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Two-fold singularities in nonsmooth dynamics — higher dimensional analogues

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Abstract

When a system of ordinary differential equations is discontinuous along some threshold, its flow may become tangent to that threshold from one side or the other, creating a *fold* singularity, or from both sides simultaneously, creating a *two-fold* singularity. The classic two-fold exhibits intricate local dynamics and accumulating sequences of local bifurcations, and is by now rather well understood, but it is just the simplest of an infinite hierarchy two-folds and multi-folds in higher dimensions. These arise when a system is discontinuous along multiple intersecting thresholds, and the induced *sliding* flows on those thresholds becomes tangent to their intersections. We show here, surprisingly, that these higher dimensional analogues of the two-fold reduce to the equations of the classic two-fold, providing the first step into their study and a new tool to understand higher dimensional systems with discontinuities.

Nonsmooth dynamical systems, which describe situations with discontinuities like impacts, stick events, switching, decisions, etc., are increasingly prevalent in mathematical modeling. Their general behaviour beyond two or three dimensions is little understood at present, and relies on understanding their attractors and singularities. Here we study a recently discovered singularity that provides a significant new window into such higher dimensional systems. We show how its equations reduce to those of a well understood yet novel singularity from three-dimensional systems, setting a template for an infinite hierarchy of related singularities in higher dimensions.

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1 Introduction

Piecewise-smooth dynamical systems are finding use in an ever increasing range of modeling applications, from low dimensional studies such as stick-slip in mechanical contact [17, 24, 35], switching electronics [9, 12, 30, 34], and abrupt transitions in living organisms [26, 28] or climate [3, 25, 29], to higher dimensional studies of oscillating blocks [4, 6] or gene networks [1, 5, 10, 23, 27]. There have been widespread advances in the theory in recent years (see e.g. [2, 11, 16, 21]).

In both theory and applications, interest is naturally turning to higher dimensions. In smooth systems behaviour becomes more complicated in higher dimensions because there is more dimensional freedom and very little to constrain complex behaviours. This situation is markedly worse for piecewise-smooth systems as even their local dynamics becomes rapidly more complicated in higher dimensions. Moreover, piecewise-smooth systems suffer from a curse of dimensionality: that every higher dimension brings qualitatively different singularities and bifurcations (see e.g. [15]).

One such singularity occurs when orbits of a dynamical system are tangent to thresholds along which its differential equations are discontinuous, or to intersections of those thresholds as we study here. We will show, however, that in some cases we can reduce higher dimensional singularities to the equations of their lower dimensional analogues.

When a flow is tangent to a switching threshold from both sides at the same point, forming a two-fold as in fig. 1, then even without the presence of equilibria or periodic orbits, they give rise to some of the most novel and challenging behaviour of piecewise-smooth systems (see e.g. Chapter 13 of [21], Chapter 5 §22 of [14], and [20, 31]).

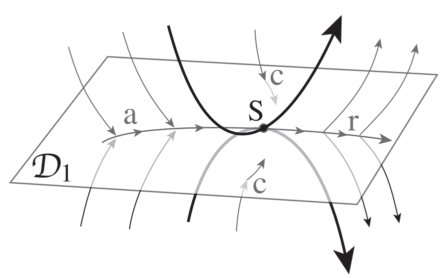

Figure 1: The '*classic*' two-fold singularity (labelled S) arises in when flows either side of a switching threshold $\mathcal{D}_1$ are both tangent to the threshold. Regions are indicated where the flow is (a) attracted to, (r) repelled from, or (c) crosses the thresholds.

In a system with multiple discontinuities that occur at multiple switching thresholds, the flow can be tangent to one or more thresholds from one or more sides. In [18] it was shown schematically that this creates more exotic kinds of two-fold, depicted in fig. 2 and fig. 3, and that surprisingly they could be reduced to the familiar equations of the classic two-fold singularity in fig. 1. These occur in a system with two switching thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$, intersecting transversally on a set $\mathcal{D}_{12}$, and the tangency occurs in the *sliding* dynamics that is induced on the switching thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$.

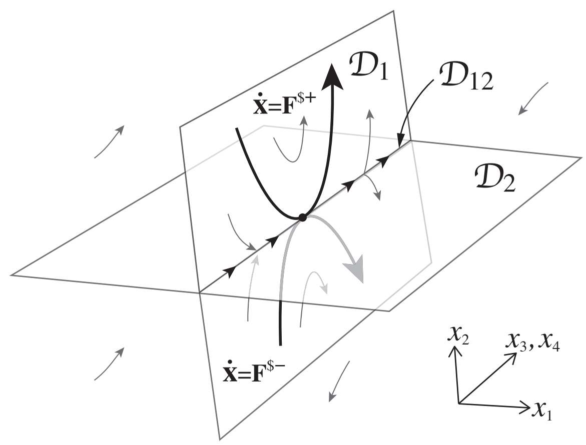

Figure 2: The co-planar two-fold: the sliding flows on half-hyperplanes of a switching threshold $\mathcal{D}_1$, are tangent to the intersection with another switching threshold $\mathcal{D}_2$. Vector fields $\mathbf{F}^{\$\pm}(\mathbf{x}) = \mathbf{F}(\mathbf{x}; \lambda^{\$\pm}, \pm 1)$ will be introduced in section 2.

The *co-planar two-fold*, fig. 2, involves the flows on two half-hyperplanes of the same threshold $\mathcal{D}_1$ being tangent to the intersection with another threshold $\mathcal{D}_2$. The *contra-planar two-fold*, fig. 3, involves the flows on two half-hyperplanes of $\mathcal{D}_1$ and $\mathcal{D}_2$ being tangent to their intersection. These two scenarios can occur generically in systems of four or more dimensions, where the thresholds $\mathcal{D}_i$ are hypersurfaces of three or more dimensional. (The term 'non-co-planar' may be more accurate than 'contra-planar', 'orthogonal' was used in [18] but 'transversal' would be more general, in any event we use 'contra-planar' as it is slightly neater in distinguishing from 'co-planar').

The systems proposed in [18] to study these singularities were toy models with certain degeneracies, and therefore not generic or structurally stable. We reveal and break those degeneracies here to obtain structurally stable models, showing the same reduction to the classic two-fold still applies, and becomes even stronger by identifying a 'singularity within the singularity' that was identified in [19] for the classic two-fold.

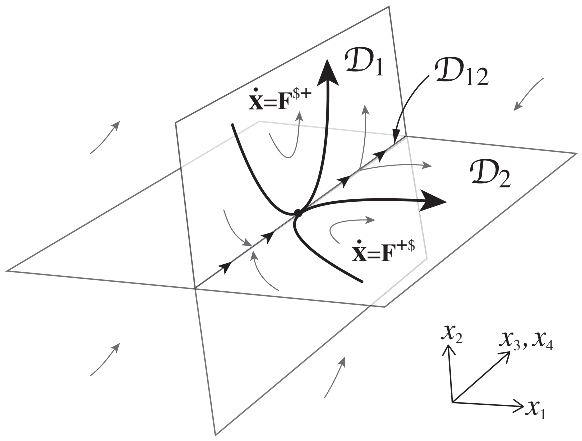

Figure 3: The contra-planar two-fold: the sliding flows on half-hyperplanes of two switching thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$, are tangent to their intersection. Vector fields $\mathbf{F}^{\$+}(\mathbf{x}) = \mathbf{F}(\mathbf{x}; \lambda^{\$+}, +1)$ and $\mathbf{F}^{+\$}(\mathbf{x}) = \mathbf{F}(\mathbf{x}; +1, \lambda^{+\$})$ will be introduced in section 2.

Consider a state $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, whose dynamics depends on discontinuous quantities known as *switching multipliers*

$$\lambda_1 = \text{sign}(x_1) \; , \qquad \lambda_2 = \text{sign}(x_2) \; , \tag{1.1}$$

with $\lambda_i \in [-1, +1]$ at $x_i = 0$, for $i = 1, 2$. Then consider the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \lambda_1, \lambda_2) \; , \tag{1.2a}$$

or written in components,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}; \lambda_1, \lambda_2) \\ g(\mathbf{x}; \lambda_1, \lambda_2) \\ h(\mathbf{x}; \lambda_1, \lambda_2) \\ i(\mathbf{x}; \lambda_1, \lambda_2) \end{pmatrix} \; , \tag{1.2b}$$

where the dot denotes the time derivative. It will also be useful to group the righthand side of (1.2) into two-vectors (omitting their arguments)

$$\underline{f} = \begin{pmatrix} f \\ g \end{pmatrix} \; , \qquad \underline{h} = \begin{pmatrix} h \\ i \end{pmatrix} \; , \tag{1.2c}$$

We will consider two different model systems defined by $\underline{f}$ and $\underline{h}$. For the *co-planar two-fold* we take the model

$$\underline{f}(\mathbf{x};\lambda_1,\lambda_2) = \tfrac{1}{2}\left\{\lambda_2-\lambda_1-\alpha(\lambda_1+\lambda_2)\right\}\underline{k} \\ \qquad +\left\{\tfrac{1}{2}\gamma\left[x_3(1-\lambda_2)+x_4(1-\lambda_1)\right]+\mu(1-\lambda_2^2)\right\}\underline{e}_2 \,, \tag{1.3a}$$

$$\underline{h}(\mathbf{x};\lambda_1,\lambda_2) = \frac{1-\lambda_2}{2\alpha\gamma}\begin{pmatrix} w+\alpha s^- \\ \alpha\gamma w \end{pmatrix} + \frac{1+\lambda_2}{2\alpha\gamma}\begin{pmatrix} \alpha v - s^+ \\ -\alpha\gamma s^+ \end{pmatrix}, \tag{1.3b}$$

and for the *contra-planar two-fold* we take

$$\underline{f}(\mathbf{x};\lambda_1,\lambda_2) = \left\{1-\tfrac{1+\alpha}{2}(\lambda_1+\lambda_2)\right\}\underline{k} + \tfrac{1}{4}\mu(1-\lambda_1)(1-\lambda_2)\underline{r} \\ \qquad + \tfrac{1}{2}\gamma\left\{(1-\lambda_2)x_3\underline{e}_1+(1-\lambda_1)x_4\underline{e}_2\right\} \,, \tag{1.4a}$$

$$\underline{h}(\mathbf{x};\lambda_1,\lambda_2) = \frac{1-\lambda_2}{2}\gamma\begin{pmatrix} s^+ \\ -v \end{pmatrix} - \frac{1-\lambda_1}{2}\gamma\begin{pmatrix} w \\ s^- \end{pmatrix}, \tag{1.4b}$$

where α and γ are arbitrary constants, but the choice

$$\gamma = -(1+\alpha)/\alpha \tag{1.5}$$

simplifies the analysis, and the vector quantities in (1.3)-(1.4) are

$$\underline{k} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.6}$$

with $r_i, v, w, s^\pm \in \mathbb{R}$, $0<\alpha<1$ (so $\gamma<-2$), and $\mu \geq 0$ is a small perturbation parameter.

The models (1.3)-(1.4) might not look intuitive at first, but they are very natural expressions to capture the geometry seen in fig. 2 and fig. 3. The $\underline{k}$ part of these expressions ensure the flows point towards the necessary parts of the switching threshold to cause sliding along them, the x_3 and x_4 terms create the tangencies or 'folds' making up a two-fold at the origin, and the μ terms break a degeneracy known from [19] to afflict any coincidence of tangencies like these, and which afflicted the first models of the singularities proposed in [18]. The $\underline{h}$ components of the vector fields are piecewise-constants chosen for the following purpose.

