ELLIPTIC GEOMETRY

by

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INTRODUCTION

Geometry, as its name implies, began as a practical science of measurement of land in ancient Egypt around 2000 B.C. It consisted at first of isolated facts of observation and crude rules for calculation until it came under the influence of Greek thought after being introduced by Thales of Miletus (640-546 B.C.). Thales helped to raise the study of geometry by abstracting the various elements from their material clothing. Geometry really began to be a metrical science in the hands of Pythagoras (about 580-500 B.C.) and his followers. Later (about 430 B.C.) Hippocrates of Chios attempted with others to give a connected and logical presentation of the science in a series of propositions based upon a few axioms and definitions. Thus by 300 B.C., the science of geometry had reached a well-advanced stage. It remained for Euclid at this time, however, to collect all the material which had already accumulated, and by adding the results of his own tremendous research, to compile and publish his famous work Elements. This book stood for many years as the model for scientific writing and gave to Euclid a prestige so great that a reputation of infallibility descended upon him which later became a distinct hindrance to future investigations.

Euclid opens his book with a list of definitions of the geometrical figures followed by a number of common notions (also called axioms) and then five postulates. An axiom or common notion was considered by Euclid as a proposition which is so self-evident that it needs no demonstration; a postulate as a proposition which, though it may not be self-evident, cannot be proved by any simpler proposition. The common notions, also five in number, deal with equalities and inequalities of magnitudes and are regarded as assumptions acceptable to all sciences and to all intelligent people. The five postulates, however, are peculiar to the science of geometry, with the famous Fifth Postulate (also known as the Parallel Postulate) playing a major role in what follows. The five postulates are:

1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be produced continously in a straight line.
3. A circle may be described with any center and any radius.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced sufficiently, meet on that side on which are the angles less than two right angles.

Euclid's fifth postulate was attacked almost immediately because it failed to satisfy the demands of Euclid's followers as a proposition acceptable without proof and also because so much was proved without using it. Indeed, in Euclid's Elements the first 26 theorems of Book 1 are proved without recourse to this questionable postulate. (Thus, Euclid's reluctance to introduce it himself until absolutely necessary provides a case for calling him the first Non-Euclidean geometer.) This discovery led to many futile attempts to prove the fifth postulate from the other four. Even today many would-be geometry students consider such a proof in quiet contradiction of a now well-established fact that other logically consistent geometries exist which admit the first four postulates but not the last. We can also see now that although these attempted proofs were in vain, they did cause a rigorous examination of the basis of geometry in particular and mathematics in general.

Some of the men who attempted to prove the fifth postulate and failed include Ptolemy, Proclus, Naseraddin, Wallis, and Saccheri[1]. Of particular interest was the method of Gerolamo Saccheri (1667-1733), an Italian Jesuit priest and Professor of Mathematics at the University of Pavia in Milan. Being quite impressed with the power of the <u>reductio ad absurdum</u> method of proof and having complete faith in the truth of the Euclidean Hypothesis, Saccheri discussed the contradictory assumptions with a definite purpose in mind. He wanted not to establish their logical possibility but to detect the logical contradictions which he was persuaded must follow from them.

The fundamental figure that Saccheri used was the isosceles birectangular quadrilateral ABCD as illustrated with $\angle A = \angle B =$ a right angle and the sides AD and BC equal. Line AB is called the base of the quadrilateral and DC is known as the summit.

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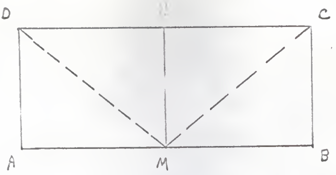
[1]Harold E. Wolfe, <u>Introduction to Non-Euclidean Geometry</u> (New York: 1930), pp. 26-33.



By letting M and N represent the midpoints of AB and DC, respectively, it is obvious that $\triangle AMD \cong \triangle BCM$.
This together with $\triangle DNM \cong \triangle MNC$ implies:
(1) $\angle ADN = \angle BCN$ (2) $MN \perp AB$ (3) $MN \perp DC$.
Saccheri's plan required the investigation of three hypotheses called appropriately the hypothesis of the right angle, the hypothesis of the obtuse angle, and the hypothesis of the acute angle. He hoped to reach contradictions with the latter two and thus to prove by trichotomy the soundness of the hypothesis of the right angle, which would lead him into Euclid's Parallel Postulate.

