

CUMULATIVE SUM TECHNIQUES

by

DAVID DONALD KRUEGER

B. S., Wisconsin State University, Oshkosh, 1965

---

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY  
Manhattan, Kansas

1967

Approved by:

  
Major Professor

LD  
2668  
RA  
1967  
K75  
C.2

TABLE OF CONTENTS

1. INTRODUCTION . . . . .	1
2. CUMULATIVE SUM SCHEMES FOR PROCESS INSPECTION . . . . .	1
2.1 Purpose of Process Inspection . . . . .	1
2.2 Calculation and Advantage of Cumulative Sums . . . . .	2
2.3 One-sided Decision Interval Schemes . . . . .	5
2.4 Two-sided Decision Interval Schemes . . . . .	19
2.5 Fraction Defective Sampling Schemes . . . . .	20
2.6 Two-sided V-mask Schemes . . . . .	22
2.7 Equivalence of V-mask and Two-sided Decision Interval Schemes . . . . .	26
2.8 Gauging in Cumulative Sum Schemes . . . . .	28
2.9 Comparison of Cumulative Sum and Shewhart Schemes . . . . .	29
3. ESTIMATION OF THE CURRENT PROCESS MEAN . . . . .	31
4. CONCLUSION . . . . .	33
ACKNOWLEDGMENTS . . . . .	34
BIBLIOGRAPHY . . . . .	35

## 1. INTRODUCTION

Cumulative sums have been used for a number of years, but not until recently have the techniques been developed to take full advantage of this type of representation of a series of results. For example, in the simplest form cumulative sums have been used for a long time in compiling the total sales to date for a business. Another application involves the use of cumulative sums of inflow minus outflow of water from a reservoir to study the distribution of depths.

Most of the development of cumulative sum techniques has been related to industrial quality control problems. E. S. Page (1954) was the first to publish an account of this type of application when he used cumulative scores to control the mean of a process. G. A. Barnard (1959) introduced the method of superimposing a V-mask on cumulative sum charts to control quantitative variables and several articles were written extending his idea. Subsequent papers have evaluated the characteristics of quality control schemes based on these methods.

## 2. CUMULATIVE SUM TECHNIQUES APPLIED TO PROCESS INSPECTION

### 2.1 Purpose of Process Inspection

The principal aim of process inspection is to furnish information, either to assure that a process is producing its output in a specified matter or to inform that some departure from specifications has occurred so that corrective action can be taken or an investigation made. In many cases information from the inspection will indicate the type of action that

is to be taken. A secondary aim of process inspection is to show improvements in a process, since interest not only lies in how to avoid deteriorations but also in how to maintain improvements that are made. Two further aims are to provide a history of a process for later investigation and to estimate the current process mean. For the fulfillment of these aims, the method used in presenting inspection results is important.

## 2.2 Calculation and Advantages of Cumulative Sums

The longest established statistical method for process inspection is the Shewhart chart. For this system samples are taken from a process at regular intervals, and the mean values for the samples are plotted successively. If any point falls outside the action limits (usually  $3\sigma$ ), a change is assumed to have occurred and action is taken. The action is either a corrective measure or an investigation. A certain amount of work has been done to improve the sensitivity of the control chart by the use of runs. For example if the last  $k$  points fell outside of the  $2\sigma$  control limits, action is taken. This idea combines the evidence of the current sample with that of previous samples. It reaches its highest development in control charts based on cumulative sums.

The cumulative sum chart is therefore very similar to the Shewhart chart. The differences are in the type of visual records made and the criteria for taking action. Instead of plotting the sample results  $x_1, x_2, \dots$ , a reference value,  $k$ , related to the target value,  $\mu$ , is chosen and the sums  $S_r = \sum_{i=1}^r (x_i - k)$  are calculated and plotted as a time series. When the cumulative sum path deviates a specified amount,  $h$ , from the target value, action is taken. As will be shown later a similar criterion is

obtained by considering the slope of the cumulative sum path.

The most fundamental advantage of cumulative sum charts is that a change in quality can be seen more easily by visual inspection than it can on the Shewhart type chart where the results are plotted independently of each other. This is illustrated in Fig. 1. The first half of the results were obtained by random sampling from a normal population with zero mean and unit variance, and the second half of the results by random sampling from a normal population with the mean increased to 0.2.

