SELECTED TOPICS IN NON-EUCLIDEAN GEOMETRY

by

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INTRODUCTION TO NON-EUCLIDEAN GEOMETRY

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Monographs on Modern Mathematics by J.W.A. Young

Non-Euclidean Geometry is a system of geometry which is built up without the use of the fifth postulate of Euclid. This postulate reads as follows:

"If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."

Many attempts have been made to establish such a system of geometry. That, in which Lobachevsky, a Russian, was the chief writer from 1833 to 1855, remained for a time the sole type of a non-Euclidean geometry. This was known as the Lobachevskian geometry. Riemann, however, in 1854, working from the standpoint of the differential calculus, discovered a new type to which the name of Riemannian geometry has been given. These two types of non-Euclidean geometry, it seems, have been the principal types recognized by mathematicians.

In the following paragraphs Euclid's fundamentals are assumed with the exception of the parallel postulate. A more general definition of parallels than that of Euclid will be given.

PARALLEL LINES

Let PQ (Fig. 1) be any straight line, and A a point not on PQ. Through A there passes a set of lines intersecting PQ, since any point on PQ may be joined to A. Let AB be any line through A intersecting PQ. The line AL is said to be parallel to PQ at the point A, if (1) AL does not intersect PQ no matter how far produced and (2) any line in the angle opening BAL, passing through A, does intersect PQ.

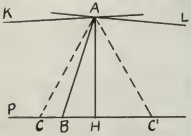


Fig. 1

Let AH be perpendicular to PQ. Then $\angle HAK = \angle HAL$, for if $\angle HAK > \angle HAL$, draw $\angle HAC = \angle HAL$ and take C' on HQ so that HC' = HC. Then the triangles HAC' and HAC are , and $\angle HAC' = \angle HAC = \angle HAL$. This is impossible since AL is parallel to HQ; hence, $\angle HAK \ngtr \angle HAL$. In like manner it can be shown that $\angle HAL \ngtr \angle HAK$. Therefore $\angle HAL = \angle HAK$.

$\angle HAL$ is called the angle of parallelism for the distance AH. The line AL is parallel to PQ at all of its points. Let AK be parallel to BQ at point A (Fig. 2).

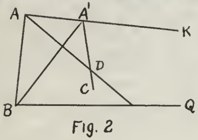


Fig. 2

Let A' be any point on AK. To show that AK is parallel to BQ at point A', connect A' and B and through A' draw any line A'C in the angle opening $BA'K$. Take D, any point on $A'C$ and connect D with A. Prolong AD until it meets BQ at F. (It will meet BQ, since AK is parallel to BQ.) Also $A'C$ will meet BQ in a point between B and F; (Veblen, Th. 17 - "A line of a plane which contains one and only one point of a side of a triangle whose vertices are in the plane contains one other point of the triangle.) That is, any line through A' in the angle opening $BA'K$ intersects BQ. But $A'K$ does not intersect BQ. Hence, it is parallel to BQ.

If a line is parallel to another line, the second is parallel to the first. Let LK (Fig. 3) be parallel to PQ.

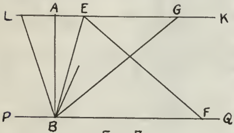


Fig. 3

Select a point B on PQ and erect a perpendicular to LK. This will meet LK at some point A since LK is parallel to PQ. Through B draw any line BC in the angle opening QBA. Construct the two angles ABE and ABD so that the $\angle ABE = \angle ABD < \frac{1}{2} \angle QBC$. Then BD = BE; hence in $\angle BEK$ draw a line EF so that $\angle BEF = \angle BDK$, and EF meets BQ since LK is parallel to PQ. Now take DG = EF, and draw BG. Then the two triangles are $\cong$ and $\angle DBG = \angle EBF$. But $\angle DBE < \angle QBC$. Therefore, $\angle EBG > \angle EBC$. Hence the line BC meets LK at some point between E and G; but BC is any line through B in the angle opening QBA, and LK and BQ do not meet. Therefore BQ is parallel to LK.

If two lines are parallel to a third line, they are parallel to each other. Case 1: (Fig. 4). Let AK and DQ be parallel to ML. To prove: AK is parallel to DQ. Draw

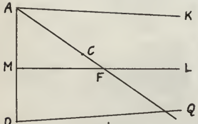


Fig. 4

AC any line through A in $\angle$DAK. Then AC will meet ML in some point F since AK is parallel to ML. CF prolonged will also meet DQ, since ML is parallel to DQ. Hence any line through A in the $\angle$ DAK meets DQ. Also AK cannot meet DQ, since it cannot meet ML. Therefore AK is parallel to DQ.

Case 2: (Fig. 5) Let AK and DQ be parallel to ML.

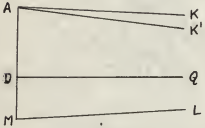


Fig. 5

To prove: AK is parallel to DQ. Draw through A a line AK' parallel to DQ. Then by case 1, AK' is parallel to ML and coincides with AK.

The Euclidean Assumption

Replace Euclid No. 5 by "Through any point in a plane there goes one and only one line parallel to a given line."

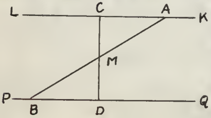


Fig. 6

Hence, (Fig. 6) $\angle$HAL = $\angle$HAK = a right angle. Take M the mid-point of AB, and draw MD perpendicular to PQ and intersecting AL in C. Then

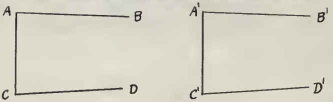
$\angle DCK$ is a right angle. The two right triangles AMC and BMD are $\cong$, and $\angle CAB = \angle ABD$. Therefore $\angle QBA$ and $\angle BAK$ together equal 2 right angles. By definition of parallels any line through A in the $\angle BAK$ meets PQ; hence the assumption is equivalent to Euclid No. 5.

The Lobachevskian Assumption

"Through any point in the plane there go two lines parallel to a given line." It follows that (Fig. 6) the $\angle QBA + \angle BAK = 2$ right angles, for if the sum is $>$ 2 right angles, we could draw in $\angle BAK$ a line not meeting BQ. This is contrary to the assumption that AK and BQ are parallel. On the other hand, if the sum of the angles = 2 right angles, we have the Euclidean assumption. Therefore the angles must be acute.

Theorem 1: Let AB and CD be two parallel lines cut by a third line AC, (Fig. 7) and let A'B' and C'D' be two other parallel lines cut by a third line A'C', and let $\angle DCA = \angle D'C'A'$.

Fig. 7



Then (1) if $A'C' = AC$, $\angle C'A'B' = \angle CAB$.

(2) if $A'C' < AC$, $\angle C'A'B' > \angle CAB$.

(3) if $A'C' > AC$, $\angle C'A'B' < \angle CAB$.

By law of converses, ("If the direct theorem is true, and the converse is true, then the opposite is true"); Let $\angle DCA = \angle D'C'A'$. Then it follows that:

Theorem 2:

(1) if $\angle C'A'B' = \angle CAB$, $A'C' = AC$.

(2) if $\angle C'A'B' > \angle CAB$, $A'C' < AC$.

(3) if $\angle C'A'B' < \angle CAB$, $A'C' > AC$.

Theorem 3: The angle of parallelism is fixed for a fixed distance, and decreases as the distance increases.

