature of **ST** and the paracomplete nature of **TS**, but we will not discuss this issue at length here.

Now, with the help of these modifications to the metainferential framework, we are in a position to describe what a metainference token and a metainference schema are, in this new sense and—most importantly—to precisely state when two metainferences are dual, which will lead us to understand when two logics are metainferentially dual. In this line, we will say that a *metainference token* of the language $\mathcal{L}$ is a pair $\langle S_1, S_2 \rangle$, where S_1 and S_2 are sets that may contain both positive and negative inference tokens of $\mathcal{L}$. So, a *metainference schema* of a language $\mathcal{L}$ is the set of all and only metainference tokens that can be obtained from one of its members—its "basic instance"—by uniformly substituting some propositional variable p in it by some formula φ. For these sorts of metainferences which can relate both positive and negative inferences, we define validity as follows.

Definition 10. *For an $\mathcal{L}$(-p, -q)-matrix $\mathcal{M}$ an $\mathcal{M}$-valuation v satisfies a metainference token $\langle S_1, S_2 \rangle$ if and only if v satisfies all the inference tokens in S_1 only if v satisfies some of the inference tokens in S_2. A metainference token $\langle S_1, S_2 \rangle$ is (locally) $\mathcal{M}$-valid if and only if all $\mathcal{M}$-valuations satisfy it.*

As before, satisfaction and validity can be defined for metainference schemata, i.e., a metainference schema is valid if and only if all its instances are. Intuitively speaking, a metainference has the following form.[14]

$$\frac{\Gamma_1 \Rightarrow^{+/-} \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow^{+/-} \Delta_n \quad \dots}{\Sigma_1 \Rightarrow^{+/-} \Pi_1 \quad \dots \quad \Sigma_m \Rightarrow^{+/-} \Pi_m \quad \dots}$$

where $\Rightarrow^{+/-}$ stands either for $\Rightarrow^+$ or for $\Rightarrow^-$.

Definition 11. *The metainferential-dual of the metainference token*

[14]Notice that in the definition S_2 can be the empty set. In this case, we take it to represent something like a meta-level falsum—intuitively speaking, representing the fact that everything (i.e., every set of positive and negative inferences) follows from S_1.

$$\frac{\Gamma_1 \Rightarrow^{+/-} \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow^{+/-} \Delta_n \quad \ldots}{\Sigma_1 \Rightarrow^{+/-} \Pi_1 \quad \ldots \quad \Sigma_m \Rightarrow^{+/-} \Pi_m \quad \ldots}$$

is the metainference token

$$\frac{\Sigma_1 \Rightarrow^{-/+} \Pi_1 \quad \ldots \quad \Sigma_m \Rightarrow^{-/+} \Pi_m \quad \ldots}{\Gamma_1 \Rightarrow^{-/+} \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow^{-/+} \Delta_n \quad \ldots}$$

where $-/+$ *means that we replace each* $\Rightarrow^+$ *by a* $\Rightarrow^-$ *and vice versa.*

Once more, a metainference schema is the metainferential-dual of another if and only if each of the metainference tokens of the former has a metainferential-dual that is a metainference token of the latter. Furthermore, a logic $\mathbf{L}_1$ is metainferential-dual to a logic $\mathbf{L}_2$ if and only if all metainference schemata valid in $\mathbf{L}_1$ have metainferential-duals that holds in $\mathbf{L}_2$. Notice that this definition already exhibits clearly the fact that we are generalizing the notion of negation duality, conceived for inferences: we flip the role of premises and conclusions and map each object—in this case, each inference—to its negation.

Having presented this novel approach, we will now proceed to discuss the extent to which **ST** and **TS**, as well as Cut and Reflexivity can be said to be metainferentially dual. In this respect, let us highlight that, happily, we have a positive answer to the question about the duality of **ST** and **TS**—as the following theorem shows.

