ANALYSIS OF A HYPERBOLIC

PARABOLOIDAL SHELL

by

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SYNOPSIS

Many articles have been published on hyperbolic parabo-
loidal shells. However to the knowledge of the author, no
complete simplified analysis covering various loading
conditions has been developed to date.

The shell analysis explained is a result of reviewing
several references using the membrane theory solution with
uniform vertical loading. By the membrane theory, no bend-
ing moment is allowed within the shell, therefore the shell
acts mainly by axial forces. Applying this theory, the
basic equations are developed for one quadrant of the hyper-
bolic paraboloid. A consideration of secondary stresses,
which cannot be included in the membrane theory solution,
is given. The basic equations and the consideration of
secondary stresses are then applied to a practical design
of an inverted, single support reinforced concrete hyper-
bolic paraboloid.

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A brief detailed drawing is presented showing the
dimensions and reinforcing resulting from the design of
the hyperbolic paraboloid.

INTRODUCTION

Increasing interest and acceptance by the general
public of shell roof construction has led to the wide
usage of reinforced concrete hyperbolic paraboloidal shells.
Through their use the designers can depart from the system
of linear members confined to three perpendicular planes
to a curvilinear system for more imaginative and graceful
structures. Some advantages other than beauty resulting
from the simple hyperbolic paraboloid are economics of
design, economics of construction, and low maintenance cost.

By membrane theory analysis, the hyperbolic paraboloid
carries a uniform load mainly by direct axial compression or
tension, uniformly stressing all material of the cross-
section, thus giving an economical design. Economy of
construction is gained by the straight-line simplicity of
forming the surface and by the repeated use of constructed
forms. Maintenance on reinforced concrete hyperbolic
paraboloid roofs is usually low. By nature of the material,
it is virtually fireproof. Also being constructed as a

single unit reduces maintenance. Other advantages[2] relating to settlement, wind and earthquake forces influence a reduction in maintenance costs.

Realizing their growing usage, the author reviewed available literature concerning the stress analysis and design of hyperbolic paraboloids in order to gain a practical working knowledge of the simpler cases. Therefore this paper contains a summary of several references which explain the stress and design analysis. When a section is developed mainly from one or two references, proper acknowledgement of each reference will be given at the beginning of the section.

METHOD OF STRESS ANALYSIS

General Definition

A structure whose thickness has a much smaller magnitude than its other two dimensions (length and width) is called a laminar structure. The shell is a laminar structure. Flugge[3] cites two examples of shells which structurally behave very differently. Other references classify the difference in structural action of shells as either a proper or an improper shell.[4]

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[2] Structural Applications of Hyperbolic Paraboloidical Shells, by Felix Candela, Proceedings, ACI, Vol. 26, January 1955, p. 397.

[3] Stresses in Shells, by W. Flugge, p. 8.

[4] Understanding the Hyperbolic Paraboloid, by Felix Candela, Architectural Record, Vol. 124 No. 1, July 1958, p. 192.

The main structural action of the improper shell is performed by bending and twisting moments combined with normal and shear forces. An example of such is a clyindrical shell formed by rolling a sheet of paper into a cylinder and securing the ends together. It is easily observed that a small lateral force applied to this shell causes considerable deformation which is resisted mainly by bending and twisting moments.

The structural action of the proper shell is performed predominately by direct stress (normal and shearing forces). An electric light bulb is an example of a proper shell. Even though its material is very thin and rather fragile, a large lateral load can be applied without any visible deformation of this shell. Since the deformation and thickness is very small, one could assume that the bending and twisting moments are almost negligible. Flugge mentions that a detailed study shows this to be true.

Surface Definition

Geometrically, the shell is bounded by a doubly curved surface. The two curved surfaces are defined as the faces of the shell. The middle surface of the shell is defined as the surface passing midway between the two faces. By knowing the shape of the middle surface and the thickness, a shell can be geometrically described.

Referring to the two shell examples given previously, the middle surface of an improper shell is developable. This means the middle surface can be formed by an initially plane flexible sheet without stretching the sheet at any point. The proper shell has a non-developable middle surface, which cannot be formed from an initially plane flexible sheet without stretching at some point.

Conditions of Equilibrium

Using the approach of Flugge[5] to define the stresses in a shell, first an x, y, z coordinate system as shown in Fig. 1, is set up to describe the position of a point on the middle surface. The x and y coordinate axis is tangent to the middle surface with the z axis being normal to the middle surface. Then an element is cut from the shell by two pairs of planes which are normal to the middle surface of the shell.