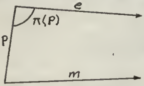


Fig. 8

(Fig. 8) Where p is fixed the angle of parallelism $\pi(p)$ is fixed; As p increases $\pi(p)$ decreases. As p decreases, $\pi(p)$ increases. $\pi(p) = 2\tan^{-1} e^{-\frac{p}{K}}$.(K = space constant).

Theorem 4: If two lines have a common perpendicular, they neither intersect nor are parallel.

(a) (Converse of Th. 4) Two straight lines which neither intersect nor are parallel have a common perpendicular.

Let LM and EF (Fig. 9) be two straight lines which neither intersect nor are parallel. To show that they

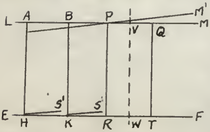


Fig. 9

have a common perpendicular; Take A and B, any two points on LM and draw AH and BK perpendicular to EF. If AH = BK, it is evident that LM and EF have a common perpendicular. Now suppose that BK < AH. Draw KS parallel to LM. Place the right angle FKB on the right angle FHA so that K falls on H, KF takes the direction of HF and KB takes the direction of HA. The point B falls at B' between H and A, B'M' and BM the same position, and KS the position HS' parallel to B'M'. Since $\angle$ FKS = $\angle$ FHS', a line parallel to KS 9and hence to LM) drawn through H lies in the angle opening FHS'. Hence HS' intersects LM and therefore B'M' intersects LM at some point P.

Draw PR perpendicular to EF. Place the right angle FHB' on the right angle FKB. Then the line PR takes the position QT, where QT is perpendicular to EF and = to PR. Now take W half way between R and T and draw WV perpendicular to EF. Fold the figure TWV on WV. Then T falls on R, TQ coincides with RP, and $\angle$ WVQ coincides with $\angle$ WVP. Hence WV is the required perpendicular.

(b) Any angle is an angle of parallelism belonging to a certain distance.

Let KAE (Fig. 10) be any given angle α . To find a distance p for which α is the angle of parallelism. Construct LAE = α , and on AK and AL take two points B and C so that AB = AC. Connect B and C and draw BL' parallel to BL, and CK' parallel

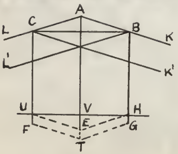


Fig. 10

to CK. Draw also CF bisecting the $\angle LCK'$, and BG bisecting the $\angle KBL'$. The figure is then symmetric with respect to the line AE.

The lines CF and BG cannot intersect, for if they did intersect at a point T, we could draw TS parallel to AL and BL' and then, since $\angle LCT = \angle L'BT$, and CT = BT, we should have $\angle STC = \angle STB$ which is impossible. Also CF and BG cannot be parallel, for if they were, since $\angle LCF = \angle L'BG$, and $\angle CNL' = \angle BNF$, we should have CN = NB and therefore $\angle NCB = \angle NBC = \angle K'CB$ which is impossible.

Since FC and BG neither intersect nor are parallel, they have a common perpendicular UV, which, by the symmetry of the figure, is also perpendicular to AE at H. UV is parallel to AK, for if UV is not parallel to AK, we could draw from each of the points U and V, a line parallel to AK and CK'. Since CU = BV, and $\angle UCK' = \angle VBK$, these two parallels would make equal angles with UV, which is impossible. Hence the angle KAE is the angle of parallelism for the distance AH.

(c) Two parallel lines approach each other continually and their distance apart eventually becomes less than any assigned quantity.

Let LK and PQ (Fig. 11) be two parallel lines, and A and B two points on LK, the point B lying from A in the direction of parallelism. From A and B draw AH and BM perpendicular to PQ. To prove: $BM \angle AH$. Take R half way between H and M, and draw RC perpendicular to PQ. Then the angle RCB is less than a right angle, since it is an angle of parallelism. Therefore the $\angle RCB < \angle RCA$. Hence if the quadrilateral RMBC is folded over on RC as an axis, the line MB takes the position HB', where $MB = HB' < HA$. Hence the lines LK and PQ continually approach each other.

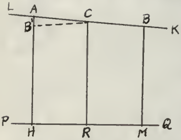


Fig. 11

To prove the second part of the theorem, let AK and HQ (Fig. 12) be any two parallel lines, and AH a perpendicular from A to HQ. Let ϵ be any assigned quantity, and lay off on AH the distance $HD < \epsilon$. Draw DL parallel to HQ and AK. Then the $\angle HDL <$ a right angle. Hence the line DE drawn from D perpendicular to AH will meet AK in some point C. From C draw CM perpendicular to HQ. Now $\angle MCD > \angle MCK$, for $\angle MCK$ is the angle of parallelism for the distance CM, and the line CD and MH neither intersect nor are parallel, since they have a common perpendicular. Hence if the quadrilateral MHDC is folded over on MC as an axis, it takes the position MH'D'C, where CK lies between

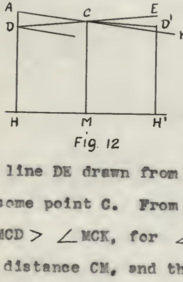


Fig. 12

CD' and MQ. Then CK meets H'D' in some point K', where H'K' $<$ H'D' $=$ HD. Hence H'K'$<\epsilon$.

(d) If two lines are not parallel, they will diverge if sufficiently far produced, and their distance apart will eventually become greater than any assigned distance.

Let AM and AN (Fig. 13) be two intersecting straight lines. Let B and D be two points on AM such that AD $>$ AB, and let BC and DE be drawn perpendicular to AN. To prove: DE $>$ BC. Suppose, if possible, that DE $=$ BC. Then a line drawn perpendicular to AN at the middle point of CE would be also perpendicular to AM, which is impossible, since AM and AN intersect. Suppose, if possible, that DE $<$ BC. Take AF less than each of the distances DE and AB, and draw FG perpendicular to AN. Then FG is $<$ AF $<$ DE. But BC $>$ DE, hence at some point K between G and C there is a perpendicular HK such that HK $=$ DE. But this is impossible. Therefore DE $>$ BC.

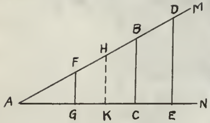


Fig. 13

To show that there is no superior limit to the length of ED, take AH (Fig. 14) so that $\angle$ MAN is the angle of parallelism for AH, and draw HL perpendicular to AM. Then AN and HL are parallel. Let "a" be any quantity, no matter how large, and

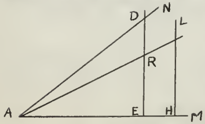


Fig. 14

take Q on HL so that HQ = 2a. Connect Q and A, and at E, a point between A and H, draw a line perpendicular to AH, intersecting AQ in R. We can take E so near H that RE will differ from HQ by as little as we please, and certainly so that RE $>$ a. But RE will intersect AN in a point D, since the angle of parallelism for AE is greater than the angle HAN. Then DE $>$ RE $>$ a. Since "A" is any positive number, there is no superior limit to the length of DE.

The Riemannian Assumption

"Through a point of the plane no line can be drawn parallel to a given line".

In other words, all lines of the pencil with its vertex at A (Fig. 1) intersect PQ. Here, the propositions of Euclid depending upon the assumption that two straight lines cannot enclose space are contradicted, except when applied to objective space in the domain of experience. We will then assume that the Euclidean assumptions, with the exception of the parallel postulate, are valid in a sufficiently restricted portion of space, that is, in a portion of space in which no straight line can be drawn of greater length than some fixed line of length M.