Theorem 12. **ST** *and* **TS** *are metainferentially dual.*

Proof. A metainference token

$$\frac{\Gamma_1 \Rightarrow^{+/-} \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow^{+/-} \Delta_n \quad \ldots}{\Sigma_1 \Rightarrow^{+/-} \Pi_1 \quad \ldots \quad \Sigma_m \Rightarrow^{+/-} \Pi_m \quad \ldots}$$

is valid in **ST** (**TS**) if and only if for every valuation v, either it does not satisfy one of the positive or negative premise-inferences, or it satisfies one of the positive or negative conclusion-inferences. It is straightforward to check that a valuation v does not satisfy a positive inference $\Gamma_i \Rightarrow^+ \Delta_i$ according to **ST** (**TS**) if and only if it satisfies the negative inference $\Gamma_i \Rightarrow^- \Delta_i$ according to **TS** (**ST**), and that it does not satisfy a negative inference $\Gamma_i \Rightarrow^- \Delta_i$ according to **ST** (**TS**) if and only if it satisfies the positive inference $\Gamma_i \Rightarrow^+ \Delta_i$ according to **TS** (**ST**). Thus, the previous metainference token is valid in **ST** (**TS**) if and only if the following metainference token, which is the metainferential dual of the one outlined above

$$\frac{\Sigma_1 \Rightarrow^{-/+} \Pi_1 \quad \ldots \quad \Sigma_m \Rightarrow^{-/+} \Pi_m \quad \ldots}{\Gamma_1 \Rightarrow^{-/+} \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow^{-/+} \Delta_n \quad \ldots}$$

is valid in **TS** (**ST**). □

In line with the corroboration of the duality holding between **ST** and **TS**, one would also expect that, just as in the inferential case, **CL** turns out to be metainferentially self-dual. This turns out to be the case, as is easy to check.

Moving on now to the question of whether or not Cut and Reflexivity are metainferentially dual, we must advance that our answer to this issue is not as nice as one would intuitively expect it to be. To understand what we mean by this, let us start by applying the previously defined notions to the simple instance of Cut, Here formulated in an even simpler form, with an empty conclusion, which is now allowed by the new framework:

$$\frac{\Rightarrow^{+} p \quad p \Rightarrow^{+}}{}$$

If a logic validates it, its dual logic must validate the following:

$$\frac{}{\Rightarrow^{-} p \quad p \Rightarrow^{-}}$$

which, strictly speaking, is not Reflexivity:

$$\frac{}{p \Rightarrow^{+} p}$$

but they are equivalent.[15] The other way around, if a logic validates Reflexivity in the following form:

$$\frac{}{p \Rightarrow^{+} p}$$

its dual logic must validate the following:

$$\frac{p \Rightarrow^{-} p}{}$$

which is not Cut. However, it is equivalent to:[16]

$$\frac{\Rightarrow^{+} p \quad p \Rightarrow^{+}}{}$$

So, as a result of our definitions, strictly speaking Cut and Reflexivity are not dual, and the duality between Cut and Reflexivity holds under equivalence (they are dual to metainferences which are logically equivalent).

One could argue that this is not enough, since any other equivalent metainference would be dual in this sense. However, this is not something peculiar to this case. Actually, as we previously said, we are developing a kind of negation duality for metainferences, and something similar happens between Excluded Middle and Explosion, regarding the notion of negation duality for inferences.

[15] Using the definition of a negative inference, a valuation v does not satisfy the dual metainference of this instance of Cut if and only if it does not satisfy $\Rightarrow^{+} p$ and it does not satisfy $p \Rightarrow^{+}$. In any logic in which this is the case, such a valuation will be a counterexample to the instance of Reflexivity $p \Rightarrow^{+} p$. And similarly in the other way around.

[16] Again, using the definition of a negative inference, a valuation v does not satisfy the dual metainference of this instance of Reflexivity if and only if it satisfies $\Rightarrow^{+} p$ and $p \Rightarrow^{+}$. In any logic in which this is the case, such a valuation will be a counterexample to the instance of the corresponding instance of Cut. And similarly in the other way around.