The forces formed by removing the element are resolved into three components positively signed as shown in Fig. 1. Referring to Fig. 2, each of the three forces shown in Fig. 1 is the resultant of either normal stresses (σ_x, σ_y), shear stresses parallel to the middle surface ($\tau_{xy} = \tau_{yx}$),
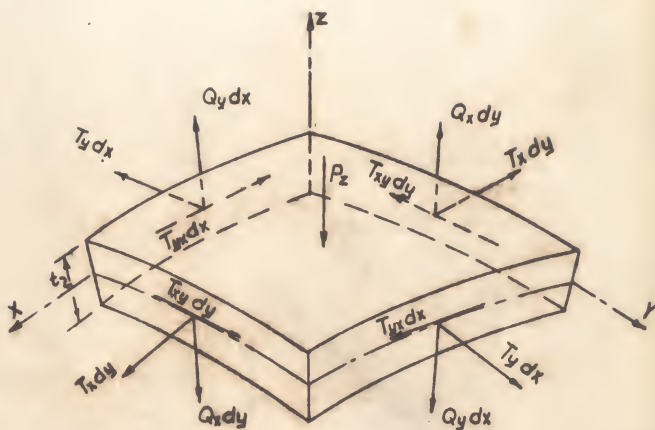
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[5] Ibid., pp. 3-9.
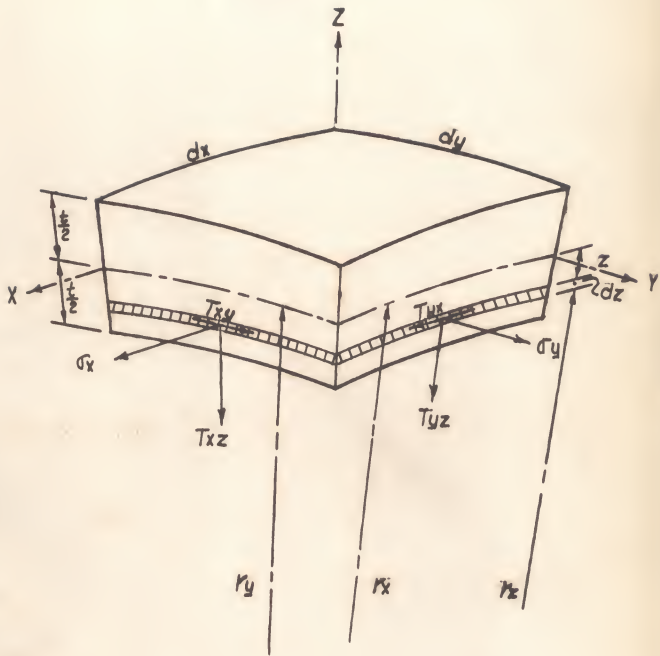
FIG.I. STRESS RESULTANTS AND LOADS

FIG.2. STRESSES ACTING ON A DIFFERENTIAL ELEMENT

and shear stresses normal to it (τ_{xz}, τ_{yz}). Therefore the three forces can be termed stress resultants.

Using the terms illustrated in Fig. 1 and Fig. 2, the stress resultants are defined as follows:[6]

$$T_x = \int_{-t/2}^{t/2} \sigma_x \left(\frac{r_y - z}{r_y}\right) dz, \qquad\qquad T_y = \int_{-t/2}^{t/2} \sigma_y \left(\frac{r_x - z}{r_x}\right) dz,$$

$$T_{xy} = \int_{-t/2}^{t/2} \tau_{xy} \left(\frac{r_y - z}{r_y}\right) dz, \qquad\qquad T_{yx} = \int_{-t/2}^{t/2} \tau_{yx} \left(\frac{r_x - z}{r_x}\right) dz,$$

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} \left(\frac{r_y - z}{r_y}\right) dz, \qquad\qquad Q_y = \int_{-t/2}^{t/2} \tau_{yz} \left(\frac{r_x - z}{r_x}\right) dz,$$

$$M_x = \int_{-t/2}^{t/2} \sigma_x \left(\frac{r_y - z}{r_y}\right) z\, dz, \qquad\qquad M_y = \int_{-t/2}^{t/2} \sigma_y \left(\frac{r_x - z}{r_x}\right) z\, dz,$$

$$M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} \left(\frac{r_y - z}{r_y}\right) z\, dz, \qquad\qquad M_{yx} = \int_{-t/2}^{t/2} \tau_{yx} \left(\frac{r_x - z}{r_x}\right) z\, dz,$$

The factors $\left(\frac{r_y - z}{r_y}\right)$ and $\left(\frac{r_x - z}{r_x}\right)$ are assumed to be equal to

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[6] Theory of Plates and Shells, by S. Timoshenko, and W. Woinowsky - Krieger, p. 429.

unity as the thickness of the shell is generally small as compared with the radii of curvature.

Since the load on the proper shell is carried mainly by normal and shearing forces, the moments may be assumed negligible in the stress analysis. This assumption leads to the membrane theory of shells. Fig. 1, therefore shows all the necessary forces acting on the sides of the shell element. By moment equations concerning the element, it can be proven[7] that $T_{xy}=T_{yx}$ and $Q_x=Q_y=0$. With these assumptions, the ten unknown stress resultants are reduced to only three unknown stress resultants (T_x, T_y, T_{xy}). From the three equations of static equilibrium, the three unknown stress resultants can be solved.

Using the same coordinate axis and transforming the actual forces of the element into forces acting on a projected area as shown in Fig. 3, considerable simplification of the analysis can be gained. The following portion of analysis follows closely that set forth by A. L. Parme.[8]

The stress resultants T_y, T_x, S ($S=T_{xy}=T_{yx}$) represent the skew forces being tangential to the element's surface. The stress resultants T_{yp}, T_{xp}, S_p represent the projected forces in the xy plane. The skew forces T_y, T_x, S are

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[7] Stresses in Shells, by W. Flugge, p. 9.