We will show that these systems represent generic models for the geometry of the co-planar and contra-planar two-folds. We will also show that they can be reduced, on the hypersurface of the switching thresholds, to the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{1+\lambda}{2}\begin{pmatrix} -y \\ -s^+ \\ v \end{pmatrix} + \frac{1-\lambda}{2}\begin{pmatrix} z \\ w \\ s^- \end{pmatrix}, \tag{1.7a}$$

where $\lambda = \text{sign}(x)$ and $v, w, s^{\pm} \in \mathbb{R}$. This system is familiar as the normal form of the classic two-fold (more precisely this is the leading order of the normal form equations that a generic system with a two-fold singularity can be locally transformed into, see [8, 14, 20], but for brevity we will refer to (1.7a) as the classic 'normal form'). On $x = 0$ where $yz > 0$ the dynamics follows an induced *sliding dynamics* (see section 2, or see [8, 14, 21]) given by

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \frac{1}{y+z} \begin{pmatrix} w & -s^{+} \\ s^{-} & v \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{1.7b}$$

and the matrix

$$\Theta = \begin{pmatrix} w & -s^{+} \\ s^{-} & v \end{pmatrix} \tag{1.7c}$$

will play a recurring role in our analysis. Generically $\det(\Theta) = vw + s^{+}s^{-}$ is nonzero. The four constants $w, v, s^{\pm}$, are used to classify the different types of two-fold singularities that can arise, see e.g. [7, 21]. By showing that the same quantities describe the higher two-folds we introduce here, we show that the same classifications will apply, giving at least a starting point for their study, but we leave in-depth classifications to future work.

The classic two-fold in fig. 1, and the co-planar and contra-planar two-folds in fig. 2 and 3, are only the start of an infinite hierarchy of coinciding tangencies in systems with many switches. Generally they may involve tangency from up to $2m$ sides of the intersection of m thresholds $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_m$ in a system with $N \geq m$ switches. They may also involve higher order (e.g. cubic, quadratic, etc.) contact between the flow and the surface. Fortunately, as we show here, the classic case forms an insightful prototype for these, and perhaps even a template for a normal form for general two-folds.

In section 2 we make a few preliminary definitions needed to characterize these models and their dynamics. Then we study the co-planar two-fold in section 3. In this case an explicit coordinate transformation brings the sliding dynamics into the familiar equations of the classic two-fold singularity. This serves as a useful primer for the more involved contra-planar case, which we take up in section 4. We show that the sliding dynamics can again be brought into the form of the equations of the classic two-fold, and that it exhibits the same key geometric features determining its dynamics. In section 3 and section 4 we show that the models (1.3)-(1.4) have the pair of folds depicted in fig. 2 and 3, but more is required to prove that they constitute generic two-folds. We delay those proofs to section 5, where we propose a working definition for higher dimensional two-folds, and give details of how these models were derived. Some concluding remarks are made in section 6.

2 Preliminaries and definitions

Consider a state $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, and define switching thresholds

$$\mathcal{D}_i = \left\{ \mathbf{x} \in \mathbb{R}^4 : \ x_i = 0 \ \right\} \ , \tag{2.1}$$

for $i = 1, 2$, which we divide into half-spaces

$$\begin{aligned} \mathcal{D}_1^{\pm} &= \left\{ \mathbf{x} \in \mathbb{R}^4 : \ x_1 = 0 < \pm x_2 \ \right\} \ , \\ \mathcal{D}_2^{\pm} &= \left\{ \mathbf{x} \in \mathbb{R}^4 : \ x_2 = 0 < \pm x_1 \ \right\} \ , \end{aligned} \tag{2.2}$$

plus their intersection $\mathcal{D}_{12} = \mathcal{D}_1 \cap \mathcal{D}_2$ given by

$$\mathcal{D}_{12} = \left\{ \mathbf{x} \in \mathbb{R}^4 : \ x_1 = x_2 = 0 \ \right\} \ . \tag{2.3}$$

A dynamical system like (1.2) defines dynamics *outside* the switching thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e. only on the regions where $x_1, x_2 \neq 0$. But it can also be used to define *sliding* dynamics on the switching thresholds (for general theory see e.g. [11, 14, 21]). If sliding occurs along $x_i = 0$, for $i = 1$ or $i = 2$, one assumes that the multiplier λ_i lies in the interval $[-1, +1]$ (called a *switching layer*), and then solves the condition $\dot{x}_i = 0$; if a value of λ_i is found satisfying this then it defines a *sliding mode* on $\mathcal{D}_i$.

A little notation will be useful. We use a superscript \$ to denote a value associated with a sliding mode, following [21]. For sliding on $\mathcal{D}_1^{\pm}$ we denote the multiplier λ_1 as $\lambda_1^{\$\pm}$, for sliding on $\mathcal{D}_2^{\pm}$ we denote the multiplier λ_2 as $\lambda_2^{\pm\$}$, and for sliding on $\mathcal{D}_{12}$ we denote the multipliers λ_1 and λ_2 as $\lambda_1^{\$\$}$ and $\lambda_2^{\$\$}$. More precisely we have the following.

Definition 2.1 (Sliding on $\mathcal{D}_i$ or $\mathcal{D}_{12}$)**.** *There exists a* **sliding mode** *at:*

- *any point* $\mathbf{x} \in \mathcal{D}_1^{\pm}$ *if*

$$\exists \ \lambda_1 = \lambda_1^{\$\pm} \in [-1, +1] \quad \textit{such that} \quad f(\mathbf{x}; \lambda_1^{\$\pm}, \pm 1) = 0 \ . \tag{2.4}$$

- *any point* $\mathbf{x} \in \mathcal{D}_2^{\pm}$ *if*

$$\exists \ \lambda_2 = \lambda_2^{\pm\$} \in [-1, +1] \quad \textit{such that} \quad g(\mathbf{x}; \pm 1, \lambda_2^{\pm\$}) = 0 \ , \tag{2.5}$$

- *any point* $\mathbf{x} \in \mathcal{D}_{12}$ *if*

$$\exists \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^{\$\$} \\ \lambda_2^{\$\$} \end{pmatrix} \in [-1, +1]^2 \quad s.t. \quad \underline{f}(\mathbf{x}; \lambda_1^{\$\$}, \lambda_2^{\$\$}) = 0 \; . \tag{2.6}$$

A useful shorthand for the vector fields in the different modes will be to associate each multiplier λ_i with an index l_i and define

$$\mathbf{F}^{l_1 l_2}(\mathbf{x}) = \mathbf{F}(\mathbf{x}; \lambda_1, \lambda_2) \quad : \quad \begin{cases} l_i = \pm & \Leftrightarrow \quad \lambda_i = \pm 1 \; , \\ l_1 = \$ & \Leftrightarrow \quad \lambda_1 = \lambda_1^{\$ l_2} \; , \\ l_2 = \$ & \Leftrightarrow \quad \lambda_2 = \lambda_2^{l_1 \$} \; . \end{cases} \tag{2.7}$$

In total we therefore have four *constituent* modes of the system outside $\mathcal{D}_1$ and $\mathcal{D}_2$, given by $\mathbf{F}^{++}$, $\mathbf{F}^{+-}$, $\mathbf{F}^{-+}$, $\mathbf{F}^{--}$, up to four possible *sliding* modes on the half-spaces $\mathcal{D}_1^{\pm}$ and $\mathcal{D}_2^{\pm}$, given by $\mathbf{F}^{\$+}$, $\mathbf{F}^{\$-}$, $\mathbf{F}^{+\$}$, $\mathbf{F}^{-\$}$, and a sliding mode on the intersection $\mathcal{D}_{12}$ given by $\mathbf{F}^{\$\$}$. These are indicated in fig. 4.

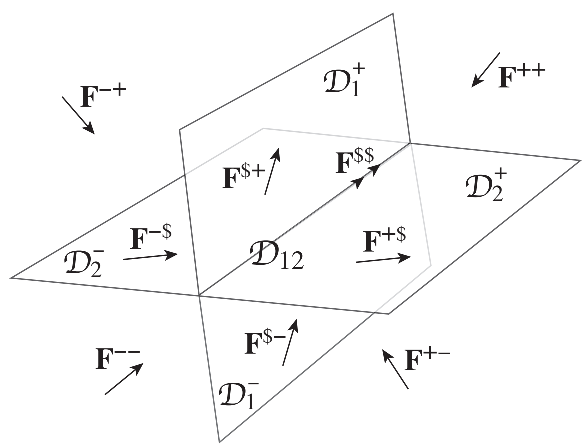

Figure 4: The modes $\mathbf{F}^{l_1 l_2}$ with $l_i = +, -, \$$, inside and outside of the switching thresholds $\mathcal{D}_1 = \mathcal{D}_1^+ \cup \mathcal{D}_1^-$, $\mathcal{D}_2 = \mathcal{D}_2^+ \cup \mathcal{D}_2^-$, and their intersection $\mathcal{D}_{12} = \mathcal{D}_1 \cap \mathcal{D}_2$.

The following definitions can also be made on $\mathcal{D}_1^{\pm}$ and $\mathcal{D}_2^{\pm}$, but we shall need them only on the intersection $\mathcal{D}_{12}$.

Definition 2.2 (Sliding manifold)**.** *The set of points defined by the condition* $f = 0$ *on* $\mathcal{D}_1^{\pm}$, $g = 0$ *on* $\mathcal{D}_2^{\pm}$, *or* $\underline{f} = 0$ *on* $\mathcal{D}_{12}$, *is called a* **sliding manifold***. For instance, on* $\mathcal{D}_{12}$ *the sliding manifold* $\mathcal{M}$ *is*

$$\mathcal{M} = \left\{ (\lambda_1, \lambda_2, x_3, x_4) \in [-1, +1]^2 \times \mathbb{R}^2 \; : \; \underline{f}(\mathbf{x}; \lambda_1, \lambda_2) = \underline{0} \right\} \; . \tag{2.8}$$

Whether $\mathcal{M}$ attracts or repels the surrounding piecewise-smooth flows is characterized conveniently by the derivatives of f or g with respect to the relevant multiplier, that is by $\frac{\partial f}{\partial \lambda_1}$ on $\mathcal{D}_1^{\pm}$, by $\frac{\partial g}{\partial \lambda_2}$ on $\mathcal{D}_2^{\pm}$, and on $\mathcal{D}_{12}$:

Definition 2.3 (Stability of sliding)**.** *The stability of a sliding mode on $\mathcal{D}_{12}$ is defined by the Jacobian*

$$J_{12} = \left. \frac{\partial(f,g)}{\partial(\lambda_1,\lambda_2)} \right|_{\$\$} , \tag{2.9}$$

with the subscript denoting that this is evaluated at $(\lambda_1, \lambda_2) = (\lambda_1^{\$\$}, \lambda_2^{\$\$})$. If J_{12} has two eigenvalues with negative/positive real part then the sliding mode is attracting/repelling, if J_{12} has one positive and one negative eigenvalue then the sliding mode has both attracting and repelling (saddle type) directions.

The in-depth theory can be found in Chapter 7 of [21] (also in [19]), but in short, because the multiplier λ_i changes from -1 to $+1$ as x_i increases through zero, the partial derivatives with respect to λ_i characterize stability along the x_i directions at $x_i = 0$. This generalizes the weaker notion — common in earlier works on nonsmooth dynamics —that sliding is attractive if the surrounding vector fields 'point towards' $\mathcal{D}$, which does not always hold in general. The simplest interpretation is that each threshold $\mathcal{D}_i$ is 'blown up' into a switching layer $\lambda_i \in [-1, +1]$, by a mapping $x_i = \varepsilon\lambda_i \in \varepsilon[-1, +1]$ with $\varepsilon \to 0$, for $i = 1$ or $i = 2$ or both. Substituting these into $\dot{x}_1 = f$ and $\dot{x}_2 = g$ gives systems describing how λ_i evolve across $[-1, +1]$,

$$\varepsilon\dot{\lambda}_1 = f(\mathbf{x}; \lambda_1, \lambda_2) \qquad \text{on} \quad x_1 = 0 , \tag{2.10a}$$

$$\varepsilon\dot{\lambda}_2 = g(\mathbf{x}; \lambda_1, \lambda_2) \qquad \text{on} \quad x_2 = 0 , \tag{2.10b}$$

noting that these are fast because $\varepsilon \to 0$, and that the righthand sides vanish on the sliding modes in definition 2.1. In this interpretation (2.9) is the Jacobian of the two-dimensional system (2.10) evaluated at one of its equilibria.

A sliding mode's linear attractivity (with respect to the dynamics on $\mathcal{M}$) vanishes where J_{12} is singular, on a subset $\mathcal{L} \subset \mathcal{M}$ that is vital in studying two-folds.

