THE STIELTJES INTEGRAL

by

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INTRODUCTION

Thomas Jean Stieltjes, born in Holland in 1856, made many significant contributions to mathematics in the last quarter of the nineteenth century. Among his interests were divergent and conditionally convergent series, Riemann's zeta function, the theory of numbers, and continued fractions. It was in connection with the latter that Stieltjes published a classical paper in 1894 entitled "Recherches sur les fractions continués". The publication contained a wealth of new ideas; among other things, it introduced a new concept of an integral, the modern "Stieltjes Integral".

Although the Stieltjes definition of the integral differs little from that of Riemann, the difference is important. This importance is due primarily to the capacity of the Stieltjes integral to take care of both sums and limits of sums. Consequently it has become an ideal tool in physical applications. Attention will be given later in this Report to applications to problems arising in physics and statistics.

Frequent reference will be made to the analogy between the Stieltjes integral and the Riemann integral. In such cases familiarity with properties of the Riemann integral and a knowledge of various proofs concerning such properties will be assumed. Similarly, some knowledge of the theory of functions of bounded variation will be assumed.

In this report the Stieltjes integral is considered only in respect to functions of a single real variable. Various extensions and modifications can be made in many of the theorems to include functions of more than one real variable and also to include functions of a complex variable.

EXISTENCE OF THE STIELTJES INTEGRAL

Definition 1.1. Let f(x) and g(x) be defined and bounded on a closed interval $[a, b]$. Let N be any subdivision of $[a, b]$: $N = (a = a_0 < a_1 < a_2 < \dots < a_n = b)$, with $a_{i-1} \leq x_i \leq a_i$, $i = 1, 2, \dots, n$. Let the norm $|N|$ be defined as the $\max_i (a_i - a_{i-1})$. Let $\Delta g_i = g(a_i) - g(a_{i-1})$. Then the Stieltjes integral of f(x) with respect to g(x) is defined:

$$I = \int_a^b f(x)dg(x) = \lim_{|N| \to 0} \sum_{i=1}^{n} f(x_i)\Delta g_i.$$

The limit may or may not exist depending on what functions are used. It is only when the limit exists that the integral is defined.

Before conditions on the functions can be discussed, it is necessary to consider some of the elementary properties of the Stieltjes integral and to prove a theorem that provides the basis for later proofs. The proofs of these properties, being almost identical with the corresponding ones for the Riemann integral, are consequently omitted.

In the following theorems k is a constant, the functions f(x) and g(x), with or without subscripts, are, respectively, continuous and monotonically increasing on $[a, b]$.

Theorem 1.1. $\int_a^b dg(x) = g(b) - g(a)$.

Theorem 1.2. $\int_a^b f(x)d[g(x) + k] = \int_a^b f(x)dg(x)$.

Theorem 1.3. $\int_a^b kf(x)dg(x) = k \int_a^b f(x)dg(x)$.

Theorem 1.4. $\int_a^b [f_1(x) + f_2(x)]dg(x) = \int_a^b f_1(x)dg(x) + \int_a^b f_2(x)dg(x)$.

Theorem 1.5. $\int_a^b f(x)d[g_1(x) + g_2(x)] = \int_a^b f(x)dg_1(x) + \int_a^b f(x)dg_2(x)$.

Theorem 1.6. $\int_a^b f(x)dg(x) = \int_a^c f(x)dg(x) + \int_c^b f(x)dg(x)$, $a < c < b$.

Theorem 1.7. $a < x < b$, $f_1(x) \leq f_2(x) \Rightarrow \int_a^b f_1(x)dg(x) \leq \int_a^b f_2(x)dg(x)$.

Theorem 1.8. $\left|\int_a^b f(x)dg(x)\right| \leq \int_a^b |f(x)|\, dg(x)$.

Theorem 1.9. $\left|\int_a^b f(x)dg(x)\right| \leq [g(b) - g(a)]\max|f(x)|$, $a \leq x \leq b$.

Theorem 1.10. $\int_a^b f(x)dg(x) = -\int_b^a f(x)dg(x)$.

