Linear Ramsey numbers

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Abstract

The Ramsey number $R_X(p,q)$ for a class of graphs X is the minimum n such that every graph in X with at least n vertices has either a clique of size p or an independent set of size q. We say that Ramsey number is *linear* in X if there is a constant k such that $R_X(p,q) \leq k(p+q)$ for all p,q. In the present paper we conjecture that Ramsey number is linear in X if and only if the co-chromatic number is bounded in X and prove a number of results supporting the conjecture.

1 Introduction

According to Ramsey's Theorem [5] for all natural p and q there exists a minimum number $R(p,q)$ such that every graph with at least $R(p,q)$ vertices has either a clique of size p or an independent set of size q.

The exact values of Ramsey numbers are known only for small values of p and q. However, with the restriction to specific classes of graphs, Ramsey numbers can be determined for all p and q. In particular, in [6] this problem was solved for planar graphs, while in [1] it was solved for line graphs, bipartite graphs, perfect graphs, P_4-free graphs and some other classes.

We denote the Ramsey number restricted to a class X by $R_X(p,q)$ and focus in the present paper on classes with a smallest speed of growth of $R_X(p,q)$. Clearly, $R_X(p,q)$ cannot be smaller than the maximum of p and q. We say that Ramsey number is *linear* in X if there is a constant k such that $R_X(p,q) \leq k(p+q)$ for all p,q.

It is not difficult to see that all classes of bounded co-chromatic number have linear Ramsey number, where the co-chromatic number of a graph G is the minimum k such that the vertex set of G can be partitioned into k subsets each of which is either a clique or an independent set. We conjecture that in the universe of hereditary classes of graphs the two notions coincide.

Conjecture 1. A hereditary graph class is of linear Ramsey number if and only if it is of bounded co-chromatic number.

A class of graphs is *hereditary* if it is closed under taking induced subgraphs. It is well known that a class of graphs is hereditary if and only if it can be characterized in terms of minimal

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forbidden induced subgraphs. Of particular interest in the present paper are *finitely defined* classes, i.e. classes defined by finitely many forbidden induced subgraphs.

In [2], it was conjectured that a finitely defined class X has bounded co-chromatic number if and only if the set of minimal forbidden induced subgraphs for X contains a P_3-free graph, the complement of a P_3-free graph, a forest and the complement of a forest. Following this conjecture, we propose a restriction of our Conjecture 1 to the case of finitely defined classes as follows.

Conjecture 2. A finitely defined class X is of linear Ramsey number if and only if the set of minimal forbidden induced subgraphs for X contains a P_3-free graph, the complement of a P_3-free graph, a forest and the complement of a forest.

In the present paper, we prove the "only if" part of the conjecture (Section 2) and determine Ramsey numbers for several classes of graphs that verify the "if" part of the conjecture (Section 3). In the rest of the present section, we introduce basic terminology and notation.

All graphs in this paper are finite, undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $x \in V(G)$ we denote by $N(x)$ the neighbourhood of x, i.e. the set of vertices of G adjacent to x. A subgraph of G induced by a subset of vertices $U \subseteq V(G)$ is denoted $G[U]$. By $\overline{G}$ we denote the complement of G and call it co-G.

A *clique* in a graph is a subset of pairwise adjacent vertices and an *independent set* is a subset of pairwise non-adjacent vertices.

By K_n, C_n and P_n we denote the complete graph, the chordless cycle and the chordless path with n vertices, respectively. Also, $G + H$ denotes the disjoint union of two graphs G and H. In particular, pG is the disjoint union of p copies of G. A star is a connected graph in which all edges are incident to the same vertex.

If a graph G does not contain induced subgraphs isomorphic to a graph H, then we say that G is H-free and call H a forbidden induced subgraph for G.

2 Classes with non-linear Ramsey number

In this section, we prove the "only if" part of Conjecture 2.

Lemma 1. *For every fixed k, the class X_k of $(C_3, C_4, \dots, C_k)$-free graphs is not of linear Ramsey number.*

Proof. Assume to the contrary that the Ramsey number for the class X_k is linear. Then there must exist a constant $t = t(k)$ such that any n-vertex graph from the class has an independent set of size at least n/t.