[8] Hyperbolic Paraboloids and Other Shells of Double Curvature, by A. L. Parme, Proceedings, ASCE September 1956, 82 ST5.
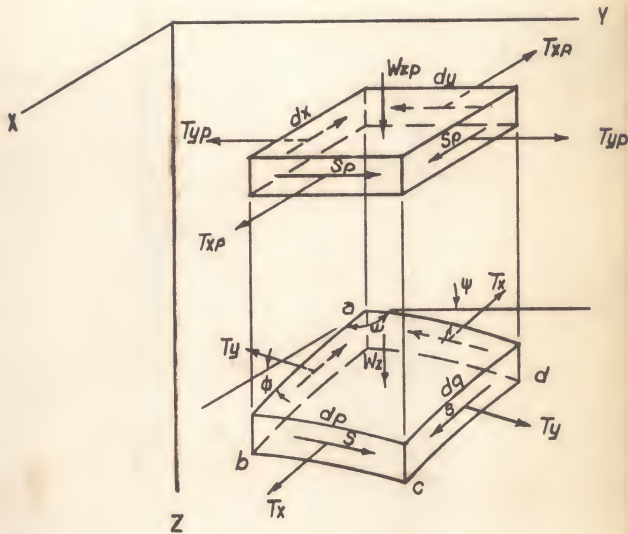
FIG. 3. SHELL ELEMENT AND ITS PROJECTION

measured in pounds per unit length of line element. By
geometry it is seen that

$$dp \cos\Psi = dy \tag{1a}$$

$$dq \cos\phi = dx \tag{1b}$$

Multiplying the normal forces by the cosine of the angle
between force and the xy plane yields the x and y components
of such. The x components are

$$T_{xp}dy = T_x \cos\phi \, dp,$$

$$S_p dx = S \cos\phi \, dq,$$

which by substitution becomes

$$T_{xp}dy = T_x \frac{\cos\phi}{\cos\Psi} \, dy,$$

$$S_p dx = S \frac{\cos\phi}{\cos\phi} \, dx,$$

and the y components are

$$T_{yp}dx = T_y \cos\Psi \, dq,$$

$$S_p dy = S \cos\Psi \, dp,$$

which by substitution becomes

$$T_{yp}dx = T_y \frac{\cos\Psi}{\cos\phi} \, dx,$$

$$S_p dy = S \frac{\cos\Psi}{\cos\Psi} \, dy.$$

The new stress resultants in terms of the projected element are

$$T_{xp} = T_x \frac{\cos \phi}{\cos \psi} \qquad (2a)$$

$$T_{yp} = T_y \frac{\cos \psi}{\cos \phi} \qquad (2b)$$

$$S_p = S. \qquad (2c)$$

Deriving the element's area dA in terms of the projected element area $dxdy$ gives

$$dA = \frac{dx}{\cos \phi} \frac{dy}{\cos \psi} \sin \omega = dxdy \frac{(1-\sin^2 \psi \, \sin^2 \phi)^{1/2}}{\cos \phi \, \cos \psi}.$$

Assuming only a vertical distributed load per unit area W_z acting on the shell element dA, the transferred distributed load per unit area W_{zp} acting on the projected element is

$$\frac{W_{zp}}{W_z} = (\frac{1-\sin^2 \psi \, \sin^2 \phi}{\cos \psi \, \cos \phi})^{1/2}. \qquad (3)$$

The load W_{zp} can be assumed to be uniform for most hyperbolic paraboloid shells of moderate rise. The $\sin^2 \psi \, \sin^2 \phi$ term can be neglected since it will become small for the moderate rise case. Using this assumption Equation (3) becomes

$$W_{zp} = W_z \frac{1}{\cos \psi \, \cos \phi}. \qquad (4)$$

With the forces acting on the element varying from the one face to another, equilibrium of forces in the x direction

expressed in terms of T_{xp}, T_{yp} and S_p gives:

$$\frac{\partial T_{xp}}{\partial x} + \frac{\partial S_p}{\partial y} = 0. \tag{5}$$

Likewise equilibrium in the y direction yields

$$\frac{\partial T_{yp}}{\partial y} + \frac{\partial S_p}{\partial x} = 0. \tag{6}$$

Equilibrium in the z direction involves all of the shell element's stress resultants as they all contain a vertical component. The vertical component of the normal force T_x is

$$T_x \sin\phi \, dp,$$

and by substituting for T_x and dp from Equations (2a) and (1a) yields

$$T_{xp} \frac{\cos\psi}{\cos\phi} \sin\phi \frac{dy}{\cos\psi} = T_{xp} \tan\phi \, dy = T_{xp}\frac{\partial z}{\partial x} \, dy.$$

The vertical component acting per unit of length along the y axis is

$$T_{xp}\frac{\partial z}{\partial x} .$$

A similar expression for the vertical component of T_y per unit of length along the x axis is found to be

$$T_{yp}\frac{\partial z}{\partial y} .$$

The vertical component of the shear force along the y axis is

$$S \, dp \, \sin\psi$$

and by substitution from Equations (2c) and (1a) gives

$$S_p \frac{dy}{\cos \psi} \sin \psi = S_p \tan \psi \ dy = S_p \frac{\partial z}{\partial y} \ dy$$

which per unit of length along the y axis is

$$S_p \frac{\partial z}{\partial y} \ .$$

Similarly, the vertical component of the shear force along the x axis is

$$S_p \frac{\partial z}{\partial x} \ .$$

Taking into account the differential increments of all the forces, the equilibrium equation in the z direction yields

$$\frac{\partial}{\partial x}(T_{xp}\frac{\partial z}{\partial x}) + \frac{\partial}{\partial y}(T_{yp}\frac{\partial z}{\partial y}) + \frac{\partial}{\partial x}(S_p\frac{\partial z}{\partial y}) + \frac{\partial}{\partial y}(S_p\frac{\partial z}{\partial x}) + W_{zp} = 0. \quad (7a)$$

