SOLUTION OF THE TORSION PROBLEM BY APPROXIMATE CONFORMAL MAPPING

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NOMENCLATURE

C	Boundary of shaft cross section.
R	Interior of C.
S	Unit circle.
U	Interior of S.
x, y	Coordinates in plane of cross section.
z	The complex variable $x + iy$.
ξ, η	Coordinates in a second plane.
ζ	The complex variable $\xi + i\eta$
$\bar{z}$	$x - iy$.
a_n	Coefficients in polynomial in ζ.
$\omega(\zeta)$	$\sum_{n=0}^{k} a_n \zeta^n$.
θ	Polar angle.
σ	$e^{i\theta}$.
r_j	$\lvert z_j \rvert$.
u, v, w	Displacement of a material point of shaft.
ϕ	Warping function.
α	Unit angle of twist of shaft.
τ_{pq}	Shearing stress on the p plane in the q direction, $p, q = x, y, z$.
G	Modulus of rigidity of shaft material.
N	Normal to C.
D	Torsional rigidity.

NOMENCLATURE concl.

I_o, D_o	Complex constants such that $D = I_o + D_o$.
M	Applied torque.
ψ	Harmonic conjugate to ϕ.
$F(z)$	$\phi(x, y) + i\psi(x, y)$.
b_n	$\sum_{r=n}^{k} r\, a_r\, \overline{a}_{r-n}$.
c_n	$\sum_{r=n}^{k} a_r\, \overline{a}_{r-n}$.
d_n	$\sum_{r=n}^{k} (r-n)\, \overline{a}_r\, a_{r-n}$.

INTRODUCTION

The problem considered in this report is torsion of an isotropic cylindrical shaft. The torsion problem was one of the first problems considered by early workers in the theory of elasticity such as Coulomb, Cauchy, and Saint-Venant. It has been treated by many writers and the general theory is well known. However, solution of the problem for irregularly shaped cylindrical shafts by the classical approach is difficult if not practically impossible in many cases.

Classically, the problem is considered as the second boundary value problem of potential theory or Neumann's problem. By introducing functions of a complex variable, the problem is reduced to the first boundary value problem of potential theory or Dirichlet's problem.

Mapping the cross section of the cylindrical shaft conformally onto the unit disc greatly simplifies solution of the problem, and a solution is obtained immediately. The original problem is thus reduced to finding the desired mapping function.

Instead of seeking the true mapping function, a polynomial is used to map the cross section onto the unit disc approximately. Using the approximate mapping function, the solution is obtained directly for a large class of cylindrical shafts.

APPROXIMATE CONFORMAL MAPPING

The function $z = \omega(\zeta)$ which maps the curve C into the unit circle S as shown in Fig. 1, and hence maps R into U, is, in general, very difficult to obtain. The object here is to develop a general expression $\omega(\zeta)$ which will map C into S, approximately.

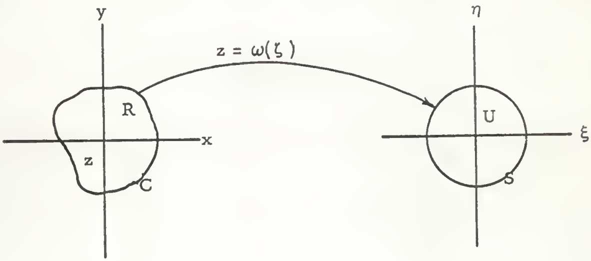

Fig. 1. Map of curve onto unit circle.

The approximating function employed is a finite polynomial of the form

$$\omega(\zeta) = \sum_{n=0}^{k} a_n \zeta^n \tag{1}$$

where the a_n are, in general, complex numbers.

The method used here to construct the mapping function is one of collocation; i. e., a procedure which forces the polynomial of equation (1) to map accurately $k + 1$ points of C into the circle S. These $k + 1$ points are called collocation points. The remaining points of C are mapped approximately into points of S.