In 1959, Erdős proved (see e.g. Theorem 11.2.2 in [4]) that X_k contains graphs of chromatic number at least k, and this proof implies that X_k contains n-vertex graphs with the independence number of order $O(n^{1-\epsilon} \ln(n))$, where ϵ depends on k, which is smaller than n/t for large n. This contradiction shows that X_k is not of linear Ramsey number. $\square$

Theorem 1. *Let X be a class of graphs defined by a finite set M of forbidden induced subgraph. If M does not contain a graph in at least one of the following four classes, then X is not of linear Ramsey number: P_3-free graphs, the complements of P_3-free graphs, forests, the complements of a forests.*

Proof. It is not difficult to see that a graph is P_3-free if and only if it is a disjoint union of cliques. The class of P_3-free graphs contains the graph $(q-1)K_{p-1}$ with $(q-1)(p-1)$ vertices with no clique of size p or independent set if size q, and hence this class is not of linear Ramsey number. Therefore, if M contains no P_3-free graph, then X_k contains all P_3-free graphs and hence is not of linear Ramsey number. Similarly, if M contains no $\overline{P}_3$-free graph, then X_k is not of linear Ramsey number.

Now assume that M contains no forest. Therefore, every graph in M contains a cycle. Since the number of graphs in M is finite, X contains the class of $(C_3, C_4, \ldots, C_k)$-free graphs for a finite value of k and hence is not of linear Ramsey number by Lemma 1. Applying the same arguments to the complements of graphs in X_k, we conclude that if M contains no complement of a forest, then X_k is not of linear Ramsey number. □

3 Classes with linear Ramsey number

In this section, we study classes of graphs defined by forbidden induced subgraphs with 4 vertices and determine Ramsey numbers for several classes in this family that verify the "if" part of Conjecture 2. All the eleven graphs on 4 vertices are represented in Figure 1.

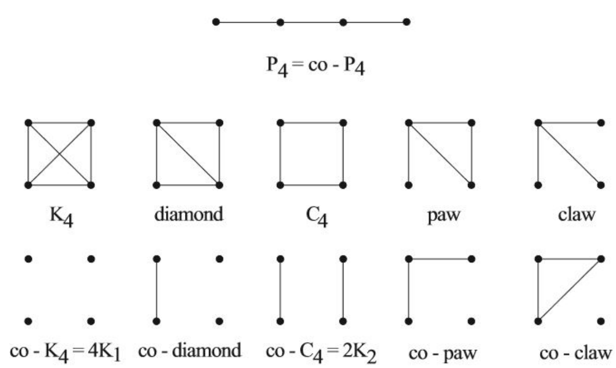

Figure 1: All 4-vertex graphs

Below we list which of these graphs are P_3-free and which of them are forests (take the complements for $\overline{P}_3$-free graphs and for the complements of forests, respectively).

- P_3-free graphs: K_4, $\overline{K}_4$, $2K_2$, co-diamond, co-claw.
- Forests: $\overline{K}_4$, $2K_2$, P_4, co-diamond, co-paw, claw.

3.1 Claw- and co-claw-free graphs

Lemma 2. *If a (claw,co-claw)-free graph G contains a $\overline{K}_4$, then it is K_3-free.*

Proof. Assume G contains a $\overline{K}_4$ induced by $A = \{a_1, a_2, a_3, a_4\}$ and suppose by contradiction that G also contains a K_3 induced by $Z = \{x, y, z\}$.

Let first A be disjoint from Z. To avoid a co-claw, each vertex of A has a neighbour in Z and hence one of the vertices of Z is adjacent to two vertices of A, say x is adjacent to a_1 and a_2. Then, to avoid a claw, x has no other neighbours in A and y has a neighbour in $\{a_1, a_2\}$, say y is adjacent to a_1. This implies that y is adjacent to a_3 (else x, y, a_1, a_3 induce a co-claw) and similarly y is adjacent to a_4. But then y, a_1, a_2, a_3 induce a claw, a contradiction.