A second advantage is the ease with which the point of change in quality can be seen. This is also illustrated in Fig. 1. In practice, the location of the change is useful in helping to discover the cause of the change.

A third advantage is the improvement in efficiency over the Shewhart chart for changes in the process mean between  $0.5$  and  $2.0\sigma$ . In this region changes can be detected approximately twice as rapidly with cumulative sum charts as with Shewhart charts, or in the same amount of time but with smaller sample sizes than Shewhart charts. This improvement will be illustrated later.

A further advantage of cumulative sum charts involves their use to confirm, with subsequent results, previous decisions concerning process changes. If the later results reinforce the earlier results in their indication of the slope of the chart, then further confidence is gained that a change has occurred. If they revert toward the target value, the suggestion may be that the process has corrected itself. Finally the use of cumulative sum charts on past history provides a useful guide as to the nature of the process variation.

Fig. 1b. Shewhart control chart

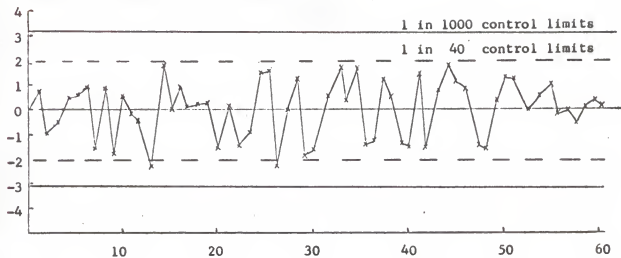


Fig. 1a. Cumulative sum chart

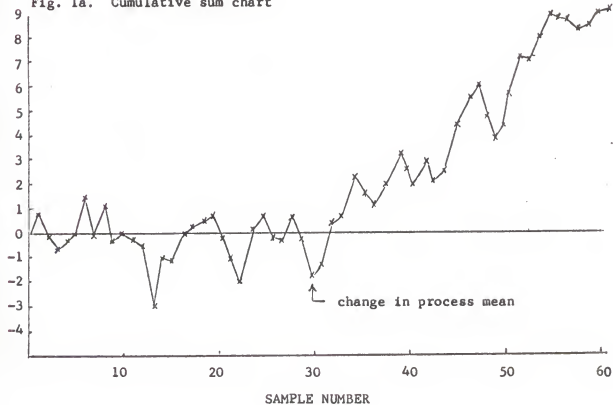


Fig. 1. Comparison between cumulative sum and Shewhart control charts. Mean of first 30 results = 0.00. Mean of second thirty results = 0.2.

### 2.3 One-sided Decision Interval Scheme

The one-sided decision interval scheme that is used with cumulative sum techniques is in essence a sequence of Wald sequential tests with boundaries at  $\underline{0}$  and  $\underline{h}$ .<sup>\*</sup> The sum  $S_r = \sum_{i=1}^r (x_i - k)$  is plotted on the chart as long as it is positive, and action is taken when the alarm value,  $\underline{h}$ , is reached. When  $S_r$  falls below zero or after action is taken the cumulative sum is started over again. The reference value  $\underline{k}$  is usually chosen so that the cumulative sum path is downward when the process is running in a satisfactory manner and upward when there is deterioration in quality. It is possible to show with the use of nomograms that the best value to use for  $\underline{k}$  is approximately half way between the acceptance quality level and the rejection quality level.

The one-sided decision chart just described, is very similar to the chart in Fig. 1a. A decision that quality has changed will be reached at the same time with both charts. An advantage of this type of plotting is that the chart is bounded in the sense that the cumulative sum does not run off the paper. Also this method is easily applicable to a table of successive results if it is desired not to plot the data. A disadvantage of this chart is that it does not present a past history of all the results, which would be needed if post-mortem investigation and parameter estimation were desired.

The alarm level,  $\underline{h}$ , can be derived mathematically if the statistical model of the production process is known, both in its normal target state

---

<sup>\*</sup> A Wald sequential test with boundaries (a,b) and initial score  $x_0$  is a procedure in which observations  $x_1, x_2, \dots$ , are taken as long as  $a \leq Z_1 \leq b$ , where  $Z_1 = Z_{1-1} + x_1$ , and  $Z_0 = z_0$ .

and all of its possible departure states. Since these conditions of full knowledge are rarely met, other means of obtaining  $h$  are needed. Average run length is a criteria frequently used.