Theorem 1.11. $\int_a^a f(x)dg(x) = 0$

With slight modifications the above theorems are also true if $g(x)$ is monotonically decreasing.

The following theorem is commonly called the "Integration by Parts Formula" for the Stieltjes integral. Practically, it offers a method of evaluation of the integral. Theoretically, it provides a basis for the development of general conditions on the functions to insure the existence of the Stieltjes integral.

Theorem 1.12. If $\int_a^b g(x)df(x)$ exists as a Stieltjes integral, then $\int_a^b f(x)dg(x)$ also exists.

Proof: Expansion of $\sum_{i=1}^{n} g(x_i) \left[f(a_i) - f(a_{i-1})\right]$ yields

$$g(x_1)\left[f(a_1) - f(a_0)\right] + g(x_2)\left[f(a_2) - f(a_1)\right] + \dots + g(x_n)\left[f(a_n) - f(a_{n-1})\right].$$

Letting $x_0 = a$ and $x_{n+1} = b$, the terms can be rearranged to yield

$$\sum_{i=1}^{n} g(x_i)\left[f(a_i) - f(a_{i-1})\right] = f(b)g(b) - f(a)g(a) - \sum_{i=0}^{n} f(a_i)\left[g(x_{i+1}) - g(x_i)\right].$$

Now taking the limit of both sides as $|N| \rightarrow 0$ yields

$$\int_a^b g(x)df(x) = f(b)g(b) - f(a)g(a) - \int_a^b f(x)dg(x).$$

The following two theorems insure the existence of the Stieltjes integral under the stated conditions.

Theorem 1.13. If, on the interval $[a, b]$, one of the functions $f(x)$ or $g(x)$ is continuous and the other monotonic, the integral $\int_a^b f(x)dg(x)$ exists.

Proof: As consequences of Theorem 1.12 and the similarity in behavior of monotonically increasing and decreasing functions, it can be assumed without loss of generality that $f(x)$ is continuous and $g(x)$ is monotonically increasing with $g(b) > g(a)$ (since the case $g(b) = g(a)$ is trivial).

The object is to find a number I which is approximated by all possible sums $\sum_{i=1}^{n} f(x_i)\,\Delta g_i$ associated with subdivisions of sufficiently small norm. For a given subdivision N, let m_i and M_i be the minimum and maximum values, respectively, of the continuous function f(x) for $a_{i-1} \leq x \leq a_i$, $i = 1, 2, \ldots, n$. For any choice of x_i such that $a_{i-1} \leq x_i \leq a_i$, $i = 1, 2, \ldots, n$, one has $m_i \leq f(x_i) \leq M_i$. Since g(x) is assumed to be monotonically increasing and $\Delta g_i \geq 0$, $i = 1, 2, \ldots, n$,

$$(1) \quad \sum_{i=1}^{n} m_i\,\Delta g_i \leq \sum_{i=1}^{n} f(x_i)\,\Delta g_i \leq \sum_{i=1}^{n} M_i\,\Delta g_i.$$

As is done in the theory of the Riemann integral, the extreme left-hand side and the extreme right-hand side of expression (1) will be referred to as the lower and upper sums, respectively, of the subdivision N, and denoted as L(N) and U(N).

In search of the desired number I, two numbers, I and J, will be defined and shown to be equal. Let I $\equiv$ sup(all lower sums) and let J $\equiv$ inf(all upper sums).

In order to establish the equality of the numbers I and J, it will be affirmed that $I \leq J$ and that if $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(2) \quad \left| \sum_{i=1}^{n} f(x_i)\,\Delta g_i - I \right| < \epsilon \quad ,$$

whenever $|N| < \delta$. By virtue of having uniform continuity of f(x) on $[a, b]$, inequalities (1) and $L(N) \leq I \leq J \leq U(N)$, the desired equality I = J and inequality (2) follow from the fact that if δ is chosen so small that $|x' - x''| < \delta$ implies

$$\left|f(x') - f(x'')\right| < \frac{\epsilon}{g(b) - g(a)} ,$$

then $|N| < \delta$ implies

$$U(N) - L(N) = \sum_{i=1}^{n} (M_i - m_i) \Delta g_i < \frac{\epsilon}{g(b) - g(a)} \sum_{i-1}^{n} \Delta g_i = \epsilon .$$

<u>Theorem 1.14.</u> If, on the interval $[a, b]$, one of the functions f(x) or g(x) is continuous and the other of bounded variation, the integral $\int_a^b f(x)dg(x)$ exists.