Let us recall that according to Definition 5 given an inference $\Gamma \Rightarrow \Delta$ its negation dual is $\neg(\Delta) \Rightarrow \neg(\Gamma)$. However, contrary to what one would expect, Excluded Middle and Explosion are negation dual, only under equivalence (in the logics $\mathbf{K_3}$ and **LP**). Therefore, under one of the most employed definitions of inferential duality—what we call negation duality) Excluded Middle and Explosion are dual only under equivalence, in the same sense as the new duality established between the instance of Cut without contexts and Reflexivity.

We would like to stay open about what should be the moral of this discussion. Perhaps, there is something special about these equivalences or, perhaps, yet another notion of metainferential duality is needed in order to capture the duality between Cut and Reflexivity.[17]

5 A word on global validity for metainferences

Throughout this article, we focused on one particular definition of metainferential validity—denoted "local"—that determined whether or not certain metainferences (like Cut and Reflexivity) hold in the context of a given logic. Thus, one may wonder how dependent on this notion our results are. In what follows we show two things. First, that the duality result can be extended to apply to a generalization of what is known as the global notion of metainferential validity implemented, e.g., in [2]. The key to this will be the fact that both concepts can be made to collapse. The second thing we will argue is that there is some technical convenience in using the local definition, as we have done up until this point.

We need to start then by providing the aforesaid generalization of the usual definition of Global validity, in order for it to fit our particular framework. The problem is that there are more than one possible way to do it, depending on how we quantify over valuations. The first choice is the most natural one, which is preservation of validity:

Definition 13. *For an $\mathcal{L}$(-p, -q)-matrix $\mathcal{M}$ a metainference token $\langle S_1, S_2 \rangle$ is globally valid if and only if either some valuation does not satisfy some inference token in S_1 or some inference token is satisfied by all valuations in S_2.*

Unfortunately, doing things this way, we face a problem. In the standard framework, the most basic link between metainferential validities consists on the fact that locally valid metainferences are a subset of the globally valid ones. This can be illustrated by the following schema, which is classically locally valid, but globally invalid under the previous definition, given that not all formulae are either classical tautologies or classical contradictions:

Thus, we offer this alternative version:

[17] Of course, a further point is related with what we are calling Cut in this section. Here, we are considering only one special instance of Cut (without context and without conclusion), so everything that we have been saying only applies to this case and we expect to expand on it in future works.

$$\frac{\Rightarrow^+}{\psi \Rightarrow^+ \quad \Rightarrow^+ \psi}$$

Definition 14. *For an $\mathcal{L}$(-p, -q)-matrix $\mathcal{M}$ a metainference token $\langle S_1, S_2\rangle$ is globally valid if and only if either some valuation does not satisfy some inference token in S_1 or all valuations satisfy some inference token in S_2.*

Notice that even though the first disjunct stays as it was in Definition 13, the second one now asks for the conclusions to be satisfied "collectively", so to speak, by every valuation. The main disadvantage is that there is nothing being preserved here, and hence, this is a somewhat more artificial concept. However, it has two redeeming qualities. The first one is that it coincides with standard global validity in the limit, single-conclusion case, and thus, it counts as a generalization thereof. The second one is that now we regain the inclusion relation: if a metainference is globally invalid in this sense, all premises are valid but there is one valuation that satisfies no conclusion, and that valuation serves as a local counterexample.

In the second place, we want to show that this relation can be turned into an identity. If one only considers metainference tokens, then the inclusion between the two concepts is strict: any token with invalid premises will be globally valid. However, if it is metainference schemata that are under analysis, things change. In the standard framework, these two notions easily collapse in languages that are rich enough—as argued in [28]. One straightforward sufficient condition for the collapse is precisely the inclusion of truth-constants for the truth-values of the semantic structure inducing the logic in question, which in the case of **TS** and **ST** can be incorporated without issue. As an illustration, consider how the following schemata, which are globally valid but locally invalid according to **TS** and **ST** respectively:

$$\frac{\Rightarrow^+ \varphi}{\psi \Rightarrow^+ \psi} \qquad \frac{\Rightarrow^+ \varphi \quad \varphi \Rightarrow^+}{\Rightarrow^+}$$

become globally invalid by the introduction of a **t**-constant $\top$ and **i**-constant λ, as they acquire as tokens the global **TS** and **ST**-counterexamples below:

$$\frac{\Rightarrow^+ \top}{p \Rightarrow^+ p} \qquad \frac{\Rightarrow^+ \lambda \quad \lambda \Rightarrow^+}{\Rightarrow^+}$$