The average run length was first defined by Page (1954) to be the average number of elements sampled before action was taken. Since then most writers have defined the average run length as the average number of samples of size  $n$  taken before some corrective action or investigation is made. The choice of the decision scheme is based on two average run lengths,  $L_a$  and  $L_r$ , the run lengths when the process is producing at acceptance and rejection quality levels respectively. Usually  $L_a$  is chosen to be of the order of 250 to 1000 samples, and  $L_r$ , 6 to 10 samples.

In some instances it may be better to relate the decision scheme directly to the amount of material produced before taking any action. When the rate of production with respect to time is constant, the amount of material produced is directly related to the time that elapses between the moment the process starts to run at a specific quality level and the moment action is taken. The average value of this time is called the average duration (Av.D.). It was shown by Ewan and Kemp (1960) that schemes could be designed to give average durations at acceptance and rejection quality levels by using a simple relationship that exists between average run length and average duration. Let  $\underline{s}$  be the time that elapses between taking samples, and let  $\underline{t}$  be the average time between the occurrence of a change and the selection of the first sample after the change. Then the average duration for quality other than acceptance quality is given by

$$Av D = (ARL - 1)s + t$$
, where ARL is the average run length at this quality.



If the probability that the process level changes in any particular sampling interval is constant with respect to time and is small relative to the sampling interval, then  $t \doteq s/2$ , and

$$\text{Av } D \doteq (\text{ARL} - \frac{1}{2})s .$$

If the first sample is taken at time  $\underline{s}$  after the process is set into operation, then at the acceptance quality level,

$$\text{Av } D = L_a \cdot s .$$

In the derivation of the average run length, Page (1954) assumed the results,  $x_i$ , were continuous variates. The rule he used was:

Take observations at regular intervals, assign a score,  $x_i$ , to the  $i$ th observation, and plot the cumulative score  $S_r = \sum_{i=1}^r x_i$  on a chart. Take action after the  $n$ th observation if  $S_n^i \geq h$ , where

$$S_n^i = \max [S_{n-1}^i + x_n, 0] , \quad n \geq 1$$

$$S_0^i = 0$$

so that  $S_n^i = 0$  whenever  $S_n < \min_{0 \leq i < n} S_i$ .

This system of scoring was chosen so that the mean sample path was downward when quality was satisfactory and upward when quality was unsatisfactory. As noted before, this scheme breaks down into a series of Wald sequential tests.

The following notations will be used in the derivation of the average run length:

$P(z)$  = probability that the Wald sequential test with initial score  $z$  and boundaries  $(0, h)$  will end on the lower boundary.

$N(z)$  = unconditional average sample number.

$N_1(z)$  = average sampling number conditional upon the test ending on the lower boundary.

$N_2(z)$  = average sampling number conditional upon the test ending on the upper boundary.

ARL = expected number of samples taken before action is taken.

Since the test is started at  $S_n^1 = 0$ ,

$$P(r \text{ acceptance tests before a rejection test}) = [P(0)]^r [1 - P(0)].$$

The expected number of acceptance tests before a rejection test is therefore

$$\begin{aligned} \sum_{r=1}^{\infty} r [P(0)]^r [1 - P(0)] &= [1 - P(0)] [P(0) + 2P^2(0) + 3P^3(0) + \dots] \\ &= P(0) + P^2(0) + P^3(0) + \dots \\ &= P(0) / [1 - P(0)]. \end{aligned}$$

With this result, the average run length can be evaluated.

$$\begin{aligned} \text{ARL} &= E[\text{number of acceptance tests before a rejection test}] \\ &\quad \cdot N_1(0) + N_2(0) \\ &= N_1(0) P(0) / [1 - P(0)] + N_2(0) \\ &= \{N_1(0) P(0) + N_2(0) [1 - P(0)]\} / [1 - P(0)]. \end{aligned}$$

However,

$$\begin{aligned}
 N(0) &= N_1(0) \cdot P(\text{test ends on lower boundary}) + N_2(0) \cdot P(\text{test ends on upper boundary}) \\
 &= N_1(0) P(0) + N_2(0) [1 - P(0)] .
 \end{aligned}$$

Therefore

$$ARL = N(0) / [1 - P(0)] . \quad (1)$$

Page stated that  $N(0)$  and  $P(0)$  could be obtained by the use of integral equations of Fredholm type. He gave the integral equation for the average sample number  $N(t)$  of a Wald sequential test with boundaries at  $(a, b)$  and initial score  $z$  as

$$N(z) = 1 + \int_a^b N(x) f(x - z) dx , \quad (2)$$

where  $f(x - z)$  represents the density function of  $x - z$ .