By differentiating the products, Equation (7a) becomes

$$T_{xp}\frac{\partial^2 z}{\partial x^2} + 2S_p\frac{\partial^2 z}{\partial x \partial y} + T_{yp} \frac{\partial^2 z}{\partial y^2} =$$

$$-(\frac{\partial T_{xp}}{\partial x} + \frac{\partial S_p}{\partial y})\frac{\partial z}{\partial x} - (\frac{\partial T_{yp}}{\partial y} + \frac{\partial S_p}{\partial x})\frac{\partial z}{\partial y} - W_{zp}. \quad (7b)$$

Substitution of Equations (5) and (6) into (7b) gives

$$T_{xp}\frac{\partial^2 x}{\partial x^2} + 2S_p\frac{\partial^2 z}{\partial x \partial y} + T_{yp}\frac{\partial^2 z}{\partial y^2} = - W_{zp}. \quad (7c)$$

The three Equations (5), (6), and (7c) are the basic membrane theory equations for this case. Using the defined shell's middle surface, a direct solution of these equations may be tried. Since we have three "dependent" variables T_{xp}, T_{yp}, S_p, each depending on two "independent" variables x, y, the solution becomes very complicated. By assuming the stress resultants described by a single stress function F of x, y, instead of T_{xp}, T_{yp}, S_p, the equations are reduced to one second order equation. The stress resultants are derived from the "Airy stress function"[9] F by differentiation, as follows:

$$T_{xp} = \frac{\partial^2 F}{\partial y^2} \tag{8a}$$

$$T_{yp} = \frac{\partial^2 F}{\partial x^2} \tag{8b}$$

$$S_p = -\frac{\partial^2 F}{\partial x \partial y} . \tag{8c}$$

By substitution, Equations (5) and (6) are satisfied and Equation (7c) reduces to

$$\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y} = -W_{zp} . \tag{9}$$

For most cases, the algebraic solution of this differential equation is quite difficult, however for a hyperbolic paraboloidal shell loaded under uniform loading, the solution is fairly simple.

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[9] Advanced Strength of Materials, by J. P. Den Hartog, p. 174.

APPLICATION OF STRESS ANALYSIS

Surface Definition

As previously stated, the hyperbolic paraboloidal shell is bounded by a doubly curved surface. This surface may be defined in two ways, either as a surface of translation or as a warped parallelogram. The surface of translation can be visualized by translating or moving a vertical parabola of upward curvature over another parabola with downward curvature. The translating parabola's plane is perpendicular to the other parabola and is at all times parallel to its original position, thus giving a saddle-shaped surface.

The second method of description is described as a warped parallelogram and is shown in Fig. 4. The remaining portion of this report, including application and design will be based on this method of description. This surface is constructed by moving a straight line along the x axis, remaining parallel to the yz plane at all times, but rotating about the x axis as the outer end of the line slides along the straight line ABC. A similar generating line can be established along the y axis. The resulting surface formed by the grid of straight lines may be described at a point by the intersection of two such lines contained in the surface.
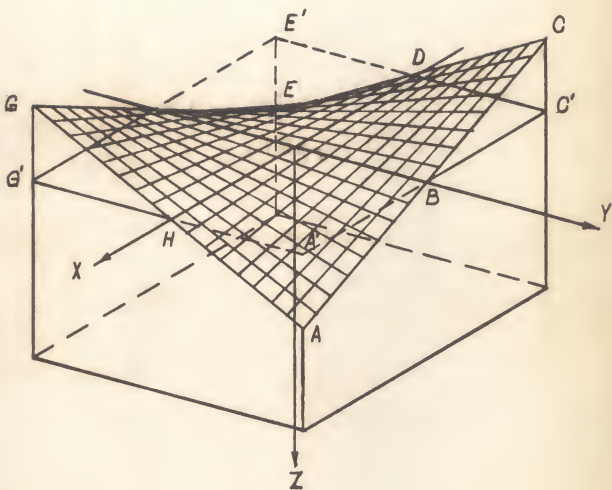
FIG. 4. SURFACE DESCRIPTION

The "warped parallelogram" connotation for the surface can be visualized as the horizontal plane A'C'E'G' is warped by vertically depressing corners A' and E' to new positions A and E, and allowing corners G' and C' to rise vertically to new positions G and C.

Stress Conditions

A basic quadrant ABOH of the "warped parallelogram" surface taken from Fig. 4 is shown in Fig. 5. Continuing as A. L. Parme[10] has done in his paper, any point on the surface can be described as a function of x, y and z. Referring to Fig. 5

$$\frac{c}{h} = \frac{y}{b} \quad \text{or} \quad c = \frac{hy}{b} \; . \tag{10a}$$

