Data-Driven Control of Linear Time-Varying Systems

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***Abstract*— An identification-free control design strategy for discrete-time linear time-varying systems with unknown dynamics is introduced. The closed-loop system (under state feedback) is parametrised with data-dependent matrices obtained from an ensemble of input-state trajectories collected offline. This data-driven system representation is used to classify control laws yielding trajectories which satisfy a certain bound and to solve the linear quadratic regulator problem - both using data-dependent linear matrix inequalities only. The results are illustrated by means of a numerical example.**

I. Introduction

Most classical contributions to modern control theory rely on the explicit knowledge of a plant model for controller design, e.g. a state-space model or transfer function. They hence fall into the category of *model-based control*. In practice, an accurate model of the system to be controlled is rarely known in advance. System models are typically derived from first principles or identified using data. Common system identification methods include, for example, prediction error and subspace methods [1]. Once a model describing the system behaviour is available, a control law can be designed in a separate step, using any "classical" technique.

Data-driven control methods, on the other hand, skip the modelling step altogether and aim to control the system directly from data. This does not only have theoretical value, but is also attractive for situations in which system identification can be difficult or time-consuming. The topic has recently attracted significant interest by the research community. In particular, with the availability of increasing computational power and novel machine learning techniques, model-free controllers using neural networks [2] and reinforcement learning [3], [4] have gained interest. However, learning from data is not a new concept in control theory and can be traced back to the work on PID tuning by Ziegler and Nichols [5] in the 1940s. Further contributions to data-driven control include model-free adaptive control [6], iterative feedback tuning [7], virtual reference feedback tuning [8] and unfalsified control [9]. For an overview of and more references on direct data-driven control the interested reader is referred to the survey papers [10] and [11], for instance.

A central question in data-driven control is how to substitute a system model with data. For linear time-invariant (LTI) systems Willems et al.'s *fundamental lemma* [12] gives an answer to that question. In brief, the result states that all possible trajectories an LTI system can produce can be parametrised by a single, finite-length input-output trajectory - provided the input sequence is exciting the system dynamics sufficiently, which is known as persistency of excitation. In [13] De Persis and Tesi use the fundamental lemma to derive a data-driven representation of LTI discrete-time systems in closed-loop with static state feedback, where the controller itself is parametrised using data only. The framework is used to formulate and solve the stabilisation and linear quadratic regulator (LQR) problems in terms of data-dependent linear matrix inequalities (LMIs). This approach is extended to the finite-horizon LQR problem in [14]. In [15] the notion of persistency of excitation is extended to multiple datasets, allowing to extract sufficient information from data that might be corrupted or missing samples. In [16] persistency of excitation is extended to certain classes of nonlinear systems. A data-enabled predictive control (DeePC) algorithm for unknown discrete-time LTI systems has been developed in [17]. Based on the fundamental lemma the open-loop dynamics and state estimate in an optimisation problem over the prediction horizon are replaced with a constraint containing only input-output data collected offline. In [18] it is shown that data-driven control and analysis are possible in certain cases in which unique system identification is not, because the data is not persistently exciting. These findings further emphasise the potential of direct data-driven control methods compared to the classical identification and model-based design approach.

Linear time-varying (LTV) systems appear in many real life applications, for instance due to changing aerodynamic coefficients in high-speed aircraft or changing parameters in electrical circuits or chemical plants. It is also common to treat nonlinear systems as LTV systems for controller design by linearising around a trajectory or time-varying operating points [19]. Consequently, increasing attention has been given to system identification of LTV systems. According to Kearney et al. [20] plant model identification techniques for time-varying systems can be divided into four groups: quasi-time-invariant methods, adaptive methods, temporal expansion methods and ensemble methods. While quasi-time-invariant methods are only applicable to slowly varying systems, adaptive and temporal expansion methods require prior knowledge of the system structure or the nature of the time-variation. Hence, those techniques only require a single time sequence of input-output data to identify the plant model. However, if the *a priori* information is imprecise or the time-variation occurs at a frequency similar

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to the sampling rate, the above methods have difficulties identifying accurate system models [21]. Ensemble methods do not rely on prior knowledge of the system structure or parameter variation and are hence the preferred choice for many applications. However, they require multiple input-output sequences capturing the same underlying time-varying behaviour. This enables the use of standard LTI identification techniques for each point in time across the ensemble of responses, see for example [20], [22], [21] and [23]. While there may be practical difficulties in obtaining ensemble data for some systems, there are a variety of applications for which such data can easily be gathered, examples include nonlinear systems linearised along a particular trajectory and periodically varying systems [21].