After studying the hypothesis of the acute angle and arriving at a long sequence of propositions and corollaries which were to become classical theorems in Hyperbolic Geometry, Saccheri weakly concluded that the hypothesis leads to the absurdity that there exist two straight lines which, when produced indefinitely, merge into one straight line and have a common perpendicular at infinity. Since he attempted a second proof later with no greater success, it was evident that Saccheri was dubious himself about his conclusions. Indeed, had he suspected that he had not reached a contradiction but had uncovered a new concept, the discovery of Non-Euclidean Geometry would have been made almost a century earlier than it was.

It is Saccheri's investigation of the hypothesis of the obtuse angle which really interests us and which hopefully will shed some light on this subject of Elliptic Geometry.

Saccheri disposed of the hypothesis of the obtuse angle by reading too much into Euclid's Second Postulate. Just as others before him, Saccheri assumed this postulate implied that the straight line was infinite. This in turn leads to a proof of Proposition 16 (which states that the exterior angle of a triangle is greater than either of the opposite and interior angles) from Book 1 of Elements which is used to show that the hypothesis of the obtuse angle implies the hypothesis of the right angle.

The crux of the contradiction, of course, lies in assuming that the straight line is infinite in length under the hypothesis of the obtuse angle. It was Riemann (1826-1866) who first realized that these assumptions were incompatible and substituted for the implication that the straight line is infinite the more general idea that it is unbounded or endless. The difference between the infinite and the unbounded he puts in the following words:

"In the extension of space construction to the infinitely great, we must distinguish between unboundedness and infinite extent; the former belongs to the extent relations, the latter to the measure relations. The unboundedness of space possesses a greater empirical certainity than any external experience, but its infinite extent by no means follows from this."[2]

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[2]Roberto Bonola, Non-Euclidean Geometry (New York: 1955), pp. 142.

Using this interpretation of Postulate Two, one can construct geometrical systems with just as much logical basis as Euclidean Geometry. In attempting to visualize the straight lines of these systems, it will help to form an analogy with the great circles of a sphere. As we know, these particular circles (and their arcs) constitute the geodesics of the sphere. That is to say, the shortest distance between any two points on a sphere is along the arc of a great circle passing through those two points. There are other properties which great circles on a sphere share with straight lines on a plane, but there also exist distinct differences. For example, these "lines" are endless, but not infinite; two points, in general, determine a line, but they can also be so situated so as to have an infinite number of "lines" drawn through them. Also, we see that two "lines" always intersect in two points and enclose a space. Finally, we must note that however, convenient this analogy (or any analogy) might be, one has to be careful in applying or carrying it too far. Relying too closely upon such an analogy will often lead researchers astray in their work. For example, from the preceding representation of elliptic geometry, one might get the "idea" that all lines in elliptic geometry are curved. In truth, however, elliptic straight lines are just as "straight" as Euclidean straight lines. We have only used curved lines as a graphic picture of something we may otherwise have not been able to visualize. Thus, in spite of some shortcomings, an analogy can give many needed insights into our study.

With these reservations in mind, another form of visualization of elliptic geometry can be constructed by considering a bundle of straight lines and planes through a point O. If we call a straight line of the bundle a "point" in elliptic space and a plane of the bundle a "line", we can easily see how the following well-known theorems from Euclidean geometry can be modified to represent something in the elliptic geometry.[3]

E1. Two lines through O uniquely determine a plane through O.

"E1." Two "points" uniquely determine a "line".

E2. Two planes through O intersect always in a single line through O.

"E2." Two "lines" intersect always in a single "point".

E3. All the planes through O perpendicular to a given plane α through O pass through a fixed line λ through O, which is orthogonal to every line through O lying in α.

"E3." All the "lines" perpendicular to a given "line" α pass through a fixed "point" A, which is orthogonal to every "point" lying in α.

The analogy can be carried further, but we can easily see that elliptic geometry can certainly be represented by the geometry of a bundle of lines and planes.

Euclid's Fifth Postulate, while under attack for previously mentioned reasons, is also known to be quite unwieldy to work with, as anyone who even reads it should agree. To alleviate matters

[3]D. M. Y. Sommerville, The Elements of Non-Euclidean Geometry (London: 1914), pp. 90.

somewhat, there have been many substitute statements which are essentially equivalent to the Fifth, but are also considerably simpler in statement and comprehension.

One such substitute, and the most commonly used of all statements of this class, is known as Playfair's Axiom after the geometer by the same name. Playfair's Axiom is as follows:

> Through a given point not on a given line can be drawn <u>one and only one</u> line which is parallel to the given line.

This axiom can readily be shown to be equivalent to the Fifth Postulate; but more importantly, it also lends itself much more easily as a characteristic postulate of a particular system than does the Fifth. This can be seen by deleting the words "one and only one" from Playfair's Axiom and substituting the phrase "more than one" in their place. This new axiom will lead one into the realm of Hyperbolic Geometry and is thus known as the characteristic postulate of that theory.

