THE EIGENVALUE PROBLEM FOR NATURAL FREQUENCY OF UNIFORM BEAM WITH LINEARLY VARYING AXIAL LOAD

by

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NOMENCLATURE

q	Uniformly distributed axial load per unit length
P_1, P_2	Constant end loads
L	Length of beam
z	Axial coordinate
y	Lateral deflection
t	Time
δ	Variation
T	Kinetic energy
V	Total potential energy
m	Density
E	Modulus of elasticity
I	Moment of inertia
U	Strain energy of bending
W	Potential energy of the external forces
ω^2	Separation constant
$x = \frac{z}{L}$	Non-dimensional axial variable
$B = \frac{P_1}{qL}$	Ratio of end load P_1 to total distributed loads
$F = \frac{qL^3}{EI}$	Distributed axial loading parameter
n, r	Subscripts (integers)
f	Natural frequency of vibrating beam
$R = \frac{\omega^2 mL^4}{EI}$	Eigenvalue

INTRODUCTION

A knowledge of the effects of longitudinal inertia force due to thrust on the vibration and stability characteristics is important in the design of a variety of aerodynamic vehicles and their components especially high speed aircraft, missiles and rockets, where inertia and friction drag forces manifest themselves as axial loads. In recent years much attention has been given to the effects of axial loads created by the thrust on the natural frequencies and stability characteristics of uniform beams. Timoshenko [1] presented the natural frequencies and buckling loads for uniform beams subjected to constant end loads. Krieger [2] and Burgreen [3] determined the effect of an axial force on the free vibration of hinged bars and columns. They considered the distance constant between two ends instead of usual theoretical assumption that the load on the beam remains constant. Tu and Handelman [4] considered the eigenvalue problem of lateral vibrations of a beam under initial linear axial strees. Beal [5] gave some results of the effect of longitudinal acceleration on the natural frequencies due to constant and pulsating thrusts. Tyler and Rouleau [6] considered an analytical method for determining the deflection, bending moments and stabilities of beam-columns having uniformly distributed axial loads by the Airy integral function. Dalley [7] considered the natural frequencies of vibration of a uniform free-free beam. Recently, Laird and Fauconneau [8, 9] have investigated the bounds for the natural frequencies of a simply supported beam and a clamped ends beam with carrying linearly distributed axial loads. The upper bounds have been obtained by the Rayleigh-Ritz method and the lower bounds by Kato's method and the method of intermediate. True values can be predicted from the upper and lower bounds.

In this report, the problem of stability of a uniform beam with linearly distributed axial loads and with constant end loads will be studied. However, the purpose of this report is to obtain an exact solution to the problem, instead of finding the upper and lower bounds. The natural frequencies for simply supported and clamped, as well as for the free-free beam, are obtained by Frobenius' method. Finally, the results are compared with those obtained by Laird and Fauconneau [9].

EQUATION OF MOTION OF BEAM WITH UNIFORMLY DISTRIBUTED AXIAL LOADS

A beam of uniform cross section, subjected to constant end loads P_1, P_2 and a uniformly distributed axial loading intensity q, is investigated. As shown in Fig. 1, P_1 and q are taken to be positive when they are acting in the positive direction of the z-axis. End load P_2 acts at the other end such that the force system is in equilibrium.

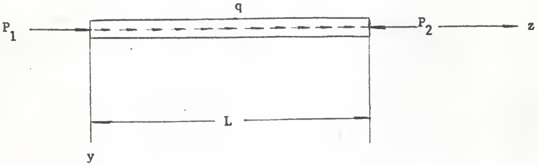

Fig. 1. Loading Condition

The equation of motion of the beam can be derived from Hamilton's principle, which is formally expressed by

$$\int_{t_0}^{t_1} (\delta T - \delta V)dt = \delta \int_{t_0}^{t_1} (T - V)dt = 0 , \tag{1}$$

where T represents kinetic energy, V the potential energy of the system and δ the first variation.

Kinetic energy T can be expressed as

$$T = \frac{1}{2} \int_0^L m(\frac{y}{t})^2 dz , \tag{2}$$

where m denotes the density of the material and y is the lateral deflection of the beam. V can be broken into two parts: the potential energy of the internal forces, or the strain energy of the system, expressed as

$$U = \frac{1}{2} EI \int_0^L \left(\frac{\partial^2 y}{\partial z^2}\right)^2 dz \, , \tag{3}$$

and the potential energy of the external forces W, given by

$$W = -\frac{1}{2} \int_0^L (P_1 + qz)\left(\frac{\partial y}{\partial z}\right)^2 dz \, . \tag{4}$$

Variations of T, U and W are evaluted as

$$\delta T = \frac{1}{2} \int_0^L m\left[\frac{\partial y}{\partial t} + \frac{\partial}{\partial t}(\delta y)\right]^2 dz - \frac{1}{2} \int_0^L m\left(\frac{\partial y}{\partial t}\right)^2 dz$$

$$= \frac{1}{2} \int_0^L m\left[2 \frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y)\right] dz$$

$$= \int_0^L m\left[\frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y)\right] dz \, .$$

$$\delta U = \frac{1}{2} EI \int_0^L \left[\frac{\partial^2 y}{\partial z^2} + \frac{\partial^2}{\partial z^2}(\delta y)\right]^2 dz - \frac{1}{2} EI \int_0^L \left(\frac{\partial^2 y}{\partial z^2}\right)^2 dz$$

$$= EI \int_0^L \frac{\partial^2 y}{\partial z^2} \frac{\partial^2}{\partial z^2}(\delta y) \, dz \, ,$$

$$\delta W = -\int_0^L \frac{1}{2}(P_1 + qz)\left[\frac{\partial y}{\partial z} + \frac{\partial}{\partial z}(\delta y)\right]^2 dz + \int_0^L \frac{1}{2}(P_1 + qz)\left(\frac{\partial y}{\partial z}\right)^2 dz$$

$$= -\int_0^L (P_1 + qz)\left[\frac{\partial y}{\partial z} \frac{\partial}{\partial z}(\delta y)\right] dz$$

and

$$\delta V = \delta U + \delta W$$

$$= EI \int_0^L \frac{\partial^2 y}{\partial z^2} \frac{\partial^2}{\partial z^2}(\delta y)\, dz - \int_0^L (P_1 + qz)\frac{\partial y}{\partial z} \frac{\partial}{\partial z}(\delta y)\, dz .$$