(a) All lines perpendicular to the same straight line meet in a point at a constant distance from the straight line. (Fig. 15) Let LK be any straight line and A and B any two points upon it. By the Riemannian hypothesis AO and BO, perpendicular to LK, meet in a point O. Since it is conceivable that the perpendiculars may meet more than once, we may assume that the two perpendiculars have no common point on the segment AO or BO. It is assumed also

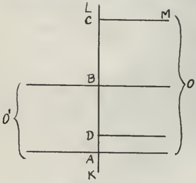


Fig. 15

that the triangle ABO lies in the restricted portion of space mentioned above, so that in particular only one straight line can be drawn from O to any point of the segment AB. Since the $\angle BAO = \angle ABO$, BO = AO. Construct $\angle BOM = \angle AOB$. Then by the Riemannian hypothesis the line OM meets LK in a point C. The triangle BOC has two angles and an included side congruent respectively to two angles and the included side of the triangle AOB.

Hence $\angle BCO = \angle ABO$ = a right angle, and OC = OB = OA. By repeating this demonstration, we prove that if P is a point on LK such that $AP = m \cdot AB$, where m is a positive integer, the line OP is perpendicular to LK at P, and PO = AO. But only one perpendicular can be drawn to LK at P. Hence this perpendicular passes through O. Now take D, so that $AB = n \cdot AB$, where n is a positive integer, and draw a line perpendicular to LK at D. If this perpendicular should intersect either BO or AO at a point O', in the segments BO or AO, then BO and AO would also intersect at O' which is contrary to the hypothesis.

Hence this perpendicular passes through O, and DO = AO.

It follows that if P is any point on LK such that $AP = \frac{m}{n}AB$, where m and n are positive integers, the perpendicular to LK at P passes through O, and PO = AO. Also, since by hypothesis, only one straight line can be drawn from P to O, the line PO is perpendicular to LK.

Now let P' be a point such that AP' = AB, where is an irrational number. Take P such that $AP = \frac{m}{n}AB$, draw OP and OP' and let $\frac{m}{n}$ pass through rational values approaching as a limit. $\angle AP'O = \lim \angle APO$; $P'O = \lim PO$. But APO is always a right angle and PO is always equal to AO. Hence $\angle AP'O$ is a right angle, and P'O = AO. The theorem is thus proved for the line LK. If L'K' is any other line we may take A' and B', any two points on it, and draw the perpendiculars A'O' and B'O', intersecting at O'. Take AB on LK so that AB = A'B'. The two triangles ABO and A'B'O' are congruent, and A'O' = AO. The distance AO is therefore independent of the line LK or of the position of the point A on the line. Place $AO = \Delta$. A corollary of the theorem is that all straight lines are of constant length. It is evident that, if P is any point on AB, $\frac{AP}{AB} = \frac{\angle AOP}{\angle AOB}$.

Now if $\angle AOP = 2\pi$, the line OA coincides with OP, and AP becomes L, the total length of the line. Then the length L equals $\frac{2\pi AB}{\angle AOB}$.

(b) All lines which pass through a point O meet again in a point O' such that the distance O' is constant.

Let O (Fig. 15) be any point, and OA any line through O. Take OA = Δ , and draw LK perpendicular to AO. Let OB be any other line through O, intersecting LK in B. Then OB is perpendicular to LK. Prolong AO to O', so that AO' = AO, and draw O'B. The triangles AOB and AO'B are congruent since two sides and the included angle of one are equal respectively to two sides and the included angle of the other. Hence $\angle$ABO' = $\angle$ABO and they are right angles, and O'B = OB = OA. Therefore the line OBO' is a straight line, and OO' = 2Δ.

Since all the lines are of finite length, any line through O returns through O' to O. Two cases are usually considered.

First, the point O' may coincide with O. The total length of a straight line is then 2Δ, and any two lines have only one point in common.

Secondly, the point O' may be distinct from O, but the lines OO' continued through O' meet again in O. The total length of a line is then 4Δ , and the two lines meet in two points. The Riemannian geometry, in this case, is the same as the geometry on the surface of a sphere.

The Sum of the Angles of a Triangle

Consider any triangle ABC (Fig. 16). Take E, the mid-point of AB, F, the mid-point of AC, and draw a straight line EF. From A, B, and C draw the lines AG, BK, and CL perpendicular to EF. In the right triangles AEG and EBK,

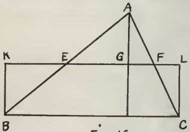


Fig. 16

EA = EB, and $\angle GEA = \angle BEK$. Hence the two triangles are congruent, and BK = AG, and $\angle KBE = \angle GAE$. Similarly, the right triangles AGF and FLC are congruent, and AG = CL. Also $\angle FCL = \angle GAF$. If we define equivalent figures as those which may be divided into parts which are congruent in pairs, it appears that the triangle ABC is equivalent to the quadrilateral BCLK. Also, the sum of the angles of the triangle ABC is equal to the sum of the angles KBC and LCB of the quadrilateral BCLK.

This quadrilateral BCLK has two right angles, L and K, and two equal sides, KB and LC, adjacent to the right angles and opposite to each other. Such a figure we shall call an isosceles-birectangular-quadrilateral. Thus the study of the angles and area of a triangle can be made by use of an equivalent isosceles-birectangular-quadrilateral, or an I.B.Q.

Let ABCD (Fig. 17) be an I.B.Q. with right angles at A and B. Let AB be called the base, CD the summit, and C and D the summit angles of the I.B.Q. Take L the mid-point of the base, and draw LK perpendicular to the base. Fold LBDK on LK as an axis. It is clear that the point D falls on C. Hence the summit angles of an I.B.Q. are equal.

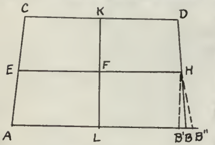


Fig. 17

Also LK is perpendicular to CD at its mid-point K, and the quadrilateral LBDK has three right angles. Through H, the mid-point of LK, draw EF perpendicular to LK. Fold HFDK on HF as an axis. The point D will fall at B', B, or B" according as KD is less than, equal to, or greater than LB. In these three cases the angle D is greater than, equal to, or less than the angle B, respectively. Hence each summit angle of an I.B.Q. is less than, equal to, or greater than, a right angle, according as the summit of the quadrilateral is greater than, equal to, or less than the base:

From the former discussion, it follows that:

1. In the Euclidean geometry each summit angle of an I.B.Q. is equal to a right angle.

2. In the Lobachevskian geometry each summit angle of an I.B.Q. is less than a right angle.

3. In the Riemannian geometry each summit angle

of an I.B.Q. is greater than a right angle.

NON-EUCLIDEAN TRIGONOMETRY

The definitions of the trigonometric functions based on Euclid are not available in the non-Euclidean geometries. So Lobachevsky constructed a "limit surface" or horisphere, on which the Euclidean geometry and trigonometry are valid at the same time that the Lobachevskian geometry is valid on the plane. But a more general method, which has also the advantage of operating entirely in the plane, must be used for the Riemannian geometry, and also applied to the Lobachevskian geometry.

Beginning with

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

the trigonometric functions are defined as follows:

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}$$

$$\cos x = \frac{e^{xi} + e^{-xi}}{2}$$

$$\tan x = \frac{1}{i}\,\frac{e^{xi} - e^{-xi}}{e^{xi} + e^{-xi}}$$

where $i = \sqrt{-1}$. These formulas obey all the formulas of trigonometry, and if x is real, they are real. If x is pure imaginary, the above formulas lead to the hyperbolic functions which are:

$$-i\sin ix = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

$$\cos ix = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

$$-i\tan ix = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \tanh x$$

If x is real, the hyperbolic functions are real. If the $\cos x < 1$, x is real; if the $\cos x > 1$, x is pure imaginary, except for the multiples of the period 2 which may always be added.