The equivalent integral equation for  $P(z)$  is determined in the following manner.

$$\begin{aligned}
 P(z) &= \int_{-\infty}^{\infty} P(z|X=x) f(x) dx , \\
 &= \int_{-\infty}^{-z} P(z|X=x) f(x) dx + \int_{-z}^{h-z} P(z|X=x) f(x) dx \\
 &\quad + \int_{h-z}^{\infty} P(z|X=x) f(x) dx .
 \end{aligned}$$

For the first integral,  $P(z|X=x) = 1$ , since for  $-\infty = x = -z$ ,  $z + x = 0$ .

By a change of variable,  $x' = x + z$ , the second integral becomes

$$\int_0^h P(x') f(x' - z) dx' .$$

In the third integral,  $P(z | X=x) = 0$ , since the cumulative sum would be greater than  $h$ . Therefore,

$$P(z) = \int_{-\infty}^{-z} f(x) dx + \int_0^h P(x) f(x-z) dx . \quad (3)$$

From these results, the average run length can be calculated as the ratio of two integral equations.

With the following modifications of his rule, Page was able to arrive at a single integral equation for the average run length.

1. Take observations and assign scores as before.
2. Take action if either

$$\begin{aligned} & \text{(a) } S_n \geq h \text{ and } S_i > 0, \text{ for all } i = 1, 2, \dots, n-1 \\ \text{or} & \text{ (b) } S_n - \min_{0 \leq i < n} S_i \geq h, \end{aligned}$$

where  $S_0 = z$ ,  $0 \geq z > h$ , and  $S_n = S_{n-1} + x_n$ . This modification changes the original rule only near its start. The equation is

$$L(z) = 1 + L(0) F(-z) + \int_0^h L(x) dF(x-z), \quad (4)$$

where  $F(x)$  is the distribution function of a single score,  $x$ .

As was previously stated, the average run length, devised by Page, assumed a continuous random variable,  $x$ . Ewan and Kemp (1960) extended his idea to include equations which can be formulated when  $x$  is a discrete variate, taking on only integer values. In these equations  $\underline{z}$ ,  $\underline{h}$ , and  $\underline{k}$  are restricted to be integral valued.

For a discrete variate  $x$ , let  $f(x)$  represent the probability of obtaining the value  $x$ , and let  $F(\infty) = \sum_0^{\infty} f(x)$ . Then,

$$P(z) = F(k - z) + \sum_{y=1}^{n-1} P(y) f(y + k - z), \quad (5)$$

$$N(z) = 1 + \sum_{y=1}^{n-1} N(y) f(y + k - z), \quad (6)$$

$$\text{and } L(z) = 1 + L(0) F(k - z) + \sum_{y=1}^{n-1} L(y) f(y + k - z). \quad (7)$$

Kemp (1958) devised a method for obtaining approximate solutions to equations (2) and (3) for  $N(z)$  and  $P(z)$  when  $\underline{x}$  was a continuous normal random variable. Ewan and Kemp (1960) then generalized this method for use with both continuous and discrete functions.

(i) For a continuous variate  $x$ ; if  $f(x)$  is the probability of obtaining the value  $\underline{x}$ , let

$$F(\alpha) = \int_{-\infty}^{\alpha} f(x) dx,$$

$$M(\alpha) = \int_{-\infty}^{\alpha} x f(x) dx,$$

$$G(\alpha) = \int_{-\infty}^{\alpha} e^{ux} f(x) dx,$$

where  $\underline{u}$  is a real non-zero root of the equation,

$$\int_{-\infty}^{\infty} e^{ux} f(x) dx = e^{uk}.$$

(ii) For a discrete variate, let

$$F(\alpha) = \sum_0^{\alpha} f(x),$$

$$M(\alpha) = \sum_0^{\alpha} x f(x),$$

$$G(\alpha) = \sum_0^{\alpha} e^{wx} f(x),$$

where  $w$  is a real non-zero root of the equation,

$$\sum_0^{\alpha} e^{ux} f(x) = e^{uk}.$$

The results Ewan and Kemp found hold for  $x$  in both the continuous and discrete cases, with integer values of  $h$  and  $k$ . The two approximations are:

$$P(z) \doteq \{ [P(h) - P(0) e^{wh}] + [P(0) - P(h)] e^{wz} \} / (1 - e^{wh}), \quad (8)$$

$$N(z) \doteq \{ N(h) - N(0) e^{wh} + h / (m - k) + \\ [N(0) - N(h) - h / (m - k)] e^{wz} - z \} / (m - k)(1 - e^{wh}), \quad (9)$$

where,

$$P(0) \doteq \frac{\begin{vmatrix} F(k) & K_2 \\ F(k-h) & K_4 \end{vmatrix}}{\begin{vmatrix} K_1 & K_2 \\ K_3 & K_4 \end{vmatrix}}, \quad N(0) \doteq \frac{\begin{vmatrix} B_1 & K_2 \\ B_2 & K_4 \end{vmatrix}}{\begin{vmatrix} K_1 & K_2 \\ K_3 & K_4 \end{vmatrix}}, \quad (10)$$

and,

$$K_1 = 1 + \{ [F(h+k)^* - F(k)] e^{wh} - e^{-wk} [G(h+k)^* - G(k)] \} / (1 - e^{wh}),$$

$$K_2 = \{ e^{-wk} [G(h+k)^* - G(k)] - [F(h+k)^* - F(k)] \} / (1 - e^{wh}),$$

$$K_3 = \{ F(k)^* - F(k-h) - e^{-wk} [G(k)^* - G(k-h)] \} e^{wh} / (1 - e^{wh}),$$

$$K_4 = 1 - \{F(k)^* - F(k-h) - e^{-w(k-h)} [G(k)^* - G(k-h)]\} / (1 - e^{wh}) ,$$

$$B_1 = 1 - \{K_2 h + M(h+k)^* - M(k) - k[F(h+k)^* - F(k)]\} / (m - k) ,$$

$$B_2 = 1 - \{(1 - K_4)h + M(k)^* - M(k-h) - (k-h)[F(k)^* - F(k-h)]\} / (m-k) .$$

In these equations,  $F(t)^* = F(t)$  for a continuous variate and  $F(t)^* = F(t-1)$  for a discrete variate. This is also true for  $G(t)^*$  and  $M(t)^*$ .

The procedure to obtain  $P(0)$  and  $N(0)$  is as follows:

- (a) Estimate  $P(0)$  and  $N(0)$  by equations (10).
- (b) Obtain approximate values of  $P(z)$  and  $N(z)$  for  $z = 1, 2, \dots, (h-1)$ , by equations (8) and (9).

(c) If  $x$  is a discrete variate, substitute the approximate values for  $P(0), P(1), \dots, P(h-1)$  into equation (5) and recalculate  $P(h)$ . Substitute this value and the values of  $P(0), P(1), \dots, P(h-2)$  into equation (5) and recalculate  $P(h-1)$ . Do this for all  $z = 0, 1, \dots, h-1$ . Repeat the procedure until no change in  $P(0)$  is obtained.  $N(0)$  may be found by using equation (6) in the same manner.

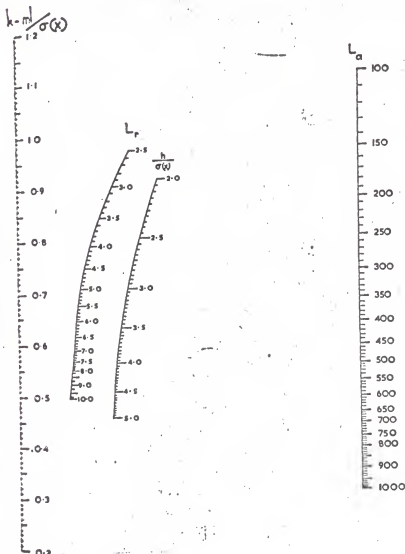
If  $x$  is a continuous variate, the linear equations are formulated from the integral equations satisfied by  $P(z)$  and  $N(z)$  using methods of quadrature. Once  $P(0)$  and  $N(0)$  are known, the average run length can be obtained from equation (1), i.e.,

$$L(0) = N(0) / (1 - P(0)) .$$

Ewan and Kemp (1960) obtained the average run lengths of a number of schemes for a normal variate with unit variance and mean,  $\underline{m}$ , by the above approximation. From these initial results they constructed a nomogram, which is shown in Fig. 2. The average run lengths of a wide variety of

schemes can be obtained at both acceptance and rejection quality levels from the nomogram. In Fig. 2,  $\sigma(x) = \sigma / \sqrt{n}$  is the standard deviation of the samples, and  $n$  is the size of the samples. If  $m_a$  is the process mean at acceptance quality level, the average run length at this point is obtained by placing a ruler on the nomogram so that it joins the known