Definition 2.4 (Non-hyperbolic set)**.** *The* **non-hyperbolic** *set of $\mathcal{M}$ is the curve*

$$\mathcal{L} = \{(\lambda_1, \lambda_2, x_3, x_4) \in \mathcal{M} \ : \ \det J_{12} = 0\} \tag{2.11}$$

The relevance of $\mathcal{L}$ is that $\mathcal{M}$ ceases to be invariant along it in the full system, so orbits can enter or leave $\mathcal{M}$ along $\mathcal{L}$. To discover what happens to solutions at $\mathcal{L}$ one must look at the full system, not only the sliding dynamics on $\mathcal{M}$.

This curve $\mathcal{L}$ typically lies transverse to the vector field $\mathbf{F}$, but a 'singularity inside the singularity' is created where the vector field $(\dot{x}_3, \dot{x}_4) = (h, i)$ lies tangent to the projection of $\mathcal{L}$ onto the (x_3, x_4) plane. This point was called the *star singularity* in [19].

Definition 2.5 (Star singularity)**.** *A* **star singularity** *is a point where the sliding vector field* $(\dot{x}_3, \dot{x}_4)$ *on* $\mathcal{M}$ *lies tangent to the projection of* $\mathcal{L}$ *on the* (x_3, x_4) *plane.*

This is a paraphrasing of the definition in [19]. The relevance of the star singularity lies in the fact that, since the remaining part $(\dot{x}_1, \dot{x}_2) = (f, g)$ vanishes in sliding, the projection of $(\dot{x}_3, \dot{x}_4) = (h, i)$ onto $\mathcal{M}$ defines the sliding dynamics. Note that the values of λ_i given by definition 2.1 vary as $\mathbf{x}$ evolves in a sliding mode, so from this we can calculate an induced dynamics on λ_i in sliding. We will only need this on the intersection. So on $\mathcal{D}_{12}$ assume there is a sliding mode with $(\lambda_1, \lambda_2) = (\lambda_1^{\$\$}, \lambda_2^{\$\$}) \in [-1, +1]^2$. In these modes the sets $f = g = 0$ remain invariant, so we must have $\dot{f} = \dot{g} = 0$, and differentiating by chain rule in the space of $(\lambda_1, \lambda_2, x_3, x_4)$ gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix} = \frac{\partial(f, g)}{\partial(\lambda_1, \lambda_2)} \begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} + \frac{\partial(f, g)}{\partial(x_3, x_4)} \begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} . \tag{2.12}$$

Re-arranging and using $(\dot{x}_3, \dot{x}_4) = (h, i)$, the multipliers evolve as follows.

Definition 2.6 (Induced dynamics of the multipliers)**.** *In a sliding mode on* $\mathcal{D}_{12}$ *the switching multipliers* (λ_1, λ_2) *evolve as*

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = -{J_{12}}^{-1} \frac{\partial(f, g)}{\partial(x_3, x_4)} \begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} . \tag{2.13}$$

So we see that the induced dynamics in (2.13) is infinite along $\mathcal{L}$ because J_{12} is singular there, giving a denominator $\det J_{12} = 0$. However, a point can arise on $\mathcal{L}$ where the induced dynamics is indefinite, but finite, if the other terms in (2.13) also vanish on $\mathcal{L}$ to give $(\dot{\lambda}_1, \dot{\lambda}_2) = \frac{1}{0}(0, 0)$. This is what happens at the star singularity for the classic two-fold, and we shall see it also happens here for the co-planar two-fold, but not the contra-planar two-fold. When this happens it becomes possible for solutions to evolve across $\mathcal{L}$ between the different branches of $\mathcal{M}$ either side of it, see [19], but study of these solutions is beyond our scope here.

These are all key elements required to study sliding dynamics in higher dimensions, particularly at two-fold singularities.

3 The co-planar two-fold

We begin with the dynamical system (1.2) with (1.3). Outside the switching thresholds, the vector fields of this piecewise smooth system are given by

$$\underline{f}^{++}(\mathbf{x}) = \underline{f}(\mathbf{x};+1,+1) = -\alpha\underline{k}\ , \tag{3.1a}$$

$$\underline{f}^{+-}(\mathbf{x}) = \underline{f}(\mathbf{x};+1,-1) = -\underline{k} + \gamma x_3\underline{e}_2\ , \tag{3.1b}$$

$$\underline{f}^{-+}(\mathbf{x}) = \underline{f}(\mathbf{x};-1,+1) = \underline{k} + \gamma x_4\underline{e}_2\ , \tag{3.1c}$$

$$\underline{f}^{--}(\mathbf{x}) = \underline{f}(\mathbf{x};-1,-1) = \alpha\underline{k} + \gamma(x_3+x_4)\underline{e}_2\ , \tag{3.1d}$$

and the piecewise-constants $\underline{h}^{\pm\pm} = \underline{h}(\mathbf{x};\pm1,\pm1)$ evaluate as

$$\begin{array}{ll} i^{++} = i^{-+} = -s^+\ , & h^{++} = h^{-+} = \frac{\alpha v - s^+}{\alpha\gamma}\ , \\ i^{--} = i^{+-} = w\ , & h^{--} = h^{+-} = \frac{w+\alpha s^-}{\alpha\gamma}\ . \end{array} \tag{3.1e}$$

Note that μ does not appear in these despite being in (1.3a), because $\mu(1-\lambda_2^2) = 0$ for $\lambda_2 = \pm1$ (making this a *hidden* term, see [19, 21]). Let us now derive the sliding dynamics of this system on the switching thresholds for $\mu = 0$, before showing how it reduces to the classic two-fold, and lastly studying the degeneracy of the $\mu = 0$ system and its perturbation.

3.1 Sliding dynamics

Let us first derive the sliding dynamics of the system (1.2) with (1.3) on the switching thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$, to see that it defines a two-fold singularity. We proceed with $\mu = 0$.

It will be useful to transform the (x_3, x_4) coordinates to

$$\eta = x_4\ , \qquad \zeta = \gamma x_3 - \tfrac{1}{\alpha}x_4\ . \tag{3.2}$$

Lemma 3.1. *There exist sliding modes on $\mathcal{D}_1$ with dynamics given by*

$$\begin{pmatrix} \dot{x}_2 \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \frac{1+\lambda_2}{2}\begin{pmatrix} -\eta \\ -s^+ \\ v \end{pmatrix} + \frac{1-\lambda_2}{2}\begin{pmatrix} \zeta \\ w \\ s^- \end{pmatrix}\ . \tag{3.3}$$

There are no sliding modes on $\mathcal{D}_2$ in a neighbourhood of the origin.

Proof. First let us show that there are no sliding modes on $\mathcal{D}_2$. By (2.5), sliding on $x_2 = 0$ would require $\dot{x}_2 = 0$ for some $\lambda_2 = \lambda_2^{\pm\$}$, for which (1.3) gives

$$\begin{aligned} 0 = \dot{x}_2 &= g(\mathbf{x}; \pm 1, \lambda_2^{\pm\$}) \\ &= \tfrac{1-\alpha}{2}\lambda_2^{\pm\$} \mp \tfrac{1+\alpha}{2} + \mathcal{O}(x_i) \\ \Rightarrow \quad \lambda_2^{\pm\$} &= \pm\tfrac{1+\alpha}{1-\alpha} + \mathcal{O}(x_i) \ , \end{aligned} \tag{3.4}$$

whose modulus is strictly greater than unity near the origin, and therefore by (2.5) does not define a valid sliding mode there.

On the threshold $\mathcal{D}_1$ there do exist sliding modes. By (2.4) these must satisfy $\dot{x}_1 = 0$ on $x_1 = 0$, for some $\lambda_1 = \lambda_1^{\$\pm} \in [-1, +1]$, with signs corresponding to lying on $\mathcal{D}_1^{\pm}$. They satisfy

$$\begin{aligned} 0 = \dot{x}_1 = f(\mathbf{x}; \lambda_1^{\$\pm}, \pm 1) = -\tfrac{1+\alpha}{2}\lambda_1^{\$\pm} \pm \tfrac{1-\alpha}{2} \\ \Rightarrow \quad \lambda_1^{\$\pm} = \pm\frac{1-\alpha}{1+\alpha} \ , \end{aligned} \tag{3.5}$$

which has modulus less than unity for all $\alpha > 0$. Substituting back into (1.3), we have sliding dynamics on $\mathcal{D}_1^{\pm}$ given by $(\dot{x}_1, \dot{x}_2) = \underline{f}(\mathbf{x}; \lambda_1^{\$\pm}, \pm 1)$ where

$$\underline{f}^{\$+}(\mathbf{x}) = \begin{pmatrix} 0 \\ -\eta \end{pmatrix} , \qquad \underline{f}^{\$-}(\mathbf{x}) = \begin{pmatrix} 0 \\ \zeta \end{pmatrix} , \tag{3.6}$$

on $\mathcal{D}_1^{+}$ and $\mathcal{D}_1^{-}$ respectively.

Turning to the (x_3, x_4) coordinates, from (1.3) on $\mathcal{D}_1^{\pm}$, substituting into (1.3) the multipliers $\lambda_1 = \lambda_1^{\$\pm}$ from (3.5) with $\lambda_2 = \pm 1$ gives

$$\begin{pmatrix} \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \frac{1+\lambda_2}{2}\begin{pmatrix} -s^{+} \\ v \end{pmatrix} + \frac{1-\lambda_2}{2}\begin{pmatrix} w \\ s^{-} \end{pmatrix} . \tag{3.7}$$

Combining (3.6) and (3.7) gives the result. □

Clearly the two vector fields in (3.6) vanish where $\eta = 0$ and $\zeta = 0$ respectively, and these are the folds of the sliding vector fields $\underline{f}^{\$+}(\mathbf{x})$ and $\underline{f}^{\$-}(\mathbf{x})$, where they are tangent to the intersection $\mathcal{D}_{12}$ as depicted in fig. 2, i.e. where $f^{\$+}(\mathbf{x}) = 0$ and $f^{\$-}(\mathbf{x}) = 0$. Note these are not fixed points because the components $(\dot{\eta}, \dot{\zeta}) = (i, \gamma h - \frac{1}{\alpha} i)$ do not typically vanish.

Then consider the dynamics on the intersection $\mathcal{D}_{12}$ itself.

Lemma 3.2. *There exist sliding modes on $\mathcal{D}_{12}$ with dynamics given by*

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \frac{\Theta}{\eta+\zeta} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} , \tag{3.8}$$

where Θ is the 2×2 matrix defined in (1.7c)*.*

Proof. The sliding dynamics on the intersection $\mathcal{D}_{12}$, is given according to (2.6) by solving $\dot{x}_1 = \dot{x}_2 = 0$, hence $f(\mathbf{x}; \lambda_1^{\$\$}, \lambda_2^{\$\$}) = g(\mathbf{x}; \lambda_1^{\$\$}, \lambda_2^{\$\$}) = 0$, for some $(\lambda_1, \lambda_2) = (\lambda_1^{\$\$}, \lambda_2^{\$\$}) \in [-1, +1]^2$. From (1.3), the solution of this is

$$\lambda_1^{\$\$} = \frac{1-\alpha}{1+\alpha}\lambda_2^{\$\$} , \qquad \lambda_2^{\$\$} = \frac{\zeta-\eta}{\zeta+\eta} . \tag{3.9}$$

These sliding modes only exist where $|\lambda_1^{\$\$}|, |\lambda_2^{\$\$}| < 1$, and noting that we defined $0 < \alpha < 1$, this implies $0 < \frac{1-\alpha}{1+\alpha} < 1$, which implies $|\lambda_1^{\$\$}| < |\lambda_2^{\$\$}| < 1$ for $\eta\zeta > 0$, hence these sliding modes exist where $\eta\zeta > 0$.

Substituting (3.9) into (1.3) gives (3.8). □

Notice from (3.9) that $\lambda_2^{\$\$} = +1$ on $\eta = 0$, and $\lambda_2^{\$\$} = -1$ on $\zeta = 0$, coinciding with the pair of folds on $\mathcal{D}_1$ from lemma 3.1.

We now have the following.