Substitution of δT and δV into Eq. (1) yields

$$\int_{t_0}^{t_1}\left\{\int_0^L \left[m \frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y) - EI \frac{\partial^2 y}{\partial z^2} \frac{\partial^2}{\partial z^2}(\delta y) + (P_1 + qz)\frac{\partial y}{\partial z} \frac{\partial}{\partial z}(\delta y)\right] dz\right\} dt = 0. \quad (5)$$

Integrating by parts, the first term of Eq. (5) yields

$$\int_{t_0}^{t_1}\int_0^L m\frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y)\, dz\, dt = \int_0^L \int_{t_0}^{t_1} m\frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y)\, dt\, dz$$

$$= \int_0^L \left[m \frac{\partial y}{\partial t}(\delta y)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} m \frac{\partial^2 y}{\partial t^2}(\delta y)\, dt \; dz . \quad (6)$$

The first term of Eq. (6) vanishes automatically because variation δy vanishes at $t = t_0$ and t_1; then

$$\int_{t_0}^{t_1}\int_0^L m\frac{\partial y}{\partial t} \frac{\partial}{\partial t}(\delta y)\, dz\, dt = - \int_{t_0}^{t_1}\int_0^L m\frac{\partial^2 y}{\partial t^2}(\delta y)\, dz\, dt . \quad (7)$$

The second term of Eq. (5) yields

$$\int_{t_0}^{t_1}\int_0^L EI \frac{\partial^2 y}{\partial z^2} \frac{\partial^2}{\partial z^2}(\delta y)\, dz\, dt$$

$$= \int_{t_0}^{t_1} EI\left\{\left[\frac{\partial^2 y}{\partial z^2} \frac{\partial}{\partial z}(\delta y)\right]_0^L - \int_0^L \frac{\partial^3 y}{\partial z^3} \frac{\partial}{\partial z}(\delta y)\, dz\right\} dt$$

$$= EI \int_{t_0}^{t_1}\left\{\left[\frac{\partial^2 y}{\partial z^2} \frac{\partial}{\partial z}(\delta y)\right]_0^L - \left[\frac{\partial^3 y}{\partial z^3}(\delta y)\right]_0^L + \int_0^L \frac{\partial^4 y}{\partial z^4}(\delta y)\, dz\right\} dt , \quad (8)$$

and the last term of Eq. (5) gives

$$\int_{t_0}^{t_1}\int_0^L (P_1 + qz)\frac{\partial y}{\partial z}\frac{\partial}{\partial z}(\delta y)\,dz\,dt$$

$$= \int_{t_0}^{t_1}\left\{\left[(P_1 + qz)\frac{\partial y}{\partial z}(\delta y)\right]_0^L - \int_0^L \frac{\partial}{\partial z}\left[(P_1 + qz)\frac{\partial y}{\partial z}\right](\delta y)\,dz\right\}dt \;. \qquad (9)$$

Substitution of Eqs. (7), (8), (9) into Eq. (5) yields

$$\int_{t_0}^{t_1}\int_0^L\left\{m\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial z^4} + \frac{\partial}{\partial z}\left[(P_1 + qz)\frac{\partial y}{\partial z}\right]\right\}(\delta y)\,dz\,dt$$

$$+ \int_{t_0}^{t_1}\left[EI\frac{\partial^2 y}{\partial z^2}\frac{\partial}{\partial z}(\delta y)\right]_0^L dt - \int_{t_0}^{t_1}\left[EI\frac{\partial^3 y}{\partial z^3}(\delta y) + (P_1 + qz)\frac{\partial y}{\partial z}(\delta y)\right]_0^L dt = 0 \;. \qquad (10)$$

A sufficient condition for the vanishing of the left hand side of Eq. (10) is the vanishing of each part of the corresponding equation. The vanishing of the first integration for arbitrary δy is the vanishing of its integrand, which yields the Euler differential equation:

$$m\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial z^4} + \frac{\partial}{\partial z}\left[(P_1 + qz)\frac{\partial y}{\partial z}\right] = 0 \;. \qquad (11)$$

The vanishing of the 2nd and 3rd integrations gives the required natural boundary conditions, which are discussed as follow*:

(i) Simply supported beam

The deflection and moment must vanish at the end points. Therefore, the boundary conditions are

$$y = 0$$

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* See reference [15]

$$\frac{\partial^2 y}{\partial z^2} = 0 \tag{12}$$

(ii) Clamped beam

The deflection and its derivative must vanish at a clamped end. The boundary conditions are

$$y = 0$$
$$\frac{\partial y}{\partial z} = 0 \tag{13}$$

(iii) Free-free beam

The moment and shear must vanish at a free end. The boundary conditions become

$$\frac{\partial^2 y}{\partial z^2} = 0$$
$$\frac{\partial^3 y}{\partial z^3} + \frac{1}{EI}(P_1 + qz)\frac{\partial y}{\partial z} = 0 \tag{14}$$

SOLUTION OF EQUATION OF MOTION

Equation (11) is a linear partial differential equation with variable coefficients. The solution is assumed to be of the form

$$y(z,t) = X(z)\cdot\bar{T}(t) .$$

Upon substitution into Eq. (11) this yields

$$\frac{EI}{m}\,\frac{\frac{\partial^4 X}{\partial z^4} + \frac{\partial}{\partial z}(P_1 + qz)\frac{\partial X}{\partial z}}{X} = -\frac{\frac{\partial^2 \bar{T}}{\partial t^2}}{\bar{T}} = \omega^2 . \tag{15}$$

The expression on the left hand side is a function of z only and the expression on the right is a function of t only, hence they are both equal to a constant ω^2. Thus Eq. (15) yields two ordinary differential equations,

$$\frac{d^4X}{dz^4} + \frac{1}{EI}\frac{d}{dz}\left[(P_1 + qz)\frac{dX}{dz}\right] - \frac{\omega^2 mX}{EI} = 0 \tag{16}$$

and

$$\frac{d^2\bar{T}}{dt^2} + \omega^2\bar{T} = 0 \tag{17}$$

The type of transverse motion of the beam depends on the value of ω^2. Thus the solution to Eq. (11) falls into the following three cases:

(1) $\omega^2 > 0$ The solution to Eq. (17) is $\bar{T}(t) = A\cos\omega t + B\sin\omega t$. This is oscillatory motion, which is stable.

(2) $\omega^2 = 0$ The solution to Eq. (17) is $\bar{T}(t) = At + B$. This gives the static buckling case.

(3) $\omega^2 < 0$ The solution to Eq. (17) is $\bar{T}(t) = A\sinh\omega t + B\cosh\omega t$. This represents a growth of T when t increases, which is unstable.