<u>Proof:</u> As a consequence of Theorem 1.12, it can be assumed that f(x) is continuous and g(x) is of bounded variation on $[a, b]$. If g(x) is of bounded variation, it can be expressed as the difference of two monotonically increasing functions[1]. Let $g(x) = g_1(x) - g_2(x)$. Consequently, by Theorem 1.13 and Theorem 1.3, both $\int_a^b f(x)dg_1(x)$ and $-\int_a^b f(x)dg_2(x)$ exist.

Hence, by Theorem 1.5,

$$\int_a^b f(x)dg_1(x) - \int_a^b f(x)dg_2(x) = \int_a^b f(x)d\left[g_1(x) - g_2(x)\right] = \int_a^b f(x)dg(x).$$

In the previous theorems, existence of the integral was established if at least one of the functions was continuous. The continuity

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[1] R. L. Jeffery, <u>The Theory of Functions of a Real Variable</u>, p. 121.

restriction is sufficient but not necessary. However it is necessary that the two functions not have common discontinuities, as is illustrated by the following theorem.

Theorem 1.15. If f(x) and g(x) both have a non-removable discontinuity at a point c, where $a < c < b$, then $\int_a^b f(x)dg(x)$ does not exist.

Proof: For $a < c < b$ and c a non-removable discontinuity for g(x), let N be a subdivision not containing c and let $a_{k-1} < c < a_k$. Then $|\Delta g_k|$ can always be made greater than or equal to Q, a fixed positive number. Also, for two suitable numbers, x_k and x_k', on $[a_{k-1}, a_k]$,

$$\left|f(x_k) - f(x_k')\right| \geq R,$$

with R some fixed positive number, regardless of how small $a_k - a_{k-1}$ is made. Hence, for any $\delta > 0$, there is a subdivision N for which $|N| < \delta$, and for which there exist sums $\sum f(x_i)\Delta g_i$ and $\sum f(x_i')\Delta g_i$ which differ numerically by at least QR. Consequently, since the limit does not exist, the Stieltjes integral does not exist.

PROPERTIES OF THE STIELTJES INTEGRAL

The following theorems establish some of the most important properties of the Stieltjes integral. In addition, the differential nature of the symbol $dg(x)$ is established, and the conversion of certain Stieltjes integrals to Riemann integrals is discussed.

Theorem 2.1. If $f(x)$ is defined and $g(x)$ is differentiable at every point of a closed interval $[a, b]$ and if $f(x)$ and $g'(x)$ are Riemann integrable there, then $f(x)$ is Stieltjes integrable with respect to $g(x)$ there and

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

Proof: Let $I = \int_a^b f(x)dg(x)$. The Stieltjes definition of integrability provides that for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(x_i) \left[g(a_i) - g(a_{i-1})\right] - I \right| < \epsilon ,$$

whenever $|N| < \delta$.

Now, since $g(x)$ is differentiable at every point of the closed interval, the law of the mean is applicable and as a consequence

$$\left| \sum_{i=1}^{n} f(x_i)g'(x_i')\Delta x_i - I \right| < \epsilon ,$$

with $x_i < x_i' < x_{i-1}$. As $f(x)$ and $g'(x)$ are integrable on $[a, b]$, and ϵ is arbitrary, an application of Bliss' Theorem[2] yields

$$\lim_{|N|\to 0} \sum_{i=1}^{n} f(x_i)g'(x_i')\Delta x_i = \int_a^b f(x)g'(x)dx.$$

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[2]John M. H. Omsted, Real Variables, p. 149.