Theorem 3.3. *The co-planar two-fold reduces on $\mathcal{D}_1$ to the equations of the classic two-fold singularity given by* (1.7)*.*

Proof. This follows directly because the systems (3.3) on $\mathcal{D}_1^{\pm}$ and (3.8) on $\mathcal{D}_{12}$ are clearly equivalent, in the coordinates (x_2, η, ζ), to the equations of the classic two-fold (1.7) in coordinates (x, y, z).

□

The regions of sliding are represented in fig. 5, with each threshold $\mathcal{D}_i$ 'blown up' into a switching layer $\lambda_i \in [-1, +1]$ (we can think of this blow up of $x_i = 0$ as the mapping $x_i = \varepsilon\lambda_i \in \varepsilon[-1, +1]$ with $\varepsilon \to 0$ for $i = 1, 2$, see [21]). The sliding manifold $\mathcal{M}$ from (2.8) (but expressed in η, ζ, coordinates) is then revealed inside the layers, namely

$$\mathcal{M} = \left\{ (\lambda_1, \lambda_2, \eta, \zeta) \in [-1, +1]^2 \times \mathbb{R}^2 \ : \ \lambda_1 = \lambda_1^{\$\$}, \lambda_2 = \lambda_2^{\$\$} \right\} \tag{3.10}$$

on $\mathcal{D}_{12}$, and exists only where $\eta\zeta > 0$. (On $\mathcal{D}_1^{\pm}$ we have, similarly, sliding manifolds $\{(\lambda_1, x_2, \eta, \zeta) \in [-1, +1] \times \mathbb{R}^3 : \lambda_1 = \lambda_1^{\$\pm}\}$). When we restrict to just the three dimensions of $\mathcal{D}_1$, on the right of fig. 5, we can see regions in

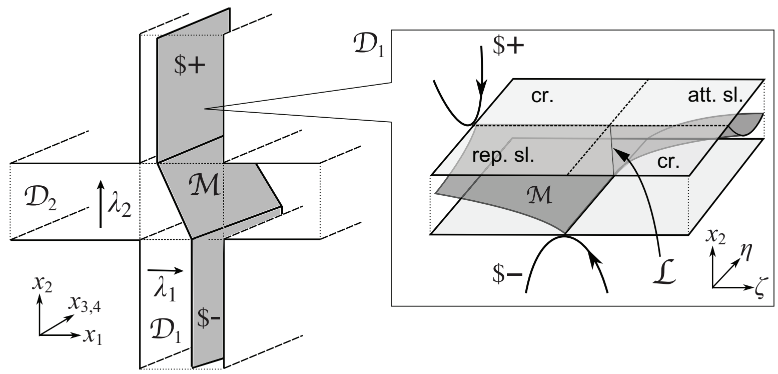

Figure 5: A representation of the sliding modes on $\mathcal{D}_1$ with multipliers $\lambda_1^{\$\pm}$ on $\mathcal{D}_1^{\pm}$ and $\lambda_1^{\$\$}$ on $\mathcal{D}_{12}$. Each switching threshold $\mathcal{D}_i$ has been 'blown up' into a layer on which the multiplier λ_i transitions over $[-1,+1]$, revealing the sliding manifold $\mathcal{M}$. To the right we represent the dynamics in (x_2,η,ζ) space on $\mathcal{D}_1$, equivalent to the classic two-fold, with 'cr' denoting crossing regions and '$att/rep.sl.$' denoting attracting/repelling sliding regions.

which the dynamics can cross between $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$, and where it slides on the switching intersection $\mathcal{D}_{12}$ between them.

As described in section 2, the attractivity of these sliding modes is given by the Jacobian (2.9) applied to (1.3), which is (calculating in (x_3,x_4) coordinates)

$$J_{12} = \tfrac{1}{2}\begin{pmatrix} -1-\alpha & 1-\alpha \\ -1-\alpha-\gamma\eta & 1-\alpha-\zeta-\frac{1}{\alpha}\eta \end{pmatrix}, \tag{3.11}$$

with determinant

$$\det(J_{12}) = \tfrac{1}{4}(1+\alpha)\,(\eta+\zeta) \tag{3.12}$$

and eigenvalues

$$e_\pm = -b \pm \sqrt{b^2 - \det J_{12}}\,, \qquad b = \tfrac{2\alpha^2+\alpha\gamma x_3}{4\alpha}\,. \tag{3.13}$$

If we restrict to $x_3 < -2\alpha/\gamma$ (noting $\gamma < 0$, so this region includes the origin) then $b > 0$. For $\eta,\zeta > 0$ we then have $\det J_{12} > 0$ and $e_\pm < 0$, so the sliding modes are attracting, while for $x_4, \gamma x_3 - \frac{1}{\alpha}x_4 < 0$ we have $\det J_{12} < 0$ and $e_- < 0 < e_+$, so the sliding modes are of saddle type, near the origin.

A number of things happen at the two-fold singularity, i.e. at $\eta = \zeta = 0$. Firstly the values (3.9) of the multipliers λ_i are undefined. The vector term in (3.8) defines an equilibrium at $\eta = \zeta = 0$, but the prefactor $\frac{1}{\eta+\zeta}$ also vanishes there, making the dynamics there ill-defined. Lastly the Jacobian (3.12) vanishes at the singularity. These are defining problems of typical two-fold singularities, and they occur here in precisely that same manner as for the classic two-fold (see [19, 21]).

All of this can also be understood in terms of the sliding manifold $\mathcal{M}$ given by (3.10). $\mathcal{M}$ is a two-dimensional hyperboloid (saddle-shaped) sheet that twists over in the space of $(\lambda_1, \lambda_2, \eta, \zeta)$, leaving the layer $(\lambda_1, \lambda_2) \in [-1, +1]^2$ along the folds $\eta = 0$ and $\zeta = 0$. This surface $\mathcal{M}$ has a unique point at each coordinate (η, ζ) for which sliding occurs, but an infinity of points at $\eta = \zeta = 0$, and these form a straight line constituting the non-hyperbolic set $\mathcal{L}$ from definition 2.4, shown in fig. 5. The fact that this line lies in the (λ_1, λ_2) plane (i.e. at $x_3 = x_4 = 0$) constitutes a degeneracy, associated with the multipliers (λ_1, λ_2) being ill-defined there. To break the degeneracy requires a perturbation of the model that deforms $\mathcal{L}$ out of the plane.

3.2 Breaking the degeneracy

The fact that the sliding mode and its dynamics are undefined at the singularity is a well established issue of the classic two-fold that was first resolved in [19]. For the co-planar two-fold we can resolve this in precisely the same way.

To resolve the degeneracy requires introducing a term that perturbs the dynamics on $\mathcal{D}_2$ only, and not outside it (known as a *hidden* term, see [19, 21]). This is done by adding a term proportional to $1 - \lambda_2^2$ orthogonal to the direction of the fold, that is in the x_2 component, provided by setting $\mu \neq 0$ in (1.3). This does not affect the dynamics on $\mathcal{D}_1^{\pm}$ since $\lambda_2 = \pm 1$ there so $\mu(1 - \lambda_2^2) = 0$, and for small μ this does not affect $\mathcal{D}_2^{\pm}$ as there are no sliding modes there.

Thus the perturbation only affects dynamics on $\mathcal{D}_{12}$. Re-deriving the sliding modes there with $\mu > 0$, by solving again $\dot{x}_1 = \dot{x}_2 = 0$ for some $(\lambda_1, \lambda_2) = (\lambda_1^{\$\$}, \lambda_2^{\$\$})$ (as we did to obtain (3.9)), we find

$$\lambda_1^{\$\$} = \frac{1 - \alpha}{1 + \alpha} \lambda_2^{\$\$} \,, \qquad \lambda_2^{\$\$} = -\frac{\zeta + \eta}{4\mu} \pm R \,, \tag{3.14}$$

where

$$R = \sqrt{1 + \frac{\zeta - \eta}{2\mu} + \left(\frac{\zeta + \eta}{4\mu} \right)^2} \,. \tag{3.15}$$

These sliding modes are now well-defined at the origin, and the '$\pm$' roots give two branches of a continuous family of solutions. Using the convention $\sqrt{x} = |x|^{1/2} e^{(i\,\mathrm{Arg}\,x)/2}$, for $\mu \to 0$ on $\eta, \zeta > 0$ the '+' solution in (3.15) reduces to (3.9) while the '$-$' solution diverges, and vice versa on $\eta, \zeta < 0$, so the '+' and '$-$' roots give the perturbation of the sliding modes on $\eta, \zeta > 0$ and $\eta, \zeta < 0$ respectively. Both '$\pm$' roots give the continuation of these branches of solutions through $0 < \eta < 2\mu$ and $-2\mu < \zeta < 0$, where the sliding manifold $\mathcal{M}$ now bulges at it twists over near the origin (as can be verified by series expansion of (3.15) or by graphing the surface, noting that $\lambda_2^{\$\$} = +1$ on $\eta = 0$ and $\lambda_2^{\$\$} = -1$ on $\zeta = 0$, and that $R = 0$ touches $\eta = 0$ at $\zeta = -2\mu$ and touches $\zeta = 0$ at $\eta = 2\mu$).

The η and ζ dynamics, obtained by substituting (3.16) into (1.3), is now

$$\begin{pmatrix} \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} i^{\$\$} \\ \gamma h^{\$\$} - \frac{1}{\alpha} i^{\$\$} \end{pmatrix} = \tfrac{1}{2}\Theta \begin{pmatrix} 1 \mp R + \frac{\zeta+\eta}{4\mu} \\ 1 \pm R - \frac{\zeta+\eta}{4\mu} \end{pmatrix} \tag{3.16}$$

constituting the perturbation of (3.8), where Θ is the matrix from (1.7c).

The sliding manifold $\mathcal{M}$ as written in (3.10) now has a well-defined value (or at most two values, one on each sheet) for each (η, ζ), including at the origin. Evaluating the Jacobian (2.9) applied to (1.3) now gives

$$\det J_{12} = (1 + \alpha)\mu R \,, \tag{3.17}$$

so the non-hyperbolic set (2.11) is now

$$\mathcal{L} = \{(\lambda_1, \lambda_2, x_3, x_4) \in \mathcal{M} \;:\; R = 0\,\} \;. \tag{3.18}$$

This is a curve with a well-defined value for each (η, ζ), on which the multipliers $(\lambda_1^{\$\$}, \lambda_2^{\$\$})$ are well-defined, though the induced dynamics (from definition 2.6) on the multipliers is not well-defined on $\mathcal{L}$ because $\det J_{12}$ in (3.17) vanishes. However, there is now a new singularity that occurs along $\mathcal{L}$, the star singularity from definition 2.5.

Theorem 3.4. *The co-planar two-fold system has a star singularity on* $(x_1, x_2) = (0, 0)$ *at* $\eta = \mu(1 - \tau_*)^2$, $\zeta = -\mu(1 + \tau_*)^2$, *if* $-1 \le \tau_* \le +1$, *where* τ_* *is a constant dependent on* v, w, s^+, s^-, *and where* $R = 0$.

Proof. By definition 2.5, a star singularity is a point where the sliding dynamics on $\mathcal{M}$ lies tangent to the projection of $\mathcal{L}$ on the (x_3, x_4) plane. To see this parameterize the set $R = 0$, say by letting $\frac{\zeta+\eta}{4\mu} = \tau$ for some parameter $\tau \in [-1, +1]$, then solving $R = 0$ gives $\eta - \zeta = 2\mu(1 + \tau^2)$, implying

$$(\eta(\tau), \zeta(\tau)) = \big(\; \mu(1 + \tau)^2 \;,\; -\mu(1 - \tau)^2 \;\big) \;. \tag{3.19}$$

The tangent vector $(\eta'(\tau), \zeta'(\tau)) = 2\mu(1+\tau, 1-\tau)$ then takes the same direction as (3.16) where

$$\tau = \tau_* := \frac{s^- - s^+ \pm \sqrt{(w-v)^2 + 4s^+s^-}}{v - w + s^+ + s^-} \, . \tag{3.20}$$

Clearly whether zero, one, or both of these values of τ_* lie in $-1 \leq \tau_* \leq +1$ depends on the values of s^+, s^-, v, w, giving zero, one, or two star singularities; we will not study the individual cases here.