If A and Z are not disjoint, they have at most one vertex in common, say $a_4 = z$. Again, to avoid a co-claw, each vertex in $\{a_1, a_2, a_3\}$ has a neighbour in $\{x, y\}$ and hence, without loss of generality, x is adjacent to a_1 and a_2. But then x, a_1, a_2, a_4 induce a claw, a contradiction again. □

Lemma 3. *The maximum number of vertices in a (claw,co-claw,K_4,$\overline{K}_4$)-free graph is 9.*

Proof. Let G be a (claw,co-claw,K_4,$\overline{K}_4$)-free graph and let x be a vertex of G. Denote by A the set of neighbours and by B the set of non-neighbours of x. Clearly, A contains neither triangles nor anti-triangles, since otherwise either a K_4 or a claw arises. Therefore, A has at most 5 vertices, and similarly B has at most 5 vertices.

If $|A| = 5$, then $G[A]$ must be a C_5 induced by vertices, say, a_1, a_2, a_3, a_4, a_5 (listed along the cycle). In order to avoid a claw or K_4, each vertex of A can be adjacent to at most 2 vertices of B, which gives rise to at most 10 edges between A and B. On the other hand, to avoid a co-claw, each vertex of B must be adjacent to at least 3 vertices of A. Therefore, B contains at most 3 vertices and hence $|V(G)| \leq 9$. Similarly, if $|B| = 5$, then $|V(G)| \leq 9$.

It remains to show that there exists a (claw,co-claw,K_4,$\overline{K}_4$)-free graph with 9 vertices. This graph can be constructed as follows. Start with a C_8 formed by the vertices $v_1, v_2, \ldots, v_8$. Then create a C_4 on the even-indexed vertices v_2, v_4, v_6, v_8 (listed along the cycle) and a $\overline{C}_4$ on the odd-indexed vertices v_1, v_3, v_5, v_7 (listed along the cycle in the complement). Finally, add one more vertex adjacent to the odd-indexed vertices. It is now a routine matter to check that the resulting graph is (claw,co-claw,K_4,$\overline{K}_4$)-free. □

Theorem 2. *For the class A of (claw,co-claw)-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = \max(\lfloor (5a - 3)/2 \rfloor, \lfloor (5b - 3)/2 \rfloor),$$

unless $a = b = 4$ in which case $R_A(a, b) = 10$.

Proof. According to Lemma 2, the class of (claw,co-claw)-free graphs is the union of three classes:

- the class X of (claw,K_3)-free graphs,
- the class Y of (co-claw,$\overline{K}_3$)-free graphs and
- the class Z of (claw,co-claw,K_4,$\overline{K}_4$)-free graphs.

Clearly, $R_A(a,b) = \max(R_X(a,b), R_Y(a,b), R_Z(a,b))$.

Since K_3 is forbidden in X, we have $R_X(a,b) = R_X(3,b)$, Also, denoting by B the class of claw-free graphs, we conclude that $R_X(3,b) = R_B(3,b)$. As was shown in [1], $R_B(3,b) = \lfloor (5b-3)/2) \rfloor$. Therefore, $R_X(a,b) = \lfloor (5b-3)/2) \rfloor$. Similarly, $R_Y(a,b) = \lfloor (5a-3)/2) \rfloor$.

In the class Z, for all $a, b \geq 4$ we have $R_Z(a,b) = 10$ by Lemma 3. Moreover, if additionally $\max(a,b) \geq 5$, then $R_Z(a,b) < \max(R_X(a,b), R_Y(a,b))$. For $a = b = 4$, we have $R_Z(4,4) = 10 > 8 = \max(R_X(4,4), R_Y(4,4))$. Finally, it is not difficult to see that $R_Z(3,b) \leq R_X(3,b)$ and $R_Z(a,3) \leq R_X(a,3)$, and hence the result follows. □

3.2 Diamond- and co-diamond-free graphs

Lemma 4. *If a (diamond,co-diamond)-free graph G contains a $\overline{K}_4$, then it is bipartite.*

Proof. Assume G contains a $\overline{K}_4$. Let A be any maximal (with respect to inclusion) independent set containing the $\overline{K}_4$ and let $B = V(G) - A$. If B is empty, then G is edgeless (and hence bipartite). Suppose now B contains a vertex b. Then b has a neighbour a in A (else A is not maximal) and at most one non-neighbour (else a and b together with any two non-neighbours of b in A induce a co-diamond).