The collocation points are chosen as the points of intersection of rays (See Fig. 2)

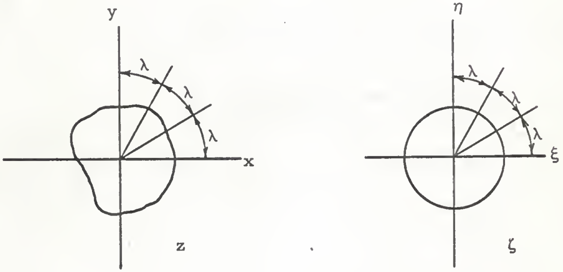

Fig. 2. Choice of collocation points.

whose arguments are $n\lambda$, $n = 0, 1, \ldots, k$ with the two curves C and S. The angle λ is chosen so that $(k+1)\lambda = 2\pi$. This is expressed in equation form as

$$z_j = \sum_{n=0}^{k} a_n \sigma_j^n , \tag{2}$$

where

$$z_j = |z_j| e^{ij\lambda} = r_j e^{ij\lambda} \text{ and } \sigma_j = e^{ij\lambda} ,$$

so Eq. (2) is

$$r_j e^{ij\lambda} = \sum_{n=0}^{k} a_n e^{ijn\lambda} . \tag{3}$$

Multiplying both sides of Eq. (3) by $e^{-ijm\lambda}$ yields

$$r_j e^{ij\lambda} \; e^{-ijm\lambda} = \sum_{n=0}^{k} a_n e^{ijn2} \; e^{-ijm\lambda}. \tag{4}$$

Summing both sides of Eq. (4) with respect to j yields

$$\sum_{j=0}^{k} r_j e^{ij(1-m)\lambda} = \sum_{j=0}^{k} \sum_{n=0}^{k} a_n \; e^{ij(n-m)\lambda}, \quad \text{or}$$

$$\sum_{j=0}^{k} r_j e^{ij(1-m)\lambda} = \sum_{n=0}^{k} \sum_{j=0}^{k} e^{ij(n-m)\lambda}. \tag{5}$$

The change in order of summation is permitted since only finite sums are involved.

The sum on j in the right-hand side of Eq. (5) is simply a finite geometric series and can be expressed in closed form as

$$\sum_{j=0}^{k} e^{ij(n-m)\lambda} = \frac{1 - e^{i(n-m)(k+1)\lambda}}{1 - e^{i(n-m)\lambda}}$$

for all n, m such that $n \neq m$. Since $\lambda = \frac{2\pi}{k+1}$,

$$e^{i(n-m)(k+1)\lambda} = e^{i(n-m)(k+1)\frac{2\pi}{k+1}} = e^{i(n-m)2\pi} = 1.$$

Therefore $\displaystyle\sum_{j=0}^{k} e^{ij(n-m)\lambda} = 0$ for $n \neq m$.

For $n = m$, $\displaystyle\sum_{j=0}^{k} e^{ij(n-m)\lambda} = \sum_{j=0}^{k} 1 = k + 1.$

Thus Eq. (5) becomes

$$\sum_{j=0}^{k} r_j \, e^{ij(1-m)\lambda} = a_m (k+1), \quad \text{or}$$

$$a_m = \frac{1}{k+1} \sum_{j=0}^{k} r_j \, e^{ij(1-m)\lambda}, \quad m = 0, 1, \ldots, k. \tag{6}$$

Equation (6) is also valid for negative m.

Thus the approximate mapping function is completely determined by knowing r_j, $j = 0, 1, \ldots, k$.

The shape of curve which can be mapped as above is not arbitrary. The following limitations must be placed on the curve C:

1. The region bounded by C must be simply connected.

2. There must exist at least one point of the cross section, Z_o, such that every ray originating at Z_o intersects the curve C once and only once. Such a curve is said to be starlike with respect to Z_o.

The method used above for determining a mapping function which maps C onto S, approximately, has a serious weakness which must be considered.

As is shown in most texts[1] on theory of functions, the function which maps C onto S is unique. In constructing the mapping function one is at liberty to fix only one point of S and the direction of one tangent to S.

These requirements have been violated here, so one may not expect

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[1] Titchmarsh, The Theory of Functions, p. 209.

to obtain a mapping function which is arbitrarily accurate simply by taking a large number of collocation points.

In using the method developed here, one must exercise great caution. When the polynomial has been determined, several points on S should be mapped into the z plane. Using these points, one may estimate the shape of the curve, C', which is actually being mapped onto S.

In obtaining an approximate solution of the torsion problem, it is required only that C' lie close to C. Therefore one may find the polynomial $\omega(\zeta)$ and estimate the shape of the curve C'. If the two curves are approximately the same, a reasonable solution of the torsion problem may be obtained through use of the development which follows.

If C' is not similar to C, other methods[2] of constructing the polynomial $\omega(\zeta)$ must be employed.