Direct data-driven control approaches for LTV systems are also available in the literature. In [24] a model-free control strategy for linear parameter-varying systems is introduced. A dual-loop iterative algorithm to solve the finite-horizon LQR problem for continuous-time LTV systems is presented in [25]. In [26] and [27] data-driven iterative extremum seeking approaches are used to find the optimal open-loop control sequence for unknown discrete-time LTV systems. In [28] a model-free approach using reinforcement learning is introduced for LTV systems to find approximate solutions to the finite-horizon LQR problem. Finally, Baros et al. [29] extend the DeePC method to capture changing system conditions by updating the data matrices online.

In this paper the direct data-driven control framework presented in [13] (for LTI systems) is extended to LTV systems. The closed-loop system is directly parametrised using data, skipping the explicit system identification step, and the control design does not require any iterative procedures or machine learning. While in [13] the closed-loop LTI system is parametrised using a single sufficiently long input-state data sequence, the approach presented herein uses multiple input-state sequences capturing the same time-varying behaviour, similar to what is common in ensemble methods for LTV system identification. At each time step input-state data from an ensemble of experiments is used to obtain a model-free parametrisation of the system in closed-loop with state feedback.

The remainder of this paper is organised as follows. In Section II some preliminaries are provided and notation used throughout the paper is defined. The data representation of the closed-loop LTV system under state feedback is introduced in Section III. In Section IV the data parametrisation is used to characterise - in terms of data-dependent LMIs - state feedback control laws which ensure the trajectories of the closed-loop system satisfy a certain bound throughout a finite time window of interest. In Section V the data parametrisation is used to formulate and solve the LQR problem - again by means of data-dependent LMIs. A numerical example is presented in Section VI, before some concluding remarks are provided in Section VII.

Notation. The set of real numbers is denoted by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of natural numbers by $\mathbb{N}$. I_n indicates the $n \times n$ identity matrix. Given a matrix A, $\mathbf{Tr}(A)$ denotes its trace. We write $A \succ 0$ ($A \succeq 0$) to denote that A is positive definite (positive semi-definite). Given a signal $z : \mathbb{Z} \to \mathbb{R}^\sigma$ the sequence $\{z(k), z(k+1), \ldots, z(k+T)\}$ is denoted by $z_{[k,k+T]}$ with $k, T \in \mathbb{Z}$. Given a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes its Euclidean norm and given a matrix $M \in \mathbb{R}^{m \times n}$, $\|M\|$ denotes the norm of M induced by the Euclidean norm.

II. Preliminaries

Consider a LTV system, described by discrete-time dynamics

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control input and $A(k)$ and $B(k)$ denote the time-varying dynamics and input matrices of appropriate dimensions, respectively.

Assume the system (1) is controllable and the full state is accessible. We are interested in representing the system using data for time instances $k = 0, 1, \ldots, T-1$, with $T \in \mathbb{N}$. Let

$$u_{d,j,[0,T-1]}, \;\; x_{d,j,[0,T]},$$

represent input-state data collected during an experiment j, for $j = 1, 2, \ldots, L$, with $L \in \mathbb{N}$. Each experiment j captures the same time-varying behaviour. This is similar to the data sequences typically required in ensemble methods for system identification of LTV systems. We introduce the matrices

$$X(k) = [x_{d,1}(k), x_{d,2}(k), \ldots, x_{d,L}(k)],$$

for $k = 0, 1, \ldots T$, and

$$U(k) = [u_{d,1}(k), u_{d,2}(k), \ldots, u_{d,L}(k)],$$

for $k = 0, 1, \ldots T-1$, which combine the data from all L experiments at each time step. Note that

$$\begin{aligned} X(k+1) &= A(k)X(k) + B(k)U(k) \\ &= [A(k) \;\; B(k)] \begin{bmatrix} X(k) \\ U(k) \end{bmatrix}. \end{aligned} \tag{2}$$

Remark 1. Considering the special case in which the time-variation of system (1) is periodic, multiple experiments can be replaced by one sufficiently long experiment capturing L periods.