If we place $\cos mx = f(x)$, $f(x)$ satisfies the functional equation $f(x - y) - f(x-y) = 2f(x)f(y)$.

Conversely, if $f(x)$ is a continuous function of x, satisfying the above equation, then $f(x) = \cos mx$, m being a constant, real or imaginary.

The sine and cosine of an acute angle may be defined as follows; (The extension to angles of any size is then made as in the ordinary trigonometry.)

Let A (Fig. 18) be an acute angle x in the right triangle ABC, and BC the side opposite x, and let AB become infinitesimal while the angle x remains constant.

Problem: To prove

1. That $\lim\limits_{AB\to 0} \frac{AC}{AB}$ = a definite number.
2. That $\lim\limits_{AB\to 0} \frac{AC}{AB} = f(x)$.
3. That $f(x)$ is a continuous function.

4. That f(x) satisfies the equation

$$f(x - y) - f(x - y) = 2 \cdot f(x) \cdot f(y).$$

5. That $f(x) = \cos mx$.

(1 and 2 may be assumed without demonstration.)

Proof:

3. Take B' on the extension of CB beyond B, and let Δx be the measure of $\angle BAB'$. If Δx is infinitesimal, then BB' is infinitesimal as compared with AB, by (Th. 2, p. 49, Coolidge: If in a triangle whereof one angle is constant, a second angle may be made as small as desired, the side opposite this angle will be infinitesimal as compared to the other sides of the triangle).

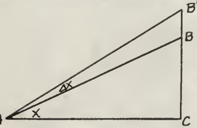


Fig. 18

The difference between two sides of a triangle (in pan geometry) is less than the third side; then $AB' - AB < BB'$. But BB' is infinitesimal. Hence AB and AB' may differ by an infinitesimal amount, and $\frac{AB'}{AC} - \frac{AB'}{AC}$ will become and remain less than any assigned number. Therefore f(x) is continuous.

4. Suppose (Fig. 19) we have two angles KAL and LAX such that each is less than a right angle, and $\angle LAX$ is $> \angle KAL$. Let $\angle LAX = x$, and the $\angle KAL = y$. Take F on AL, and find D so that AF = AD, and the $\angle FAL = \angle LAD$. AD is within

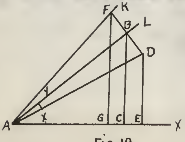


Fig. 19

the $\angle x$. These points will certainly exist if AF be very small. Join F and D, and let FD intersect AL in point B. Through F, B, and D, draw perpendiculars to AX, meeting it in G, C, and E, respectively, which points are sure to exist if AF be very small. Let us consider Fig. 20 a "close-up" of part of Fig. 19. Suppose there is a midpoint of CE at C'. Erect a perpendicular to CE at C', intersecting BD at B'. Lay off $CB'' = C'B'$. Then the figure $CC'B'B''$ is an I.B.Q. whose summit is $B'B''$. As AD $\to 0$ as a limit, FD and BB' each $\to 0$ as a limit. Hence $\angle BAB' \to 0$ as a limit, and BB' is an infinitesimal of higher order than AB. But AB and CD are infinitesimals of the same order; hence BB' is of higher order than AD. As AD, AB and DF $\to$ 0 as a limit, the $\angle B'BB''$ is finite, and $\angle B'B''C$ approaches a right angle. Therefore $\angle BB'B''$ is finite and BB' is an infinitesimal of the same order as $B'B''$, for the ratio $\frac{BB'}{B'B''}$ is a finite number. $B'B''$ and CC' being the summit and base of an I.B.Q. are infinitesimals of the same order. Therefore CC' is an infinitesimal of same order as BB' and hence of higher order than AD.

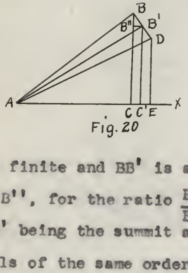


Fig. 20

Then, $\frac{AE}{AD} = f(x - y) + \epsilon_1$

$$\frac{AC}{AD} = \frac{AC}{AB} \cdot \frac{AB}{AD} = f(x) \cdot f(y) + \epsilon_2$$

$$\frac{AG}{AF} = f(x + y) + \epsilon_3 \text{, but } AF = AD, \therefore \frac{AG}{AD} = f(x+y) + \epsilon_3$$

$$\frac{AG+AE}{2} = AC + \epsilon_4 \quad , \; (\epsilon_4 = CC')$$

$$AG + AE = 2AC + 2\epsilon_4$$

$$\frac{AG}{AD} + \frac{AE}{AD} = 2\,\frac{AC}{AD} + 2\,\frac{\epsilon_4}{AD}$$

$$f(x + y) + \epsilon_3 + f(x - y) + \epsilon_1 = 2f(x)f(y) + 2\epsilon_2 + \frac{2\epsilon_4}{AD}.$$

As $AD \rightarrow 0$, ϵ_1, ϵ_2 and ϵ_3 become 0, and since $\epsilon_4 = CC'$, an infinitesimal of higher order than AD, $\frac{2\epsilon_4}{AD}$ also becomes 0, and we have $f(x + y) + f(x - y) = 2\, f(x) \cdot f(y)$.

5. $f(x + y) = f(x) + \frac{f'(x)y}{\lfloor\underline{1}} + \frac{f''(x)y^2}{\lfloor\underline{2}} + \frac{f'''(x)y^3}{\lfloor\underline{3}} + \ldots.$

$f(x - y) = f(x) - \frac{f'(x)y}{\lfloor\underline{1}} + \frac{f''(x)y^2}{\lfloor\underline{2}} - \frac{f'''(x)y^3}{\lfloor\underline{3}} + \ldots.$

Adding,

$$f(x + y) + f(x - y) = 2f(x) + \frac{2f''(x)y^2}{\lfloor\underline{2}} - \frac{2f''''(x)y^4}{\lfloor\underline{4}} + \ldots.$$

Dividing by $2f(x)$,

$$\frac{f(x + y) + f(x - y)}{2f(x)} = 1 + \frac{f''(x)}{f(x)\lfloor\underline{2}}\, y^2 + \frac{f^{IV}(x)}{f(x)\lfloor\underline{4}}\, y^4 + \ldots$$

$$\frac{f(x+y)+f(x-y)}{2f(x)} = f(y).$$

$$\therefore\ f(y) = 1 + \frac{f''(x)}{f(x)\ \lfloor 2} y^2 + \frac{f^{IV}(x)}{f(x)\ \lfloor 4} y^4 + \ldots.$$

$\therefore\ \dfrac{f''(x)}{f(x)} =$ a constant k, $\dfrac{f^{IV}(x)}{f(x)} =$ a constant also.

$f''(x) = kf(x)$ or $\dfrac{d^2 f(x)}{dx^2} = kf(x)$; $f'''(x) = kf'(x)$;

$f^{IV}(x) = k^2 f(x)$.

$\dfrac{f^{IV}(x)}{f(x)} = k^2$, and $\dfrac{f^{VI}(x)}{f(x)} = k^3$.