REVIEW OF THE TORSION PROBLEM

The torsion problem for isotropic cylindrical shafts was first solved by Saint-Venant. His method of solution is called the semi-inverse method. In it, he made the following assumptions concerning displacement of material points in the shaft: $u = -\alpha z y$, $v = \alpha z x$ and $w = \alpha\, \phi(x, y)$ where, as shown in Fig. 3, the axis of the shaft is taken to be the Z axis with the X and Y axes in the plane of the cross

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[2] Kantorovich and Krylov, <u>Approximate Methods of Higher Analysis</u>, Chapt. V.

section. u, v and w are the rectangular components of displacement of a point (x, y, z) in the shaft in the x, y and z directions, respectively.

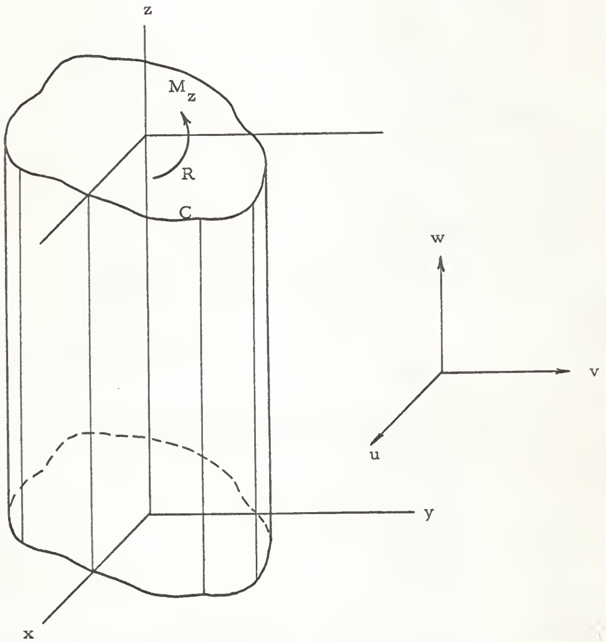

Fig. 3. Shaft under consideration.

Using these assumptions, the stresses are given by

$$\tau_{zy} = G\alpha\left(x + \frac{\partial \phi}{\partial y}\right),$$

$$\tau_{zx} = G\alpha\left(-y + \frac{\partial \phi}{\partial x}\right),$$

$$\tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0$$

where G is the modulus of rigidity and α is the unit angle of twist in the shaft.

Furthermore, the equations of equilibrium will be satisfied if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

throughout the cross section. In order that the lateral surface of the shaft be free of stress, it must be such that

$$\frac{d\phi}{dN} = y \cos(N, x) - x \cos(N, y) \tag{7}$$

on the boundary of the cross section.

It can be shown that a measure of the torsional rigidity is given by

$$D = \iint_R \left(x^2 + y^2 + x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right) dx\, dy,$$

where $M = GD\alpha$.

A function $\psi(x, y)$ such than

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{8),(9}$$

is now introduced. Since

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

ψ is also harmonic in R.

The condition that ϕ satify Eq. (7) on C is reduced to the condition that ψ satisfy the condition $\psi = \frac{1}{2}(x^2 + y^2)$ on C. (10)

This problem is commonly called the Dirichlet Problem. It is this form of the torsion problem that is solved with the aid of conformal mapping.

Consider $\phi(x, y)$ and $\psi(x, y)$ to be functions of the complex variable and define

$$F(z) = \phi(x, y) + i\,\psi(x, y)$$

$F(z)$ is an analytic function in R, since Eqs. (8) and (9), which define ψ, are the Cauchy-Riemann conditions exactly.

The analytic function $z = \omega(\zeta)$ of the complex variable $\zeta = \xi + i\,\eta$ is introduced which maps the region R in the z plane conformally onto the unit disc U in the zeta plane. The function $F(z)$ then becomes

$$f(\zeta) = F(\omega(\zeta)) = \phi^*(\xi, \eta) + i\,\psi^*(\xi, \eta),$$

and the condition of Eq. (10) becomes

$$\psi(x, y) = \frac{1}{2}\, z\,\bar{z} \qquad \text{on C, or}$$

$$\psi^*(\xi, \eta) = \frac{1}{2}\, \omega(\zeta)\,\bar{\omega}(\bar{\zeta}) \qquad \text{on S,}$$

where $\bar{\omega}(\bar{\zeta}) = \bar{\omega}(\bar{\zeta})$.