III. Data-driven system representation

Consider system (1) in closed-loop with a state feedback control law of the form $u(k) = K(k)x(k)$. Namely, consider the closed-loop system

$$x(k+1) = \Big(A(k) + B(k)K(k)\Big)x(k). \tag{3}$$

In the following statement the matrices defined in Section II are used to parametrise the closed-loop system using data.

Corollary 1. Suppose the rank condition

$$\text{rank} \begin{bmatrix} X(k) \\ U(k) \end{bmatrix} = n + m, \tag{4}$$

holds for $k = 0, 1, \ldots, T-1$. Then, the closed-loop system (3) can equivalently be represented as

$$x(k+1) = X(k+1)G(k)x(k), \tag{5}$$
$$u(k) = U(k)G(k)x(k), \tag{6}$$

where $G(k) \in \mathbb{R}^{L\times n}$ satisfies

$$\begin{bmatrix} I_n \\ K(k) \end{bmatrix} = \begin{bmatrix} X(k) \\ U(k) \end{bmatrix} G(k), \tag{7}$$

for $k = 0, 1, \ldots, T-1$.

Proof. The proof is analogous to the proof of Theorem 2 in [13], using the alternative time-varying data matrices defined in Section II. □

While Corollary 1 is similar to the results presented in [13, Theorem 2] and [14, Section 3.2], the main difference between those formulations and the formulation presented herein is that the data matrices $X(k)$ and $U(k)$ are time-varying. This is a result of the time-varying system dynamics (1). Moreover, it is the reason why an ensemble of experiments is required for the data-based representation, rather than a single input-state trajectory collected during one experiment as in [13] and [14].

Remark 2. In the LTI case considered in [13] the equivalent rank condition to the condition (4) is satisfied if the input sequence is persistently exciting of order $n+1$. Since Willems et al.'s fundamental lemma [12] is specific to LTI systems, no similar result is available in the LTV case. Note, however, that the condition (4) can always be verified if the system state is accessible, as assumed herein, in our consideration of state feedback control problems. Note that a necessary condition for (4) to hold is $L \geq n+m$. In most cases the rank condition is satisfied if the number of experiments L is chosen accordingly and the input sequence for each experiment is randomly generated.

IV. State Feedback Stabilisation

In Corollary 1 the sequence of control gains $K(k)$ is parametrised using data through the identity (7). Hence, the matrices $G(k)$ can be seen as decision variables for identification-free controller design. For instance, we can search for a matrix sequence $G(k)$ that guarantees the trajectories of the closed-loop system (3) satisfy a certain bound for all $k = 0, 1, \ldots, T-1$. Analogous to the stabilisation problem in the LTI case (see [13]) this problem can be formulated in terms of linear matrix inequalities.

Theorem 1. Consider system (1) and suppose the rank condition (4) holds for data gathered during an ensemble of experiments. Then, any sequences of matrices $Y(k)$, $P(k)$ satisfying

$$\begin{bmatrix} P(k+1) - I_n & X(k+1)Y(k) \\ Y(k)^\top X(k+1)^\top & P(k) \end{bmatrix} \succeq 0, \tag{8a}$$

$$X(k)Y(k) = P(k), \tag{8b}$$

$$\eta I_n \preceq P(k) \preceq \rho I_n, \tag{8c}$$

for $k = 0, 1, \ldots, T-1$, where $\eta \geq 1$ and $\rho > \eta$ are finite constants, are such that the trajectories of the system (1) in closed-loop with

$$K(k) = U(k)Y(k)P(k)^{-1}, \tag{9}$$

satisfy

$$\|x(k)\| \leq \sqrt{\frac{\rho}{\eta}} \left(1 - \frac{1}{\rho}\right)^{\frac{k}{2}} \|x(0)\|, \tag{10}$$

for $k = 0, 1, \ldots, T-1$.

Remark 3. In the limit as $T \to \infty$ the result of Theorem 1 gives a data-driven characterisation of stabilising state feedback controllers.

V. Optimal Control

In this section, the result of Corollary 1 is used to formulate the finite-horizon LQR problem as an identification-free optimisation problem. First, the solution to the LQR problem is revisited, before the problem is reformulated as a covariance selection problem. Finally, the data-representation is introduced and it is shown that the finite-horizon LQR problem for LTV systems is equivalent to a semi-definite programme with data-dependent LMI constraints.