Then $f(y) = 1 + \dfrac{ky^2}{\lfloor 2} + \dfrac{k^2 y^4}{\lfloor 4} + \dfrac{k^3 y^6}{\lfloor 6} + \ldots$; Let $k = -a^2$

Then $f(x) = 1 - \dfrac{(ay)^2}{\lfloor 2} + \dfrac{(ay)^4}{\lfloor 4} - \dfrac{(ay)^6}{\lfloor 6} + \ldots$,

But this is cos (ay);

$\therefore$ f = cosine function.

Hence we have $\lim \dfrac{AC}{AB} = \cos mA$; since $AC < AB$, m is real, and if we take the system of measurement of angle by which a right angle $= \dfrac{\pi}{2}$, $m = 1$; hence $\lim \dfrac{AC}{AB} = \cos A$. Also $\dfrac{BC}{AB} = \sin A$.

It may be shown also that if CD is the summit and AB the base of an I.B.Q., $\dfrac{CD}{AB}$ approaches a limit as $AB \to 0$, and that the limit is a continuous function of "a", satisfying the above functional equation.

Hence, $\lim \dfrac{CD}{AB} = \cos ma$.

In the Lobachevskian geometry, $CD > AB$, and m is pure imaginary. In this case we place $m = \frac{i}{k}$, where k is real, and $\lim \frac{CD}{AB} = \cos \frac{ia}{k} = \frac{a}{k}$.

In the Riemannian geometry, $CD < AB$, and m is real. In this case we place $m = \frac{1}{k}$, and have $\lim \frac{CD}{AB} = \cos \frac{a}{k}$. The constant k depends upon the unit of distance used.

Without details of proof, we shall now assume the general formulas in Young's Monographs as follows: (ABC being a right triangle, C = a right angle, $AB = c$, $AC = b$, $BC = a$)

1. $\cos mc = \cos ma \cdot \cos mb$
2. $\sin ma = \sin mc \cdot \sin A$
3. $\tan ma = \tan mc \cdot \cos B$
4. $\cos A = \cos ma \cdot \sin B$
5. $\sin mb = \sin mc \cdot \sin B$
6. $\tan mb = \tan mc \cdot \cos A$
7. $\cos B = \cos mb \cdot \sin A$

in which, in the Lobachevskian geometry, we replace m by $\frac{i}{k}$; and in the Riemannian geometry, by $\frac{1}{k}$.

NON EUCLIDEAN ANALYTIC GEOMETRY

If x and y are assumed arbitrarily to be the coordinates of a point P, using the coordinate axes OX and OY, there is not necessarily a corresponding point P' in the

Lobachevskian geometry, since perpendiculars at points on the axes may be parallel or non-intersecting.

If we take the polar coordinates of P with the rectangular coordinates, between the two sets there exist, in either the Riemannian or the Lobachevskian geometry, the relations:

$$\tan mx = \tan mr \cos \theta, \tag{1}$$

$$\tan my = \tan mr \sin \theta \tag{2}$$

whence $\tan^2 mx + \tan^2 my = \tan^2 mr$.

The equation of a straight line becomes

$$\tan mx \cos\alpha + \tan my \sin\alpha = \tan mp \tag{3}$$

The distance between two points P and P becomes

$$\cos m P_1P_2 = \frac{1 + \tan mx_1 \tan mx_2 + \tan my_1 \tan my_2}{\sqrt{1 + \tan^2 mx_1 + \tan^2 my_1} + \sqrt{1 + \tan^2 mx_2 + \tan^2 my_2}} \tag{4}$$

The angle between two lines becomes

$$\phi = \cos^{-1} \frac{\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 + \tan mp_1 \tan mp_2}{\sqrt{1 + \tan^2 mp_1} \quad \sqrt{1 + \tan^2 mp_2}} \tag{5}$$

In the Riemannian geometry, instead of x and y, we will use the new coordinates ξ and η, where

$$\xi = k \tan \frac{x}{k}, \quad \eta = k \tan \frac{y}{k} \tag{6}$$

and in the Lobachevskian geometry,

$$\xi = -ik \tan \frac{ix}{k} = k \tanh \frac{x}{k}, \eta = -ik \tan \frac{iy}{k} = k\tanh\frac{y}{k}. \tag{7}$$

(1) To obtain the equation of the straight line in the Lobachevskian geometry, substitute (7) in (3) above, and the equation becomes $\xi \cos\alpha + \eta \sin\alpha = k \tanh \frac{p}{k}$ (8)

This may be written $A\xi + B\eta + C = 0$, where (9)

$$\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}},\ \text{Sin}\,\alpha = \frac{B}{\sqrt{A^2 + B^2}},\ k \tanh \frac{p}{k} = \frac{-C}{\sqrt{A^2 + B^2}} \qquad (10)$$

(since $\sin^2\alpha + \cos^2\alpha = 1$, and in a right triangle whose sides are A and B, and whose hypotenuse is C, the above relations exist).

Now if p is real, $\tan \frac{p}{k} < 1$, and $\tanh^2 \frac{p}{k} < 1$, and from (9) $\frac{C^2}{k^2(A^2 + B^2)} < 1$, $C^2 < k^2(A^2 + B^2)$.

Conversely, equation (9) represents a straight line provided $C^2 < k^2(A^2 + B^2)$, for then $\alpha = \cos^{-1} \frac{A}{\sqrt{A^2 + B^2}}$ and

$$p = k \tan^{-1} \frac{-C}{\sqrt{A^2 + B^2}}.$$

2. To obtain the formula for the distance between points P_1 and P_2 in the Lobachevskian geometry, substitute $m = \frac{1}{k}$ in (4) above, and we have

$$\cos \frac{1P_1P_2}{k} = \frac{1 + \tan \frac{ix_1}{k} + \tan \frac{ix_2}{k} + \tan \frac{iy_1}{k} \tan \frac{iy_2}{k}}{\sqrt{1 + \tan^2 \frac{ix_1}{k} + \tan^2 \frac{iy_1}{k}}\ \sqrt{1 + \tan^2 \frac{ix_2}{k} + \tan \frac{iy_2}{k}}}$$

$$\cosh \frac{P_1P_2}{k} = \frac{-K^2 + \xi_1\xi_2 + \eta_1\eta_2}{\sqrt{-K^2 + \xi_1^2 + \eta_1^2}\ \sqrt{-K^2 + \xi_2^2 + \eta_2^2}}$$

$$\cosh \frac{P_1P_2}{k} = \frac{-(K^2 - \xi_1\xi_2 - \eta_1\eta_2)}{i\sqrt{K^2 - \xi_1^2 - \eta_1^2}\ i\sqrt{K^2 - \xi_2^2 - \eta_2^2}}$$

$$\cosh \frac{P_1P_2}{k} = \frac{K^2 - \xi_1\xi_2 - \eta_1\eta_2}{\sqrt{K^2 - \xi_1^2 - \eta_1^2}\ \sqrt{K^2 - \xi_2^2 - \eta_2^2}} \qquad (11)$$

If in equation (11) we place $\xi_1 = \xi$, $\eta_1 = \eta$, $\xi_2 = \xi + d\xi$, $\eta_2 = \eta + d\eta$, and $P_1P_2 = ds$, it becomes, as far as infinitesimals of the second order are concerned:

$\cosh \frac{P_1P_2}{k}$ becomes $\cosh \frac{ds}{k} = 1 + \frac{1}{2} \frac{(ds)^2}{k^2} +$ (higher order) ....