As shown by I. S. Sokolnikoff[1],

$$f(\zeta) = \frac{1}{2\pi} \int_S \frac{\omega(\sigma)\,\bar{\omega}(\bar{\sigma})\, d\sigma}{\sigma - \zeta}, \tag{11}$$

where $\sigma = e^{i\theta}$ is the complex variable ζ taken on the unit circle S

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[1] Sokolnikoff, Mathematical Theory of Elasticity, pp. 151-154.

and the ζ in Eq. (11) is any point interior to S. But since $f(\sigma)$ as determined by Eq. (11) is not well defined, define

$$f(\sigma) = \lim_{\zeta \to \sigma} \left(\frac{1}{2\pi}\right) \int_S \frac{\omega(\sigma)\,\overline{\omega}(\overline{\sigma})\, d\sigma}{\sigma - \zeta} . \tag{12}$$

Further, it is shown that $D = I_o + D_o$ where

$$I_o = -\frac{i}{4} \int_o^{2\pi} \left[\overline{\omega}(\overline{\sigma})\right]^2 \omega(\sigma) \frac{d\,\omega(\sigma)}{d\theta}\, d\theta \quad \text{and} \tag{13}$$

$$D_o = -\frac{1}{4} \int_o^{2\pi} \left[f(\sigma) + \overline{f}(\overline{\sigma})\right] \frac{d\left[\omega(\sigma)\,\overline{\omega}(\overline{\sigma})\right]}{d\theta}\, d\theta . \tag{14}$$

Finally, Sokolnikoff shows that at $z = \omega(\zeta)$,

$$\tau_{zx} - i\,\tau_{zy} = G\,\alpha \left[\frac{f'(\zeta)}{\omega'(\zeta)} - i\,\overline{\omega}(\overline{\zeta})\right] . \tag{15}$$

Thus if $\omega(\zeta)$ is known, $f(\zeta)$ is determined from Eq. (11) and the problem can be solved for D, τ_{zx} and τ_{zy}.

APPLICATION OF APPROXIMATE MAPPING TO THE TORSION PROBLEM

To find the torsional rigidity of the bar it is necessary to evaluate the integrals of Eqs. (13) and (14).

With $\omega(\sigma)$ represented as a polynomial, $\omega(\sigma) = \sum_{n=0}^{k} a_n \sigma^n$ the product $\omega(\sigma)\ \overline{\omega}(\overline{\sigma})$ is written as

$$\omega(\sigma)\,\overline{\omega}(\overline{\sigma}) = \left[\sum_{n=0}^{k} a_n \sigma^n\right] \left[\sum_{n=0}^{k} \overline{a}_n \sigma^{-n}\right] .$$

This product reduces to

$$\omega(\sigma)\,\overline{\omega}(\overline{\sigma}) = \sum_{n=0}^{k} c_n \sigma^n + \sum_{n=1}^{k} \overline{c}_n \sigma^{-n}$$

where $$c_n = \sum_{r=n}^{k} a_r \bar{a}_{r-n}.$$

Equation (11) becomes

$$f(\zeta) = \frac{1}{2\pi}\left[\sum_{n=0}^{k} \int_S \frac{c_n \sigma^n d\sigma}{\sigma - \zeta} + \sum_{n=1}^{k} \int_S \frac{\bar{c}_n d\sigma}{\sigma^n(\sigma - \zeta)}\right]. \tag{16}$$

The integrals of the first sum in Eq. (16) are integrable by the Cauchy Integral Formula to

$$\frac{1}{2\pi}\sum_{n=0}^{k} \int_S \frac{c_n \sigma^n d\sigma}{\sigma - \zeta} = i \sum_{n=0}^{k} c_n \zeta^n .$$

By the use of the residue theorem, it is easily shown that the integral of the second sum vanishes. Thus,

$$f(\zeta) = i \sum_{n=0}^{k} c_n \zeta^n . \tag{17}$$

Next $\dfrac{d\omega(\sigma)}{d\theta} = \dfrac{d}{d\theta}\left(\sum_{n=0}^{k} a_n \sigma^n\right) = i \sum_{n=0}^{k} n\, a_n \sigma^n$, and

$$\bar{\omega}(\bar{\sigma}) \frac{d\omega(\sigma)}{d\theta} = i \left(\sum_{n=0}^{k} b_n \sigma^n + \sum_{n=1}^{k} d_n \sigma^{-n}\right),$$

where $b_n = \sum_{r=n}^{k} r\, a_r \bar{a}_{r-n}$ and $d_n = \sum_{r=n}^{k} (r-n) \bar{a}_r a_{r-n}$.