Altering Playfair's Axiom in the opposite sense gives:

> Through a given point not on a given line can be drawn <u>no</u> line which is parallel to the given line.

This obviously will simplify into the <u>Characteristic Postulate</u> of Elliptic Plane Geometry:

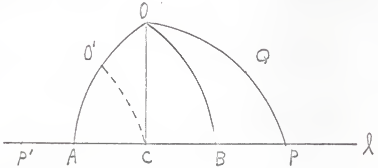
POSTULATE: <u>Two straight lines always intersect one another</u>.

With this postulate and the previously mentioned assumption that the straight line is not infinite, but just endless, we proceed to the development of Elliptic Plane Geometry.

PROPERTIES OF LINES AND SURFACES

To expedite our investigations we will proceed under two assumptions that help to free us from small technical details in a great many proofs that follow. The first such assumption will be that line segments are undirected. That is, segments AB and BA are identical, since no direction is associated with either of them. The second assumption will be the validity of all theorems from Euclidean geometry that are not dependent upon either the Parallel Postulate or the concept of an infinite line. This will allow us to skip over much of Elliptic Geometry which is simply a repetition of Euclidean Geometry. The various congruence theorems fall in this category and will play a major role in the proof of some of our succeeding work.

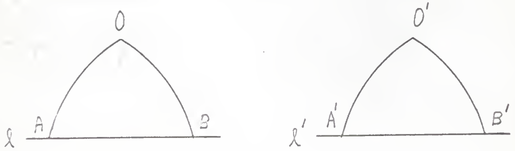
With these ideas in mind, let A and B be any two points on a given line ℓ. The perpendiculars at A and B to ℓ must intersect by reason of the Characteristic Postulate. Call this point of intersection O.



Since $\angle OAB = \angle OBA =$ right angle, we have OA = OB. At O make $\angle BOQ = \angle AOB$ and produce OQ to cut the line ℓ at P. Then AB = BP and $\angle OPA$ is a right angle by congruent triangles AOB and BOP.

By repeation of this construction, we can show that if P is a point on AB produced through B such that $AP = m \cdot AB$, the line OP is perpendicular to ℓ and equal to OA and OB. The same holds for points on AB produced through A such that $BP' = m \cdot AB$. In all cases, m is a positive integer. Likewise let C be a point on AB such that $AB = m \cdot AC$. The perpendicular at C to ℓ must pass through the point O, since if it met OA at O', the above argument shows that O'B must be perpendicular to ℓ and coincide with OB. It follows that if P is any point on the line ℓ such that $AP = \frac{m}{n} \cdot AB$, where m and n are two positive integers, then OP is perpendicular to the line and equal to OA and OB. The case when the ratio AP : AB is not rational is deduced from the above by using a limiting process on the infinite decimal representation of the irrational number.

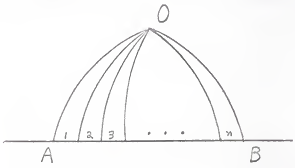
Thus all points on the line are included in this argument, so that the perpendiculars at all points of the line ℓ pass through the same point. Now let ℓ' be another line and A', B' two points upon it such that segment AB = A'B'.



The perpendiculars at A' and B' meet in a point O'. Then the triangles AOB and A'O'B' are congruent by virtue of having an angle - side - angle identical in measure. Thus it follows that

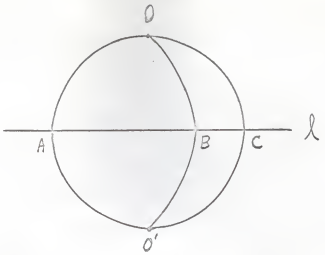
$O'A' = OA$, and we have shown that the perpendiculars at all points on any line meet at a point which is at a constant distance from the line. The point at which they all meet is called the Pole of the Line and the constant distance will be denoted by q. Thus every ray emanating from the pole of the line is perpendicular to the line.

Given any two points (A and B) in the plane, we can construct at least one line which contains the points. Construct perpendiculars to the line at A and B. These perpendiculars meet at the pole of the line containing A and B.



Divide AB into n equal parts and construct perpendiculars at the division points. Any two of the small triangles are congruent by virtue of having an angle - side - angle combination which is equal. Since this procedure is valid for any n, we arrive at two very important conclusions: (1) the distance between any two points is proportional to the angle formed at the pole of the line containing these two points and (2) the measure of the area of this figure is proportional to that same angle.

Next consider the figure OAB where O is the pole of line AB. Extend OA to O', where $q = O'A = OA$, and then construct O'B.