Assume B has two adjacent vertices, say b_1 and b_2. Since $|A| \geq 4$ and each of b_1 and b_2 has at most one non-neighbour in A, there are must be at least two common neighbours of b_1 and b_2 in A, say a_1, a_2. But then a_1, a_2, b_1, b_2 induced a diamond. This contradiction shows that B is independent and hence G is bipartite. □

Lemma 5. *A co-diamond-free bipartite graph containing at least one edge is either a simplex (a bipartite graph in which every vertex has at most one non-neighbour in the opposite part) or a $K_{s,t} + K_1$ for some s and t.*

Proof. Assume $G = (A, B, E)$ is a co-diamond-free bipartite graph containing at least one edge. Then G cannot have two isolated vertices, since otherwise an edge together with two isolated vertices create an induced co-diamond.

Assume G has exactly one isolated vertex, say a, and let $G' = G - a$ Then any vertex $b \in V(G')$ is adjacent to every vertex in the opposite part of G'. Indeed, if b has a non-neighbour c in the opposite part, then a, b, c together with any neighbour of b (which exists because b is not isolated) induce a co-diamond. Therefore, G' is complete bipartite and hence $G = K_{s,t} + K_1$ for some s and t.

Finally, suppose G has no isolated vertices. Then every vertex $a \in A$ has at most one non-neighbour in B, since otherwise any two non-neighbours of a in B together with a and any neighbour of a (which exists because a is not isolated) induced a co-diamond. Similarly, every vertex $b \in B$ has at most one non-neighbour in A. Therefore, G is a simplex. □

Lemma 6. *The maximum number of vertices in a (diamond,co-diamond,$K_4, \overline{K}_4$)-free graph is 9.*

Proof. Let G be a (diamond,co-diamond,$K_4, \overline{K}_4$)-free graph and x be a vertex of G. Denote by A the set of neighbours and by B the set of non-neighbours of x. Then $G[A]$ is (P_3, K_3)-free, else G contains either a diamond or a K_4. Since $G[A]$ is P_3-free, every connected component of $G[A]$ is a clique and since this graph is K_3-free, every connected component has at most 2 vertices. If it least one of the components of $G[A]$ has 2 vertices, the number of components is at most 2 (since otherwise a co-diamond arises), in which case A has at most 4 vertices. If all the components of $G[A]$ have size 1, the number of components is at most 3 (since otherwise a $\overline{K}_4$ arises), in which case A has at most 3 vertices. Similarly, B has at most 4 vertices and hence $|V(G)| \leq 9$.

To conclude the proof, we observe that the graph on 9 vertices described in the proof of Lemma 3 is (diamond,co-diamond,$K_4, \overline{K}_4$)-free. □

Theorem 3. *For the class A of (diamond,co-diamond)-free graphs and $a, b \geq 3$,*

$$R_A(a, b) = \max(2a - 1, 2b - 1),$$

unless $a, b \in \{4, 5\}$, in which case $R_A(a, b) = 10$, and unless $a = b = 3$, in which case $R_A(a, b) = 6$.

Proof. According Lemma 4, in order to determine the value of $R_A(a, b)$, we analyze this number in three classes: the class X of co-diamond-free bipartite graphs, the class Y of the complements of graphs in X and the class Z of (diamond,co-diamond,$K_4, \overline{K}_4$)-free graphs (the classes of edgeless and complete graphs can obviously be ignored).

In the class X of co-diamond-free bipartite graphs, $R_X(a, b) = 2b-1$, since every graph in this class with at least $2b-1$ contains an independent set of size b, while the graph $K_{b-1,b-1}$ contains neither an independent set of size b nor a clique of size $a \geq 3$. Similarly, $R_Y(a, b) = 2a - 1$.