Likewise,

$$\frac{z}{c} = \frac{x}{a} \; . \tag{10b}$$

Substituting into Equation (10b), the value of c found in Equation (10a) gives

$$z = \frac{cx}{a} = (\frac{hy}{b})\frac{x}{a} = xy \; (\frac{h}{ba}) \; .$$

Letting $k = \frac{h}{ab}$,

$$z = kxy. \tag{11}$$

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[10] Hyperbolic Paraboloids and Other Shells of Double Curvature, by A. L. Parme, Proceedings, ASCE September 1956, 82 ST5.
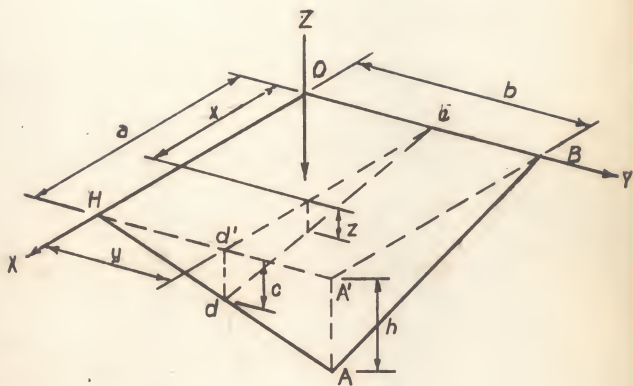
FIG. 5. BASIC QUADRANT

Upon substituting Equation (11) into Equation (9), the
second differential of Equation (11) equals zero and the
remainder yields

$$- 2 \frac{\partial^2 \overline{F}}{\partial x \partial y} k = - W_{zp}$$

which by Equation (8c) reduces to

$$S_p = \frac{W_{zp}}{2k} . \tag{12}$$

Using the above value for S_p, it is easily seen that

$$\frac{\partial S_p}{\partial x} = 0$$

and

$$\frac{\partial S_p}{\partial y} = 0.$$

Substituting these values into Equations (5) and (6) yields

$$T_{xp} = T_{yp} = 0. \tag{13}$$

By Equations (12) and (13), it can be observed that the
edges of a hyperbolic paraboloid shell resist only a uniform
tangential shear, since the normal forces equal zero.

Using Mohr's circle, this pure shear resolves into
principal stresses of equal and opposite magnitude acting
on sections at 45 degrees to the shear plane (see Fig. 6).
Identical parabolic arches are formed by the sections taken
45 degrees to the coordinate axis as shown in Fig. 6. The
parabolic arches or parabolas parallel to OB curve downward
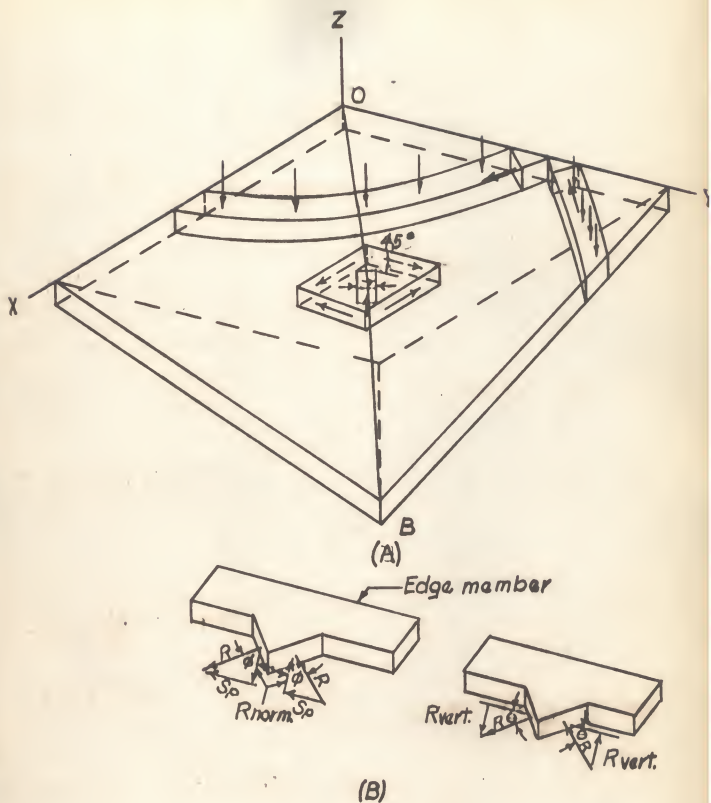while those at right angles to OB curve upward.

(A)

(B)

FIG. 6. PARABOLIC ARCHES

Assuming the uniform load is equally divided between the two sets of perpendicular arches as shown in Fig. 6, the arches parallel to OB exert an outward thrust R on the edge members by the nature of their curvature, while those perpendicular to OB exert an inward thrust R.

Since the arches have curvature, they exert both vertical and horizontal forces at their ends on the edge members. The net effect as shown in Fig. 6-b is that the normal force components to the edge member cancel out and the vertical force components cancel out leaving only the pure shear acting along the edge member. Fig. 7 shows the net result of the pure shear acting on the edge members, or in this case, the edge beams, for two quadrants forming a portion of a complete structure.

Secondary Stresses

In using the membrane theory of analysis, only the equilibrium of forces has been considered, neglecting the compatibility between stresses and strains. The following discussion on secondary stresses is based mainly on P.C.A.'s publication of "Elementary Analysis of Hyperbolic Paraboloid Shells".[11]

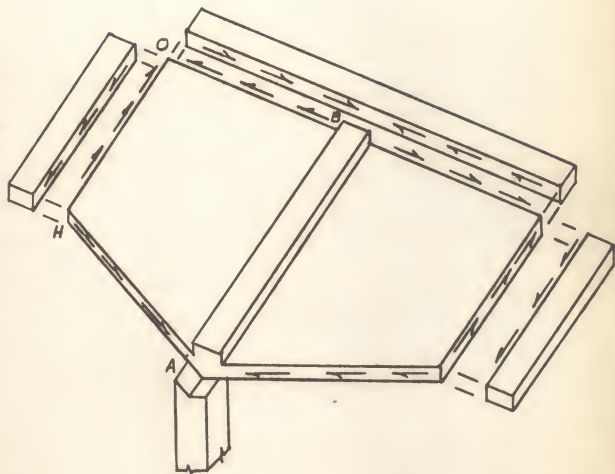[11] Elementary Analysis of Hyperbolic Paraboloid Shells, Portland Cement Association, 1960.