$\square$

The star singularity inherits the ambiguity of the two-fold in the perturbed system. To explain this consider the induced dynamics of (λ_1, λ_2) from definition 2.6. First, we know that the Jacobian J_{12} is singular at the star singularity by (3.17). The remaining term needed in (2.13) to calculate the induced dynamics is, setting $R = 0$ and calculating derivatives with respect to x_3 and x_4,

$$\begin{aligned} \tfrac{\partial(f,g)}{\partial(x_3,x_4)} \cdot \begin{pmatrix} h \\ i \end{pmatrix} &= \tfrac{\gamma}{2} \begin{pmatrix} 0 & 0 \\ 1 - \lambda_2^{\$\$} & 1 - \lambda_1^{\$\$} \end{pmatrix} \begin{pmatrix} h^{\$\$} \\ i^{\$\$} \end{pmatrix} \\ &= -\tfrac{1}{4} \begin{pmatrix} 0 & 0 \\ -1 + \frac{\zeta+\eta}{4\mu} & 1 + \frac{\zeta+\eta}{4\mu} \end{pmatrix} \Theta \begin{pmatrix} 1 + \frac{\zeta+\eta}{4\mu} \\ 1 - \frac{\zeta+\eta}{4\mu} \end{pmatrix} . \end{aligned} \tag{3.21}$$

A straightforward calculation shows that this vanishes precisely at the star singularity, hence the induced dynamics in sliding becomes $(\dot{\lambda}_1, \dot{\lambda}_2) = \frac{1}{0}(0,0)$ there. The same occurs in the classic two-fold. In the classic case the fact that $(\dot{\lambda}_1, \dot{\lambda}_2)$ is of indefinite value, rather than being infinite as it is along the rest of $\mathcal{L}$, permits solutions to travel between the two sheets of $\mathcal{M}$ via the star singularity [19, 21]. A comparison of the analysis above with that of the classic two-fold suggests the same occurs for co-planar two-folds, but further study is beyond our scope here.

In the following section we tackle the contra-planar case using a similar methodology to that above, but requiring a different choice of coordinates to obtain the classic normal form, and a different kind of perturbation to break the degeneracy and find the star singularity.

4 The contra-planar two-fold

Take the dynamical system (1.2) with (1.4). Outside the switching thresholds the vector fields of this piecewise smooth system are given by

$$\underline{f}^{++}(\mathbf{x}) = \underline{f}(\mathbf{x};+1,+1) = -\alpha\underline{k}\ , \tag{4.1a}$$
$$\underline{f}^{+-}(\mathbf{x}) = \underline{f}(\mathbf{x};+1,-1) = \underline{k} + \gamma x_3\underline{e}_1\ , \tag{4.1b}$$
$$\underline{f}^{-+}(\mathbf{x}) = \underline{f}(\mathbf{x};-1,+1) = \underline{k} + \gamma x_4\underline{e}_2\ , \tag{4.1c}$$
$$\underline{f}^{--}(\mathbf{x}) = \underline{f}(\mathbf{x};-1,-1) = (2+\alpha)\underline{k} + \mu\underline{r}\ , \tag{4.1d}$$

and the piecewise-constants $\underline{h}^{\pm\pm} = \underline{h}(\mathbf{x};\pm1,\pm1)$ evaluate as

$$\begin{aligned}(h^{++},i^{++},h^{--},i^{--}) &= -\gamma(0,0,w-s^+,v+s^-)\ ,\\ (h^{-+},i^{-+},h^{+-},i^{+-}) &= -\gamma(w,s^-,-s^+,v)\ .\end{aligned} \tag{4.1e}$$

As in the previous section let us now derive the sliding dynamics of this system on the switching thresholds for $\mu = 0$, then show how it reduces to the classic two-fold, and lastly study the degeneracy at $\mu = 0$ and its perturbation.

The analysis henceforth follows as closely as possible that of section 3.

4.1 Sliding dynamics

To see that the system (1.4) defines a two-fold singularity let us first derive its sliding dynamics on $\mathcal{D}_1$ and $\mathcal{D}_2$. We let $\mu = 0$.

Lemma 4.1. *There exist sliding modes on $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ with dynamics given by*

$$\begin{pmatrix}\dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix}g^{\$+}(\mathbf{x})\\ h^{\$+}\\ i^{\$+}\end{pmatrix} = \tfrac{1}{1+\alpha}\begin{pmatrix}-x_4\\ w\\ s^-\end{pmatrix} \quad \text{on}\ \ \mathcal{D}_1^+\ , \tag{4.2a}$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix}f^{+\$}(\mathbf{x})\\ h^{+\$}\\ i^{+\$}\end{pmatrix} = \tfrac{1}{1+\alpha}\begin{pmatrix}-x_3\\ -s^+\\ v\end{pmatrix} \quad \text{on}\ \ \mathcal{D}_2^+\ . \tag{4.2b}$$

There are no sliding modes on $\mathcal{D}_1^-$ or $\mathcal{D}_2^-$ in a neighbourhood of the origin.

Proof. Let us first show that no sliding takes place on $\mathcal{D}_1^-$ or $\mathcal{D}_2^-$. By (2.4), a sliding mode on $\mathcal{D}_1^-$ would satisfy $f(\mathbf{x};\lambda_1^\$,-1) = 0$, which using (1.4) has solution $\lambda_1^\$ = 1 + 2\frac{1+\gamma x_3}{1+\alpha}$, and a sliding mode on $\mathcal{D}_2^-$ would satisfy $g(\mathbf{x};-1,\lambda_2^\$) = 0$, which using (1.4) has solution $\lambda_2^\$ = 1 + 2\frac{1+\gamma x_4}{1+\alpha}$. Both of these clearly lie outside $[-1,+1]$ for small x_3, x_4, and therefore fail to define valid sliding modes near the origin. So we turn instead to the remaining regions of the thresholds $\mathcal{D}_1$ and $\mathcal{D}_2$.

If sliding takes place along $\mathcal{D}_1^+$, on which $\lambda_2 = +1$, then according to (2.4) we must have $\dot{x}_1 = 0$ and hence $f = 0$. Solving (1.4) to find the value of $\lambda_1 = \lambda_1^{\$+}$ that gives $f = 0$ implies

$$\lambda_1^{\$+} = \frac{1-\alpha}{1+\alpha} \; , \tag{4.3}$$

and substituting back into (1.4) gives sliding dynamics

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}; \lambda_1^{\$+}, +1) \\ g(\mathbf{x}; \lambda_1^{\$+}, +1) \end{pmatrix} = \begin{pmatrix} 0 \\ -x_4 \end{pmatrix} . \tag{4.4}$$

Similarly, if sliding takes place along $\mathcal{D}_2^+$, on which $\lambda_1 = +1$, then by (2.5) we must have $\dot{x}_2 = 0$ and hence $g = 0$. Solving (1.4) to find the value of $\lambda_2 = \lambda_2^{+\$}$ that gives $g = 0$ implies

$$\lambda_2^{+\$} = \frac{1-\alpha}{1+\alpha} \; , \tag{4.5}$$

and substituting back into (1.4) gives sliding dynamics

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}; +1, \lambda_2^{+\$}) \\ g(\mathbf{x}; +1, \lambda_2^{+\$}) \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \end{pmatrix} . \tag{4.6}$$

For the (x_3, x_4) components of the vector field we have $(\dot{x}_3, \dot{x}_4) = \underline{h}^{l_1 l_2}$ which, on $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ respectively, are given by

$$\begin{aligned} \begin{pmatrix} h^{\$+} \\ i^{\$+} \end{pmatrix} &= \tfrac{1}{2}(1+\lambda_1^{\$+}) \begin{pmatrix} h^{++} \\ i^{++} \end{pmatrix} + \tfrac{1}{2}(1-\lambda_1^{\$+}) \begin{pmatrix} h^{-+} \\ i^{-+} \end{pmatrix} , \\ \begin{pmatrix} h^{+\$} \\ i^{+\$} \end{pmatrix} &= \tfrac{1}{2}(1+\lambda_2^{+\$}) \begin{pmatrix} h^{++} \\ i^{++} \end{pmatrix} + \tfrac{1}{2}(1-\lambda_2^{+\$}) \begin{pmatrix} h^{+-} \\ i^{+-} \end{pmatrix} . \end{aligned} \tag{4.7}$$

Substituting in the multipliers from (4.3) and (4.5), and the constants from (1.4b), gives

$$\begin{aligned} \begin{pmatrix} h^{\$+} \\ i^{\$+} \end{pmatrix} &= \tfrac{1}{1+\alpha} \left\{ \begin{pmatrix} h^{++} \\ i^{++} \end{pmatrix} + \alpha \begin{pmatrix} h^{-+} \\ i^{-+} \end{pmatrix} \right\} = \begin{pmatrix} w \\ s^- \end{pmatrix} , \\ \begin{pmatrix} h^{+\$} \\ i^{+\$} \end{pmatrix} &= \tfrac{1}{1+\alpha} \left\{ \begin{pmatrix} h^{++} \\ i^{++} \end{pmatrix} + \alpha \begin{pmatrix} h^{+-} \\ i^{+-} \end{pmatrix} \right\} = \begin{pmatrix} -s^+ \\ v \end{pmatrix} . \end{aligned} \tag{4.8}$$

Putting together (4.4), (4.6), and (4.8), gives the result (4.2). □

By (4.3) and (4.5), the sliding flows have folds, where the flow is tangent to the intersection $\mathcal{D}_{12}$, where $g(\mathbf{x};\lambda_1^{\$+},+1) = -x_4 = 0$ on $\mathcal{D}_1^+$ and $f(\mathbf{x};+1,\lambda_2^{+\$}) = -x_3 = 0$ on $\mathcal{D}_2^+$

Now let us turn to the intersection $\mathcal{D}_{12}$.

Lemma 4.2. *There exist sliding modes on $\mathcal{D}_{12}$ with dynamics given, to leading order, by*

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} h^{\$\$} \\ i^{\$\$} \end{pmatrix} = \frac{\Theta}{x_3 + x_4} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \tag{4.9}$$

where Θ is the 2×2 matrix defined in (1.7c)*.*

Proof. To find the sliding dynamics on $\mathcal{D}_{12}$, we must solve

$$(\dot{x}_1, \dot{x}_2) = \underline{f}(\mathbf{x};\lambda_1^{\$\$},\lambda_2^{\$\$}) = \underline{0}$$

for the multipliers $\lambda_1^{\$\$},\lambda_2^{\$\$}$. It will be useful to define

$$x^{\pm} = x_3 \pm x_4 , \tag{4.10}$$

using which we find

$$\lambda_1^{\$\$} = 1 + \frac{2\gamma^{-1}x_3}{x^+ - \frac{1}{\alpha}x_3x_4} , \tag{4.11a}$$

$$\lambda_2^{\$\$} = 1 + \frac{2\gamma^{-1}x_4}{x^+ - \frac{1}{\alpha}x_3x_4} . \tag{4.11b}$$

By definition, sliding modes only exist where $|\lambda_1^{\$\$}|,|\lambda_2^{\$\$}| \le 1$. For $0<\alpha<1$ we have $-1/2 < \gamma^{-1} < 0$, and note that if we assume $x_3, x_4 < \alpha$, then

$$\begin{aligned} x_3x_4 > 0 \quad &\Rightarrow \quad 0 < \frac{x_j}{x^+ - \frac{1}{\alpha}x_3x_4} < 1 \\ &\Rightarrow \quad 0 < 1 + 2\gamma^{-1} < \lambda_i^{\$\$} < 1 , \end{aligned} \tag{4.12}$$

for $i=1,2$, $j=3,4$, such that $\lambda_1^{\$\$}=1$ along $x_3=0$ and $\lambda_2^{\$\$}=1$ along $x_4=0$. For $x_3x_4<0$ at least one of $\lambda_1^{\$\$}$ and $\lambda_2^{\$\$}$ lies outside $[-1,+1]$. Thus sliding takes place on $x_3x_4>0$ (restricting our attention to $x_3,x_4<\alpha$), with boundaries on $x_3=0$ and $x_4=0$, coinciding with the folds found in section 4.1.