Then, from the triangles OAB and O'AB, it follows that $\angle O'BA = \angle OBA$ = a right angle. Thus OB = BO' = q and they are parts of the same straight line. Also, AO' produced through O' must intersect AB at a point C since every ray from O' is perpendicular to the line l. Thus OC will also be perpendicular to AB. This shows that OAO' produced returns to O and the line is closed or re-entrant and thus is finite and of length 4q. It should be noted, however, that the line is still endless or unbounded in our system.

Assuming for the moment then that O and O' are two distinct points, every line has <u>two</u> poles. Also, any two lines intersect in two points and have a common perpendicular. The figure that these two lines enclose is called a digon, or biangle, each side of which has length 2q. The angle between the two lines at their point (or points) of intersection is called the angle of the digon. Such a simple figure is impossible to form in the other spaces.

Another contradiction with the other systems is seen in that two points do not always determine a unique straight line. For example, we see that through the two poles of a line an infinite number of lines can be drawn, just as through the two ends of a diameter of a sphere an infinite number of great circles can be drawn.

It should also now be clear why Euclid's proof of I-16[4] is not valid in this geometry. The proof of proposition I-16, uses an argument that depends upon producing a line an amount dependent upon the "size" of the triangle. Thus, in light of our restricted definition of the line, we can only conclude that the exterior angle of a triangle is greater than either of the interior and opposite angles only when the corresponding median is less than q. If this median is equal to q, the exterior angle is equal to the angle considered; if it is greater than q, the exterior angle is less than the interior angle considered. Since I-16 is in turn essential to the proof of I-27 (which states that if a transversal cuts two straight lines and makes the alternate angles equal, then the two straight lines are parallel), it is now evident why in this geometry that theorem does not hold.

Of course, if I-27 did hold, then by the construction implied by that proposition there would exist at least one parallel to a line through any point outside it. Obviously, in "limited" regions of the plane, I-16 does hold and various theorems dependent upon it are true. The case given above when the median of a triangle is less than q is such an example.

We should note that in stating the previous few remarks

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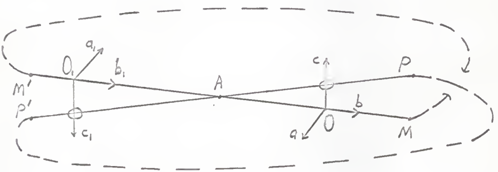
[4]The exterior angle of a triangle is greater than either of the opposite and interior angles.

we have assumed that the point O is a different point from O'. If the two points coincide, then the plane of this geometry has a wholly different character. For example, the length of a straight line is now 2q instead of 4q. Also, if two points P, Q are given on the plane along with any arbitrary straight line, we can pass from P to Q by a path which does not leave the plane, and yet does not cut the line. In other words, the plane is not divided by its lines into two parts.

Imagine a set of three rectangular lines Oabc with Ob on the line AM and Oc always cutting the fixed line AP. (Remember that M' is the same point as M and that P' coincides with P. We have emphasized this point by drawing curved dashed lines between M and M' and between P and P'. However, these lines intersect in only one point, A.)



As O moves along AM extended (in the direction indicated by the arrow Ob) it will eventually return to A. But now Oc_1 is turned downwards and Oa_1 points to the left instead of to the right. Thus the point c has moved in the plane PAM and come to the other side of the line AM as c_1 without actually crossing it.[5]

A more concrete example of this peculiarity is given by what is called a Leaf (or Sheet, or Strip) of Möbius, which consists of a band of paper twisted 180° and with its ends joined. A line traced along the center of the band will return to its starting point, but on the opposite surface of the sheet. Thus the two sides of the sheet are continuously connected.

Obviously then, the essential difference between the two planes is that in the one, the plane has the characteristics of a two-sided surface, and in the other it has the characteristics of a one-sided surface. The first plane is usually called the spherical or double elliptic plane; the second is usually called the elliptic or single elliptic plane. Although the geometries which can be developed on both of these planes are referred to as Riemann's (Non-Euclidean) Geometries, it seems likely that he had only the double elliptic plane in mind as he did his work. The single elliptic plane and its peculiar distinctions were first brought to light by the German mathematician Felix Klein in his publications during the 1870's. (It was Klein who attached the now usual nomenclature to the three geometries; the geometry of Lobachewsky and Bolyai he called Hyperbolic, that of Riemann Elliptic, and that of Euclid Parabolic. The names were suggested by the fact that a straight line contains two infinitely distant points under the Hypothesis of the Acute Angle, none under the Hypothesis of the Obtuse Angle, and only one under the Hypothesis

[5] Sommerville, op. cit., p. 91.

of the Right Angle.)