Then $\dfrac{K^2-\xi_1\xi_2-\eta_1\eta_2}{\sqrt{K^2-\xi_1^2-\eta_1^2}\sqrt{K^2-\xi_2^2-\eta_2^2}} = \dfrac{K^2-\xi(\xi+d\xi)-\eta(n+dn)}{\sqrt{K^2-\xi^2-\eta^2}\sqrt{K^2-(\xi+d\xi)^2-(n+dn)^2}}$

Then $1+\frac{1}{2}\frac{(ds)^2}{K^2}+\cdots = \dfrac{(K^2-\xi^2-\eta^2)-(\xi d\xi+ndn)}{\sqrt{K^2-\xi^2-\eta^2}\sqrt{(K^2-\xi^2-\eta^2)-2(\xi d\xi+nd\eta)-(d\xi^2+d\eta^2)}}$

For convenience, let $(k^2-\xi^2-\eta^2) = u$, $(\xi d\xi+nd\eta) = v$, $(d\xi^2+d\eta^2) = w$. Then

$$1+\frac{1}{2}\frac{(ds)^2}{k^2}+\cdots = \frac{u-v}{\sqrt{u}\sqrt{u-2v-w}}+\cdots = \sqrt{\frac{(u-v)^2}{u(u-2v-w)}}+\cdots = \sqrt{\frac{u^2-2uv+v^2}{u^2-2uv-uw}}$$

$$= \left[1+\frac{v^2+uw}{u^2-2uv-uw}\right]^{\frac{1}{2}}+\cdots\cdots$$

But v^2 is an infinitesimal of second order; uw is also an infinitesimal of second order; u^2 is finite. Therefore the fraction is $\frac{\text{an inf}}{\text{a finite}}$. and is < 1, so we may expand the expression by the binomial theorem. Thus

$$1+\frac{1}{2}\frac{(ds)^2}{k^2}+\cdots\cdot = 1 + \frac{1}{2}\cdot 1^{-\frac{1}{2}}\left(\frac{v^2+uw}{u^2-2uv-uw}\right)+\overset{\text{(higher order)}}{\cdots\cdots\cdots\cdots}$$

$$= 1+\tfrac{1}{2}(v^2+uw)(u^2-2uv-uw)^{-1}-\cdots\cdots\cdots\cdots$$

Again, the expression $(u^2-2uv-uw)^{-1}$ may be expanded, (since u^2 is finite, and $(2uv+uw)$ is an infinitesimal of second order, and $< u^2$), giving $(u^2)^{-1}-(-1)(u^2)^{-2}(2uv+uw)+\ldots$ Thence $1+\frac{1}{2}\frac{(ds)^2}{k^2} = 1+\frac{1}{2}(v^2+uw)\left(\frac{1}{u^2}+\frac{2uv-\cancel{uw}}{\cancel{u^4}}\right)-\cdots\cdots\cdots)$

$$\frac{(ds)^2}{k^2} = \frac{v^2+uw}{u^2} = \frac{(\xi d\xi+ndn)^2+(K^2-\xi^2-n^2)(d\xi^2+d\eta^2)}{(K^2-\xi^2-\eta^2)^2}$$

$$= \frac{\xi^2d\xi^2+2\xi d\xi\cdot ndn+n^2dn^2+K^2(d\xi^2+d\eta^2)-\xi^2d\xi^2-\xi^2dn^2-n^2d\xi^2-n^2dn^2}{(K^2-\xi^2-\eta^2)^2}$$

$$= \frac{K^2(d\xi^2+d\eta^2)-(\eta^2d\xi^2-2nd\xi\cdot\xi dn+\xi^2dn^2)}{(K^2-\xi^2-\eta^2)^2}$$

$$= \frac{\kappa^2(d\xi^2+d\eta^2)-(\eta d\xi - \xi d\eta)^2}{(\kappa^2-\xi^2-\eta^2)^2}$$

Hence $$ds = \frac{\kappa\sqrt{\kappa^2(d\xi^2+d\eta^2)-(\eta d\xi-\xi d\eta)^2}}{(\kappa^2-\xi^2-\eta^2)^2} \tag{12}$$

In terms of polar coordinates, $\xi = k \cdot \tanh\frac{p}{k} \cos\theta$; $\eta = k \tanh\frac{p}{k} \sin\theta$.

$$d\xi = \cancel{k} \cdot \operatorname{sech}\frac{p}{k} \cdot \frac{1}{\cancel{k}} dp \cdot \cos\theta - k \cdot \tanh\frac{p}{k} \cdot \sin\theta \cdot d\theta$$

$$d\eta = \cancel{k} \cdot \operatorname{sech}\frac{p}{k} \cdot \frac{1}{\cancel{k}} dp \cdot \sin\theta - k \cdot \tanh\frac{p}{k} \cdot \cos\theta \cdot d\theta$$

$$d\xi^2 = \operatorname{sech}^4\tfrac{p}{\kappa} dp^2 \cdot \cos^2\theta - 2\kappa \cdot \operatorname{sech}^2\tfrac{p}{\kappa} dp \cdot \tanh\tfrac{p}{\kappa} \cdot \sin\theta \cdot \cos\theta d\theta + \kappa^2\tanh^2\tfrac{p}{\kappa} \cdot \sin^2\theta d\theta^2$$

$$d\eta^2 = \operatorname{sech}^4\tfrac{p}{\kappa} dp^2 \cdot \sin^2\theta + 2\kappa \operatorname{sech}^2\tfrac{p}{\kappa} dp \cdot \tanh\tfrac{p}{\kappa} \cdot \sin\theta \cdot \cos\theta d\theta + \kappa^2\tanh^2\tfrac{p}{\kappa} \cos^2\theta d\theta^2$$

Add; $d\xi^2 + d\eta^2 = \operatorname{sech}^4\frac{p}{\kappa} dp^2 + \kappa^2\tanh^2\frac{p}{\kappa} d\theta^2$

$$\eta d\xi = \kappa\tanh\tfrac{p}{\kappa} \cdot \operatorname{sech}^2\tfrac{p}{\kappa} dp \cdot \sin\theta \cdot \cos\theta - \kappa^2\tanh^2\tfrac{p}{\kappa} \sin^2\theta d\theta$$

$$\xi d\eta = \kappa\tanh\tfrac{p}{\kappa} \cdot \operatorname{sech}^2\tfrac{p}{\kappa} dp \cdot \sin\theta \cos\theta + \kappa^2\tanh^2\tfrac{p}{\kappa} \cos^2\theta d\theta$$

Subtract; $\eta d\xi - \xi d\eta = -k^2\tanh^2\frac{p}{k} d\theta$

$(\eta d\xi - \xi d\eta) = k^4\tanh^4\frac{p}{k} d\theta^2$. $\xi^2 = k^2\tanh^2\frac{p}{k} \cos^2\theta$, $\eta^2 = k^2\tanh^2\frac{p}{k} \sin^2\theta$; $(\xi^2 + \eta^2) = k^2\tanh^2\frac{p}{k}$, $k^2 - \xi^2 - \eta^2 = k^2 - k^2\tanh^2\frac{p}{k} = k^2\operatorname{sech}^2\frac{p}{k}$.

$$ds = \frac{\kappa\sqrt{\kappa^2(\operatorname{sech}^4\frac{p}{\kappa}dp^2+\kappa^2\tanh^2\frac{p}{\kappa}d\theta^2)^2 - \kappa^4\tanh^4\frac{p}{\kappa}d\theta^2}}{\kappa^2\operatorname{sech}^2\frac{p}{\kappa}}$$

$$= \frac{\cancel{\kappa^2}\sqrt{\operatorname{sech}^4\frac{p}{\kappa}dp^2+\kappa^2\tanh^2\frac{p}{\kappa}d\theta^2[1-\tanh^2\frac{p}{\kappa}]}}{\cancel{\kappa^2}\operatorname{sech}^2\frac{p}{\kappa}}$$

$$= \sqrt{\frac{sech^4\frac{p}{k}dp+k^2\sinh^2\frac{p}{k}d\theta^2 sech^2\frac{p}{k}\sec^2\frac{p}{k}}{\sec^4\frac{p}{k}}}$$