The (x_3,x_4) components of the dynamics are found by substituting the multipliers (4.11) into (1.4), giving $\underline{f}^{\$\$}=0$ and

$$\begin{aligned}\binom{\dot{x}_3}{\dot{x}_4}=\binom{h^{\$\$}}{i^{\$\$}}&=\tfrac{\Theta}{x^+-x_3x_4/\alpha}\binom{x_3}{x_4}\\&=\tfrac{\Theta}{x_3+x_4}\binom{x_3}{x_4}\left\{1+\mathcal{O}\left(\tfrac{x_3x_4}{x^+}\right)\right\}\ ,\end{aligned}\tag{4.13}$$

where Θ is the matrix defined in (1.7c), and where x_3x_4/x^+ is small in the regions of sliding since $x_3x_4>0$. To leading order this gives (4.9). □

To compare this to the classic two-fold, let us define a coordinatization of $\mathcal{D}_1^+\cup\mathcal{D}_2^+$ given by

$$u=\begin{cases}-x_2 & \text{for } \ x_2>0=x_1\ ,\\ x_1 & \text{for } \ x_1>0=x_2\ .\end{cases}\tag{4.14}$$

With this we can obtain the classic two-fold equations within the three-dimensional space of $\mathcal{D}_1^+\cup\mathcal{D}_2^+\cup\mathcal{D}_{12}$.

Theorem 4.3. *The contra-planar two-fold reduces on $\mathcal{D}_1^+\cup\mathcal{D}_2^+$ to the equations of the classic two-fold singularity given by* (1.7)*.*

Proof. From (4.2), in the (u,x_3,x_4) coordinates we have

$$\begin{pmatrix}\dot{u}\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\frac{1+\lambda}{2}\begin{pmatrix}-x_3\\ -s^+\\ v\end{pmatrix}+\frac{1-\lambda}{2}\begin{pmatrix}x_4\\ w\\ s^-\end{pmatrix}\ ,\tag{4.15}$$

where $\lambda=\operatorname{sign}(u)$. The sliding dynamics on $u=0$, i.e. on $\mathcal{D}_{12}$, was already given in (4.13). Thus we see that the three dimensional piecewise smooth system (4.15), and the sliding dynamics (4.13) on the switching threshold between them, correspond in the coordinates (u,x_3,x_4) to the equations of the classic two-fold in the coordinates (x,y,z) in (1.7).

□

Calculating the Jacobian (2.9) to determine the attractivity of these sliding modes, we have

$$J_{12}=\tfrac{1}{2}\gamma\begin{pmatrix}\alpha & \alpha-x_3\\ \alpha-x_4 & \alpha\end{pmatrix}\ .\tag{4.16}$$

This has determinant

$$\det\left(J_{12}\right)=\tfrac{1}{4}\gamma^2(\alpha x^+-x_3x_4)\tag{4.17}$$

and eigenvalues

$$e_{\pm} = \tfrac{1}{2}\alpha\gamma\left(1 \pm \sqrt{1 - \tfrac{x^+}{\alpha} + \tfrac{x_3 x_4}{\alpha^2}}\right) \ , \tag{4.18}$$

implying (recalling $\gamma < 0$) that the sliding manifold is attracting (two negative real eigenvalues) for $x_3, x_4 > 0$ and saddle-like (as one of these eigenvalues becomes positive) for $x_3, x_4 < 0$, in a neighbourhood of the origin. At the origin $x_3 = x_4 = 0$ the determinant vanishes so J_{12} is singular, and the sliding modes (4.11) are ill-defined, so we see that this constitutes the non-hyperbolic set $\mathcal{L}$ from definition 2.4.

The sliding manifold $\mathcal{M}$ is as usual

$$\mathcal{M} = \left\{(\lambda_1, \lambda_2, x_3, x_4) \in [-1, +1]^2 \times \mathbb{R}^2 \ : \ \lambda_1 = \lambda_1^{\$\$}, \lambda_2 = \lambda_2^{\$\$} \right\} \tag{4.19}$$

on $\mathcal{D}_{12}$, and exists only where $x_3 x_4 > 0$. As for the co-planar case, $\mathcal{M}$ is a two-dimensional hyperboloid sheet that twists over in the space of $(\lambda_1, \lambda_2, x_3, x_4)$, leaving the layer $(\lambda_1, \lambda_2) \in [-1, +1]^2$ along the folds $x_3 = 0$ and $x_4 = 0$. This surface $\mathcal{M}$ has a unique point at each coordinate (x_3, x_4) for which sliding occurs, but an infinity of points at $x_3 = x_4 = 0$, and this is precisely the set $\mathcal{L}$ on which $\det(J_{12}) = 0$. Again, the fact that this line lies in the (λ_1, λ_2) plane (i.e. at $x_3 = x_4 = 0$) constitutes a degeneracy, associated with the multipliers (λ_1, λ_2) being ill-defined there, and to break the degeneracy requires a perturbation that deforms $\mathcal{L}$ out of the plane.

4.2 Breaking the degeneracy

Above we found that the sliding mode and its dynamics were ill-defined at the singularity, as we found for the co-planar case in section 3.1. For the co-planar case we could resolve this by introducing a term proportional to $1 - \lambda_2^2$, which vanishes on $\mathcal{D}_1^{\pm}$ (as it vanishes for any $x_2 \neq 0$) and so does not affect the crucial sliding dynamics there, and only has effect on $\mathcal{D}_{12}$. For the contra-planar case we require a perturbation instead that does not affect the crucial sliding dynamics on $\mathcal{D}_1^+$ or $\mathcal{D}_2^+$, and this is done with a term proportional to $(1 - \lambda_1)(1 - \lambda_2)$, which vanishes on $\mathcal{D}_1^+$ or $\mathcal{D}_2^+$, and so again only has an effect on $\mathcal{D}_{12}$ (and some effect on $\mathcal{D}_1^-$ and $\mathcal{D}_2^-$, but there is no sliding on those surfaces and a small perturbation has no qualitative effect there). This is given by the $\mu \underline{r}$ term in (1.4).

We proceed in a similar manner to section 4.1 to re-derive the sliding modes on the intersection $\mathcal{D}_{12}$ with $\mu \neq 0$. The analysis on $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ in section 4.1 is unaffected by taking μ nonzero.

Take (1.4) and solve $\underline{f} = 0$ on $x_1 = x_2 = 0$ to find the multipliers $\lambda_1 = \lambda_1^{\$\$}$ and $\lambda_2 = \lambda_2^{\$\$}$, which become

$$\begin{aligned}
\lambda_{1,\pm}^{\$\$} &= 1 + \tfrac{\alpha\gamma^2 x^+ + \alpha\mu r_{12} - \gamma^2 x_3 x_4 \pm \mu\alpha\gamma^2 R}{\mu\gamma(\alpha r_{12} - x_4 r_1)} \\
&= 1 + \tfrac{1}{\gamma} + \tfrac{\gamma}{r_{12}}(X^+ \pm R) + \mathcal{O}\left(\mu\right) \ , \qquad (4.20a) \\
\lambda_{2,\pm}^{\$\$} &= 1 - \tfrac{\alpha\gamma^2 x^+ - \alpha\mu r_{12} - \gamma^2 x_3 x_4 \pm \mu\alpha\gamma^2 R}{\mu\gamma(\alpha r_{12} + x_3 r_2)} \\
&= 1 + \tfrac{1}{\gamma} - \tfrac{\gamma}{r_{12}}(X^+ \pm R) + \mathcal{O}\left(\mu\right) \ , \qquad (4.20b)
\end{aligned}$$

where $r_{12} = r_1 - r_2$ and $x^\pm = x_3 \pm x_4$, with

$$\begin{aligned}
R &= \sqrt{(\tfrac{x^+}{\mu} + \tfrac{r_{12}}{\gamma^2})^2 - \tfrac{4r_{12}x_3}{\gamma^2\mu} + \tfrac{2x_3x_4(\mu r_1 + \mu r_2 - \gamma^2 x^+)}{\alpha(\gamma\mu)^2} + (\tfrac{x_3x_4}{\alpha\mu})^2} \\
&= \sqrt{(X^+)^2 - 2\tfrac{r_{12}X^-}{\gamma^2} + \tfrac{r_{12}^2}{\gamma^4}} + \mathcal{O}\left(\mu\right) \ . \qquad (4.20c)
\end{aligned}$$

The expansions in (4.20) are made for small μ and $x_i = \mathcal{O}\left(\mu\right)$, by introducing local variables

$$x_3 = \mu X_3 \ , \qquad x_4 = \mu X_4 \ , \qquad x^\pm = \mu X^\pm \ . \qquad (4.21)$$

These sliding modes are well-defined at the origin and the '$\pm$' roots give two branches of a continuous family of solutions. Again using the convention $\sqrt{x} = |x|^{1/2} e^{(i\,\mathrm{Arg}\,x)/2}$, the '$+$' and '$-$' roots give the perturbation of the sliding modes on $x_3, x_4 < 0$ and $x_3, x_4 > 0$ respectively. The continuation of these branches of solutions now passes through the region where $-r_{12}/\gamma^2 < x_4 < 0$ and $0 < x_3 < r_{12}/\gamma^2$, given by both '$\pm$' roots in (4.20) (as can be verified by graphing the surface, noting that $\lambda_1^{\$\$} = +1$ on $x_3 = 0$ and $\lambda_2^{\$\$} = +1$ on $x_4 = 0$, and that $R = 0$ touches $x_3 = 0$ at $x_4 = -r_{12}/\gamma^2$ and touches $x_4 = 0$ at $x_3 = r_{12}/\gamma^2$).

The sliding dynamics is given, substituting (4.20) into (1.4), by

$$\begin{aligned}
\mu\begin{pmatrix} \dot{X}_3 \\ \dot{X}_4 \end{pmatrix} &= \underline{h}^{\$\$} = \tfrac{\gamma}{2}\Theta\begin{pmatrix} \lambda_1^{\$\$} - 1 \\ \lambda_2^{\$\$} - 1 \end{pmatrix} \\
&= \tfrac{1}{2r_{12}}\Theta\begin{pmatrix} r_{12} + \gamma^2(X^+ \pm R) \\ r_{12} - \gamma^2(X^+ \pm R) \end{pmatrix} + \mathcal{O}\left(\mu\right) \ , \qquad (4.22)
\end{aligned}$$

using Θ from (1.7c).

The stability of these sliding modes is described by the Jacobian J_{12} from (2.9), whose determinant is

$$\det J_{12} = \mp\tfrac{1}{4}\alpha\gamma^2\mu R + \mathcal{O}\left(\mu^2\right) \ , \qquad (4.23)$$

implying the sliding modes have node-like attractivity on the branch in $X_3, X_4 > 0$ and saddle-like attraction/repulsion on the branch in $X_3, X_4 < 0$, with respect to the (x_1, x_2) dynamics.

This determinant now vanishes on the curve

$$\mathcal{L} = \left\{ (\lambda_1, \lambda_2, x_3, x_4) \in [-1, +1]^2 \times \mathbb{R}^2 \;:\; \lambda_i = \lambda_i^{\$\$},\; R = 0 \right\} \;, \tag{4.24}$$

with $i = 1$ and 2, where

$$0 = R^2 = (X^+)^2 - 2\tfrac{r_{12} X^-}{\gamma^2} + \tfrac{r_{12}^2}{\gamma^4} + \mathcal{O}\left(\mu\right) \;. \tag{4.25}$$

The set of sliding modes and the curve $\mathcal{L}$ are depicted in fig. 6.