Fig. 2. Nomogram from which average run length values can be determined when  $x$  is normally distributed.





points  $h / \sigma(x)$  and  $|k - m_a| / \sigma(x)$ , and reading off the run length from the scale on the right hand side of the diagram. In a similar manner the average run length at  $m_r$ , the process mean at rejection quality level, is obtained from the line joining  $|k - m_r| / \sigma(x)$  and  $h / \sigma(x)$ .

The usual procedure for devising a scheme is to specify average run lengths at rejection and acceptance quality levels and find  $\underline{n}$ ,  $\underline{k}$ , and  $\underline{h}$ . Since there are only two equations,  $|k - m| / \sigma(x)$  and  $h / \sigma(x)$ , and three definable variables, several different combinations of the value  $\underline{n}$ ,  $\underline{k}$ , and  $\underline{h}$  would give the same average run length. It is possible to show with the use of the nomogram that there are advantages to using a central reference value for  $\underline{k}$  of approximately  $\frac{1}{2}(m_a + m_r)$ . For example, fixed sample size schemes with the same average run length at the acceptance quality level have a minimum average run length at the rejection quality level when  $k \doteq \frac{1}{2}(m_a + m_r)$ . Also if a scheme is to have certain values for the average run length at both acceptance and rejection quality levels, the level of sampling is a minimum when  $k \doteq \frac{1}{2}(m_a + m_r)$ .

For a fixed average run length, if the sample size is plotted against the reference value, there will be a flat minimum in the region of  $k \doteq \frac{1}{2}(m_a + m_r)$ . Therefore as long as the reference value is in the region of the central value, there will be a negligible difference in sample size. Thus in actual use, the reference value may be rounded off to simplify the arithmetic necessary to operate a scheme.

In order to design a scheme with specified values of  $L_a$  and  $L_r$ , use  $k \doteq \frac{1}{2}(m_a + m_r)$  as the reference value. The values of  $h$  and  $n$  appropriate to this reference value can be determined by placing a ruler across the nomogram so that it joins the points  $L_a$  and  $L_r$ . The values  $|m_a - m_r| 2 \sqrt{n} / \sigma$

and  $h\sqrt{n}/\sigma$  are then read off. From these values,  $\underline{h}$  and  $\underline{n}$  may easily be calculated. If the values so obtained are convenient for use in practice, no additional work is required and the details of the sampling scheme are complete. If they are not convenient, the nomogram can be used to devise an alternative scheme with approximately the same values of  $\underline{h}$ ,  $\underline{k}$ , and  $\underline{n}$ .

For example, suppose that it is desired to design a scheme for a process which produces with acceptable quality as long as  $m_a = 4.00$ , and produces unacceptable material when the process mean becomes equal to or exceeds 4.50. The standard deviation of the process is equal to 1.00, and it is required to have  $L_a = 500$  and  $L_r = 5.00$ . From Fig. 2,  $|k-m|\sqrt{n}/\sigma$  is equal to 0.74. If the central value for  $k$  is used, this formula gives a value for  $n = 8.76$ . When rounded to the nearest integer the result is  $n = 9$ . Recalculating,  $|k-m|\sqrt{n}/\sigma = 0.75$ . This result, along with  $L_r = 5$ , produces the values  $L_a = 560$  and  $h\sqrt{n}/\sigma = 3.2$ . Therefore,  $h = 3.2\sigma/\sqrt{n} = 1.06$ . If it is desired to keep  $L_a = 500$ , then  $|k-m|\sqrt{n}/\sigma = 0.75$ , and  $L_a = 500$  may be used to obtain  $L_r = 4.9$  and  $h\sqrt{n}/\sigma = 3.125$ . With this scheme,  $h = 3.125\sigma/\sqrt{n} = 1.04$ .