What we want to know in order to extend our duality result is whether the collapse also holds in the modified framework. The answer is that it does. The introduction of truth constants allows us to have, for each locally invalid schema, an instance where the premises are valid and the conclusion is not. The trick in the proof—as can be guessed from the example above—consists in mimicking the assignments of the local counterexample with formulas that get that value in every valuation. The proof in the new framework works in the exact same way.

So for the remainder of this section, let us grant that our base language is indeed enriched with all truth-constants. Thus, the following follows from Theorem 12:

Theorem 15. **ST** *and* **TS** *are globally metainferentially dual.*

Proof. Let S be a metainference schema which is globally invalid in **ST** (**TS**). Thus, there is a token S_t which is globally invalid in **ST** (**TS**). By the inclusion of global invalidity in local invalidity, S_t is also locally invalid in **ST** (**TS**). Thus, by the theorem 12, its dual S_t^d is locally invalid in **TS** (**ST**). Thus, the dual of S, S^d has a locally invalid instance in **TS** (**ST**). Hence, because of the collapse of local and global schemata, S^d has a globally invalid instance in **TS** (**ST**). Hence, S^d is globally invalid in **TS**. □

Whence, unless one has a motive against considering schemata, or against enriching the language, the choice between local and global validity is mostly a matter of convenience. People have certainly put forward a variety of reasons against those things. For instance, [12] believe the focus should be made on metainference tokens. And although we do think schemata are of crucial importance, in the context of our present discussion, we believe there is some weight in favor of working with local validity, the way we did.

First of all, we want to point out that one reason to be interested in duals at all has to do with the fact that the information one learns about something can often be turned into information about that thing's dual. Thus for example, if we know the logic $\mathbf{K_3}$ lacks tautologies, and that **LP** is its dual, we can infer that **LP** lacks contradictions, which is the dual property. Of course, this will not hold for any property whatsoever, but if we are dealing with logical duality, it is reasonable to expect that logical properties will be preserved.

However, even though Theorem 15 guarantees duality for schemata, Global validity is not preserved token-to-token. In particular, if a schema is globally invalid, then it will have tokens which are globally valid in **TS** but their duals are invalid in **ST**.[18]

Put another way, the counterexample relation between tokens and schemata is not preserved through duality. The reason is that most metainference schemata have what we will call "vacuous tokens"—i.e., metainference tokens where global validity is preserved merely because the premises of the metainference are invalid inferences. Take as an illustration the **ST**-invalid schema to the left, and its **TS**-invalid dual to the right:

$$\frac{\Rightarrow^+ \varphi}{\Rightarrow^+ \psi} \qquad \frac{\Rightarrow^- \psi}{\Rightarrow^- \varphi}$$

Even though the following token to the left is a global **ST**-counterexample to the first schema, the **TS**-counterexample to its dual is not the token to the right, which is vacuously **TS**-valid :

[18] Recall that, in the proof, the invalid instance need not be the same.

$$\frac{\Rightarrow^{+} p \vee \neg p}{\Rightarrow^{+} q} \qquad \frac{\Rightarrow^{-} q}{\Rightarrow^{-} p \vee \neg p}$$

To get a **TS**-counterexample, one necessarily needs to use a truth constant:

$$\frac{\Rightarrow^{-} \bot}{\Rightarrow^{-} p}$$

Summing up these reflections, we can take away the following conclusions. First, that using Local metainferential validity—instead of Global validity—allows us to nicely pair metainferences such that duality holds not only for metainference schemata, but also point-wise for metainference tokens. This might come as no surprise, since Local validity makes finer-grained distinctions among metainference tokens which are globally valid in a vacuous way, and is thus better suited to work with tokens. Second, that at least from a formal point of view, there is no strong dependence of the duality results on the local definition, at least for cases where we have truth-constants and are comparing schemata.