Introducing the following non-dimensional parameters

$$x = \frac{z}{L}, \quad B = \frac{P_1}{qL}, \quad F = \frac{qL^3}{EI}, \quad \text{and} \quad R = \frac{\omega^2 mL^4}{EI}, \tag{18}$$

Eq. (16) now becomes

$$\frac{d^4X}{dx^4} + F\frac{d}{dx}\left[(B + x)\frac{dX}{dx}\right] - Rx = 0 . \tag{19}$$

In view of the difinition of parameter B, it is clear that for a given distributed load q, the following cases may occur:

(1) $B > 0$: the beam is entirely in compression.

(2) $-1 < B < 0$: the beam is partly in tension and partly in compression.

(3) $B \leq -1$: the beam is entirely in tension since the tensile end load P_1 is larger than the distributed loads.

In the last case, the problem of elastic stability does not exist.

Determination of natural frequencies and the mode shapes involves the solution of the differential equation specified by Eq. (19) with appropriate boundary conditions. By the well-known method of Frobenius [12,13] , a power series

$$X = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots \quad (20a)$$

can be assumed as the general solution of Eq. (19). Thus

$$X' = \sum_{r=0}^{\infty} a_r r x^{r-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots, \quad (20b)$$

$$X'' = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} = [2]_2 a_2 + [3]_2 a_3 x + [4]_2 a_4 x^2 + \cdots \quad (20c)$$

$$X''' = \sum_{r=0}^{\infty} a_r r(r-1)(r-2) x^{r-3} = [3]_3 a_3 + [4]_3 a_4 x + [5]_3 a_5 x^2 + \cdots \quad (20d)$$

and

$$X'''' = \sum_{r=0}^{\infty} a_r r(r-1)(r-2)(r-3) x^{r-4} = [4]_4 a_4 + [5]_4 a_5 x + [6]_4 a_6 x^2 + \cdots, \quad (20e)$$

where $[r]_n = r(r-1)(r-2)\cdots(r-n+1)$ denotes the descending factor.

Substitution of Eqs. (20) into Eq. (19) yields

$$[4]_4 a_4 + [5]_4 a_5 x + [6]_4 a_6 x^2 + [7]_4 a_7 x^3 + \cdots$$

$$+ FB\left\{[2]_2 a_2 + [3]_2 a_3 x + [4]_2 a_4 x^2 + [5]_2 a_5 x^3 + \cdots\right\}$$

$$+ F\left\{[2]_2 a_2 x + [3]_2 a_3 x^2 + [4]_2 a_4 x^3 + \cdots\right\}$$

$$+ F\left\{a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right\}$$

$$- R\left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right\} = 0 .$$

Collection of like powers of x results in

$$\left\{[4]_4 a_4 + FB[2]_2 a_2 + Fa_1 - Ra_0\right\}$$

$$+ \left\{[5]_4 a_5 + FB[3]_2 a_3 + F[2]_2 a_2 + 2Fa_2 - Ra_1\right\} x$$

$$+ \left\{[6]_4 a_6 + FB[4]_2 a_4 + F[3]_2 a_3 + 3Fa_3 - Ra_2\right\} x^2$$

$$+ \left\{[7]_4 a_7 + FB[5]_2 a_5 + F[4]_2 a_4 + 4Fa_4 - Ra_3\right\} x^3$$

$$+ \cdots = 0 . \tag{21}$$

Since this equation must hold for all x, the coefficients of like powers of x vanish, and $(n)_2 + n = n(n-1+1) = n^2$. Thus

$$[4]_4 a_4 + FB[2]_2 a_2 + Fa_1 - Ra_0 = 0$$

$$[5]_4 a_5 + FB[3]_2 a_3 + F[2]^2 a_2 - Ra_1 = 0$$

$$[6]_4 a_6 + FB[4]_2 a_4 + F[3]^2 a_3 - Ra_2 = 0$$

$$[7]_4 a_7 + FB[5]_2 a_5 + F[4]^2 a_4 - Ra_3 = 0$$

$$\vdots$$

$$[r]_4 a_r + FB[r-2]_2 a_{r-2} + F[r-3]^2 a_{r-3} - Ra_{r-4} = 0 \tag{22}$$

$$\vdots$$

Substituting the coefficients $a_k (k=4,5,6,\cdots)$, which by Eqs. (22) can be written in terms of the coefficients a_0, a_1, a_2 and a_3, into Eq. (20a), the general solution of Eq. (15) is

$$X = a_0 X_0 + a_1 X_1 + a_2 X_2 + a_3 X_3 = \sum_{i=0}^{3} a_i X_i (F,B,R,x) \ , \tag{23}$$

where a_0, a_1, a_2 and a_3 are arbitrary constants which will be determined by appropriate boundary conditions. Also

$$X_0(F,B,R,x) = 1 + 0 + 0 + 0 + \frac{R}{[4]_4} x^4 + \cdots \ , \tag{24a}$$

$$X_1(F,B,R,x) = x + 0 + 0 - \frac{Fx^4}{[4]_4} + \frac{Rx^5}{[5]_4} + \cdots \ , \tag{24b}$$

$$X_2(F,B,R,x) = x^2 + 0 - \frac{FB[2]_2}{[4]_4} x^4 - \frac{F[2]^2}{[5]_4} x^5 + \cdots \tag{24c}$$

and

$$X_3(F,B,R,x) = x^3 + 0 - \frac{FB[3]_2}{[5]_4} x^5 - \frac{F[3]^2}{[6]_4} x^6 + \cdots \ . \tag{24d}$$

The recursion relationship for Eqs. (24) is

$$b_n = \frac{Rb_{n-4} - F[n-3]^2 b_{n-3} - FB[n-2]^2 b_{n-2}}{[n]_4}$$

for n 4 .

SPECIAL PROBLEM

A. Simply Supported Beam

A uniform cross section beam with simply supported ends is considered. The boundary conditions for a simply supported beam are

$$X(0) = X''(0) = X(1) = X''(1) = 0 . \tag{26}$$

Substitution of Eq. (23) into Eq. (26) gives

$$a_0 = 0 , \tag{27a}$$

$$a_2 = 0 , \tag{27b}$$

$$a_1 X_1(F,B,R,1) + a_3 X_3(F,B,R,1) = 0 \tag{27c}$$

and

$$a_1 X_1''(F,B,R,1) + a_3 X_3''(F,B,R,1) = 0 . \tag{27d}$$

In order to obtain a nontrival solution of Eqs. (27), the determinant of the coefficients of a_0, a_1, a_2 and a_3 must vanish. This yields the characteristic equation

$$X_1(F,B,R,1) \cdot X_3''(F,B,R,1) - X_3(F,B,R,1) \cdot X_1''(F,B,R,1) = 0 . \tag{28}$$

For given values of F and B, Eq. (28) is a polynomial function of R and was solved by IBM 1620 computer with method of trial and error. The ratio of convergence was quite rapid; no more than 100 terms were needed. The eigenvalue R in Eq. (28) was obtained by trial and error for given loading parameters F and B.