Since the absolute value of the difference between $\int_a^b f(x)dg(x)$ and $\int_a^b f(x)g'(x)dx$ can be made less than any positive ϵ , the two integrals must be equal. Hence the theorem is established.

Theorem 2.1 provides a ready way of evaluating a certain type of Stieltjes integral in terms of a Riemann integral. If $g(x) = x$, a special case occurs, resulting in $\int_a^b f(x)dg(x) = \int_a^b f(x)dx$, where the integral on the right is the ordinary Riemann integral.

As in the Riemann theory of integration, there are two very useful mean value theorems for the Stieltjes integral. As corollaries, the familiar laws of the mean for Riemann integrals will be obtained.

<u>Theorem 2.2.</u> (First Mean Value Theorem) If $f(x)$ is continuous and $g(x)$ is monotonically increasing on $[a, b]$, then there exists a point t', $a \leq t' \leq b$, such that

$$(1) \qquad \int_a^b f(x)dg(x) = f(t')\left[g(b) - g(a)\right].$$

<u>Proof:</u> Since $g(x)$ and $f(x)$ are integrable over $[a, b]$ and since $f(x)$ is bounded over $[a, b]$, let $M = \sup\left[f(x)\right]$ and $m = \inf\left[f(x)\right]$ for $a \leq x \leq b$. Then

$$m\left[g(b) - g(a)\right] \leq \int_a^b f(x)dg(x) \leq M\left[g(b) - g(a)\right].$$

Hence there exists a p such that $m \leq p \leq M$ and

$$\int_a^b f(x)dg(x) = p\,[g(b) - g(a)]\,.$$

Since f(x) is continuous over $[a, b]$, there exists a point t', on $[a, b]$, such that $p = f(t')$. Thus

$$\int_a^b f(x)dg(x) = f(t')\,[g(b) - g(a)]$$

or, when written in integral form,

$$\int_a^b f(x)dg(x) = f(t') \int_a^b dg(x).$$

Corollary 2.2. Let f(x) and h(x) be continuous on $[a, b]$ and let $h(x) \geq 0$ on $[a, b]$. Then there exists a t', $a \leq t' \leq b$, such that

$$(2) \qquad \int_a^b f(x)h(x)dx = f(t') \int_a^b h(x)dx.$$

Proof: Set $g(x) = \int_a^x h(t)dt$, $a \leq x \leq b$. By Theorem 2.2 equations (1) and (2) are equivalent. Clearly g(x) is non-decreasing since $h(x) \geq 0$, and the conclusion of the Corollary is immediate.

Theorem 2.3. (Second Mean Value Theorem) Let f(x) and g(x) be defined on $[a, b]$ with f(x) monotonically increasing on $[a, b]$ and g(x) continuous over $[a, b]$. Then there exists a t' on $[a, b]$ such that

$$\int_a^b f(x)dg(x) = f(a)\,[g(t') - g(a)] + f(b)\,[g(b) - g(t')]\,.$$

Proof: By Theorem 1.12 $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ both exist. The Integration by Parts Formula yields

$$\int_a^b f(x)dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)df(x).$$

By the First Mean Value Theorem there exists a t' on $[a, b]$ such that

$$\int_a^b g(x)df(x) = g(t') \left[f(b) - f(a)\right].$$

Thus

$$\int_a^b f(x)dg(x) = f(b)g(b) - f(a)g(a) - g(t') \left[f(b) - f(a)\right]$$

$$= f(a) \left[g(t') - g(a)\right] + f(b) \left[g(b) - g(t')\right].$$

When expressed in terms of integrals,

$$\int_a^b f(x)dg(x) = f(a) \int_a^{t'} dg(x) + f(b) \int_{t'}^b dg(x).$$

Corollary 2.3. If f(x) and h(x) are continuous on $[a, b]$ and if f(x) is monotonically increasing on $[a, b]$, then there exists a t', $a \leq t' \leq b$, such that

(3) $$\int_a^b f(x)h(x)dx = f(a) \int_a^{t'} h(x)dx + f(b) \int_{t'}^b h(x)dx.$$

Proof: If g(x) is defined as in Corollary 2.2, the result follows immediately from Theorem 2.3. Equation (3) is sometimes referred to as the Weierstrass form of Bonnet's Theorem[3].