In the brief outline of Elliptic Geometry presented here, we are trying to restrict ourselves as much as possible to those properties common to both the single and double elliptic planes, with occasional ventures into the particular characteristics of each if the problem warrants it. Having looked at the distinguishing traits of the line in each space, we turn to an investigation of the simple figures common to both planes.

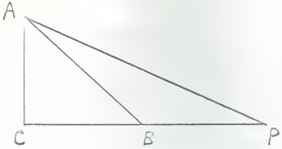
PROPERTIES OF TRIANGLES AND QUADRILATERALS

The initial reaction would be to assume that the digon (or biangle) mentioned earlier is the most basic figure involving straight lines. We note, however, that this figure exists only in the double elliptic plane. By design, the single elliptic plane's property of having any two points <u>uniquely</u> determining a single line disallows the construction that the digon requires.

With this in mind we turn our attention to the triangle and some of its more interesting properties under the hypotheses of Elliptic Geometry.

THEOREM: In any triangle which has one of its angles a right angle, each of the other two angles is less than, equal to, or greater than a right angle if and only if the side opposite it is less than, equal to, or greater than q, respectively.

PROOF: Let angle C in triangle ABC be a right angle. Let P be the pole of the side AC.



Then P lies upon BC and PC = q. Construct AP. Then ∢PAC is a right angle. Thus:

CB > CP if and only if ∢BAC > ∢PAC = a right angle,

CB = CP if and only if ∢BAC = ∢PAC = a right angle,

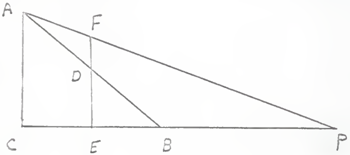
CB < CP if and only if ∢BAC < ∢PAC = a right angle.

Therefore the theorem is proved.

Next consider any right-angled triangle ABC in which C is the right angle.

THEOREM: In any right-angled triangle the sum of the angles is greater than two right angles.

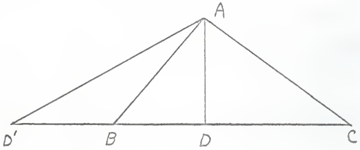
PROOF: If either of the legs of the right triangle is greater than or equal to q, the sum of the angles is greater than two right angles by the above theorem. If both sides are less than q, draw DE perpendicular to the side BC, where D is the midpoint of the hypotenuse.



Let P be the pole of DE. Thus EP = q. Produce ED to F, so that ED = DF. Construct AF and PF. Then the triangles ADF and DEB are congruent by virtue of having a side - angle - side combination equal in measure. Thus $\sphericalangle$ AFD = $\sphericalangle$ DEB = a right angle. Thus A, F, and P are colinear. However, we know that $\sphericalangle$ PAC > a right angle, since CP > EP = q. But $\sphericalangle$ PAC = $\sphericalangle$ CAB + $\sphericalangle$ DAF = $\sphericalangle$ CAB + $\sphericalangle$ DBE. Therefore, the sum of the angles at A and B in the right-angled triangle ABC is greater than a right angle in this case as well as in the others.

THEOREM: The sum of the angles of any triangle is greater than two right angles.

PROOF: Let ABC be any triangle. If at least one of the angles is a right angle, then the theorem follows from the preceding theorem. If two of the angles are obtuse, the theorem is obviously true. Thus we need only consider the case when two of the angles are acute. Let $\sphericalangle$ABC and $\sphericalangle$ACB be acute.



From A draw AD perpendicular to BC. Then D must lie on the segment BC, for, if it did not, then altitude AD' would have to be both greater than and less than q at the same time by virtue of being the side of two right triangles with opposite obtuse and acute angles respectively at the same time. Thus from the previous theorem, $\sphericalangle$ ABC + $\sphericalangle$ BAD > a right angle and

$\sphericalangle$ DAC + $\sphericalangle$ ACD > a right angle. Therefore, it follows that the sum of the angles of the triangle ABC is greater than two right angles.

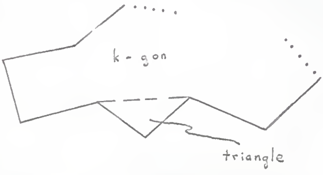
The amount by which the sum of the angles of a triangle exceeds two right angles is called the excess of the triangle.

COROLLARY: The sum of the angles of every quadrilateral is greater than four right angles.

PROOF: Since any quadrilateral can be divided into two triangles by either diagonal, this corollary follows readily from the previous theorem.