The eigenvalues for the first five modes of the simply supported beam are presented in Table 1. The Rayleigh-Ritz upper bounds, the lower bounds

Table 1. Eigenvalue for the simply supported beam

B = 0.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
0.00	1	97.40909	97.40909	97.40909	97.40909
	2	1558.545	1558.545	1558.545	1558.545
	3	7890.136	7890.136	7890.136	7890.136
	4	24936.73	24936.73	24936.73	24936.73
	5	60880.68	60880.68	60880.68	60880.68
2.00	1	87.48401	87.48399	86.82118	87.47622
	2	1519.022	1519.022	1509.258	1518.916
	3	7801.265	7801.265	7759.954	7798.981
	4	24778.77	24778.77	24668.74	24740.64
	5	60633.89	60633.89	60403.19	60441.32
4.00	1	77.44337	77.44335	76.09688	77.42758
	2	1479.411	1479.412	1459.780	1479.201
	3	7712.310	7712.309	7629.280	7707.814
	4	24620.72	24620.73	24399.70	24544.67
	5	60387.02	60387.03	59923.86	60001.91
6.00	1	67.27963	67.27964	65.20801	67.25670
	2	1439.716	1439.715	1410.094	1439.409
	3	7623.273	7623.271	7498.086	7616.639
	4	24462.60	24462.60	24120.60	24348.82
	5	60140.07	60140.08	59442.65	59562.45

Table 1. Eigenvalue for the simply supported beam

B = 0.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
8.00	1	56.98461	56.98461	54.11895	56.95145
	2	1399.940	1399.940	1360.181	1399.538
	3	7534.153	7534.152	7366.345	7525.454
	4	24304.39	24304.40	23858.37	24153.09
	5	59893.04	59893.05	58959.51	59122.92
10.00	1	46.54933	46.54932	42.78417	46.51453
	2	1360.085	1360.384	1310.021	1359.597
	3	7444.952	7444.952	7234.029	7434.263
	4	24146.10	24146.11	23585.90	23957.49
	5	59645.92	59645.93	58474.38	58683.34
12.00	1	35.96395	35.96393	31.14474	35.92827
	2	1320.157	1320.157	1259.591	1319.593
	3	7355.672	7355.674	7101.106	7343.061
	4	23987.74	23987.74	23312.42	23762.01
	5	59398.72	59398.74	57987.22	58243.69
14.00	1	25.21769	25.21764	19.12343	25.17796
	2	1280.161	1280.160	1208.868	1279.623
	3	7266.316	7266.317	6967.549	7251.870
	4	23829.29	23829.29	23037.61	23566.66
	5	59151.45	59151.46	57497.99	57803.98

Table 1. Eigenvalue for the simply supported beam

B = 0.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
16.00	1	14.29866	14.29859	6.61749	14.25562
	2	1240.103	1240.103	1157.825	1239.414
	3	7175.883	7176.885	6833.32	7160.673
	4	23670.76	23670.76	22761.53	23371.45
	5	58904.09	58904.10	57006.62	57364.20
18.00	1	3.193809	3.193706	- - -	3.148092
	2	1199.990	1199.990	1106.434	1199.233
	3	7087.378	7087.379	6698.383	7069.476
	4	23512.15	23512.16	22484.12	23176.36
	5	58656.65	58656.66	56513.06	56924.36
18.50	1	0.386901	0.386779	- - -	0.341591
	2	1189.955	1189.954	1093.528	1189.187
	3	7064.990	7064.991	6664.534	7046.678
	4	23472.49	23472.49	22414.56	23127.62
	5	58594.78	58594.79	56389.33	56814.39

Table 1. Eigenvalue for the simply supported beam

B = 1.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
1.00	1	82.59065	82.59067	82.28879	82.58765
	2	1499.316	1499.316	1494.533	1499.263
	3	7756.884	7756.885	7736.428	7755.743
	4	24699.84	24699.84	24645.15	24680.75
	5	60510.55	60510.55	60395.69	60414.26
3.00	1	52.86138	52.86139	52.14043	52.85093
	2	1380.795	1380.795	1366.927	1380.634
	3	7490.320	7490.321	7429.579	7486.927
	4	24226.01	24225.96	24062.55	24158.90
	5	59770.24	59770.22	59426.05	59481.41
5.00	1	22.98678	22.98678	22.15408	22.96599
	2	1262.199	1262.199	1239.950	1261.929
	3	7223.677	7223.679	7123.531	7218.113
	4	23752.11	23752.11	23480.69	23657.17
	5	59029.85	59029.87	58456.83	58548.49
6.00	1	7.984866	7.984859	7.241968	7.961275
	2	1202.877	1202.877	1176.730	1202.567
	3	7090.327	7090.329	6970.827	7083.696
	4	23515.12	23515.13	23190.04	23401.35
	5	58659.63	58659.64	57972.37	58082.01

Table 1. Eigenvalue for the simply supported beam

B = 1.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
6.25	1	4.226834	4.226826	3.524225	4.197706
	2	1188.045	1188.045	1160.955	1187.721
	3	7056.987	7056.989	6932.686	7050.098
	4	23455.87	23455.88	23117.41	23337.41
	5	58567.07	58567.08	57851.28	57965.38
6.50	1	0.4656114	0.0656002	- - -	0.4387692
	2	1173.212	1173.212	1145.193	1172.879
	3	7023.645	7023.647	6894.559	7016.496
	4	23396.62	23396.63	23044.79	23273.46
	5	58474.51	58474.52	57730.19	57848.76