Many of the theorems from Riemann integration can be extended to the Stieltjes integral. One such extension results in the following theorem which is analogous to the familiar Duhamel's theorem for Riemann integrals.

Theorem 2.4. If f(x) and g(x) are continuous on $[a, b]$, h(x) monotonically increasing on $[a, b]$ and $N = \{a = x_0 < x_1 < \dots < x_n = b\}$ is a subdivision of $[a, b]$ with $x_{k-1} \leq r_k \leq x_k$, $x_{k-1} \leq s_k \leq x_k$ for $k = 1, 2, \dots, n$, then

$$\lim_{|N| \to 0} \sum_{k=1}^{n} f(r_k) g(s_k) \left[h(x_k) - h(x_{k-1})\right] = \int_a^b f(x) g(x) dh(x).$$

Proof: Let $\sigma_1 = \lim_{|N| \to 0} \sum_{k=1}^{n} f(r_k) g(s_k) \left[h(x_k) - h(x_{k-1})\right]$ and

$$\sigma_2 = \lim_{|N| \to 0} \sum_{k=1}^{n} f(s_k) g(s_k) \left[h(x_k) - h(x_{k-1})\right].$$

The expressions, σ_1 and σ_2, differ only in respect to having a different argument for f(x). Since f(x) is continuous on $[a, b]$, it is also uniformly continuous on $[a, b]$. Hence $|f(r_k) - f(s_k)| < \epsilon$ if δ is chosen sufficiently small. Therefore,

$$\left|\sigma_1 - \sigma_2\right| = \left|\sum_{k=1}^{n} \left[f(r_k) - f(s_k)\right] \left[g(s_k)\right] \left[h(x_k) - h(x_{k-1})\right]\right|$$

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[3]David V. Widder, Advanced Calculus, p. 138.

$$\left|\sigma_1 - \sigma_2\right| < \epsilon \sum_{k=1}^{n} |g(s_k)| \left[h(x_k) - h(x_{k-1})\right]$$

$$< \epsilon \left[h(b) - h(a)\right] \max_{a \leq x \leq b} \left|g(x)\right| .$$

Hence $\lim_{|N| \to 0} \sigma_1 = \lim_{|N| \to 0} \sigma_2 = \int_a^b f(x)g(x)dh(x)$. Thus the theorem is established. If $h(x) = x$, the theorem reduces to the conventional form of Duhamel's principle.

THE PROBLEM OF MOMENTS

In "Recherches sur les fractions continués" Stieltjes proposed and solved completely the following problem which he called the "Problem of Moments"[4]:

Find a bounded non-decreasing function $g(x)$ on the interval $(0, \infty)$ such that its moments,

$$\int_0^\infty x^n dg(x), \quad n = 0, 1, 2, \ldots ,$$

have a prescribed set of values

$$\int_0^\infty x^n dg(x) = U_n , \quad n = 0, 1, 2, \ldots . \tag{1}$$

Stieltjes frequently used such concepts as mass and stability from mechanics in solving analytical problems. In accordance

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[4] J. A. Shohat and J. D. Tamarkin, <u>The Problem of Moments</u>, p. vii.

with this, $dg(x)$ can be considered as a mass distributed over $[x, x + dx]$ so that $\int_0^x dg(t)$ represents the mass distribution over the segment $[0, x]$. Then

$$\int_0^\infty x\,dg(x) \quad \text{and} \quad \int_0^\infty x^2 dg(x)$$

represent, respectively, the first (statical) moment and the second moment (moment of inertia) with respect to 0 of the total mass $\int_0^\infty dg(x)$ distributed over the real semi-axis $[0, \infty)$.

Generalizing, Stieltjes called $\int_0^\infty x^n dg(x)$ the n-th moment, with respect to 0, of the given mass distribution characterized by the function $g(x)$.