6 An alternative: duality via translations

Up to now, we have been focused on our proposal about metainferential duality which ultimately led us to establish the duality between **ST** and **TS**. In this respect, one may wonder whether similar results could be established through the implementation of some representation theorems recently proved for these substructural logics. The aim of this section is to explore this alternative, and to assess its advantages and disadvantages, in relation to our own proposal defended above. Given that the results we use in this section are proved for a framework without negative inferences and with metainferences with single conclusions, in what follows we will work within that framework.

For this purpose, we will examine the prospects of combining two sorts of technical results. On the one hand, the previously discussed fact that **LP** and $\mathbf{K_3}$ are negation duals of each other. On the other hand, the deep connection between **ST** and **LP** (and, concomitantly, between **TS** and $\mathbf{K_3}$) noticed by [22] and [2], and reconstructed in [12]. The latter refers to the fact that—under somehow reasonable *translations*—it is possible to correlate the (locally) valid metainferences of **ST** exactly with the valid inferences of **LP**, and similarly for **TS** and $\mathbf{K_3}$. These representation theorems can be proved by several means, with some minor differences in the way different authors present this issue—all irrelevant to our point. For this purpose, here we will work with the transformations called τ and ρ in [12], which we alternatively call τ_1 and τ_2.

As shown in the aforementioned works, the translation function τ_1 can be used to correlate every metainference with an inference. That is to say, to transform each metainference into an inference, by translating its set of premise-inferences into a set of premise-formulae, and its conclusion-inference into a conclusion-formula.

Definition 16. *The inference-correlate of a metainference token* $\langle\{s_1, ..., s_n\}, s\rangle$ *is the inference token* $\langle\{\tau_1(s_1), ...\tau_1(s_n)\}, \tau_1(s)\rangle$

Interestingly, with these tools in hand (or some slight variation thereof), it was recently shown that the set of valid metainferences of **ST** can be essentially understood in terms of the set of valid inferences of **LP**—for each valid metainference of **ST** has a corresponding valid inference-correlate that is valid in **LP**. Similar results hold, in a straightforward manner, for **TS** and $\mathbf{K_3}$.[19]

Fact 17. *A metainference token is locally valid in* **ST** *(respectively, in* **TS***) if and only if its inference-correlate is valid in* **LP** *(respectively, in* $\mathbf{K_3}$*).*

Given the duality holding between **LP** and $\mathbf{K_3}$ one might expect to use this fact, in addition to the representation results above, to establish the desired duality between **ST** and **TS**—and, possibly, between Cut and Reflexivity.

However, if we want to pursue this line of reasoning, we immediately face a technical problem. We cannot use τ_2 to define the metainference-correlate of an inference as straightforwardly as we used τ_1 to define the inference-correlate of a metainference—i.e., in a point-wise fashion. As we discussed earlier, the reason is simply that frameworks for metainferences and inferences are usually different. Thus, we need to adjust either the definition of duality for inferences or the translation functions, in order to guarantee that we will always have one object in the conclusions of metainferences. We detail below one way of doing this.

Definition 18. *The metainference-correlate of an inference token* $\Gamma \Rightarrow \Delta$ *is the metainference token* $\langle\tau_2(\bigwedge\Gamma), \tau_2(\bigvee\Delta)\rangle$*, if* $\Delta \neq \emptyset$*, or* $\langle\tau_2(\bigwedge\Gamma), \emptyset \Rightarrow \emptyset\rangle$*, if* $\Delta = \emptyset$*.*

Given the way conjunction and disjunction work within the strong Kleene algebra, this transformation preserves the desired connection between these four non-classical logics, as is straightforward to see.