Thus $$\left[\bar{\omega}(\bar{\sigma})\right]^2 \omega(\sigma) \frac{d\omega(\sigma)}{d\theta} = i\left[\sum_{n=0}^{k} c_n \sigma^n + \sum_{n=1}^{k} \bar{c}_n \sigma^{-n}\right]\left[\sum_{n=0}^{k} b_n \sigma^n + \sum_{n=1}^{k} d_n \sigma^{-n}\right].$$

Also, $$\frac{d}{d\theta}\left[\omega(\sigma)\,\overline{\omega}(\overline{\sigma})\right] = \frac{d}{d\,\theta}\left[\sum_{n=0}^{k} c_n\,\sigma^{n} + \sum_{n=1}^{k} \overline{c}_n\,\sigma^{-n}\right]$$

$$= i\left[\sum_{n=0}^{k} n\,c_n\,\sigma \;-\; \sum_{n=1}^{k} n\,\overline{c}_n\,\sigma^{-n}\right],$$

and

$$\left[f(\sigma) + \overline{f}(\overline{\sigma})\right]\frac{d}{d\,\theta}\left[\omega(\sigma)\,\overline{\omega}(\overline{\sigma})\right] = -\left[\sum_{n=0}^{k} c_n\,\sigma^{n} - \sum_{n=0}^{k} \overline{c}_n\,\sigma^{-n}\right]\left[\sum_{n=1}^{k} n\,c_n\,\sigma^{n} - \sum_{n=1}^{k} n\,\overline{c}_n\,\sigma^{-n}\right]$$

Using these expressions, Eq. (13) and (14) are

$$I_o = \frac{1}{4}\int_0^{2\pi}\left[\sum_{n=0}^{k} c_n\,\sigma^{n} + \sum_{n=1}^{k} \overline{c}_n\,\sigma^{-n}\right]\left[\sum_{n=0}^{k} b_n\,\sigma^{n} + \sum_{n=1}^{k} d_n\,\sigma^{-n}\right] d\theta, \tag{18}$$

$$D_o = \frac{1}{4}\int_0^{2\pi}\left[\sum_{n=0}^{k} c_n\,\sigma^{n} - \sum_{n=0}^{k} \overline{c}_n\,\sigma^{-n}\right]\left[\sum_{n=1}^{k} n\,c_n\,\sigma^{n} - \sum_{n=1}^{k} n\,\overline{c}_n\,\sigma^{-n}\right] d\theta. \tag{19}$$

Since $\sigma = e^{i\theta}$,

$$\int_0^{2\pi} \sigma^{n}\,d\theta = \frac{1}{n\,i}\left(e^{i\,n\,2\pi} - e^{i\,o}\right) = 0,\; n \neq 0.$$

Therefore, only constants in the integrand of Eqs. (18) and (19) yield non-zero terms, and

$$I_o = \frac{\pi}{2}\left[c_o\,b_o + \sum_{n=1}^{k}\left(c_n\,d_n + \overline{c}_n\,b_n\right)\right] \tag{20}$$

$$D_o = -\,\pi\sum_{n=1}^{k} n\,c_n\,\overline{c}_n \tag{21}$$

The stresses in terms of constants are given by Eq. (15),

where $$f'(\zeta) = i \sum_{n=1}^{k} n\, c_n\, \zeta^{n-1}, \quad \omega'(\zeta) = \sum_{n=1}^{k} n\, a_n\, \zeta^{n-1}$$

and $$\overline{\omega}(\overline{\zeta}) = \sum_{n=0}^{k} \overline{a}_n\, \zeta^{-n}.$$

Thus at the point $z = \omega(\zeta)$,

$$\tau_{zx} - i\,\tau_{zy} = G\,\alpha\, i \left[\frac{\sum_{n=1}^{k} n\, c_n\, \zeta^{n-1}}{\sum_{n=1}^{k} n\, a_n\, \zeta^{n-1}} - \sum_{n=0}^{k} \overline{a}_n\, \zeta^{-n} \right].$$

SOLUTION OF A SIMPLE PROBLEM

As an illustrative problem, a shaft whose cross section is bounded by a cardioid is considered. The cardioid is chosen as an example because the function which maps it into a circle is a finite polynomial, so that the method developed here should yield correct results.