Stieltjes made the solution of the Moments-Problem (1) dependent upon the nature of the continued fraction corresponding to the integral

$$(2) \qquad I(z, g) = \int_0^\infty \frac{dg(y)}{z + y} \sim \frac{U_0}{z} - \frac{U_1}{z^2} + \frac{U_2}{z^3} - \frac{U_3}{z^4} + \ldots$$

$$\sim \frac{1|}{|a_1 z} + \frac{1|}{|a_2} + \frac{1|}{|a_3 z} + \frac{1|}{|a_4} + \ldots$$

and upon the closely related "associated" continued fraction

$$(3) \qquad \frac{\lambda_1|}{|z + c_1} - \frac{\lambda_2|}{|z+c_2} - \frac{\lambda_3|}{|z+c_3} - \ldots$$

derived from (2) by "contraction":

$$z - \frac{r|}{|1} - \frac{s|}{|z - t} = z - r - \frac{rs}{z - (s + t)}$$

Making use of the theory of continued fractions, he showed that in (2) all the a_i are positive.

He further showed that this necessary condition is also sufficient for the existence of a solution of the Problem of Moments (1). In terms of the given sequence $\{U_n\}$ this condition is equivalent to the positiveness of the following determinants:

$$\Delta_n = \begin{vmatrix} U_0 & U_1 & \cdots & U_n \\ U_1 & U_2 & \cdots & U_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ U_n & U_{n+1} & & U_{2n} \end{vmatrix} \qquad n = 0, 1, 2, \ldots$$

$$\Delta_{n_1} = \begin{vmatrix} U_1 & U_2 & \cdots & U_n & U_{n+1} \\ U_2 & U_3 & \cdots & U_{n+1} & U_{n+2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ U_{n+1} & U_{n+2} & & U_{2n} & U_{2n+1} \end{vmatrix} \qquad n = 0, 1, 2, \ldots$$

If the solution is unique, the problem is referred to as a "determinate" Moment-Problem. If there is more than one solution, in which case there are infinitely many solutions, the Moment-Problem is said to be "indeterminate". Stieltjes further gave an effective construction for certain solutions of the Moment-Problem which in the indeterminate case turns out to possess important minimal properties. Here the denominators of the successive approximants to the continued fractions (2) and (3) play an important role. In passing, Stieltjes introduced an important new proposition dealing with the convergence of series of functions of a complex variable (Stieltjes-Vitali Theorem) which leads to a complete solution of the problem of convergence of the continued fraction (2) in the complex plane. Here Stieltjes showed that the Moment-Problem (1) is determinate or indeterminate depending on whether the continued fraction (2) is convergent or divergent.

PHYSICAL APPLICATIONS

As was indicated earlier in this Report, the Stieltjes integral is very useful in the definition of certain physical concepts which involve a combination of discrete distributions and continuous distributions. In particular, distributions of mass can be expressed in terms of a Stieltjes integral.

As an illustration of this, assume the physical notion of mass is undefined in our mathematical system. It should be noted that the mathematical situation about to be described can be closely approximated by a physical one in which mass is well defined. A particle can be approximated by a small pellet of matter, and a curve with a mass distribution can be realized by a fine wire of heavy material. The masses of these physical objects can be determined by weighing.

Consider a plane curve whose equations can be given parametrically, the arc length s being the parameter:

$$x = x(s) \quad \text{and} \quad y = y(s).$$

Assume that $x(s)$ and $y(s)$ are both continuous on $[0, L]$ where L is the total length of the curve. The position of a point on the curve can be determined by a single coordinate s. A particle on the curve is to be thought of as a quantity of mass situated at a geometrical point of the curve. It may be defined mathematically as follows:

Definition 4.1. A particle of mass, m, at a point, s, of the curve $x = x(s)$, $y = y(s)$, is the number pair (s, m).