$= \sqrt{dp^2 + k^2\sin^2\frac{p}{k}\,d\theta^2}$; whence the circumference (13) of the circle $p = a$, since $da^2 = 0$, is $C = \int ds = \int_0^{2\pi} k \sinh \frac{a}{k}\, d\theta$ $= k \sinh \frac{a}{k}\int_0^{2\pi} d\theta = k \sinh \frac{a}{k}(2\pi - 0)$. $\therefore C = 2\pi k \cdot \sinh \frac{a}{k}$, but $\sinh \frac{a}{k} = \frac{e^{\frac{a}{k}} - e^{-\frac{a}{k}}}{2}$;

Thence $C = k\pi(e^{\frac{a}{k}} - e^{-\frac{a}{k}})$.

3. To apply the formula for the angle $\emptyset$ between two straight lines to the Lobachevskian geometry, in (5) substitute $\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}$, $m = \frac{1}{k}$;

Thence

$$\cos\emptyset = \frac{\frac{A_1A_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}} + \frac{B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}} + \tan\frac{ip_1}{k}\tan\frac{ip_2}{k}}{\sqrt{1+\tan^2\frac{ip_1}{k}}\sqrt{1+\tan^2\frac{ip_2}{k}}}$$

Multiply both numerator and denominator by $(-ik)$,

$$\cos\emptyset = \frac{\frac{-k^2(A_1A_2+B_1B_2)}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}} + (-ik\tan\frac{ip_1}{k})(-ik\tan\frac{ip_2}{k})}{\sqrt{-k^2+(-ik\tan\frac{ip_1}{k})^2}\sqrt{-k^2+(-ik\tan\frac{ip_2}{k})^2}}$$

But $(-ik \cdot \tan\frac{ip}{k}) = k\tanh\frac{p}{k} = \frac{-C}{\sqrt{A^2 + B^2}}$.

$$\text{so } \cos\emptyset = \frac{\frac{-k^2(A_1A_2+B_1B_2)}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}} + \frac{C_1C_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}}{\sqrt{-k^2+\frac{C_1^2}{A_1^2+B_1^2}}\sqrt{-k^2+\frac{C_2^2}{A_2^2+B_2^2}}}$$

$$= \frac{\frac{-k^2(A_1A_2+B_1B_2)+C_1C_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}}{\frac{1}{\sqrt{A_1^2+B_1^2}}\sqrt{k^2(A_1^2+B_1^2)-C_1^2}\cdot\frac{1}{\sqrt{A_2^2+B_2^2}}\sqrt{k^2(A_2^2+B_2^2)-C_2^2}}$$

$$\therefore \cos\emptyset = \frac{k^2(A_1A_2+B_1B_2) \mp C_1C_2}{\sqrt{k^2(A_1^2+B_1^2)-C_1^2}\sqrt{k^2(A_2^2+B_2^2)-C_2^2}} \tag{14}$$

In the Riemannian geometry, the equation for the straight line becomes $\cos - \sin = k \tan \frac{p}{k}$; (15)

for distance between points

$$\cos \frac{PP}{K} = \frac{K^2 + \xi_1\xi_2 + \eta_1\eta_2}{\sqrt{K^2+\xi_1^2+\eta_1^2}\ \sqrt{K^2+\xi_2^2+\eta_2^2}} \tag{16}$$

and for the angle ϕ, $\cos \phi = \frac{k^2(A_1A_2 + B_1B_2 + C_1C_2)}{\sqrt{k^2(A_1^2+B_1^2)+C_1^2}\sqrt{k^2(A_2^2+B_2^2)+C_2^2}}$ (17)

SUMMARY OF FORMULAS

1. In an I.B.Q. whose summit is CD, and base, AB, and whose side AC is a given length "a", $\lim \frac{CD}{AB} = \cos ma$, hence in L.G. where $m = \frac{i}{k}$, $\lim \frac{CD}{AB} = \cos \frac{ia}{k} = \cosh \frac{a}{k}$. (1)

2. If ABC be a triangle with a right angle at C, and $AB = c$, $AC = b$, $BC = a$, for the L.G. we have

$$\cosh \frac{c}{k} = \cosh \frac{a}{k} \cosh \frac{b}{k} \tag{2}$$

$$\sinh \frac{a}{k} = \sinh \frac{c}{k} \sin A \tag{3}$$

$$\tanh \frac{a}{k} = \tanh \frac{c}{k} \cos B \tag{4}$$

$$\cos A = \cosh \frac{a}{k} \sin B \tag{5}$$

$$\sinh \frac{b}{k} = \sinh \frac{c}{k} \sin B \tag{6}$$

$$\tanh \frac{b}{k} = \tanh \frac{c}{k} \cos A \tag{7}$$

$$\cos B = \cosh \frac{b}{k} \sin A \tag{8}$$

3. If ABC be any triangle with vertices A, B, and C, and the opposite sides a, b, and c, respectively,

$$\cos ma = \cos mc \cdot \cos mb - \sin mb \cdot \sin mc \cdot \cot A,$$

hence in the L.G. where $m = \frac{i}{k}$,

$$\cosh \frac{a}{k} = \cosh \frac{c}{k} \cdot \cosh \frac{b}{k} \cdot \cosh \frac{b}{k} - \sinh \frac{c}{k} \cdot \cot A. \quad (9)$$

4. If we use new coordinates ξ and η, in place of x and y, where $\xi = -ik \tan \frac{ix}{k} = k \tanh \frac{x}{k}$, and $\eta = -ik \tan \frac{iy}{k}$ $= k \tanh \frac{y}{k}$, the equation of the straight line in L.B. becomes

$$\xi \cos\alpha + \eta \sin\alpha = k \tanh \frac{p}{k} \quad (10)$$

$$\text{or } a\xi + b\eta + c = 0 \quad (11)$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, and

$$k \tanh \frac{p}{k} = \frac{-c}{\sqrt{a^2+b^2}}. \quad (12)$$

5. Distance between points $P_1 : (\xi_1, \eta_1)$ and $P_2 : (\xi_2, \eta_2)$ is measured by $\cosh \frac{P_1P_2}{k} = \frac{K^2 - \xi_1\xi_2 - \eta_1\eta_2}{\sqrt{K^2 - \xi_1^2 - \eta_1^2}\,\sqrt{K^2 - \xi_2^2 - \eta_2^2}}$ (13)

6. The angle $\emptyset$ between two lines is measured by

$$\cos \emptyset = \frac{K^2(a_1a_2 + b_1b_2) - c_1c_2}{\sqrt{K^2(a_1^2 + b_1^2) - c_1^2}\,\sqrt{K^2(a_2^2 + b_2^2) - c_2^2}} \quad (14)$$

7. Let $P(\xi, \eta)$ be any point on a Lobachevskian plane, (r, θ) its polar coordinates, where r is always positive.

Then

$$\left.\begin{aligned} \xi &= k \tanh \frac{r}{k} \cos \theta \\ \eta &= k \tanh \frac{r}{k} \sin \theta \\ \xi^2 + \eta^2 &= k^2 \tanh^2 \frac{r}{k} < k^2 \end{aligned}\right\} \quad (15)$$

ξ and η may be taken as ordinary Cartesian coordinates upon a Euclidean plane, i.e.; a plane on which the Euclidean geometry is assumed to hold. Then to P on the Lobachevskian plane corresponds a point P' on the Euclidean plane, and P' lies inside the circle $\xi^2 + \eta^2 = k$, called the fundamental circle.