The polynomial which maps the cardioid $r = 2(1 + \cos\alpha)$ onto the unit circle is $z = 1 + 2\zeta + \zeta^2$. It is anticipated then that three collocation points will be sufficient to determine the mapping function.

Using the IBM 1620 computer and the FORGO programs (See Appendix) with data for three collocation points, and instructions to compute stress at four points on the boundary, the results given in Table 1 were obtained. As was expected, these results are correct.

Data for thirty six collocation points and instructions to compute stress at four points were employed in the same manner and, to four decimal places, the results were the same as with three collocation points.

Input Data:

r_j	j
4.0	0
1.0	1
1.0	2

$k = 2$ $\quad I_S = 3$

(instruction to calculate stress at 4 points)

Output:

$I_o = 1.0996 \times 10^2$

$D_o = -5.6549 \times 10$

$\tau_{zx}(j)$	$\tau_{zy}(j)$	j
-1.5000	-1.5000	1
-3.2787×10^6	2.7322×10^6	2
1.5000	-1.5000	3
-8.2000×10^{-7}	2.5000	4

Table 1. Results for cardioid.

SHAFT WITH SQUARE CROSS SECTION

As an example of a problem to which the method developed in this report does not apply, torsion of a shaft with square cross section was considered.

Using data for a two by two-inch square with eight and thirty-two collocation points, and Program 1 (See Appendix), the a_n's were found.

With the a_n's determined above and Program 3 (See Appendix), twenty points of the unit circle with argument less than $\pi / 4$ were mapped into the z plane.

It was desired that the resulting curve in the z plane be close to the vertical straight line portion of the square in the first quadrant of the z plane. However, upon plotting the curves (See Fig. 4),

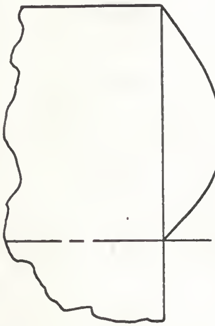

Fig. 4(a)

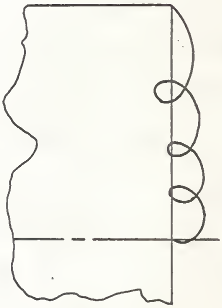

Fig. 4(b)

Fig. 4. Plot of curves associated with square.

it is seen that for eight collocation points (See Fig. 4a) the curve departs considerably from the square. For thirty-two collocation points (See Fig. 4b), the curve departs violently from the square and is further not a simple curve.

It may be concluded, therefore, that the method of approximate conformal mapping developed in this report is not applicable in this case.

CONCLUSIONS

The torsion problem has been reduced to the problem of finding a function which maps the boundary of the cross section, c, conformally onto the unit circle. Since such a function is difficult to obtain, an approximate conformal mapping is determined.

A polynomial with undetermined coefficients is assumed to be the desired function, and the coefficients are found through use of Eq. (6).

In using this polynomial, one must be cautious. After the coefficients of the polynomial have been determined, a rough plot of the curve which is actually mapped onto the unit circle should be made. If this curve, call it c' , is close to the curve c, then a reasonable solution of the problem for torsional rigidity, displacement and stress is to be expected.

If, however, c' is not close to c, then other means of determining the coefficients of the polynomial must be used.