In the class Z of (diamond,co-diamond,$K_4, \overline{K}_4$)-free graphs, for all $a, b \geq 4$ we have $R_Z(a, b) = 10$ by Lemma 6. Moreover, if additionally $\max(a, b) \geq 6$, then $R_Z(a, b) < \max(R_X(a, b), R_Y(a, b))$. For $a, b \in \{4, 5\}$, we have $R_Z(a, b) = 10 > \max(R_X(a, b), R_Y(a, b))$. Also, $R_Z(3, 3) = 6$ (since $C_5 \in Z$) and hence $R_Z(3, 3) > \max(R_X(3, 3), R_Y(3, 3))$ Finally, by direct inspection one can verify that Z contains no K_3-free graphs with more than 6 vertices and hence for $b \geq 4$ we have $R_Z(3, b) \leq R_X(3, b)$. Similarly, for $a \geq 4$ we have $R_Z(a, 3) \leq R_Y(a, 3)$. Thus for all values of $a, b \geq 3$, we have $R_A(a, b) = \max(2a-1, 2b-1)$, unless $a, b \in \{4, 5\}$, in which case $R_A(a, b) = 10$, and unless $a = b = 3$, in which case $R_A(a, b) = 6$. □

3.3 $2K_2$- and C_4-free graphs

Theorem 4. *For the class A of ($2K_2, C_4$)-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = a + b.$$

Proof. Let G be a ($2K_2, C_4$)-free graph with $a + b$ vertices. If G is C_5-free, then it is a split graph and hence it contains either a clique of size a or an independent set of size b.

If G contains a C_5, then every vertex $u \notin V(C_5)$ is either complete to the C_5 or anticomplete to it. Indeed, assume that u is adjacent to $v_1 \in V(C_5)$ and a non-adjacent to $v_2 \in V(C_5)$. Then u is not adjacent to v_3 (else $G[u, v_1, v_2, v_3] = C_4$), adjacent to v_4 (else $G[u, v_1, v_3, v_4] = 2K_2$), adjacent to v_5 (else $G[u, v_3, v_4, v_5] = C_4$). But then $G[u, v_5, v_2, v_3] = 2K_2$. This contradiction shows that if u is adjacent to v_1, then it is also adjacent to v_2 and hence to v_3 and hence to v_4 and hence to v_5.

We denote by U the set of vertices complete to the cycle C_5 and by W the set of vertices anticomplete to the C_5. Then U is a clique, since otherwise a C_4 arises, and W is an independent set, since otherwise a $2K_2$ arises. We have $|U| + |W| = a + b - 5$ and hence either $|U| \geq a - 2$ or $|W| \geq b - 2$. In the first case, U together with any two adjacent vertices of the cycle C_5 create a clique of size a. In the second case, W together with any two non-adjacent vertices of the cycle create an independent set of size b. This shows that $R_A(a, b) \leq a + b$.

For the inverse inequality, we construct a graph G with $a + b - 1$ vertices as follows: G consists of a cycle C_5, an independent set W of size $b - 3$ anticomplete to the cycle and a clique U of size $a - 3$ complete to both W and $V(C_5)$. It is not difficult to see that the size of a maximum clique in G is $a - 1$ and the size of a maximum independent set in G is $b - 1$. Therefore, $R_A(a, b) \geq a + b$. □

3.4 $2K_2$- and diamond-free graphs

Lemma 7. *If a $(2K_2, diamond)$-free graph G contains a K_4, then G is a split graph partitionable into a clique C and an independent set I such that every vertex of I has at most one neighbour in C.*

Proof. Let G be a $(2K_2, diamond)$-free graph containing a K_4. We extend the K_4 to any maximal (with respect to inclusion) clique and denote it by C. Also, denote $I = V(G) - C$.

Assume a vertex $a \in I$ has two neighbours b, c in C. It also has a non-neighbour d in C (else C is not maximal). But then a, b, c, d induce a diamond. This contradiction shows that any vertex of I has at most one neighbour in C.