COROLLARY: The sum of the angles of an n-gon is greater than (n-2) times two right angles for $n \geq 3$.

PROOF: The proof of this corollary is by mathematical induction on n, the number of sides of the n-gon. Since the 3-gon is a triangle, we have already proven this statement for $n = 3$. Assume, then, that the sum of the angles of a k-gon is greater than (k-2) times two right angles. Now look at a (k + 1)-gon. Pick any two vertices such that the line segment constructed to connect these two vertices lies within the polygon and such that it divides the (k + 1)-sided figure into a triangle and a k-sided polygon.



Then the sum of the angles of the (k + 1)-sided figure is equal to the sum of the angles of the k-sided figure and the angles of the triangle. However, since the sum of the angles of the k-sided figure is greater than (k-2) times two right angles by the induction hypothesis and since the sum of the angles of the triangle is greater than two right angles, we see that the sum of the angles of the (k + 1)-sided figure is greater than $(k-2)\pi + \pi = (k-2+1)\pi = [(k+1)-2]\pi$, where π represents two right angles. Thus by mathematical induction the corollary holds for all $n \geq 3$.

We can now generalize our definition of the excess of a n-gon to be the sum of the angles of the polygon minus $(n-2)\pi$.

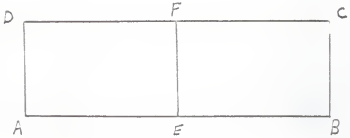
A quadrilateral of particular interest is the birectangular, isosceles quadrilateral known as Saccheri's Quadrilateral as mentioned previously. The distinguishing features of this figure can be summarized in one general theorem.

THEOREM: The line joining the midpoints of the base and the summit of a Saccheri Quadrilateral is perpendicular to both of them, and the summit angles are equal and obtuse.

PROOF: The only question remaining to be answered is whether the summit angles are obtuse. The proof of the remainder of the theorem is given in the introduction of the paper. Therefore, since the sum of the angles of every quadrilateral is greater than four right angles, the equal summit angles must be obtuse and the theorem holds. Incidentally, this proof shows that

Elliptic Geometry does indeed correspond to Saccheri's work with the Hypothesis of the Obtuse Angle.

Another quardilateral of interest is the trirectangular quadrilateral also known as Lambert's Quadrilateral after the German geometer J. H. Lambert (1729 - 1777). Like Saccheri before him, he also came close to the discovery of Non-Euclidean Geometry. He chose this particular quadrilateral as his fundamental figure and proposed three hypotheses in which the fourth angle was in turn a right, an obtuse, and an acute angle. The similarity of his hypotheses and his work to that of Saccheri is evidenced even further by the realization that the Saccheri Quadrilateral can be constructed by adjoining two congruent Lambert Quadrilaterals. That is, in the Saccheri Quadrilateral ABCD below

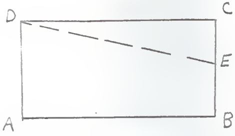


where EF is the line segment joining the midpoints of the base and the summit, we can see two congruent Lambert Quadrilaterals, AEFD and EBCF.

The interesting characteristics of a Lambert Quadrilateral are given in the following theorem.

THEOREM: In a trirectangular quadrilateral (Lambert Quadrilateral) the fourth angle is obtuse and each side adjacent to this angle is smaller than the side opposite.

PROOF: Let ABCD be a Lambert Quadrilateral with right angles at A, B, and D.



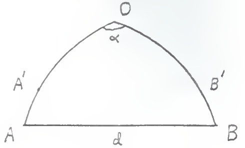
Since the sum of the angles of any quadrilateral exceeds four right angles, then the angle at C must be obtuse.

Assume that one of the sides adjacent to this angle is greater than the side opposite. Without loss of generality, we can assume that BC > AD. Construct BE on BC such that BE = AD. Then we have that ∢ADE = ∢BED since ADEB would be a Saccheri Quadrilateral. However, ∢ADE is less than a right angle, resulting in the sum of the angles of ADEB being less than four right angles, an obvious contradiction. Next assume that BC = AD. Then ∢ADC = ∢BCD = a right angle since ABCD would again be a Saccheri Quadrilateral. We proved, however, that ∢BCD is obtuse. This second contradiction now allows us to conclude that BC is indeed less than AD. In general terms, this means that a side adjacent to the obtuse angle is smaller than the side opposite and the theorem is proved.

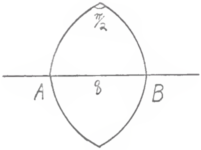
MEASUREMENTS IN THE ELLIPTIC PLANE

To complete our brief look at the properties of the simple figures in the Elliptic plane, we shall investigate some of the problems dealing with measurements of lengths of line segments, of angle measure, and of area.