Fact 19. *An inference token is valid in* **LP** *(respectively, in* $\mathbf{K_3}$*) if and only if its metainference-correlate is locally valid in* **ST** *(respectively, in* **TS***).*

With these auxiliary results in mind we may now proceed to detail a somewhat derivative notion of duality, defined through transformation or translation functions. According to this approach, for every metainference token, its dual according to the previous translations is the metainferential-correlate of the dual of its inference-correlate.

Definition 20. *The translation-dual of a metainference token* $\langle\{s_1, \ldots, s_n\}, s\rangle$ *is the metainference token* $\langle\tau_2(\neg\tau_1(s)), \tau_2(\neg\tau_1(s_1) \vee ... \vee \neg\tau_1(s_n))\rangle$*.*

Putting all these facts together, we can finally connect **ST** with **TS**.

[19]The proofs of the following four facts can be easily adapted from [2], and [4]

Theorem 21. *A metainference token is valid in* **ST** *(respectively, in* **TS***) if and only if its translation-dual is valid in* **TS** *(respectively, in* **ST***).*

Let us mention a few reservations regarding some of the drawbacks that this result has—all stemming from the fact that it took us such a roundabout way to finally get to the desired destination.

Surely, the method is cumbersome and inelegant, as it depends too heavily on the linguistic resources of the language. But this is not the only inconvenience—that could be chalked up to the language, and not to the theory of duality. However, the finitary nature of both inferences and metainferences is characteristic of the previously described notion of metainferential duality. With respect to the former, inferences can only have finitely many premises, as well as finitely many conclusions. In this vein, it shall be noted that this could be a desirable feature of logical frameworks, sought either proof-theoretically or semantically, i.e., through compactness. However, the present conception of duality assumes this property by *fiat*, rather than arguing for it or showing that it is somehow induced by other aspects of the formalism. With regard to the latter (that is, with regard to metainferences) it is less clear whether the finite character thereof is desirable or not. In any case, we would like to underline that the present framework does not allow for a choice in this respect—thus being rather sub-optimal from the point of view of having neutral playgrounds to work in.

More importantly, as we are trying to shed light on the connection between **ST** and **TS**, translation-duality renders rather unhappy results regarding Cut and Reflexivity. To wit, if we take one of the most simple (in the logical sense) instances of Cut.

$$\frac{\Rightarrow p \quad p \Rightarrow}{\Rightarrow}$$

Notice that, according to the previous definitions, its translation-dual not only is hardly reminiscent of Reflexivity, but worse, it is not a structural property anymore—instead, it is a property of negation, or of the interaction between negation and disjunction.

$$\frac{\Rightarrow \bot}{\Rightarrow \neg p \vee \neg\neg p}$$

This seems quite unpleasant for someone trying to have an accurate understanding of metainferential duality.[20] As highlighted by an anonymous reviewer, it still is true that Cut and Reflexivity are duals up to logical equivalence—i.e., that the dual of each is logically equivalent to the other.

[20]In fact, this asymmetry is not particular to Cut, but present when we look at the dual of every metainference which has more than one metainference-premise. A quick inspection of the definitions shows that these sorts of metainferences will have duals with only one metainference-conclusion, full of negations and disjunctions. However, if the notion of duality that we are looking for will somehow illuminate the duality between structural features understood as schematic metainferences where no connective is explicitly mentioned, then the outcome of this method is clearly unsatisfactory.

Given that this is just the same result for our novel notion of metainferential duality detailed in Section 4, one may wonder why our account could be said to be superior to the translation approach. The answer to this is that the translation account seems too focused on **ST** and **TS**, relying too much on the aforementioned representation theorems for said substructural logics. It is unclear, though, how well would this generalize to other substructural logics, for which we do not have translation results of any sort. Indeed, excessively focusing on this attempt (although somewhat efficient for the case of **ST** and **TS**) could prove to leave out, e.g., the case of the duality between the smallest non-transitive and the smallest non-reflexive logic. Given these considerations, we think it is best to focus on our proposal for metainferential duality, which is independent of object-language resources and representation results.