Definition 4.2. A distribution of mass on the curve $x = x(s)$, $y = y(s)$, is a function, $M(s)$ such that

$$M(0) = 0, \quad M(s) \text{ is non-decreasing on } [0, L].$$

The mass of the segment of the curve between any two points $s = a$ and $s = b$ $(0 \leq a < b \leq L)$ is

$$M(b) - M(a). \tag{1}$$

If, for example, the distribution consists entirely of n particles

(2) $$(s_k, m_k), \quad k = 1, 2, \ldots, n$$

where $0 < s_1 < s_2 < \ldots < s_n \leq L$, then

(3)
$$\begin{aligned} M(s) &= 0 & 0 \leq s < s_1 \\ &= m_1 & s_1 \leq s < s_2 \\ &= m_1 + \ldots + m_{n-1} & s_{n-1} \leq s < s_n \\ &= m_1 + \ldots + m_n & s_n \leq s \leq L\ . \end{aligned}$$

That is, $M(s)$ is a step function with jump m_k at the point s_k. Adopting the convention that a particle situated at the point b of Definition 4.2 is to belong to the segment (a, b) and a particle at a is not to belong, the mass of the segment (a, b) is given by (1) where $M(s)$ is described by equations (3). The total mass of the wire is $M(L)$. The mass of the particle at s_k is

$$m_k = M(s_k) - M(s_{k-1}), \quad k = 1, 2, \ldots, n.$$

<u>Definition 4.3.</u> The density of distribution $M(s)$, at a point a, is $M'(a_+)$, whenever this right-hand derivative exists.

This latter definition conforms to the intuitive notion of density. Average density of a wire is thought of as mass per unit length. The average density of the arc (a, b) of Definition 4.2 is

$$\frac{M(b) - M(a)}{b - a},$$

and the limit of this as b approaches a is $M'(a_+)$. For a continuous distribution, the total mass is the integral of the density

$$M(L) = \int_0^L M'(s)ds.$$

This follows from the fact that M(s) is absolutely continuous. For an arbitrary distribution, another formula can be written using the Stieltjes integral

$$M(L) = \int_0^L dM(s).$$

Proceed next to the determination of the moment of inertia about an axis of a mass distribution. For the set (2) it is

$$I = \sum_{k=1}^{n} m_k r_k^2,$$

where r_k is the distance of the particle (s_k, m_k) from the axis. Now proceeding to find the moment of inertia about the x-axis of a distribution M(s) on the curve $x = x(s)$, $y = y(s)$, let the points $\{s_k\}_0^n$ be a subdivision N of the interval $0 \leq s \leq L$. If a total mass is divided into separate parts, the moment of inertia of the whole is the sum of the moments of inertia of the parts. Therefore the moment of inertia desired will be

$$I = \sum_{k=1}^{n} I_k ,$$

where I_k is the moment of inertia of the arc (s_{k-1}, s_k). Let

$$y(s_k'') = \max_{s_{k-1} \leq s \leq s_k} |y(s)| ,$$

and

$$y(s_k') = \min_{s_{k-1} \leq s \leq s_k} |y(s)| .$$

The mass of the arc (s_{k-1}, s_k) is $M(s_k) - M(s_{k-1})$. If this mass were concentrated in a particle at s_k' or s_k'', mass would have been moved nearer to or farther from the x-axis, respectively. The moment of inertia is increased if mass is moved farther from the axis. Consequently

$$\sum_{k=1}^{n} y^2(s_k') \left[M(s_k) - M(s_{k-1})\right] \leq \sum_{k=1}^{n} I_k \leq \sum_{k=1}^{n} y^2(s_k'') \left[M(s_k) - M(s_{k-1})\right] .$$

By Theorem 1.13 both extremes of these inequalities approach the same limit as $|N| \to 0$. Hence

$$I = \int_0^L y^2(s)dM(s).$$

To illustrate this consider the following example.