Choose any two points (A and B) in the Elliptic Plane and construct a line segment AB which connects them.



Construct $AA' \perp AB$ and $BB' \perp AB$. Lines AA' and BB' extended, intersect in at least one point O, the pole of the line AB. From our previous work we know that the distance between two points (A and B) is proportional to the angle formed at the pole of the line containing these two points. That is to say, if we let d denote the measure of line segment AB and let α denote the measure of the angle at the pole O, then $d = c\alpha$ where c is some constant. For convenience we chose the unit of line such that $q = \frac{\pi}{2} k$ and the unit of angle such that a right angle measures $\pi/2$. Now look at the digon with angles of $\pi/2$.



From the above we had $d = c\alpha$. The distance d is now given by q and α is equal to a right angle. Thus $q = c\frac{\pi}{2}$. Now since the unit of line was chosen such that $q = k\frac{\pi}{2}$, we have that $c = k$.

Therefore we arrive at the defining relationship given by $d = k\alpha$ where $k = \frac{2}{\pi} q$. Thus if the length of the segment of line included between two rays from its pole is given by x, then the angle between these rays will be given by $\frac{x}{k}$ and conversely.

We should observe that two points have two distances, that is, d and $4q-d$, although these might be equal. We see also that two lines have two angles, α and $2\pi - \alpha$. (These second measures in each case would be $2q-d$ and $\pi - \alpha$, respectively in the Single Elliptic Plane.)

Just as we have in the previous biangle that the distance between A and B is proportional to the angle at the vertex, we also have that the measure of the area of a biangle is proportional to that same angle. Again by choosing a convenient unit of measure so that a biangle with angle $\frac{\pi}{2}$ has a unit of area given by $k^2\pi$, we arrive at the following relationship between the area of a biangle and the angle α at its vertex:

$$A(\text{Biangle}) = 2k^2\alpha .$$

By looking at the digon with angles 2π, we note that the area of the entire plane is given by:

$$A(\text{Plane}) = (2k^2)\ (2\pi) = 4\pi k^2 .$$

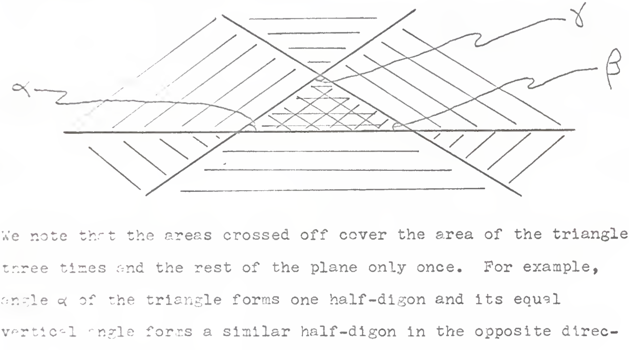
(Once again we note that this result holds for the Double Elliptic Plane only. The Single Elliptic Plane has a total area one-half the previous value.) Obviously, then, the Elliptic Plane has a limited, finite area. This result should not be too surprising for it fits in nicely with the concept of a line of finite length which we have previously discussed.

An important theorem which we now have the tools to prove is as follows:

THEOREM: The area of a triangle is proportional to its excess.

PROOF: We will give a proof of this theorem for the Single Elliptic Plane. A similar proof exists for the Double Plane where all the values would be double what they are here.

As we have seen before, two lines enclose an area proportional to the angle α between them. (Think of the lines as forming the vertex for one-half a digon. Remember that full digons do not exist in the Single Elliptic Plane.) This area is given by $k^2\alpha$. We also know that the area of the entire Single Elliptic Plane is $2k^2\pi$. In the following figure, we mark the areas enclosed by the triangle with angles α, β, γ.



We note that the areas crossed off cover the area of the triangle three times and the rest of the plane only once. For example, angle α of the triangle forms one half-digon and its equal vertical angle forms a similar half-digon in the opposite direction. Thus the total area of the plane taken up by these two half-digons would be $2k^2\alpha$. Like results hold for angles β and γ.

We have, therefore,

$$2k^2\alpha + 2k^2\beta + 2k^2\gamma = 2k^2\pi + 2\Delta$$

where Δ represents the area of the triangle. Thus

$$2k^2 (\alpha+\beta+\gamma) = 2k^2\pi + 2\Delta$$

or

$$\Delta = k^2 (\alpha+\beta+\gamma - \pi).$$

However, $(\alpha+\beta+\gamma - \pi)$ is known as the excess of the triangle, as defined earlier. Therefore the area of a triangle is indeed proportional to the excess of the sum of its angles over two right angles.