Finally, assume two vertices $a, b \in I$ are adjacent. Since each of them has at most one neighbour in C and $|C| \geq 4$, there are two vertices $c, d \in C$ adjacent neither to a nor to b. But then a, b, c, d induce a $2K_2$. This contradiction shows that I is independent and completes the proof. □

Lemma 8. *Let G be a $(2K_2, diamond, K_4)$-free graph containing a K_3. Then G is 3-colorable.*

Proof. Denote a triangle K_3 in G by $T = \{a, b, c\}$, and for any subset $U \subseteq \{a, b, c\}$ let V_U be the subset of vertices outside of T such that $N(v) \cap T = U$ for each $v \in V_U$. Then

- $V_{a,b,c} = \emptyset$, since G is K_4-free.
- $V_{a,b} = V_{ac} = V_{bc} = \emptyset$, since G is diamond-free.
- $V_a, V_b, V_c, V_\emptyset$ are independent sets, since G is $2K_2$-free. For the same reason, every vertex of $V_\emptyset$ is isolated.

Then each of the following three sets $\{a\} \cup V_b$, $\{b\} \cup V_c$ and $\{c\} \cup V_a \cup V_\emptyset$ is independent and hence G is 3-colorable. □

The above two lemmas reduce the analysis to $(2K_2, K_3)$-free graphs. In order to characterize this class, let us say that G^* is an extended G (also known as a blow-up of G) if G^* is obtained from G by replacing the vertices of G with independent sets.

Lemma 9. *If G is a $(2K_2, K_3)$-free graph, then it is either a chain graph or an extended $C_5 + K_1$.*

Proof. If G is C_5-free, then it is a $2K_2$-free bipartite graph, i.e. a chain graph. Assume now that G contains a C_5 induced by a set $S = \{v_0, v_1, v_2, v_3, v_4\}$. To avoid an induced $2K_2$ or K_3, any vertex $u \notin S$ must be either anticomplete to S or have exactly two neighbours on the cycle of distance 2 from each other, i.e. $N(u) \cap S = \{v_i, v_{i+2}\}$ for some i (addition is taken modulo 5). Moreover, if $N(u) \cap S = \{v_i, v_{i+2}\}$ and $N(w) \cap S = \{v_j, v_{j+2}\}$, then

- if $i = j$ or $|i - j| > 1$, then u is not adjacent to w, since G is K_3-free.
- if $|i - j| = 1$, then u is adjacent to w, since G is $2K_2$-free.

Clearly, every vertex $u \notin S$, which is anticomplete to S, is isolated, and hence G is an extended $C_5 + K_1$. □

Theorem 5. *Let A be the class of $(2K_2, diamond)$-free graphs. Then*

- *for $a = 3$, we have $R_A(a, b) = \lfloor 2.5(b - 1) \rfloor + 1$,*
- *for $a = 4$, we have $R_A(a, b) = 3b - 2$,*
- *for $a \geq 5$, we have $R_A(a, b) = 3b - 2$ if $a < 2b$ and $R_A(a, b) = a + b - 1$ if $a \geq 2b$.*

Proof. As before, we split the analysis into several subclasses of A.

For the class X of $(2K_2, diamond)$-free graphs containing a K_4 and $a \geq 5$, we have $R_X(a, b) = a + b - 1$. Indeed, every split graph with $a + b - 1$ vertices contains either a clique of size a or an independent set of size b and hence $R_X(a, b) \leq a + b - 1$. On the other hand, the split graph with a clique C of size $a - 1$ and an independent set I of size $b - 1$ with a matching between C and I belongs to X and hence $R_X(a, b) \geq a + b - 1$.

For the class Y of 3-colorable $(2K_2, diamond)$-free graphs and for $a \geq 4$ we have $R_Y(a, b) = 3b - 2$. Indeed, a 3-colorable graph with $3b - 2$ vertices contains an independent set of size b and hence $R_Y \leq 3b - 2$. On the other hand, consider the graph G constructed from $b - 1$ triangles $T_i = \{a_i, b_i, c_i\}$ $(i = 1, 2, \ldots, b - 1)$ such that for all $j > i$,

- a_i is adjacent to b_j,
- b_i is adjacent to c_j,
- c_i is adjacent to a_j.

It is not difficult to see that G is 3-colorable $(2K_2, diamond)$-free graph with $3b - 3$ vertices containing neither a clique of size $a \geq 4$ nor an independent set of size b. Therefore, $R_Y \geq 3b - 2$.