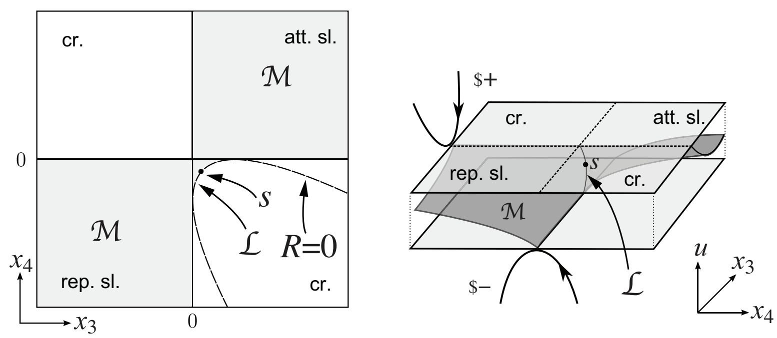

Figure 6: A representation of the regions where the sliding modes on $\mathcal{D}_{12}$ exist in (x_3, x_4) space, in the perturbed system. In the degenerate system sliding occurred on $x_3 x_4 > 0$, but now this perturbs near the origin onto the curve $\mathcal{L}$. The representation in (u, x_3, x_4) space is shown right, with $u = 0$ blown up into a layer representing $\lambda_1 \in [-1, +1]$ or $\lambda_2 \in [-1, +1]$, where the sliding manifold $\mathcal{M}$ from (4.19) is a two-sheeted curved surface that turns over at $\mathcal{L}$, where $\mathcal{M}$ meets the hypersurface $R = 0$. One sheet is attracting (att.), the other saddle-like (rep.). Also marked on $\mathcal{L}$ is 'S' denoting the star singularity which we find in theorem 4.4.

As for the co-planar two-fold, we have now resolved the two-fold into a curve $\mathcal{L}$ within the space of the multipliers $(\lambda_1, \lambda_2) \in [-1, +1]^2$ and coordinates $(x_3, x_4) \in \mathbb{R}^2$ on $x_1 = x_2 = 0$. That set connects two sheets of sliding modes, one attracting, and one with a repelling direction. Again there can exist a distinguished point — the star singularity — where the sliding vector field (4.22) lies tangent to the projection of $\mathcal{L}$ onto the (x_3, x_4) plane.

Theorem 4.4. *The contra-planar two-fold system has a star singularity on* $(x_1, x_2) = (0, 0)$ *at* $x_3 = \frac{\mu r_{12}}{4\gamma^2}(1 + \tau_*)^2$, $x_4 = -\frac{\mu r_{12}}{4\gamma^2}(1 - \tau_*)^2$ *if* $-1 \le \tau_* \le +1$, *where* τ_* *is a constant dependent on* w, v, s^+, s^-, *and where* $R = 0$.

Proof. Parameterize the set $\mathcal{L}$, say by letting $\frac{\gamma^2}{r_{12}}X^+ = \tau$ for some parameter $\tau \in [-1, +1]$, then solving $R = 0$ gives $X^- = \frac{r_{12}}{2\gamma^2}(1+\tau^2)$, implying

$$X_3(\tau) = \tfrac{r_{12}}{4\gamma^2}(1+\tau)^2\ , \qquad X_4(\tau) = -\tfrac{r_{12}}{4\gamma^2}(1-\tau)^2\ . \tag{4.26}$$

The tangent vector $(X_3'(\tau), X_4'(\tau)) = \frac{r_{12}}{2\gamma^2}(1+\tau, 1-\tau)$ lies along the same direction as (4.22) where

$$\tau = \tau_* := \tfrac{s^- - s^+ \pm \sqrt{(w-v)^2 - 4s^+s^-}}{v - w - s^+ - s^-}\ , \tag{4.27}$$

thus defining a *star* singularity according to definition 2.5. As in the co-planar case, whether zero, one, or both of these values of τ_* lie in $-1 \leq \tau_* \leq +1$ depends on the values of s^+, s^-, v, w, giving zero, one, or two star singularities, but we will not study the individual cases here.

□

Again the star singularity inherits the ambiguity of the two-fold in the perturbed system, but in a slightly different manner compared to the co-planar or classic two-folds.

At the end of section 3 we saw that the induced dynamics of (λ_1, λ_2) (given by definition 2.6) in the sliding mode became zero-over-zero at the star singularity, because (3.21) vanished there as well as $\det J_{12}$. In this case we also have $\det J_{12} = 0$, since the star singularity lies on $\mathcal{L}$, but now

$$\frac{\partial(f,g)}{\partial(x_3,x_4)} \cdot \begin{pmatrix} h \\ i \end{pmatrix} = \tfrac{1}{2}\gamma \begin{pmatrix} 1-\lambda_2^{\$\$} & 0 \\ 0 & 1-\lambda_1^{\$\$} \end{pmatrix} \cdot \begin{pmatrix} h^{\$\$} \\ i^{\$\$} \end{pmatrix} \tag{4.28}$$

does not vanish there. To do so, the matrix $\frac{\partial(f,g)}{\partial(x_3,x_4)}$ would have to be singular (note in (3.21) that this matrix is singular by virtue of one row being zero), and for contra-planar two-folds this appears to be atypical.

This improves on an observation from the degenerate models in [18] that there seemed to be a fundamental difference in the singularity obtained from the co-planar and contra-planar two-folds. We see here that the difference is in whether the induced sliding dynamics on the multipliers (λ_1, λ_2) at the star singularity is merely indefinite as in the co-planar case, or infinite as in the contra-planar case. This means, for the contra-planar case, we cannot establish the existence or not of solutions that travel between the attracting and repelling branches of sliding — this will require further work, but is beyond our scope here.

5 A general definition for two-fold singularities in higher dimensions

The models (1.3) and (1.4) for the co-planar and contra-planar two-folds are derived by starting with series expansions for small $\mathbf{x}$, of each of the four constituent vector fields $\mathbf{F}^{l_1l_2}$ in which l_1 and l_2 each take '+' or '−' values,

$$\mathbf{F}^{l_1l_2}(\mathbf{x}) = \mathbf{F}_0^{l_1l_2} + \sum_{i=1}^{4} x_i \mathbf{F}_{0,i}^{l_1l_2} + \mathcal{O}\left(x_i^2\right) \ , \tag{5.1}$$

in terms of constants $\mathbf{F}_0^{l_1l_2} = \mathbf{F}^{l_1l_2}(\mathbf{0})$ and $\mathbf{F}_{0,i}^{l_1l_2} = \frac{\partial}{\partial x_i}\mathbf{F}^{l_1l_2}(\mathbf{0})$.

Then assume there exists a two-fold at the origin, of co-planar or contra-planar type, and attempt to omit any terms that could be considered 'higher order', in the sense that their absence does not destroy the two-fold structure or introduce any degeneracies. Below in definition 5.1 we provide the definition used to test these criteria. The constants $\underline{f}_0^{l_1l_2}$ must be chosen to give sliding on the appropriate switching thresholds, the $\underline{f}_{0,i}^{l_1l_2}$ terms for $i = 3, 4$, are chosen to give appropriate folds, the fields $\underline{h}_0^{l_1l_2}$ can be chosen to fit the equations of the classic two-fold (1.7), and all higher order terms in x_i neglected. The μ perturbations in $\underline{f}$ were chosen to break the degeneracy of the non-hyperbolic set $\mathcal{L} \subset \mathcal{M}$.

Rather than describe step-by-step this lengthy process to reduce (5.1) to (1.3) and (1.4), let us instead generalize the definition of generic two-fold singularities to higher dimensional analogues, along with non-degeneracy conditions, and show how the models (1.3) and (1.4) satisfy it.

We will propose this definition for an n-dimensional system with discontinuities along m different thresholds. Let a piecewise-smooth system switch between different vector fields or *modes*, across different thresholds $\mathcal{D}_i$ comprising a *switching threshold* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_m$, for some $m \in \mathbb{N}$. Each of the sub-thresholds $\mathcal{D}_i$ is assumed to be a smooth manifold. We then say such a system in $\mathbf{x} \in \mathbb{R}^n$ switches between modes

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \lambda_1, ..., \lambda_m) = \mathbf{F}^{l_1...l_m}(\mathbf{x}) \ , \tag{5.2}$$

where each $\mathbf{F}^{l_1...l_m}$ is a smooth vector valued function of $\mathbf{x}$, labelled by indices

$$\left.\begin{array}{lclcl} \lambda_i = +1 & \Leftrightarrow & l_i = + & \text{if} & \sigma_i(\mathbf{x}) > 0 \\ \lambda_i = -1 & \Leftrightarrow & l_i = - & \text{if} & \sigma_i(\mathbf{x}) < 0 \\ \lambda_i = \lambda_i^{\$} & \Leftrightarrow & l_i = \$ & \text{if} & \sigma_i(\mathbf{x}) = 0 \end{array}\right\} \ , \tag{5.3}$$

in terms of some smooth scalar function $\sigma_i(\mathbf{x})$ where

$$\mathcal{D}_i = \{\mathbf{x} \in \mathbb{R}^n \ : \ \sigma_i(\mathbf{x}) = 0\} \tag{5.4}$$

for $i = 1, 2, \dots, m$. We can assume for simplicity that each function $\mathbf{F}^{l_1 \dots l_m}(\mathbf{x})$ is defined for all $\mathbf{x}$, but they must at least be defined on the closure of the regions (given by (5.3)) on which they prescribe the modes (5.2).

As throughout this paper, a label '\$' indicates that $\mathbf{x}$ lies on the indicated switching threshold $\mathcal{D}_i$ and that sliding occurs there, and then $\mathbf{F}$ prescribes *sliding* motion along that $\mathcal{D}_i$ (or along the intersection of multiple $\mathcal{D}_i$s if there are multiple labels '\$'). These *sliding vector fields* may or may not exist at any given $\mathbf{x}$, because sliding along a threshold $\mathcal{D}_i$ occurs if and only if there exists $\lambda_i \in [-1, +1]$ such that $\mathbf{F}(\mathbf{x}; ..., \lambda_i, ...) \cdot \nabla\sigma_i(\mathbf{x}) = 0$. Sliding occurs along the intersection of multiple thresholds $\mathcal{D}_i$ if multiple such conditions are satisfied for different $i = 1, ..., m$.

A two-fold can then be defined as follows.

Definition 5.1. *A two-fold singularity is a point* $\mathbf{x} = \mathbf{x}_* \in \mathcal{D}$ *where*

$$0 = \sigma_i(\mathbf{x}_*) = \mathbf{F}^{l_1 \dots l_i \dots l_m}(\mathbf{x}_*) \cdot \nabla\sigma_i(\mathbf{x}_*) \ , \tag{5.5}$$

for two different modes, either for some $i \in \{1, \dots, , m\}$ *in the two modes* $l_i = \pm$, *or two different* $i, i' \in \{1, \dots, , m\}$ *in any pair of modes* $l_i, l_{i'} = \pm$. *(Each remaining index must be fixed as either a '+', '−', or '\$'). At* $\mathbf{x}_*$*:*

- *(i) the tangencies defined by* (5.5) *are quadratic (i.e. and not higher order, e.g. cusps), so* $[\mathbf{F}^{l_1 \dots l_i \dots l_m}(\mathbf{x}_*) \cdot \nabla]^2 \sigma_i(\mathbf{x}_*) \neq 0$ *for the mode in* (5.5)*;*
- *(ii) the conditions* (5.5) *are satisfied by only two modes at* $\mathbf{x}_*$*;*
- *(iii) let* S_1, S_2, *denote the two sets of points on which* (5.5) *are satisfied in each of the two modes, then* S_1, S_2, *are smooth manifolds at* $\mathbf{x}_*$ *and intersect each other transversally at* $\mathbf{x}_*$*;*
- *(iv) the vector field* $\mathbf{F}^{l_1 \dots l_m}(\mathbf{x}_*)$ *in any mode is nonzero, and moreover the convex hull of these modes does not contain zero;*
- *(v) any intersections of manifolds* $\mathcal{D}_i$ *at* $\mathbf{x}_*$ *are transversal.*

For example, for a system with two switches condition (iv) becomes

$$0 \notin \text{hull}\left[\mathbf{F}^{++}, \mathbf{F}^{+-}, \mathbf{F}^{-+}, \mathbf{F}^{--}\right] \ , \tag{5.6}$$

where hull [..] denotes the convex hull of the vectors in its argument. Using the vector fields in (1.2), if the constituent fields $\mathbf{F}^{\pm l_2}$ are tangent to $x_1 = 0$ at a point $\mathbf{x}_p = (0, x_2, x_3, x_4)$, then

$$f^{\pm l_2}(\mathbf{x}_p) = 0 \ , \tag{5.7}$$

and $\mathbf{x}_p$ is a classic two-fold singularity as in fig. 1.