Two important corollaries follow from this theorem.

COROLLARY: Two triangles having the same excess have the same area.

COROLLARY: The areas of two polygons are to each other as their excesses.

The proof of the first corollary comes directly from the relation $\Delta = k^2 (\alpha+\beta+\gamma - \pi)$ given in the previous proof. The second corollary follows by the realization that any polygon can be triangulated, and then the sums of the areas of the triangles can be compared.

CONCLUSION

With the basic tools we have developed here, one is able to extend the scope of the theory of Elliptic Geometry into areas such as trigonometry,[6] analytic geometry,[7] solid geometry, and other familiar grounds that were once considered to be solely within the realm of ordinary Euclidean Geometry. Such advanced topics obviously cannot be covered adequately in the small amount of space remaining.

Another phase of Elliptic Geometry (or for that matter, any form of geometry) which could be studied in detail is its consistency. That is to say, we wish to be sure that the geometry which we are developing will never lead us into a contradiction, regardless of how far or in what direction we desire to continue our study.

Most tests of this consistency have been tests of comparison. That is, an analogy is usually found which would represent the system to be tested in some form within another better known system. For example, Carslaw[8] sets up an analogy whereby Elliptic Geometry is represented by a particular family of circles in the Euclidean Plane. With this analogy developed fully, he then reasons that no contradictions could possibly arise in Elliptic Geometry, for if they did, then a contradiction would also exist within a subsystem of Euclidean Geometry. This justification of the consistency of Elliptic Geometry could certainly be false, since no one has ever proven that such could not happen within Euclidean Geometry. However, we do accept his work since we are

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[6]Wolfe, op. cit., p. 185.

[7]Henry Parker Manning, Non-Euclidean Geometry (Boston: 1901), p. 69.

[8]H. S. Carslaw, The Elements of Non-Euclidean Plane Geometry and Trigonometry (London: 1916), p. 171.

as certain that Euclidean Geometry is consistent as we can be about any theory in existence today.

However, even if we conclude that each of the three geometries is as consistent as either of the others, there still remains the question of which geometry is really the "true" geometry. This question has no place in geometry as a pure science, but rather in geometry as an applied science. The answer, of course, lies with the experimenter. The fallacy is, however, that the researcher cannot make measurements of an exact enough nature to give himself the answer. If he could, a simple measurement of the sum of the angles of any triangle would tell us immediately which geometry is "true"--if such a thing can be said.

The pivotal element in most applications which would have a choice such as ours is convenience. Measurement of space is no exception. We shall conclude this presentation of our subject with a quotation by the French geometer Poincaré:

"What then are we to think of the question: Is Euclidean Geometry true? It has no meaning. We might as well ask if the metric system is true, and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true then another; it can only be more convenient. Now, Euclidean Geometry is, and will remain, the most convenient; first, because it is the simplest, and it is so not only because of our mental habits or because of the kind of intuition that we have of Euclidean space; it is the simplest in itself, just as a polynomial of the first degree is simpler than a polnomial of the second degree; secondly, because it sufficiently agrees with the properties of natural solids, those bodies which we compare and measure by means of our senses."[9]

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[9] *Ibid.*, p. 174.

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ELLIPTIC GEOMETRY

by

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B. S., Kansas State University, 1965

AN ABSTRACT OF A MASTER'S REPORT

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requirements for the degree

MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
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The purpose of this report was to make a basic study of Elliptic Geometry. This study began with an introduction to the historical development as initiated by Euclid's statement of his five postulates and continued by the later work of Saccheri and Lambert. It remained for Riemann (1826-1866), however, to realize fully the area now known as Elliptic Geometry. Indeed, it was Riemann who discovered that the postulate dealing with extending a line made just as much sense if we considered the line as being unbounded, but not infinite. With this reservation in mind, the characteristic postulate of Elliptic Plane Geometry was introduced:

Two straight lines always intersect one another.

By using this postulate in our development of the properties of lines and surfaces, it was found that straight lines were re-entrant and that they had a constant finite length. It was found that the total area of the plane was dependent upon the assumption of one or two distinct poles for every line (which led to Single and Double Elliptic Geometry respectively), but in each case this still meant that the plane had a constant finite area.

In the section on triangles and quadrilaterals it was shown among other things that the sum of the angles of a triangle was always greater than two right angles. The amount by which this sum exceeded two right angles was called the excess of the triangle and was shown to be proportional to the area of the triangle. Similar results were noted for an n-gon.

Finally, the consistency of Elliptic Geometry and the manner in which it could be used to describe the "true" nature of space is discussed in the report.