It was shown by Ewan and Kemp (1960) that if  $\underline{x}$  is a Poisson variate with mean  $\underline{m}$ , the average run lengths of a scheme can be obtained by using equations (2) to (6) in which

$$F(\alpha) = \sum_0^{\alpha} m^x e^{-m} / x!,$$

$$M(\alpha) = m F(\alpha - 1),$$

$$e^{-wk} G(\alpha) = \sum_0^{\alpha} (me^w)^x \exp(-me^w) / x!,$$

where  $\underline{w}$  is the real non-zero root of the equation

$$m(e^u - 1) = uk .$$

The notation is modified so that  $P(z,k,h)$ ,  $N(z,k,h)$ , and  $L(z,k,h)$  are used to denote  $P(z)$ ,  $N(z)$ , and  $L(z)$  respectively. Also the following approximate relationships are used to determine the average run lengths of a number of schemes with different values of  $\underline{h}$  and  $\underline{k}$ :

$$P(z,k,h) = P(z-1, k-1, h+1) \quad (7)$$

$$P(z,k,h) = P(z+1, k, h+1) . \quad (8)$$

If  $P(z,k,h)$  is known for  $\underline{z}$  between  $\underline{0}$  and  $\underline{h}$ , accurate values of  $P(z+1,k,h+1)$  can be obtained, with the help of equation (8), by direct substitution in the set of equations generated by equation (2).

Tables 1 and 2 presented by Ewan and Kemp (1960) give the values for  $\underline{h}$  and  $\underline{k}$  and  $m_a$  of a number of schemes for average run lengths at acceptance quality level of 500, and for average run lengths at the rejection quality levels of 3 and 7. It was found from these tables and other calculations that the schemes for which the sample size is smallest are those with reference values which are in the neighborhood of the central value. The reference values shown in these tables are around the central value. It is not possible to interpolate in these tables for non-integral values of  $\underline{h}$  and  $\underline{k}$ .

In order to design a scheme with specified values of  $L_a$  and  $L_r$ , find  $R = RQL / AQL$ , where RQL and AQL are respectively the rejection and acceptance quality levels. If there is more than one scheme with this value of  $\underline{R}$  in the table, choose the one for which  $m_a$  is a minimum. If there are no schemes with this value of  $\underline{R}$ , choose the scheme with the nearest value of  $\underline{R}$ . The values for  $\underline{h}$  and  $\underline{k}$  are then obtained from the table. In order to determine the amount of material to examine, divide  $m_a$  by the acceptance

$R$	$h$	$k$	$m_a$	$m_r$	$R$	$h$	$k$	$m_a$	$m_r$
4-64	2	1	0.22	1-02	1-65	9	5	3-84	6-33
3-41	3	1	0.39	1-33	1-64	10	5	3-98	6-52
3-04	4	1	0.52	1-58	1-60	8	6	4.40	7.02
2-90	5	1	0.62	1-80	1-59	9	6	4-60	7-30
2-80	6	1	0.70	1-96	1-58	10	6	4.75	7.50
2-79	7	1	0.77	2-15	1-57	11	6	4.89	7.70
2-43	4	2	1-01	2-45	1-55	9	7	5-33	8-26
2-29	5	2	1-18	2-70	1-54	10	7	5-51	8-46
2-21	6	2	1-31	2-89	1-53	11	7	5-67	8-65
2-15	7	2	1-43	3-07	1-52	9	8	6-07	9-24
2-12	8	2	1-53	3-24	1-50	10	8	6-28	9-45
2-10	9	2	1-62	3-41	1-50	11	8	6-45	9-65
2-07	5	3	1-75	3-62	1-49	12	8	6-58	9-83
1-97	6	3	1-95	3-84	1-48	10	9	7-05	10-44
1-89	7	3	2-13	4-03	1-47	11	9	7-25	10-65
1-87	8	3	2-26	4-22	1-47	12	9	7-38	10-83
1-85	9	3	2-17	4-40	1-46	10	10	7-81	11-40
1-79	7	4	2-79	5-00	1-44	11	10	8-05	11-61
1-76	8	4	2-96	5-20	1-44	12	10	8-20	11-81
1-74	9	4	3-10	5-39	1-42	11	11	8-85	12-61
1-72	10	4	3-21	5-52	1-42	12	11	9-00	12-81
1-67	8	5	3-68	6-16	...	...	...	...	...

Table 1. Values of  $m_a$ ,  $m_r$ ,  $R = (m_r/m_a)$ ,  $h$  and  $k$  for schemes with A.R.L. = 500 at A.Q.L. and A.R.L. = 7 at R.Q.L. for a Poisson variate.