A. The finite-horizon LQR problem

Consider system (1) and suppose we are interested in finding the optimal control sequence $\{u^*(0), u^*(1), \ldots, u^*(N-1)\}$ as a function of the state, which minimises the quadratic cost function

$$J\left(x(k), u(k)\right) = x(N)^\top Q_f x(N) + \sum_{k=0}^{N-1} \left(x(k)^\top Q(k)x(k) + u(k)^\top R(k)u(k)\right), \tag{11}$$

over the time horizon $N \in \mathbb{N}$, starting from the initial condition $x(0) = x_0$, with $Q_f = Q_f^\top \succeq 0$, $Q(k) = Q(k)^\top \succeq 0$ and $R(k) = R(k)^\top \succ 0$, for $k = 0, 1, \ldots, N-1$. Namely, consider the minimisation problem

$$\begin{aligned} \min_{\mu_k} \quad & J\left(x(k), u(k)\right) \\ \text{s.t.} \quad & x(k+1) = A(k)x(k) + B(k)u(k), \\ & x(0) = x_0, \\ & u(k) = \mu_k\left(x(0), x(1), \ldots, x(k)\right), \\ & \forall k \in \{0, 1, \ldots, N-1\}. \end{aligned} \tag{12}$$

Lemma 1 ([30, Chapter 4.1]). The solution to problem (12) exists and is unique. The optimal control sequence is given by the state feedback law

$$u^*(k) = K^*(k)x(k), \tag{13}$$

with the time-varying gain matrix $K^*(k)$ given by

$$K^*(k) = -\left(R(k) + B(k)^\top P(k+1)B(k)\right)^{-1} \times B(k)^\top P(k+1)A(k),$$

where the symmetric and positive-definite matrix $P(k)$ is the solution of the difference Riccati equation

$$\begin{aligned} P(k) &= Q(k) + A(k)^\top P(k+1)A(k) \\ &- A(k)^\top P(k+1)B(k)\left(R(k) + B(k)^\top P(k+1)B(k)\right)^{-1} \\ &\times B(k)^\top P(k+1)A(k), \end{aligned} \quad (14)$$

for $k = 0, 1, \ldots, N-1$, with $P(N) = Q_f$.

B. Formulation as a covariance selection problem

A variety of problems arising in systems and control can be reduced to convex programmes involving LMIs. Consequently, many numerical solvers have been developed to solve those problems efficiently (see e.g. [31]). In order to derive a convex programming formulation of the optimal control problem involving data-dependent LMIs an equivalent formulation of the LQR problem - the covariance selection problem (see [32]) - is considered.

Lemma 2 ([14, Sections 2.1,3.1]). Solving (12) is equivalent to solving the optimisation problem

$$\min_{\mathcal{S},\mathcal{K},\mathcal{Z}} \quad \mathbf{Tr}\left(Q_f S(N)\right) + \sum_{k=0}^{N-1}\Big(\mathbf{Tr}\left(Q(k)S(k)\right) + \mathbf{Tr}\left(Z(k)\right)\Big) \quad (15a)$$

s.t.

$$S(0) \succeq I_n, \quad (15b)$$

$$S(k+1) - I_n - \left(A(k) + B(k)K(k)\right)S(k)\left(A(k) + B(k)K(k)\right)^\top \succeq 0, \quad (15c)$$

$$Z(k) - R(k)^{1/2}K(k)S(k)K(k)^\top R(k)^{1/2} \succeq 0, \quad (15d)$$

for $k = 0, 1, \ldots, N-1$, with

$$\begin{aligned} \mathcal{S} &:= \{S(1), S(2), \ldots, S(N)\}, \\ \mathcal{Z} &:= \{Z(0), Z(1), \ldots, Z(N-1)\}, \\ \mathcal{K} &:= \{K(0), K(1), \ldots, K(N-1)\}. \end{aligned}$$

The optimal gain sequence for the feedback law (13) is given by $\mathcal{K}$.

Note that the proof in [14] is presented for LTI systems, however, all arguments also hold for time-varying dynamics and time-varying LQR weight matrices.