Table 1. Eigenvalue for the simply supported beam
$B = -0.5$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
1.00	1	97.39548	97.39550	97.05219	97.38885
	2	1558.533	1558.534	1553.621	1558.481
	3	7890.123	7890.125	7869.443	7888.977
	4	24936.71	24936.71	24881.71	24917.63
	5	60880.66	60880.67	60765.39	60784.37
5.00	1	97.06945	97.06947	95.02765	97.05359
	2	1558.264	1558.265	1532.551	1558.000
	3	7889.865	7889.867	7783.922	7884.291
	4	24936.45	24936.46	24656.89	24841.50
	5	60880.40	60880.41	60296.88	60399.04
10.00	1	96.05006	96.05006	91.43544	96.01405
	2	1557.425	1557.425	1506.462	1556.936
	3	7889.057	7889.059	7677.593	7878.347
	4	24935.66	24935.66	24368.81	24747.01
	5	60879.61	60879.61	59699.54	59917.04
20.00	1	91.96428	91.96419	78.64063	91.91397
	2	1554.071	1554.070	1446.067	1553.239
	3	7885.830	7885.830	7451.941	7866.169
	4	24932.48	24932.48	23756.87	24560.22
	5	60876.46	60876.46	58451.08	58951.90

Table 1. Eigenvalue for the simply supported beam

$B = -0.5$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
30.00	1	85.12538	85.12513	56.42424	85.06259
	2	1548.503	1548.501	1375.513	1547.459
	3	7880.455	7880.454	7210.208	7853.494
	4	24927.18	24927.17	23095.63	24376.41
	5	60871.19	60871.19	57129.12	57985.23
40.00	1	75.48948	75.48896	21.68015	75.42380
	2	1540.754	1540.751	1292.824	1539.589
	3	7872.940	7872.937	6950.046	7840.212
	4	24919.78	24919.76	22387.13	24195.62
	5	60863.84	60863.81	55727.85	57017.05
50.00	1	62.99520	62.99428	- - -	62.93328
	2	1530.869	1530.863	1195.846	1529.658
	3	7863.296	7863.290	6669.137	7826.211
	4	24910.26	24910.21	21611.65	24017.91
	5	60854.38	60854.33	54239.88	56047.35
60.00	1	47.56379	47.56229	- - -	47.49907
	2	1518.906	1518.898	1082.880	1517.698
	3	7851.536	7851.525	6364.444	7811.384
	4	24898.63	24898.56	20767.33	23843.33
	5	60842.82	60842.74	52657.41	55076.16

Table 1. Eigenvalue for the simply supported beam

B = - 0.5

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
70.00	1	29.09930	29.09698	- - -	29.03900
	2	1504.937	1504.925	951.7511	1503.762
	3	7837.675	7837.658	6034.095	7795.619
	4	24884.90	24884.80	19863.03	23671.88
	5	60829.17	60829.05	50971.82	54103.51
80.00	1	7.488954	7.485497	- - -	7.427070
	2	1489.045	1489.030	801.0849	1487.923
	3	7821.731	7821.708	5689.257	7778.816
	4	24869.08	24868.95	18858.86	23503.62
	5	60813.43	60813.26	49175.77	53129.41
81.00	1	5.150067	5.146474	- - -	5.085385
	2	1487.354	1487.338	785.6676	1486.242
	3	7820.022	7819.999	5652.000	7777.073
	4	24867.38	24867.24	18753.30	23486.96
	5	60811.74	60811.57	48988.74	53031.92
82.00	1	2.778307	2.7745732	- - -	2.713758
	2	1485.644	1485.629	768.9310	1484.539
	3	7818.294	7818.270	5615.842	7775.319
	4	24865.67	24865.52	18646.77	23470.35
	5	60810.03	60809.85	48801.56	52934.43

by Kato's method and the method of intermediate, which were given by Laird and Fauconneau [9], are also included to faciliate comparison of these methods.

Analysis of Table 1 indicates that the Rayleigh-Ritz upper bound is the best approximate method. For B=0 and 1, the upper bounds actually coincide with the results obtained. The Rayleigh-Ritz results become a little higher when B has a negative value and F has a large value and the beam is extremely close to buckling. One explanation for this behavior is that the actual mode of a simply supported beam is always easy to assume in using Rayleigh-Ritz method.

The effect of the distributed axial loads on the first frequency of the simply supported beam is shown in Fig. 2. The ratio of the first frequency of the loaded beam to that of the unloaded beam is taken as the axis of the ordinate. The distributed loading parameter F is taken as the axis of abscissa. The curves are plotted with various values of B. The critical value of the axial distributed loads is given at the intersection of the curves with the horizontal axis.

B. Clamped Beam

The boundary conditions for a clamped beam in non-dimensional form can be expressed as

$$X(0) = X(1) = X'(0) = X'(1) = 0 \; . \tag{29}$$

Substitution of Eq. (23) into Eq. (29) yields

$$a_0 = 0 \; , \tag{30a}$$

$$a_1 = 0 \; , \tag{30b}$$

$$a_2X_2(F,B,R,1) + a_3X_3(F,B,R,1) = 0 \tag{30c}$$

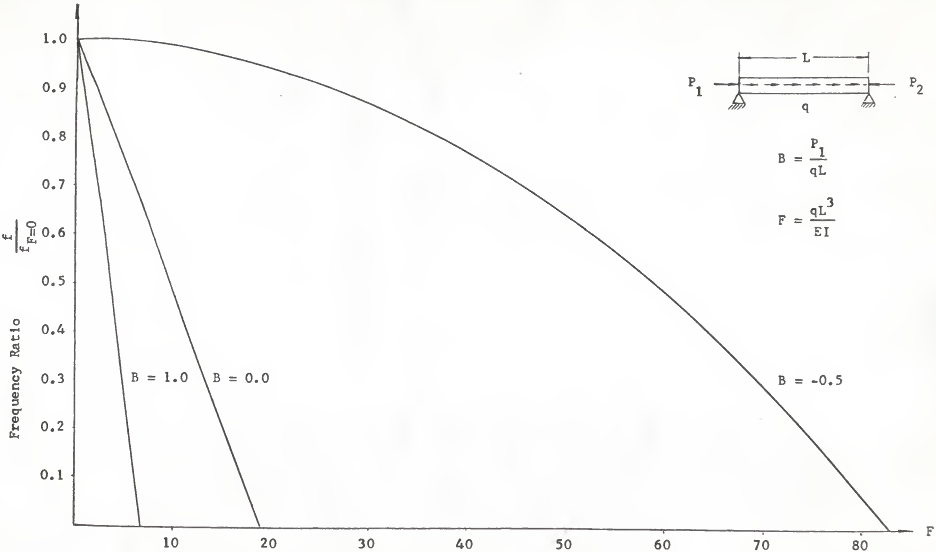

Figure 2. Effect of Axial Loads on the Fundamental Frequency of a Simply Supported Beam.

and

$$a_2X_2'(F,B,R,1) + a_3X_3'(F,B,R,1) = 0 . \tag{30d}$$

From the set of Eqs. (30), the characteristic equation for clamped beam is derived as

$$X_2(F,B,R,1)\cdot X_3'(F,B,R,1) - X_3(F,B,R,1)\cdot X_2'(F,B,R,1) = 0 . \tag{31}$$

For given values F and B, the eigenvalues R are calculated by trial and error as in previous case.