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REFERENCES

1. Hille, E., *Analytic Function Theory*. Boston: Ginn and Company, 1959.

2. Kantorovich, L., and V. Krylov, *Approximate Methods of Higher Analysis*. New York: Interscience Publishers, 1958.

3. Love, A., *The Mathematical Theory of Elasticity*. New York: Dover, 1944.

4. Sokolnikoff, I., *Mathematical Theory of Elasticity*. New York: McGraw-Hill, 1956.

5. Titchmarsh, E., *The Theory of Functions*. London: Oxford University Press, 1939.

APPENDIX

PROGRAM 1

```
      TORSION PROBLEM
      DIMENSION AR(90),AI(90),BR(90),BI(90),CR(90),CI(90)
      DIMENSION DR(90),DI(90),R(90)
    1 DO 10 J=1,90
      AR(J)=0.
      AI(J)=0.
      BR(J)=0.
      BI(J)=0.
      CR(J)=0.
      CI(J)=0.
      DR(J)=0.
      DI(J)=0.
   10 R(J)=0.
      READ,K
    4 READ, T,J,N
      IF(N) 6,5,5
    5 R(J+1)=T
      GO TO 4
    6 CONTINUE
      KP = K+1
      Q=K
      DL=(2.*3.1415927)/(Q+1.)
      DO 102 M=1,KP
      DM=M-1
      DO 101 J=1,KP
      DJ=J-1
      APR=(1./(Q+1.))*R(J   )*COSF(DJ*(1.-DM)*DL)
      API=(1./(Q+1.))*R(J   )*SINF(DJ*(1.-DM)*DL)
      AR(M   )=AR(M   )+APR
  101 AI(M   )=A I(M   )+API
  102 CONTINUE
      DO 132 N= 1,KP
      DN=N-1
      DO 131 J= N, KP
      JNP = J-N+1
      DJ = J-1
      BPR=DJ*(AR(J   )*AR(JNP   )+AI(J   )*AI(JNP   ))
      BPI=DJ*(AI(J   )*AR(JNP   )-AR(J   )*AI(JNP   ))
      BR(N   )=BR(N   )+BPR
  111 BI(N   )=BI(N   )+BPI
      CPR=AR(J   )*AR(JNP   )+AI(J   )*AI(JNP   )
      CPI=AI(J   )*AR(JNP   )-AR(J   )*AI(JNP   )
      CR(N   )=CR(N   )+CPR
```

```
121 CI(N   )=CI(N   )+CPI
    DPR=(DJ-DN)*(AR(J   )*AR(JNP   )+AI(J   )*AI(JNP   ))
    DPI=(DJ-DN)*(AR(J   )*AI(JNP   )-AI(J   )*AR(JNP   ))
    DR(N   )=DR(N   )+DPR
131 DI(N   )=DI(N   )+DPI
132 CONTINUE
    DPR = 0.0
    DOPUR = 0.0
    TPUMR=CR(1)*BR(1)-CI(1)*BI(1)
    DO 201 J=1,K
    DJ=J
    TPR = CR(J+1)*DR(J+1)- CI(J+1)*DI(J+1)+CR(J+1)*BR(J+1)
    TPR = TPR + CI(J+1)*BI(J+1)
    DOPR=-2.*DJ*(CR(J+1)*CR(J+1)+CI(J+1)*CI(J+1))
       DOPUR = DOPUR +DOPR
201 TPUMR =TPUMR+TPR
    TR=3.1415927*.5*TPUMR
    DOR = 3.1415927*.5*DOPUR
    DPUMI=CI(1)*BR(1)+CR(1)*BI(1)
    DO 211 J=1,K
    DJ=J
    DPI=CI(J+1)*DR(J+1)+CR(J+1)*DI(J+1) +CR(J+1)*BI(J+1)
    DPI = DPI -CI(J+1)*BR(J+1)
211 DPUMI=DPUMI+DPI
    DIDBG=3.1415927*.5*DPUMI
    KA = K+1
    JA=1
    JB=2
    JC=3
    JD=4
501 FORMAT(E18.8,2I3)
600 FORMAT(I3,4E16.8)
    PUNCH,TR,DOR,DIDBG
    DO 303 J=1,KA
    I=J-1
     PUNCH 600,J,BR(J),BI(J),DR(J),DI(J)
    PUNCH 501, AR(J),I,JA
    PUNCH 501, AI(J), I, JB
    PUNCH 501, CR(J), I, JC
303 PUNCH 501, CI(J),I, JD
    GO TO 1
    END
```