The optimisation problem (15) can be transformed to a convex programme by a suitable change of variables [14]. This concept has been introduced in [33] and makes it straight forward to formulate a data-driven representation of the LQR problem in terms of a convex programme as shown in [13], [14] for LTI systems. A similar result for LTV systems is provided in the following subsection.

C. Data parametrisation

Using Lemma 2 and Corollary 1 the finite-horizon LQR problem (12) is formulated as a data-parametrised semi-definite programme in the following statement.

Theorem 2. Let the rank condition (4) hold for data from an ensemble of experiments for system (1). The optimal state feedback gain sequence $\{K^*(0), K^*(1), \ldots, K^*(N-1)\}$ minimising the LQR problem (12) is given by

$$K^*(k) = U(k)H(k)S(K)^{-1} \quad (16)$$

with $H(k)$ and $S(k)$ a solution of the optimisation problem

$$\min_{\mathcal{S},\mathcal{H},\mathcal{Z}} \quad \mathbf{Tr}\left(Q_f S(N)\right) + \sum_{k=0}^{N-1}\Big(\mathbf{Tr}\left(Q(k)S(k)\right) + \mathbf{Tr}\left(Z(k)\right)\Big) \quad (17a)$$

s.t.

$$S(0) \succeq I_n, \quad (17b)$$

$$\begin{bmatrix} S(k+1) - I_n & X(k+1)H(k) \\ H(k)^\top X(k+1)^\top & S(k) \end{bmatrix} \succeq 0, \quad (17c)$$

$$\begin{bmatrix} Z(k) & R(k)^{1/2}U(k)H(k) \\ H(k)^\top U(k)^\top R(k)^{1/2} & S(k) \end{bmatrix} \succeq 0, \quad (17d)$$

$$S(k) = X(k)H(k), \quad (17e)$$

for $k = 0, 1, \ldots, N-1$, where

$$\begin{aligned} \mathcal{S} &= \{S(1), S(2), \ldots, S(N)\}, \\ \mathcal{H} &= \{H(0), H(1), \ldots, N(N-1)\}, \\ \mathcal{Z} &= \{Z(0), Z(1), \ldots, Z(N-1)\}. \end{aligned}$$

Remark 4. The data representation and hence the above results are based on data from open-loop experiments. For rapidly diverging unstable systems in combination with large time horizons N this can lead to numerical issues, due to the large difference in magnitude of the data samples. In the LTI case similar issues have been observed (see [13]) and a solution to the issues has been provided in [15] by using multiple, short experiments and the notion of "collective persistency of excitation", in place of one long data sequence collected from a single experiment for the data-based parametrisation. This method is not applicable to the LTV case, since it is required that all experiments in the ensemble used for the data-representation cover the entire time horizon. This is necessary to capture the time-variation of the system dynamics over all time steps of interest. However, if a stabilising, but not necessarily optimal controller $\hat{K}(k)$ is known, experiments can be performed on the closed-loop system by superimposing a sufficiently informative signal $\hat{u}_d(k)$ to ensure the rank condition (4) is satisfied, i.e. $u_d(k) = \hat{K}(k)x_d(k) + \hat{u}_d(k)$, for $k = 0, 1, \ldots, N-1$. This data can then be used to find an optimal control sequence by solving (17).

VI. Numerical Example

To illustrate the efficacy of the results presented in the preceding sections consider the numerical example introduced in [28]. Namely, consider the LTV system (1) with

$$A(k) = \begin{bmatrix} 1 & 0.0025k \\ -0.1\cos(0.3k) & 1 + 0.05^{3/2}\sin(0.5k)\sqrt{k} \end{bmatrix},$$

$$B(k) = 0.05\begin{bmatrix} 1 \\ \frac{0.1k+2}{0.1k+3} \end{bmatrix}.$$

Note that the system is open-loop unstable. Suppose we are interested in finding a feedback gain $K^*(k)$ which minimises the cost function (11), with