7 Conclusions and future work

In this article, we evaluated the extent to which one of the most important notions of duality available in the literature, the so-called notion of negation duality, can be applied to illuminate certain alleged dualities between substructural logics. In doing so, we focused on the much commented "intuitive" duality between the non-transitive logic **ST** and the non-reflexive logic **TS** and, more particularly, between Cut and Reflexivity. Our aim in discussing this issue was to provide a deeper understanding of the notion of duality with respect to substructural logics. In this respect, we showed that the negation duality was unsatisfactory to elucidate the sense in which said substructural logics are dual—for these systems ended up actually being self-duals. We also inspected an indirect way of establishing this duality, through a detour on translation functions, ultimately concluding that this was unsatisfactory too.

For these reasons, we thought there was a need to develop a novel notion of duality, inspired in the notion of negation duality and considered as a metainferential version thereof, which turned out to actually require a change of framework. More concretely, we highlighted that an appropriate study of the phenomenon of duality within the realm of substructural logics required an extension of the framework that includes both positive and negative inferences, together with metainferences that allowed multiple conclusions, as well as multiple premises. Thus, within this framework, we developed a novel notion of metainferential duality, which led us to finally show the intended duality between **ST** and **TS** and—modulo equivalences—of Cut and Reflexivity.

These explorations, however, mark only the beginning of a series of questions that should be answerable with the help of the tools developed above. Some of these pertain to the duality of certain (schematic) metainferences, and of some substructural logics. So, one could consider the other structural rules like Contraction and Weakening, asking which metainferences are dual thereof. According to our notion of metainferential duality, we obtain the following duals, respectively.[21]

$$\text{Dual-Contraction:}\quad \frac{\Gamma, \varphi \Rightarrow^{-} \Delta}{\Gamma, \varphi, \varphi \Rightarrow^{-} \Delta} \qquad \frac{\Gamma \Rightarrow^{-} \varphi, \Delta}{\Gamma \Rightarrow^{-} \varphi, \varphi, \Delta}$$

$$\text{Dual-Weakening:}\quad \frac{\Gamma, \varphi \Rightarrow^{-} \Delta}{\Gamma \Rightarrow^{-} \Delta} \qquad \frac{\Gamma \Rightarrow^{-} \varphi, \Delta}{\Gamma \Rightarrow^{-} \Delta}$$

Both rules are valid metainferences in most logics, and a drastic modification in the notion of validity is needed in order to invalidate them. Thus, one still could insist on investigating whether there is an argument against these duals and develop another notion of duality. We hope to investigate these issues in future works. In a similar vein, we stressed that we hoped the above-developed tools could illuminate the extent to which other logics, were dual to each other. That is the case of subvaluationism and supervaluationism. Proving whether or not the resulting logics are metainferentially dual to each other, in our technical sense, is a deeply interesting task that we hope to tackle in the near future.

Finally, substructural logics have been recently subject to discussion also regarding the so-called generalized metainferences of arbitrarily great levels—that is, metainferences of level 1 (inferences between inferences), metainferences of level 2 (inferences between metainferences), and so on. In this respect, [4] described a "hierarchy" of systems counting with logics that coincide with Classical Logic up to a certain metainferential level n—that is, they share their metainferential validities. Moreover, [21] and [25] construct a parallel hierarchy of systems counting with logics agreeing with Classical Logic up to a certain metainferential level n, in a rather different sense—that is, by sharing their metainferential anti-validities. Whence, there is an intuitive sense in which these two hierarchies (and each of the logics of each hierarchy for each metainferential level n) are dual to each other. However, the notion of metainferential duality discussed in the previous sections cannot explain this, as it is only devised for metainferences of level 1. We conjecture, in this vein, that a generalization of the definition of metainferential duality, for any metainferential level n, and a suitable modification of the metainferential frameworks, may provide the expected results. We hope to discuss these and other issues soon.

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[21]Here we assume that the formulation of the structural rules involves only the positive inferences. An interesting generalization would be to also formulate the structural rules using negative inferences.

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