FIG. 7. SHEAR FORCES ON EDGE BEAM

As the shell becomes flatter (less rise to span ratio) axial strain within the perpendicular parabolic arches becomes important. This axial strain will set up secondary bending moments within the arches. For the usual rise, h/a = 1/5 or h/b = 1/5, the effect of axial strains is relatively small and can be safely neglected. When the shell flattens out, i.e. h/a decreases, the effect of axial strain must be investigated. Using a two-hinged parabolic arch subjected to uniform load as a comparison to the parabolic arches in the shell, the horizontal component of the reaction for a given span decreases as the ratio of rise to span decreases. This decrease in the horizontal component is coupled with a subsequent increase of axial strain, thus resulting in secondary bending moment. The limiting value of secondary bending moment would be the case of zero rise in which it would equal the simple-beam bending moment.

Fig. 8-a represents the results of a study[12] concerning secondary bending moment. This graph is also based on the fact that the curved surface can be represented by sets of perpendicular parabolic arches, whose shearing forces and normal forces on the two opposite faces can be neglected. Also the assumption that the ends of the arches are not free to move is made.
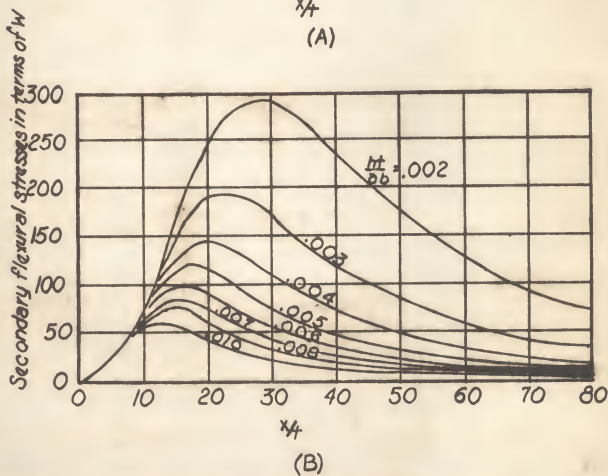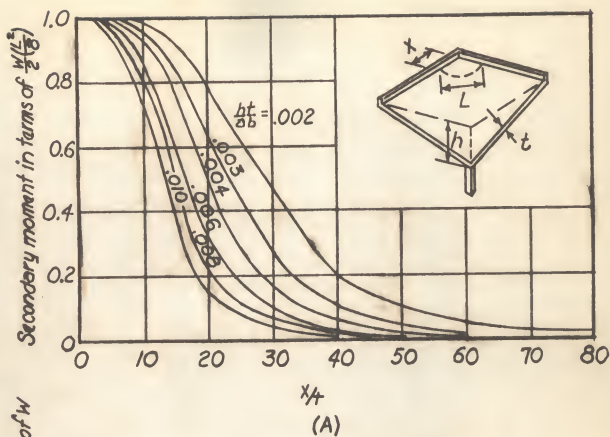
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[12] Ibid., p. 16.

FIG.8. SECONDARY MOMENTS AND STRESSES

Using the appropriate ratio of $\frac{ht}{ab}$ coupled with the dimensionless quantity $\frac{x}{t}$, the secondary bending moment can be found for a strip of L length. In studying Fig. 8-a, it is seen that as $\frac{x}{t}$ decreases to zero, the bending moment approaches the simple beam bending moment, which is not in line with the membrane theory analogy. As $\frac{x}{t}$ increases, i.e. the strip of L length is farther from the corner, the secondary bending moment decreases, the rate of which is a function of $\frac{ht}{ab}$. As the ratio of $\frac{ht}{ab}$ becomes larger, the rate of decrease of the secondary bending moment becomes larger. Using a shell thickness of 3", the normal ratio of rise to span of 1/5,

$$\frac{ht}{ab} = \frac{1(1/4)}{5 \times 5} = 0.010.$$

From Fig. 8-a, for $\frac{ht}{ab} = 0.010$, and a distance of 5 feet or more from the corner, it is observed that the secondary bending moment is becoming less important.

Fig. 8-b, representing the secondary bending moment as secondary flexural stress, brings out the importance of curvature on the magnitude of secondary stresses.[13] For example, a square shell whose thickness is 3 inches and $\frac{h}{a} = 1/5$ has a secondary stress in terms of w equal to 40, at a point 5 feet from the corner. Another square shell

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[13] Ibid., p. 16.

with the same thickness and at the same point from the corner, except having $\frac{h}{a} = 1/7$ has secondary stress in terms of w equal to 115. From comparing the two values of stress, the importance of curvature is quite evident.

DESIGN EXAMPLE

Shell Design

This design is for a square roof unit having a shape commonly called the inverted umbrella. Fig. 7 shows a partial unit with the exterior edges being horizontal. The plan dimensions are 30 x 30 feet with a 3 foot vertical rise. Since adequate reinforcement coverage controls the shell thickness rather than stress limitations, a shell thickness of 3 inches will be used. This gives a uniform dead load of 37.5 psf for the shell. Live load is equal to 30 psf plus 15% of the live load to allow for the weight of the edge beams. The total design uniform load is therefore 72.0 psf.