$$Q(k) = (0.04k + 2)\, I_2, \;\; R(k) = 5 - 0.02k, \;\; Q_f = 50 I_2,$$

over the time horizon $k = 0, 1, \ldots, N-1$, with $N = 120$. The data for the model-free representation is gathered in $L = 3$ open-loop simulations with random initial conditions and by applying a random bounded input sequence over the interval $[0, N-1]$, both generated using the MATLAB function `rand`. The data-parametrised optimisation (17) is solved using CVX [34]. For comparison, the optimal solution is also computed by solving (14) (using the model). The sequence of control gains $K^*(k)$ computed using the data-based representation (i.e. the result given in Theorem 2) coincides with the control sequence $\bar{K}(k)$, for $k = 0, 1, \ldots, N-1$, obtained by recursively solving the difference Riccati equation (14) with an average error $\|K^*(k) - \bar{K}(k)\|$ of order 10^{-8}. The time histories of the first (top plot) and second (bottom plot) components of the state of the closed-loop system with $\{K^*(0), K^*(1), \ldots, K^*(N-1)\}$ and $x_0 = [0.4411 \;\, 0.2711]^\top$ are shown in Fig. 1. The corresponding input sequence (top plot) and the gain error $\|K^*(k) - \bar{K}(k)\|$ (bottom plot) for each time instance are shown in Fig. 2.

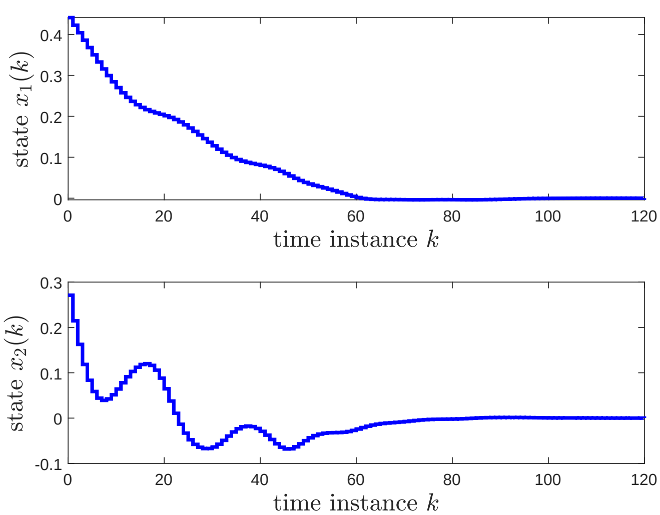

Fig. 1. The time histories of the states of the system in closed-loop with the optimal gain sequence determined from (17).

VII. Conclusion

A model-free representation of closed-loop LTV systems under state feedback is introduced. The presented approach extends the methods for parametrising LTI systems with data-dependent matrices presented in [13] to time-varying systems. The input-state data used for the data-parametrisation

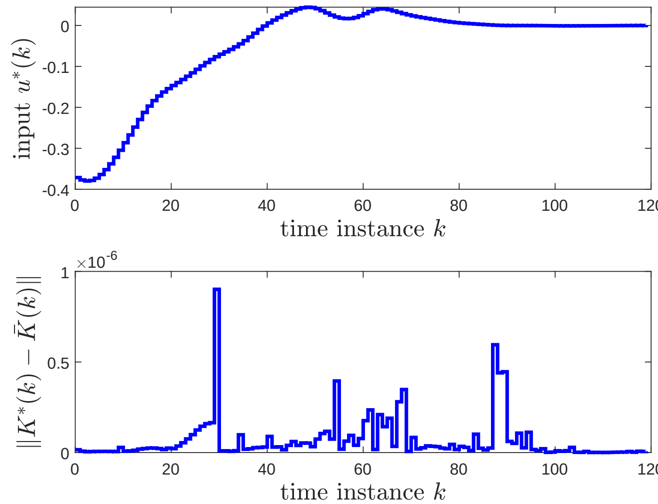

Fig. 2. The time histories of the optimal input sequence $u^*(k) = K^*(k)x(k)$ of the system (top) and the error between the optimal control gains (bottom) determined from (17) (model-free) and (14) (model-based).

is obtained from an ensemble of experiments capturing the same time-varying behaviour. Using this result, the problem of designing feedback controllers such that the closed-loop system trajectories satisfy a certain bound can be recast as a feasibility problem involving data-dependent linear matrix inequalities. Similarily, the LQR problem can be recast as a semi-definite programme involving data-parametrised LMI constraints. These data-based formulations can be solved efficiently using numerical solvers. A numerical example is provided to illustrate the efficacy of the proposed data-based methods. Problems to be addressed in future research include the extension of the results presented herein to scenarios in which the collected data is influenced by noise and the consideration of systems evolving according to nonlinear dynamics.

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