For the class Z_0 of chain graphs, we have $R_{Z_0}(a, b) = 2b - 1$, which is easy to see. Finally, in the class Z_1 of graphs each of which is an extended $C_5 + K_1$, we have $R_{Z_1}(a, b) = \lfloor 2.5(b-1) \rfloor + 1$. For an odd b, a maximum counterexample is constructed from a C_5 by replacing each vertex with an independent set of size $(b-1)/2$. This graph has $\lfloor 2.5(b-1) \rfloor$ vertices, the independence number $b - 1$ and the clique number $2 < a$. For an even b, a maximum counterexample is constructed from a C_5 by replacing two adjacent vertices of a C_5 with independent sets of size $b/2$ and the remaining vertices of the cycle with independent sets of size $b/2 - 1$. This again gives in total $\lfloor 2.5(b - 1) \rfloor$ vertices, and the independence number $b - 1$. Therefore, in the class $Z = Z_0 \cup Z_1$, we have $R_Z(a, b) = \max(R_{Z_0}(a, b), R_{Z_1}(a, b)) = \lfloor 2.5(b - 1) \rfloor + 1$.

Combining, we conclude that

- for $a = 3$, we have $R_A(a, b) = \lfloor 2.5(b-1) \rfloor + 1$,
- for $a = 4$, we have $R_A(a, b) = 3b - 2$,
- for $a \geq 5$, we have $R_A(a, b) = 3b - 2$ if $a < 2b$ and $R_A(a, b) = a + b - 1$ if $a \geq 2b$.

□

3.5 (P_4, C_4,co-claw)-free graphs

With start with a lemma characterizing the structure of graphs in this class, where we use the following well-known fact (see e.g. [3]): every P_4-free graph with at least two vertices is either disconnected or the complement to a disconnected graph.

Lemma 10. *Every disconnected (P_4, C_4,co-claw)-free graph is a collection of disjoint stars and every connected (P_4, C_4,co-claw)-free graph consists of a collection of disjoint stars plus a number of dominating vertices, i.e. vertices adjacent to all other vertices of the graph.*

Proof. Let G be a disconnected (P_4, C_4,co-claw)-free graph. Then every connected component of G is K_3-free, since a triangle in one of them together with a vertex from any other component create an induced co-claw.

Now let G be a connected graph. Since G is P_4-free, $\overline{G}$ is disconnected. Let $C^1, \ldots, C^k$ ($k \geq 2$) be co-components of G, i.e. components in the complement of G. If at least two of them have more than 1 vertex, then an induced C_4 arises. Therefore, all co-components, except possibly one, have size 1, i.e. they are dominating vertices in G. If, say, C^1 is a co-component of size more than 1, then the subgraph of G induced by C^1 must be disconnected and hence it is a collection of stars. □

Theorem 6. *For the class A of (P_4, C_4,co-claw)-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = a + 2b - 4.$$

Proof. Let G be a graph in A with $a + 2b - 5$ vertices, $2b - 2$ of which induce a matching (a 1-regular graph with $b-1$ edges) and the remaining $a-3$ vertices are dominating in G. Then G has neither a clique of size a nor an independent set of size b. Therefore, $R_A(a, b) \geq a + 2b - 4$.

Conversely, let G be a graph in A with $a + 2b - 4$ vertices. If G is disconnected, then it is bipartite and hence at least one part in a bipartition of G has size at least b, i.e. G contains an independent set of size b. If G is connected, denote by C the set of dominating vertices in G. If $|C| \geq a-1$, then either C itself (if $|C| \geq a$) or C together with a vertex not in C (if $|C| = a-1$) create a clique of size a. So, let $|C| \leq a - 2$. The graph $G - C$ is bipartite and has at least $2b - 2$ vertices. If this graph has no independent set of size b, then in any bipartition of this graph each part contains exactly $b-1$ vertices, and each vertex has a neighbour in the opposite part. But then $|C| = a - 2$ and therefore C together with any two adjacent vertices in $G - C$ create a clique of size a. □

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