<u>Example.</u> Let the parametrically represented curve be the straight line

$$y = \frac{s}{\sqrt{2}} , \quad x = 1 - \frac{s}{\sqrt{2}} , \qquad 0 \leq s \leq \sqrt{2} .$$

Let the distribution be a combination of a continuous one in which the density is proportional to the distance from the end point $s = 0$ and a discrete one consisting of the two particles $\left(\frac{1}{\sqrt{2}}, 2\right)$, $(\sqrt{2}, 4)$. More explicitly

$$M(s) = M_1(s) + M_2(s) ,$$

where $M_1(s) = \int_0^s t\,dt = \frac{s^2}{2}$;

$$\begin{aligned} M_2(s) &= 0 , & 0 \leq s < \tfrac{1}{\sqrt{2}} \ ; \\ &= 2 , & \tfrac{1}{\sqrt{2}} \leq s < \sqrt{2} \ ; \\ &= 6 , & s = \sqrt{2} \ . \end{aligned}$$

Then the moment of inertia about the x-axis is given by

$$\begin{aligned} I &= \int_0^{\sqrt{2}} \frac{s^2}{2} dM_1(s) + \int_0^{\sqrt{2}} \frac{s^2}{2} dM_2(s) \\ &= \int_0^{\sqrt{2}} \frac{s^3}{2} ds + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 2 + \frac{1}{2}(\sqrt{2})^2 4 = 5 . \end{aligned}$$

The Stieltjes integral is also used quite extensively in statistics. Again, much of its importance is due to its capacity to take care of both sums and limits of sums. As a consequence it is unnecessary to state theorems once for a discontinuous distribution and again for a continuous distribution.

To indicate other uses of the integral, suppose $F(x)$ is a distribution function. Let $g(x)$ be a continuous function in the range of F(x), which will be taken to be finite, a to b. Divide the range into n parts at the points $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. Take t_i to be on the i-th subinterval. Then

$$S = g(t_1)\left[F(x_1) - F(a)\right] + g(t_2)\left[F(x_2) - F(x_1)\right] + \ldots + g(t_n)\left[F(b) - F(x_{n-1})\right].$$

As the norm of the subdivision approaches zero, the limit can be written as the Stieltjes integral

$$\int_a^b g(x)dF(x) \ . \tag{1}$$

Then if F(x) is the distribution function of a distribution possessing a continuous frequency function, the Stieltjes integral becomes the ordinary Riemann integral

$$\int_a^b g(x)f(x)dx$$

where

$$\begin{aligned} dF(x) &= F'(x)dx \\ &= f(x)dx \quad \text{in (1)} \ . \end{aligned}$$

If F(x) is the distribution function of a discontinuous distribution, the sum S must tend to a limit. Then the Stieltjes integral includes such a summation as a particular case.

The Stieltjes integral can also be used in defining the moments of a distribution,

$$U'_r = \int_{-\infty}^{\infty} x^r dF(x) ,$$

or in defining the moments about the mean,

$$U_r = \int_{-\infty}^{\infty} (x - U'_1)^r dF(x).$$

Also it enables one to see the relationship between various test statistics when these statistics are expressed in terms of Stieltjes integrals. In this case too, much of the statistical importance of the Stieltjes integral lies in its descriptive value.

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THE STIELTJES INTEGRAL

by

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AN ABSTRACT OF A MASTER'S REPORT

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The primary purpose of this Report is to give a brief analysis of the Stieltjes integral. Although the development is restricted to functions of a single real variable, modifications and extensions can readily be made to include functions of two real variables and also functions of a complex variable. The analysis consists not only of considering conditions on the functions insuring the existence of the Stieltjes integral, but also of considering conditions under which the Stieltjes integral is not defined. Some of the special properties of the Stieltjes integral, such as the Integration by Parts Formula, are developed.

Since the Stieltjes integral is a generalization of the Riemann integral, analogous properties such as the Mean Value Theorem and Duhamel's principle are developed.

The method Stieltjes used in first introducing this particular type of integral in the paper he published on continued fractions is included. In addition, this discussion includes a consideration of the Problem of Moments as Stieltjes analyzed it.

The ability of the Stieltjes integral to include both sums and limits of sums makes it an ideal tool in physical applications. As a result, many times it is possible to take care of "distributions" that are partly discrete and partly continuous with a single formula. For this same reason the Stieltjes integral is extremely important in theoretical mathematics.