PROGRAM 2

```
      STRESS FOR TORSION PROBLEM
      DIMENSION AR(90),AI(90),CR(90),CI(90),FDR(90),FDI(90)
      DIMENSION WDR(90),WDI(90),WBR(90),WBI(90),TXDGA(90),TYDGA(90)
      READ,K ,IS
    2 READ,T,J,N
      GO TO (3,4,5,6,7),N
    3 AR(J)=T
      GO TO 2
    4 AI(J)=T
      GO TO 2
    5 CR(J)=T
      GO TO 2
    6 CI(J)=T
      GO TO 2
    7 CONTINUE
      READ,ARZ,AIZ
      ISP = IS+1
      DO 102 J=1,ISP
      DJ=J
      FDR(J)=0.
      FDI(J)=0.
      WDR(J)=0.
      WDI(J)=0.
      WBR(J)=0.
      WBI(J)=0.
      DO 101 N=1,K
      DN=N
      QP=ISP
      DL= 3.1415927*2./(QP)
      FDRP=(-1.)*DN*CR(N)*SINF ((DN-1.)*DJ*DL)
      FDRP=FDRP-DN*CI(N)*COSF((DN-1.)*DJ*DL)
      FDR(J)    =FDR(J   )+FDRP
      FDIP=DN*(CR(N)*COSF((DN-1.)*DJ*DL)-CI(N)*SINF((DN-1.)*DJ*DL))
      FDI(J)=FDI(J) +FDIP
      WDRP=DN*(AR(N)*COSF((DN-1.)*DJ*DL)-AI(N)*SINF((DN-1.)*DJ*DL))
      WDR(J) =WDR(J) +WDRP
      WDIP=DN*(AR(N)*SINF((DN-1.)*DJ*DL)+AI(N)*COSF((DN-1.)*DJ*DL))
      WDI(J)=WDI(J) +WDIP
      WBRP=AR(N)*COSF(DN*DJ*DL)-AI(N)*SINF(DN*DJ*DL)
      WBR(J)=WBR(J)+WBRP
      WBIP=(-1.)*(AI(N)*COSF(DN*DJ*DL)+AR(N)*SINF(DN*DJ*DL))
  101 WBI(J)=WBI(J)+WBIP
      WBR(J)=WBR(J)+ARZ
  102 WBI(J)=WBI(J)-AIZ
      DO 201 J=1,ISP
```

```
      TXDGA(J)=(FDR(J)*WDR(J)+FDI(J)*WDI(J))
      TXDGA(J)=TXDGA(J)/(WDR(J)*WDR(J)+WDI(J)*WDI(J))+WBI(J)
      TYDGA(J)=(FDR(J)*WDI(J)-FDI(J)*WDR(J))
      TYDGA(J)=TYDGA(J)/(WDR(J)* WDR(J)+WDI(J)*WDI(J))+WBR(J)
  201 PUNCH,J,TXDGA(J),TYDGA(J)
      END
```

PROGRAM 3

```
      PLOTTING THE APPROXIMATING CURVE
      DIMENSION AR(40), AI(40)
      READ,K,IS,T
    5 READ,R,I,J
      IP = I+1
      GO TO (10,11,12),J
   10 AR(IP ) = R
      GO TO 5
   11 AI(IP ) = R
      GO TO 5
   12 CONTINUE
      KP = K+1
      S = IS
      DO 21 L=1,IS
      DL = L
      TH = (T/S)*DL
      ZI = 0.0
      ZR = 0.0
      DO 20 N = 1,KP
      DNM = N-1
      ZRP= AR(N)*COSF(DNM*TH) - AI(N)*SINF(DNM*TH)
      ZIP= AR(N)*SINF(DNM*TH) + AI(N)*COSF(DNM*TH)
      ZR = ZRP + ZR
   20 ZI = ZI + ZIP
      PUNCH,TH,L,ZR,ZI
   21 CONTINUE
      END
```

SOLUTION OF THE TORSION PROBLEM BY
APPROXIMATE CONFORMAL MAPPING

by

EDWARD J. HAUG, JR.

B. S. Missouri School of Mines and Metallurgy, 1962

AN ABSTRACT OF A MASTER'S REPORT

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ABSTRACT

The problem considered in this report is torsion of an isotripic cylindrical shaft.

This problem is first formulated as the second boundary value problem of potential theory, i. e., the problem of determining a function which satisfies Laplace's equation and has a normal derivative prescribed on the boundary of the cross section of the shaft under consideration. The problem is then transformed to the Dirichlet problem, i. e., the problem of finding a solution of Laplace's equation which assumes prescribed values on the boundary of the unit disk. The Dirichlet problem is then solved with the aid of conformal mapping.

The function which maps the cross section of the shaft onto the unit disk is assumed to be a finite polynomial of the form $\sum_{n=0}^{k} a_n \zeta^n$. The constants of the polynomial are found by the method of collocation. In this method, $k + 1$ points of the boundary of the cross section are mapped exactly into $k + 1$ points of the unit circle. All other points of the boundary of the cross section are assumed to map approximately into the unit circle.

Using this finite polynomial, equations are developed from which the stresses and the torsional rigidity of the shaft can be calculated.