From Equation (12), the pure shear per foot acting within the parabolic arches of the shell is

$$S_p = \pm \frac{Wzp}{2k} = \pm \frac{Wzp}{2} \frac{ab}{h} = \pm \frac{72 \ (15) \ (15)}{2 \ (3)} = \pm 2700 \ \text{lb./ft.}$$

The compressive concrete stress is

$$f_c = \frac{2700}{3(12)} = \pm 75 \text{ psi.}$$

No steel is needed for concrete in the compressive parabolic arches, however a nominal amount should be used for temperature and shrinkage stresses. The tensile concrete stress is also 75 psi. Here again, no steel is theoretically needed for loading stresses, however steel will be included to take care of all tensile forces. The amount being

$$A_s = \frac{2,700}{20,000} = 0.135 \text{ sq. in. per ft.}$$

Recalling that the parabolic arches are rotated 45 degrees from the shell's edge, the reinforcing should be placed likewise. If for ease of placement, the reinforcing is oriented parallel to the shell's edge, the amount of reinforcing must be subsequently increased. For this example, the amount of reinforcement placed parallel to the edges equals

$$A_s = 0.135 \times 1.414 = 0.191 \text{ sq. in. per ft.}$$

Therefore No. 3 bars placed at 7 inches apart are sufficient. This same pattern will also be used for the compressive area.

Considering secondary stresses in the flattened corners, Fig. 8-b shows the maximum secondary stresses occurring at x/t equalling approximately 22 for

$$\frac{ht}{ab} = \frac{3 \; (1/4)}{15 \; (15)} = 0.0033.$$

For the 3 inch thickness, this gives the L length strip placed out an x distance 5'-6" from the corner. For the D.L. plus L.L. of 72.0 psf, the maximum secondary stress equals

$$f_c = \frac{175 \times 72.0}{144} = 87.5 \text{ psi.}$$

While the magnitude of flexural stress is not critical, five No. 3 bars placed 45 degrees to the edge beams will be added for tension in the immediate range of maximum flexural stress.

Edge Beam Design

According to the membrane theory analysis the horizontal edge beams are loaded by pure shear from the shell, causing tension ranging from zero at the corners to a maximum value at the center. Maximum tension in the horizontal edge beams is

$$H = 2,700 \times 15 = 40,500 \text{ lb.}$$

for which the area of steel required is

$$A_s = \frac{40,500}{20,000} = 2.025 \text{ sq. in.}$$

Two No. 6 bars and two No. 7 bars are sufficient.

Excessive corner deflection of the horizontal edge beam has been experienced for this type of structure.[14] Recalling the discussion of secondary stresses, as the shell flattens out near the corners the validity of membrane theory analysis decreases. The parabolic arch strips approach simple beams in which the horizontal edge beams take the vertical reaction. This in turn causes a downward deflection. To offset this deflection, the centroid of the edge beam is placed above the application of the shearing action from the shell. This results in an eccentric loading which tends to lift the end of the edge beam. In addition, a small amount of steel can be placed in the top of the edge beam to take tension stresses.

The sloped edge beams are designed for axial compression resulting from shearing action on both sides ranging from zero at the outer edge to a maximum at the valley of the shell. The maximum amount is

$$2H \frac{15.30}{15} = 2 \times 40,500 \times \frac{15.30}{15} = 82,620 \text{ lb.}$$

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[14] Building for Economy with Hyperbolic Paraboloids, by G. Madsen and D. Biggs, Journal ACI, Vol. 32, No. 4, pp. 373-383.

Since the sloped beam is subjected to axial compression with small eccentricity, it may be designed by column formulas. Another approach[15] to the design would be to consider the sloped beam as a flange with the shell acting as the web. This approach permits using allowable compressive stress in flexure. Since the membrane analysis does not include the effect of strains occurring in the edge beam, the conservative column formulas will be used to reduce possible strains.

The standard formula for lateral tied columns is

$$P = 0.8 \ (0.225 \ f'_c A_g + A_s f_s)$$

substituting $p_g A_g = A_s$, the above formula becomes

$$P = 0.8 \ (0.225 \ f'_c + p_g f_s) \ A_g$$

from which

$$A_g = \frac{P}{0.8(0.225 \ f'_c + p_g f_s)} \ .$$

Using a percentage of steel $p_g = 0.01$, $f_s = 20,000$ psi and $f'_c = 3,000$ psi, the required gross area at the valley for the sloped beam is

$$A_g = \frac{82,620}{540 + 20,000 \ \text{x} \ 0.01} = 111.6 \ \text{sq. in.}$$

[15] Elementary Analysis of Hyperbolic Paraboloid Shells, Portland Cement Association, 1960.