The higher dimensional analogues of this occur where the sliding vector fields are tangent to intersections of the switching thresholds. The sliding modes on $x_1 = 0$ are tangent to the surface $x_2 = 0$ at $\mathbf{x}_p = (0, 0, x_3, x_4)$ if

$$g^{\$\pm}(\mathbf{x}_p) = 0 \;, \tag{5.8}$$

forming a *co-planar* two-fold singularity as in fig. 2. (There is an equivalent case for the sliding modes on $x_2 = 0$ tangent to $x_1 = 0$ if $f^{\pm\$}(\mathbf{x}_p) = 0$). The sliding modes on $x_1 = 0 < x_2$ and $x_2 = 0 < x_1$ are tangent to the surfaces $x_2 = 0$ and $x_1 = 0$, respectively, at $\mathbf{x}_p = (0, 0, x_3, x_4)$ if

$$f^{+\$}(\mathbf{x}_p) = 0 \quad \text{and} \quad g^{\$+}(\mathbf{x}_p) = 0 \;, \tag{5.9}$$

forming a *contra-planar* two-fold singularity as in fig. 3. (There are equivalent cases on the other segments of the switching thresholds obtained by replacing $f^{+\$}(\mathbf{x}_p)$with $f^{-\$}(\mathbf{x}_p)$ and/or $g^{\$+}(\mathbf{x}_p)$ with $g^{\$-}(\mathbf{x}_p)$ in (5.9)).

The conditions *(i-v)* in definition 5.1 prohibit the most obvious possible degeneracies of a two-fold, namely for these co- and contra- planar cases:

(i) the second derivatives $\ddot{x}_i$ must not vanish when $\dot{x}_i = 0$ (otherwise they form cusp tangencies or higher), that is $\ddot{x}_1 = f^{\pm l_2}$ in (5.7), $\ddot{x}_2 = g^{\$\pm}$ in (5.8), or $\ddot{x}_1 = f^{+\$}$ and $\ddot{x}_2 = g^{\$+}$ in (5.9), must not vanish;

(ii) the conditions (5.5) are satisfied for only two indices and no others as in (5.8) or (5.9) (otherwise they form three-folds or higher);

(iii) the fold sets $x_i = 0$ and $\sigma_{i'} = 0$ for the two indices i, i', satisfying (5.5) should be transverse, that is, for any $x_1, x_2, f^{l_1 l_2}, g^{l_1 l_2}$, that vanish at the two-fold, the gradient vectors $\nabla x_1, \nabla x_2, \nabla f^{l_1 l_2}, \nabla g^{l_1 l_2}$, should be linearly independent;

(iv) the singularity does not coincide with an equilibrium, that is, $\mathbf{F}^{l_1 l_2}(\mathbf{x}_p) \neq 0$ in any mode;

(v) no two manifolds $\mathcal{D}_i$ may touch tangentially at the singularity.

A two-fold may exhibit degeneracies other than those ruled out by conditions *(i-v)* in definition 5.1, such as local bifurcations involving connections between separatrices of the local dynamics, which unfold as the constants v and w are varied. These require more detailed study of the dynamics, a task that truly reveals the intricacy of singularities in nonsmooth systems. Even for the simple two-fold in fig. 1 it took around three decades for this intricacy of the local dynamics to be unravelled and the conditions for structural stability to be fully derived, primarily across the series of works [7, 13, 14, 22, 31]. The prototypes we introduce in this paper are just the start of a similar, but hopefully more informed, study for higher dimensional two-folds.

Finally, let us show that the models (1.3)-(1.4) satisfy the conditions in definition 5.1 to give a two-fold singularity at the intersection of the switching thresholds $\mathcal{D}_1$ (where $x_1 = 0$) and $\mathcal{D}_2$ (where $x_2 = 0$).

Theorem 5.1. *The local model* (1.3) *has a co-planar two-fold singularity at the origin.*

Proof. We must show that (1.3) satisfies the conditions of the two-fold from definition 5.1, specifically the co-planar case as defined by (5.8).

Let us first verify that the sliding vector fields $\mathbf{F}^{\$\pm}$ on $\mathcal{D}_1^{\pm}$ both have folds at the origin and thus satisfy (5.8). From (3.6) we have

$$\dot{x}_2 = g^{\$\pm}(\mathbf{x}) = \begin{cases} -\eta & \text{for } \; x_2 > 0 \;, \\ \zeta & \text{for } \; x_2 < 0 \;, \end{cases} \tag{5.10}$$

both of which clearly vanish at $\eta = \zeta = 0$, so indeed the fields $\mathbf{F}^{\$\pm}$ on $\mathcal{D}_1^{\pm}$ are tangent to $\mathcal{D}_{12}$ at the origin. We must then check that these are generic folds by evaluating the conditions *(i-v)* from definition 5.1:

(i) the second derivatives of $\sigma_2 = x_2$ with respect to the sliding flows on $\mathcal{D}_1$ are

$$\begin{cases} \ddot{x}_2 = -\dot{\eta} = s^+ & \text{for } \; x_2 > 0 \;, \\ \ddot{x}_2 = \;\; \dot{\zeta} \;\; = s^- & \text{for } \; x_2 < 0 \;, \end{cases} \tag{5.11}$$

by (3.7), which clearly does not vanish since we defined $|s^{\pm}| \neq 1$. Hence these are folds and not higher order tangencies.

(ii) there exist no sliding modes on $\mathcal{D}_2$, so there can be no folds from any sliding flows there. Locally the modes $\underline{f}^{l_1 l_2}$ for $l_i = \pm$ are non-vanishing, so there are no folds from the constituent fields. Hence the fold conditions are not satisfied for any further modes.

(iii) by (3.6) the fold sets are $\eta = 0$ and $\zeta = 0$, which are clearly transversal on $x_1 = 0$ for finite $\alpha, \gamma \neq 0$.

(iv) at the singularity, in all constituent modes we have $\underline{f}^{\pm\pm}(\mathbf{x})$ proportional to $\underline{k} \neq \underline{0}$, so these are non-vanishing. The (η, ζ) motion in the sliding modes given by (3.7) is non-vanishing for typical parameter

values. It remains to consider the sliding mode given by (3.8) on the intersection $x_1 = x_2 = 0$. The determinant of the matrix Ω in (3.8) is $-\alpha^2(vw + s^+s^-)$, which is non-vanishing for typical parameters for which Θ from (1.7c) is non-singular. The term $\frac{1}{\eta+\zeta}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}$ is ill-defined at $\eta = \zeta = 0$, but letting $(\eta, \zeta) = r(\cos\theta, \sin\theta)$ this becomes

$$\frac{1}{\cos\theta+\sin\theta}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

which does not include zero for any $\theta \in [0, 2\pi)$, even as we take the limit $r \to 0$. Hence none of the modes $\mathbf{F}^{l_1 l_2}$ typically vanishes at the singularity.

(v) Clearly $\mathcal{D}_1$ and $\mathcal{D}_2$ are transversal as they are the hypersurfaces $x_1 = 0$ and $x_2 = 0$.

□

Theorem 5.2. *The local model* (4.1) *satisfies the conditions of the two-fold from definition 5.1, consistent with the* **contra-planar** *case as defined by* (5.9)*.*

Proof. We must show that (1.4) satisfies the conditions of the two-fold from definition 5.1, specifically the co-planar case as defined by (5.9). Let us first show that the tangency conditions (5.9) are satisfied. On $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ respectively, from (4.2) we have

$$f^{+\$}(\mathbf{x}) = -x_3\ , \qquad g^{\$+}(\mathbf{x}) = -x_4\ , \tag{5.12}$$

both of which clearly vanish at $x_3 = x_4 = 0$, so the fields $\mathbf{F}^{\$+}$ and $\mathbf{F}^{+\$}$ on $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ are tangent to the intersection $\mathcal{D}_{12}$ at the origin. As for the co-planar case, we must then check that these are generic folds by evaluating the conditions *(i-v)* from definition 5.1:

(i) the second derivative of $\sigma_2 = x_2$ with respect to the sliding flow on $\mathcal{D}_1^+$ is

$$\ddot{x}_2 = -\dot{x}_4 = -i^{\$+} = -s^-\ , \tag{5.13}$$

and the second derivative of $\sigma_1 = x_1$ with respect to the sliding flow on $\mathcal{D}_2^+$ is

$$\ddot{x}_1 = -\dot{x}_3 = -h^{+\$} = s^+\ , \tag{5.14}$$

by (4.8), and these clearly do not vanish for typical $s^\pm$ values. Hence these are folds and not higher order tangencies.

(ii) from lemma 4.1 there exist no sliding modes on $\mathcal{D}_1^-$ or $\mathcal{D}_2^-$, so there can be no folds from any sliding flows there. Locally the modes $\underline{f}^{l_1 l_2}$ for $l_i = \pm$ are non-vanishing, so there are no folds from the constituent fields. Hence fold conditions are not satisfied in any further modes.

(iii) by (4.2) the fold sets are $x_4 = 0$ and $x_3 = 0$, which are clearly transversal.

(iv) at the singularity, in all constituent modes we have $\underline{f}^{\pm\pm}(\mathbf{x})$ proportional to $\underline{k} \neq \underline{0}$, which are therefore non-vanishing. The (x_3, x_4) motion in the sliding modes given by $\underline{h}^{\$+}$ and $\underline{h}^{+\$}$ in (4.8) are clearly non-vanishing for typical parameter values, and the sliding mode given by $\underline{h}^{\$\$}$ in (4.13) does not vanish (by the argument in the proof of theorem 5.1(iv)) for typical parameter values. Hence none of the modes $\mathbf{F}^{l_1 l_2}$ vanishes at the singularity.

(v) Clearly $\mathcal{D}_1$ and $\mathcal{D}_2$ are transversal as they are the surfaces $x_1 = 0$ and $x_2 = 0$.

□

6 Closing Remarks

The singularities we describe here were introduced in [18] along with toy models, but these turn out to be structurally unstable because they suffer from degeneracies shown in [19] to afflict any coinciding tangencies to a switching threshold. We showed how to break the degeneracy by introducing nonlinear dependence on the switching multipliers. Then these singularities, of higher dimensional systems and involving higher codimensional intersections of switching thresholds, can be treated in the same manner as, and indeed reduced to, the well known two-fold singularities in three dimensions, and therefore studied using well developed theory found in [9, 14, 20, 21, 31, 32, 33] and references therein.

Our results are an example of how dimension reduction can be applied to high dimensional piecewise-smooth systems if they have many switching thresholds, by reducing attention to those thresholds and their intersections. In this case two singularities that occur generically in four dimensions or more reduce to three essential dimensions on the switching thresholds, within which an analogue of the classic two-fold singularity in its familiar normal form. It remains to study what happens outside these surfaces in the full four dimensional dynamics, looking to the analysis of the classic two-fold as a guide, but this is beyond our scope here.

We have not considered in detail the different dynamics that results locally depending on whether the folds are visible (the flows turn away from the switching thresholds) or invisible (the flows turn towards the surfaces), or a mixture thereof (fig. 1 shows visible folds only). The most complex in the classic case are the cases involving invisible folds, which curve towards the thresholds, leading to intricate ejection from and return to the switching thresholds and the singularity itself. For example, the number of times solutions cross the switching thresholds can change as described in [13], and families of solutions repeatedly winding around the singularity can form separatrices such as the *nonsmooth diabolo* [22]. Local bifurcations can alter whether and how many solutions travel from attracting to repelling regions of sliding (so-called *canards*) or vice versa (called *faux canards*) [8]. The correspondence between the higher dimensional two-folds and the vector fields of the classic case are just a first step in beginning to probe the generalization of these behaviours to higher dimensions.

G. Olivar first asked (at the 6th SICC Tutorial Workshop on Topics in Nonlinear Dynamics at Urbino, Italy, in 2011) whether there existed m-folds for $m > 2$, to which one of the author's (M. Jeffrey) response in the negative has been proven categorically and wonderfully naive. Clearly, given the geometry by which the two-folds studied here arise, there could also be three-folds or four-folds (where the sliding flows are tangent to the intersection from three or four segments of the switching threshold), generic in five or six dimensions, respectively, and so on if more than two surfaces intersect. These will create more complicated dynamics, still qualitatively related to the classic two-fold, not reducible to the equations of a two-fold yet analyzable by similar methods.

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