Conversely, if (ξ, η) be the coordinates of any point on the Euclidean plane,

$$\left.\begin{aligned} \cos \theta &= \frac{\xi}{\sqrt{\xi^2 + \eta^2}} \\ \sin \theta &= \frac{\eta}{\sqrt{\xi^2 + \eta^2}} \\ r &= \frac{k}{2} \log \frac{k + \sqrt{\xi^2 + \eta^2}}{k - \sqrt{\xi^2 + \eta^2}} \end{aligned}\right\} \qquad (16)$$

SOME APPLICATIONS OF THE LOBACHEVSKIAN GEOMETRY

1. Conic Sections: The path of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line is called a conic section. The fixed point is called the focus of the conic, the fixed line the directrix, and the constant ratio the eccentricity, (e).

If $e < 1$, the conic is an ellipse; if $e = 1$, the conic is a parabola, and if $e > 1$, the conic is a hyperbola.

General equation; Let the focus of a conic be (i, j)

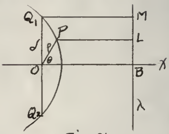


Fig. 21

and the directrix (λ) the line $x \cos\beta + y \sin\beta = p$ (Fig. 21). The distance of the moving point (x, y) from the line (λ) is $x \cos\beta + y \sin\beta - p$. Hence the equation of the conic is $\sqrt{(x - i)^2 + (y - j)^2} = e(x \cos\beta + y \sin\beta - p)$

or $\quad (x - i)^2 + (y - j)^2 = e^2(x \cos\beta + y \sin\beta - p)^2.$

If in this equation we substitute for x, y, and p the values ξ, η, and $k \tanh \frac{p}{k}$ respectively the equation becomes, in the L.G.

$$(\xi - i)^2 + (\eta - j)^2 = e^2(\xi \cos\beta + \eta \sin\beta - k \tanh \frac{p}{k})^2 \quad (1)$$

Polar equation; To find the equation of a conic in polar coordinates, let us take the focus as pole, and the line through the focus perpendicular to the directrix as polar axis. (Fig. 21) Denote by 2δ the length of Q_1Q_2 (the latus rectum); then the distance OB from the focus to the directrix λ is $\frac{\delta}{e}$, for the point Q_1 is a point on the curve. Hence by the definition

$$OQ_1 = e \cdot Q_1M$$

so that
$$OB = Q_1M = \frac{OQ_1}{e} = \frac{\delta}{e}.$$

Now assume a point $P(r, \theta)$ in a general position on the curve, and drop a perpendicular PL to the directrix.

Since $\qquad OP = e \cdot PL, \quad PL = \frac{r}{e}$,

Evidently $\qquad ON + PL = OB,$

or $\quad r \cos \theta + \frac{r}{e} = \frac{d}{e}, \quad$ or $r = \dfrac{d}{1 + e \cos \theta}$.

Making substitutions in the above from formula (16) of the summary, we have in L.G. the polar equation of the conic; $\frac{k}{2} \log \dfrac{k + \sqrt{\xi^2 + \eta^2}}{k - \sqrt{\xi^2 + \eta^2}} = \dfrac{d}{1 + \dfrac{e\xi}{\sqrt{\xi^2+\eta^2}}}.$

$$\text{or } \frac{k}{2} \log \frac{k + \sqrt{\xi^2+\eta^2}}{k - \sqrt{\xi^2+\eta^2}} = \frac{d\sqrt{\xi^2+\eta^2}}{\sqrt{\xi^2+\eta^2} + e\xi} \qquad (2)$$

2. The ellipse; The ellipse has been defined as the conic section for which $e < 1$. To obtain the cartesian equation of the curve, let us take as focus (Fig. 22) the point $(ae, 0)$, and as directrix the line $x = \frac{a}{e}$ (the point F_1 and line of Fig. 22) where a is a constant. With these assumptions the defining equation $p = ed$ becomes

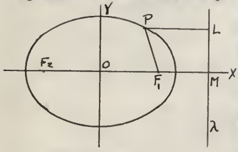


Fig. 22

$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right),$$

$$\text{or} \quad x - \cancel{2aex} + a^2e^2 + y^2 = a^2 - \cancel{2aex} + e^2x^2,$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad \text{Let } b^2 = a^2(1 - e^2),$$

and the equation becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Since $e < 1$, b^2 is positive, and b is real).

To obtain the equation of the ellipse in L.G., substitute in the above, ξ and η respectively for x and y, where $\xi = k \tanh \frac{x}{k}$, and $\eta = k \tanh \frac{y}{k}$, and the equation becomes

$$\frac{k^2 \tanh^2 \frac{x}{k}}{a^2} + \frac{k^2 \tanh^2 \frac{y}{k}}{b^2} = 1$$

or $k^2 (b^2 \tanh^2 \frac{x}{k} + a^2 \tanh^2 \frac{y}{k}) = a^2 b^2$ (3)

Center at a point (m,n); The equation

$$\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} = 1$$

becomes in L.G. $\frac{(k \tanh \frac{x}{k} - m)^2}{a^2} + \frac{(k \tanh \frac{y}{k} - n)^2}{b^2} = 1.$ (4)

Polar form; Where $P = \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $\frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} = 1.$

If (ξ , η) be the coordinates of the point P on the Euclidean plane, by formulas (16) the equation becomes

$$\frac{\left[\frac{k}{2} \log \frac{k + \sqrt{\xi^2+\eta^2}}{k - \sqrt{\xi^2+\eta^2}}\right]^2 \cdot \frac{\xi^2}{\xi^2+\eta^2}}{a} + \frac{\left[\frac{k}{2} \log \frac{k - \sqrt{\xi^2+\eta^2}}{k - \sqrt{\xi^2+\eta^2}}\right]^2 \cdot \frac{\xi^2}{\xi^2+\eta^2}}{b^2} = 1,$$

or $\frac{\left[\frac{k}{2} \log \frac{k - \sqrt{\xi^2+\eta^2}}{k - \sqrt{\xi^2+\eta^2}}\right]^2}{\xi^2 + \eta^2} \cdot \left[\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}\right] = 1$ (5)

Tangent to the ellipse at a given point of contact: In the Euclidean geometry, if the given point is (x_1, y_1) the tangent at this point on the curve is found by the formula $m = -\frac{b^2 x_1}{a^2 y_1}$. In L.G. the corresponding formula becomes $m = -\frac{b^2 \xi_1}{a^2 \eta_1}$, and the equation of the tangent is

$$\frac{\xi_1 \xi}{a^2} + \frac{\eta_1 \eta}{b^2} = 1 \qquad (6)$$

Normal to the ellipse at a given point; Since the slope of the normal at any point is the negative reciprocal of the slope of the tangent, for the normal, at the point (x_1, y_1) the slope $m_1 = \frac{a^2 y_1}{b^2 x_1}$, and the equation of the normal becomes in L. G. $\eta - \eta_1 = \frac{a^2 \eta_1}{b^2 \xi_1} (\xi - \xi_1)$. (7)

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