The eigenvalues for the first four modes of the clamped beam are listed in Table 2. The results obtained by the Rayleigh-Ritz method, by Kato's method and by the method of intermediate are also included in Table 2 to faciliate the comparison of these methods.

Examination of the results indicates the values obtained are a little less than the Rayleigh-Ritz method but are much larger than those obtained by other methods. Again, the Rayleigh-Ritz method gives the best approximate.

The effect of the distributed axial load on the first frequency of the clamped beam is shown in Fig. 3, which is plotted in the ratio of the first frequency of the loaded beam to that of the unloaded beam vs. loading parameter F with several values of B.

C. Free-free Beam

In this section, a beam of uniform cross section with both ends free is considered. The boundary conditions for a free-free beam in non-dimensional form are

$$X''(0) = X''(1) = 0 , \tag{32a}$$

$$X'''(0) = - F(B + 0)X'(0) \tag{32b}$$

Table 2. Eigenvalue for the clamped beam

$B = 0.0$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
0.0	1	500.564	500.564	500.564	500.564
	2	3803.54	3803.54	3803.54	3803.54
	3	14617.6	14617.6	14617.6	14617.6
	4	39943.8	39943.8	39943.8	39943.8
10.00	1	438.485	438.484	436.964	438.281
	2	3572.38	3572.37	3570.81	3571.05
	3	14122.2	14122.2	14119.9	14115.8
	4	39084.9	39084.9	39081.3	39069.3
20.00	1	375.192	375.189	366.109	374.825
	2	3339.36	3339.34	3320.29	3336.87
	3	13624.9	13624.9	13589.2	13612.6
	4	38224.2	38224.1	38189.7	38196.0
30.00	1	310.543	310.536	289.161	310.010
	2	3104.42	3104.38	3062.73	3100.83
	3	13125.9	13125.8	13047.8	13107.8
	4	37361.7	37361.4	37285.5	37318.9
40.00	1	244.366	244.353	200.691	243.766
	2	2867.50	2867.43	2786.55	2863.13
	3	12625.1	12624.9	12472.4	12601.6
	4	36497.3	36496.8	36364.3	36440.4

Table 2. Eigenvalue for the clamped beam

B = 0.0

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
50.00	1	176.455	176.436	110.389	175.815
	2	2628.54	2628.43	2505.85	2623.39
	3	12122.6	12122.2	11891.2	12094.1
	4	35631.2	35630.4	35422.2	35562.0
60.00	1	106.557	106.529	5.130	105.877
	2	2387.52	2387.36	2164.11	2381.38
	3	11618.4	11617.9	11249.9	11584.8
	4	34763.3	34762.2	34453.4	34679.2
70.00	1	34.354	34.315	- - -	33.672
	2	2144.43	2144.20	1801.43	2137.92
	3	11112.6	11111.8	10592.4	11074.6
	4	33893.8	33892.2	33459.1	33799.5
72.00	1	19.603	19.561	- - -	18.915
	2	2095.57	2095.33	1735.77	2088.83
	3	11011.2	11010.4	10464.7	10971.8
	4	33719.7	33717.9	33261.5	33621.8
73.00	1	12.185	12.143	- - -	11.517
	2	2071.11	2070.86	1702.82	2064.43
	3	10960.5	10959.7	10400.7	10920.7
	4	33632.5	33630.8	33097.8	33535.9

Table 2. Eigenvalue for the clamped beam

$B = 0.0$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
74.00	1	4.740	4.696	- - -	4.101
	2	2046.62	2046.37	1669.78	2039.78
	3	10909.8	10908.9	10336.5	10870.1
	4	33545.5	33545.7	32997.1	33445.2

Table 2. Eigenvalue for the clamped beam

$B = 1.0$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
5.0	1	407.759	407.758	404.794	407.663
	2	3457.42	3457.42	3451.38	3456.75
	3	13875.4	13875.4	13863.4	13872.2
	4	38656.6	38656.5	38645.4	38648.7
10.0	1	313.783	313.779	302.702	313.598
	2	3109.78	3109.76	3084.36	3108.52
	3	13132.5	13132.4	13087.2	13126.2
	4	37368.7	37368.6	37325.8	37354.4
15.0	1	218.460	218.451	192.281	218.202
	2	2760.52	2760.47	2700.07	2758.67
	3	12388.9	12388.7	12292.5	12379.5
	4	36080.3	36080.0	35975.2	36058.3
20.0	1	121.567	121.552	74.571	121.262
	2	2409.53	2409.45	2295.25	2407.26
	3	11644.6	11644.3	11482.5	11632.4
	4	34791.4	34790.8	34611.8	34761.8
22.0	1	82.309	82.292	22.824	82.001
	2	2268.63	2268.54	2135.06	2266.17
	3	11346.8	11346.4	11154.9	11333.5
	4	34275.6	34274.9	34051.7	34243.6

Table 2. Eigenvalue for the clamped beam

$B = 1.0$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
25.0	1	22.822	22.801	- - -	22.515
	2	2056.73	2056.61	1865.36	2054.05
	3	10899.9	10899.4	10574.8	10885.1
	4	33501.9	33501.0	33194.9	33465.6
25.6	1	10.832	10.810	- - -	10.502
	2	2014.26	2014.14	1814.86	2011.50
	3	10810.6	10810.0	10472.3	10795.1
	4	33347.2	33346.2	33026.9	33310.1

Table 2. Eigenvalue for the clamped beam

$B = -0.25$

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
10.0	1	469.394	469.394	467.968	469.193
	2	3687.69	3687.69	3684.62	3686.35
	3	14369.5	14369.5	14364.0	14363.1
	4	39513.9	39513.9	39508.5	39498.7
20.0	1	437.367	437.366	431.682	436.975
	2	3570.41	3570.40	3558.22	3567.84
	3	14119.7	14119.7	14097.8	14107.3
	4	39082.3	39082.2	39060.8	39051.9
40.0	1	370.529	370.523	347.856	369.834
	2	3331.41	3331.36	3283.41	3326.61
	3	13615.1	13615.0	13529.4	13591.1
	4	38213.6	38213.3	38129.0	38156.6
60.0	1	299.567	299.551	244.237	298.689
	2	3086.33	3086.22	2949.91	3079.92
	3	13103.8	13103.5	12914.7	13069.2
	4	37337.9	37337.2	37149.8	37257.5
80.0	1	223.878	223.848	116.025	222.877
	2	2835.04	2834.84	2625.53	2826.75
	3	12585.8	12585.3	12256.2	12541.8
	4	36455.1	36453.8	36081.7	36344.4