For an 18 inch wide beam, the depth required is

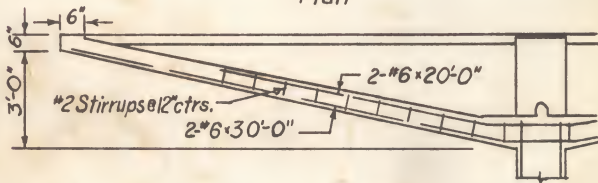$$d = \frac{111.6}{18} = 6.2 \text{ inches.}$$

Due to possible unsymmetrical loading causing increased bending stresses, an increase of depth to 9 inches will be used. The required A_S for the sloped beams equals

$$A_g = 0.01 \times 111.6 = 1.116 \text{ sq. in.}$$

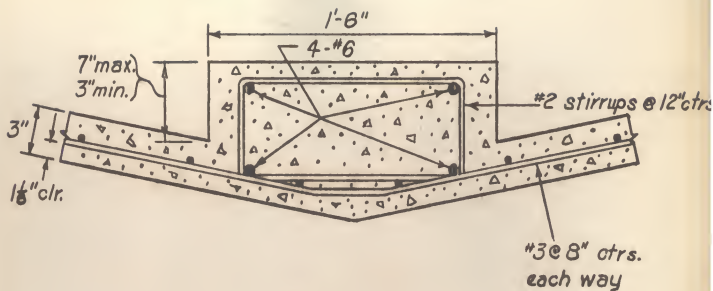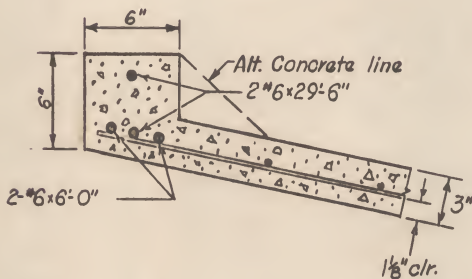Four No. 6 bars will be used.
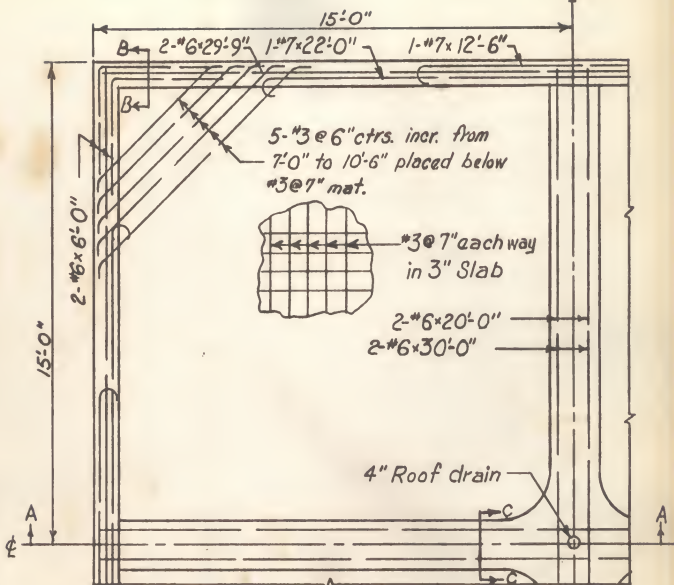
Plan

Section A-A

FIG. 9-A. DETAILED DRAWING

Section C-C

Section B-B

FIG. 9-B. DETAILED DRAWING

CONCLUSION

By membrane theory, the doubly curved surface of the hyperbolic paraboloidal shell will support a uniform load mainly by direct axial compression or tension forces. The edge beam stiffens the shell at the boundaries, and is loaded by pure shear from the shell, according to membrane theory. However, since stresses developed by this theory are dependent upon shell curvature, sections of the flatter portion of the shell near the corners tend to act as simple beams, thus developing bending stresses. Accompanying the bending stresses of the shell is a vertical reaction on the tension edge beam. This in turn produces bending moments in the edge beam.

Analysis by the membrane theory for stresses in the shell under uniform load does not involve complicated mathematics. However, under unsymmetrical loading, the mathematics becomes very complicated. With unsymmetrical loading and even uniform loading to a lesser degree, there is some question as to how much of the shell is conforming to the assumptions made for the use of membrane theory. No allowance by the membrane theory is given for compatibility of stress-strain relationships. Consequently, any deflection which the structure obtains will introduce a need for stress-strain consideration. The uniformly loaded structure

treated in this paper will have slight deflection, especially near the shell boundaries. However the stress analysis resulting from this deflection is beyond the scope of this paper.

In studying the available literature, the author failed to uncover a complete general solution of hyperbolic paraboloidal shells. Several approaches have been used.[16] Model studies with simplified mathematical analysis appears to be one of the simpler, while a more complicated approach would be to apply a combination of membrane, bending, ultimate load, and buckling theories to the particular shell being studied.

[16] Thin-shelled Structures, by D. P. Billington, Civil Engineer, December 1961, p. 57.

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ANALYSIS OF A HYPERBOLIC

PARABOLOIDAL SHELL

by

MELVIN L. BURDORF

B. S., Kansas State University, 1958

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AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Approved by:

———————————
Major Professor

Many articles have been published on hyperbolic paraboloidal shells. However to the knowledge of the author, no complete simplified analysis covering various loading conditions has been developed to date.

The shell analysis explained is a result of reviewing several references using the membrane theory solution with uniform vertical loading. By the membrane theory, no bending moment is allowed within the shell, therefore the shell acts mainly by axial forces. Applying this theory, the basic equations are developed for one quadrant of the hyperbolic paraboloid. A consideration of secondary stresses, which cannot be included in the membrane theory solution, is given. The basic equations and the consideration of secondary stresses are then applied to a practical design of an inverted, single support reinforced concrete hyperbolic paraboloid.

A brief detailed drawing is presented showing the dimensions and reinforcing resulting from the design of the hyperbolic paraboloid.