Table 2. Eigenvalue for the clamped beam

B = - 0.25

F	Order	Upper bound by Rayleigh Ritz	Eigenvalue R	Lower bound by Kato's method	Lower bound by intermediate problem
100.0	1	142.697	142.646	- - -	
	2	2577.45	2577.13	2255.24	
	3	12061.2	12060.3	11555.8	
	4	35565.4	35563.4	34990.1	
120.0	1		54.9553		
	2		2313.10		
	3		11528.8		
	4		34665.8		
130.0	1		8.28584		
	2		2178.76		
	3		11260.7		
	4		34214.5		

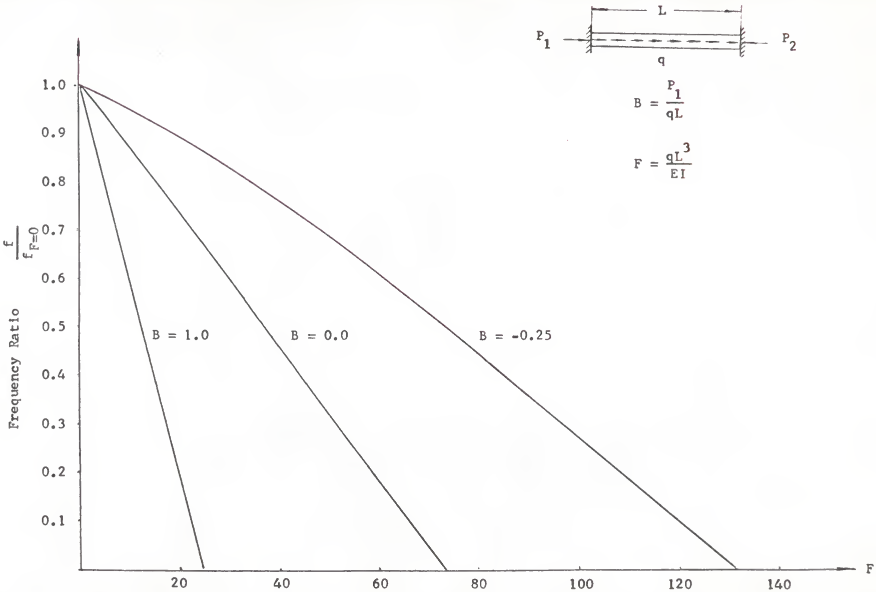

Figure 3. Effect of Axial Loads on the Fundamental Frequency of a Clamped Beam.

and

$$X'''(1) \;=\; -\,F(B+1)X'(1)\ . \tag{32c}$$

(i) For the general case when $B \neq 0$ and $F \neq 0$, substitution of Eq. (23) into Eqs. (32) yields

$$a_2 \;=\; 0\ , \tag{33a}$$

$$[3]_3 a_3 \;=\; -\,FBa_1\ , \tag{33b}$$

$$a_0X_0''(F,B,R,1) + a_1X_1''(F,B,R,1) + a_2X_2''(F,B,R,1) + a_3X_3''(F,B,R,1) \;=\; 0, \tag{33c}$$

$$\begin{aligned} &a_0X_0'''(F,B,R,1) + a_1X_1'''(F,B,R,1) + a_2X_2'''(F,B,R,1) + a_3X_3'''(F,B,R,1) \\ &\quad = -\,F(B+1)\big[a_0X_0'(F,B,R,1) + a_1X_1'(F,B,R,1) + a_2X_2'(F,B,R,1) \\ &\qquad + a_3X_3'(F,B,R,1)\big]\ . \end{aligned} \tag{33d}$$

Substitution of Eqs. (33a) and (33b) into Eqs. (33c) and (33d) gives

$$a_0X_0'' + a_3\Big[X_3'' - \frac{[3]_3}{FB}X_1''\Big] \;=\; 0, \tag{33e}$$

and

$$\begin{aligned} &a_0\big[X_0''' + F(B+1)X_0'\big] + a_3\Big[X_3''' - \frac{[3]_3}{FB}X_1''' + F(B+1)X_1' \\ &\quad - \frac{[3]_3}{FB}F(B+1)X_1'\Big] \;=\; 0\ . \end{aligned} \tag{33f}$$

Thus the characteristic equation for free-free beam is

$$\begin{aligned} &X''\,X_3 + F(B+1)X_3' - \frac{[3]_3}{FB}\big[X_1 + F(B+1)X_1'\big] \\ &\quad - \Big[X_3'' - \frac{[3]_3}{FB}X_1''\Big]\big[X_0 + F(B+1)X_0'\big] \;=\; 0\ . \end{aligned} \tag{34}$$

(ii) When B = 0, the boundary conditions of Eqs. (32) become

$$X''(0) = X''(1) = 0 , \tag{35a}$$

$$X(0) = 0 \tag{35b}$$

and

$$X(1) = -FX'(1) . \tag{35d}$$

Substitution of Eq. (23) into Eqs. (35) yields

$$a_2 = 0 , \tag{36a}$$

$$a_3 = 0 , \tag{36b}$$

$$a_0X_0''(F,B,R,1) + a_1X_1''(F,B,R,1) = 0 \tag{36c}$$

and

$$a_0\left[X_0'''(F,B,R,1) + FX_0'(F,B,R,1) + a_1\left[X_1'''(F,B,R,1) + FX_1'(F,B,R,1)\right] = 0 . \tag{36d}$$

The characteristic equation is

$$X_0''\left[X_1''' + FX_1'\right] - X_1''\left[X_0''' + FX_0'\right] = 0 . \tag{37}$$

(iii) When F = 0, B = 0, the boundary conditions, Eqs. (32), become

$$X_0''(0) = X''(1) = X'''(0) = X'''(1) = 0 . \tag{38}$$

Substitution of Eq. (23) into Eq. (38) gives

$$a_2 = 0 , \tag{39a}$$

$$a_3 = 0 , \tag{39b}$$

$$a_0X_0''(F,B,R,1) + a_1X_1''(F,B,R,1) = 0 \tag{39c}$$

and

$$a_0X_0'''(F,B,R,1) + a_1X_1'''(F,B,R,1) = 0 \; . \qquad (39d)$$

The characteristic equation is

$$X_0''X_1''' - X_1''X_0''' = 0 \; . \qquad (40)$$

The third case in this report is to present eigenvalues for a free-free beam by the same method as those obtained for a simply supported beam and a clamped beam. Translational and rotational modes corresponding to a zero natural frequency will exist if the beam is free on both ends. Using a strict definition of stability, such motion would be unstable. However, the type of instability in which failure occurs due to structural damage caused by bending is of primary importance. For this reason rigid body translation and rotation will not be considered to be unstable in a free-free beam case.

The results are presented in Table 3. The effect of the distributed axial loads on the first frequency of the free-free beam is again shown in Fig. 4, as before. When $F = 0$ and $B = 0$, this case reduces to a uniform free-free beam without axial load. Meirovitch [12] gave the same answers, $R = 500.564$ and 3803.54, for the first and second modes.

Table 3. Eigenvalues of the free-free beam

B = 0	Eigenvalue R		
F	1st mode	2nd mode	3rd mode
0.0	500.564	3803.54	14617.6
5.0	375.001	3530.10	14149.7
10.0	249.721	3255.28	13680.8
15.0	138.039	2980.76	13211.3
20.0	57.2011	2708.73	12742.0
22.5	28.5197	2574.50	12507.7
25.0	5.27823	2441.95	12273.6

B = +0.5	Eigenvalue R		
F	1st mode	2nd mode	3rd mode
1.0	450.910	3694.51	14430.7
3.0	350.721	3475.86	14056.5
5.0	250.080	3256.63	13682.0
7.0	151.181	3037.09	13307.3
9.0	59.5318	2817.60	12932.5
10.0	19.4391	2708.02	12745.1

B = 1.0	Eigenvalue R		
F	1st mode	2nd mode	3rd mode
1.0	426.031	3639.95	14337.2
3.0	275.375	3311.70	13775.9
5.0	124.763	2982.47	13214.0
6.0	52.2076	2817.70	12933.0
6.5	17.9028	2735.32	12792.5
6.75	1.53088	2694.15	12722.2

Table 3. Eigenvalues of the free-free beam

B = -0.25		Eigenvalue R	
F	1st mode	2nd mode	3rd mode
1.0	488.122	3776.26	14570.9
10.0	371.672	3528.15	14148.2
30.0	160.316	2975.04	13202.2
50.0	71.5717	2462.97	12261.3
70.0	14.0831	2033.98	11345.3
73.0	5.74743	1977.44	11211.2

B = -0.5		Eigenvalue R	
F	1st mode	2nd mode	3rd mode
1.0	500.498	3803.50	14617.6
20.0	476.067	3790.63	14608.6
40.0	421.064	3753.82	14582.1
60.0	368.555	3698.09	14538.7
80.0	329.750	3629.76	14480.6

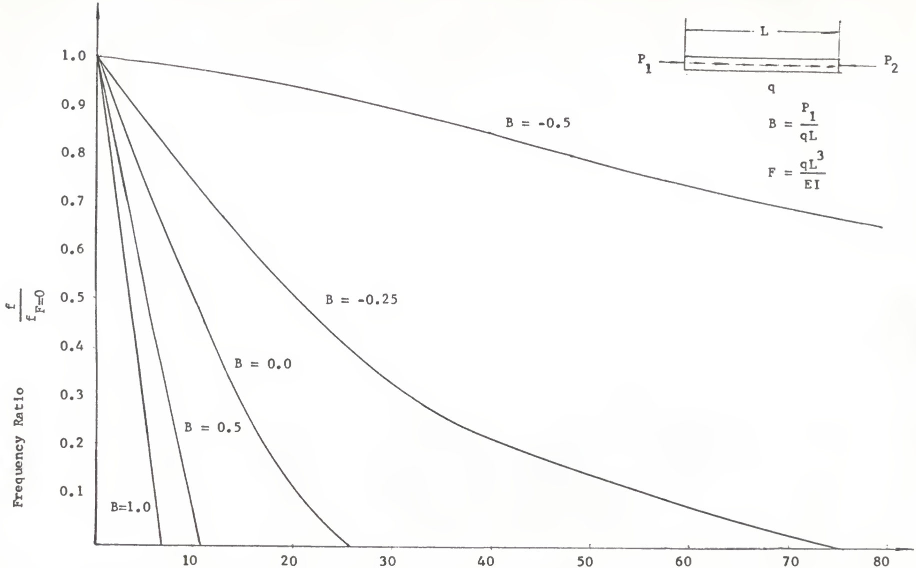

Figure 4. Effect of Axial Loads on the Fundamental Frequency of a Free-free Beam.

SUMMARY

In this report, a direct method of obtaining the natural frequencies of a uniform beam carrying linearly distributed axial loads is proposed. The governing differential equation with varying coefficients has been solved directly by Frobenius' method. The solution is in series form. It is not necessary to approach these problem by an indirect approximate method in seeking the upper and lower bounds. The eigenvalues of the simply supported, clamped and free-free beam carrying linearly distributed axial loads have been presented in Tables 1, 2, 3, for a range of loading parameters F and B.

It should be mentioned that the equations derived in this report are based on the small deflection theory by which it can be considered that the axial loads remain constant in magnitude during the beam's motion and that the supports are free to slide in the axial direction. The equations are also valid only for the lower modes because the effects of shearing deformation and rotatory inertia become [13,14] as the mode number increases.

The method presented in this report could be extended to the consideration of beams with other boundary conditions and to other problems, giving rise to differential equations with variable coefficients, such as the determination of natural frequencies and buckling loads of beams of varying cross sections. Information of this nature would be valuable to designers of high speed aerospace vehicles.

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THE EIGENVALUE PROBLEM FOR NATURAL FREQUENCY OF UNIFORM BEAM WITH LINEARLY VARYING AXIAL LOAD

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AN ABSTRACT OF A MASTER'S REPORT

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In this report the stability of a flexible uniform beam with linearly distributed axial loads and with constant end loads is considered. This problem is reduced to that of finding the eigenvalues of a fourth order ordinary differential equation with variable coefficients. By Frobenius' method the natural frequencies of vibration of the beam for bending modes are found as functions of the loading parameters. The numerical values of natural frequencies are evaluted by trial and error. The fundamental frequency vs. loading parameter curves is plotted for the simply supported, clamped and free-free beams. Values of the critical distributed axial loads are found at the intersections of these curves with the horizontal axis where the natural frequency equals zero.