$R$	$h$	$k$	$m_a$	$m_r$	$R$	$h$	$k$	$m_a$	$m_r$
7-77	2	1	0.22	1-71	2-26	5	6	3-57	8-06
5-64	3	1	0.39	2-20	2-18	6	6	3-90	8-50
5-19	4	1	0.52	2-70	2-14	7	6	4-17	9-31
4-98	5	1	0.62	3-09	2-12	8	6	4.40	9-91
4-05	3	2	0-79	3-20	2-07	6	7	4-60	9-50
3-63	4	2	1-01	3-67	2-04	7	7	4-87	9-92
3-43	5	2	1-18	4-05	2-02	8	7	5-11	10-33
3-34	3	3	1-23	4-11	2-00	6	8	5-24	10-50
2-97	4	3	1-52	4-51	1-96	7	8	5-57	10-91
2-87	5	3	1-75	5-03	1-95	8	8	5-83	11-38
2-83	6	3	1-95	5-51	1-94	9	8	6-07	11-80
2-77	7	3	2-13	5-91	1-91	7	9	6-25	11-92
2-75	4	4	2-04	5-60	1-89	8	9	6-55	12-39
2-58	5	4	2-13	6-02	1-87	9	9	6.83	12-79
2-52	6	4	2-58	6-50	1-86	7	10	6-94	12-93
2-48	7	4	2-79	6-91	1-84	8	10	7-28	13-19
2-47	8	4	2-96	7-31	1-82	9	10	7-57	13-80
2-43	5	5	2-92	7-09	1-80	8	11	8-00	14-40
2-35	6	5	3-21	7-53	1-78	9	11	8-30	14-80
2-30	7	5	3-46	7-96	1-77	10	11	8-59	15-23
2-27	8	5	3-68	8-34	...	...	...	...	...

Table 2. Values of  $m_a$ ,  $m_r$ ,  $R = (m_r/m_a)$ ,  $h$  and  $k$  for schemes with A.R.L. = 500 at A.Q.L. and A.R.L. = 3 at R.Q.L. for a Poisson variate.

quality level.

A numerical example of a Poisson variate follows: suppose it is necessary to control the level of faults in lengths of fabric. The quality is acceptable if the faults do not exceed a mean of 5 per  $10^6$  yards and is unacceptable if the mean number of faults exceeds 10 per  $10^6$  yards. It is required to have an average run length of 500 at acceptance quality and 7 at rejection quality. The decision interval, the reference value, and the number of yards of fabric to examine at one time can be determined by using Table 1. The nearest value of  $\bar{R}$  is 1.97 with a decision interval,  $h = 6$ , a reference value,  $k = 3$ , and Poisson parameter,  $m_a = 1.95$ . Therefore the number of yards of fabric which should be examined is  $(1.95 / 5) \times 10^6$  yards =  $0.39 \times 10^6$  yards.

#### 2.4 Two-sided Decision Interval Schemes

Given the two-sided schemes, (1)  $S_n = \sum_{i=1}^n (x_i - k_1)$ , and (2)  $S_n^1 = \sum_{i=1}^n (x_i - k_2)$ , it was shown by Kemp (1961) that if the two schemes are run simultaneously, then  $1/L_0 = 1/L_1 + 1/L_2$ , where  $L_0$ ,  $L_1$ , and  $L_2$  are the average run lengths of the two-sided scheme, scheme (1), and scheme (2) respectively.

This result is found by the following method. The expected number of occasions of the crossing of the upper boundary of scheme (1) when  $N$  samples are taken is  $N / L_1$ . Likewise for scheme (2) the expected number of occasions is  $N / L_2$ . It was shown by Ewan and Kemp (1960) that when one of the two cumulations crosses its action limit, the other can not be between its boundaries. Therefore the cumulations for the two schemes will not interfere with one another. When the two schemes are considered together, the

expected number of occasions of crossing is  $N/L_0 = N/L_1 + N/L_2$ . This leads to the result,  $1/L_0 = 1/L_1 + 1/L_2$ .

It is easily seen that when  $\bar{x}$  is a  $N(0,1)$  variate and  $k_1 = -k_2$ , then  $L_1 = L_2$  when the process is running at its target value. Therefore  $L_0 = \frac{1}{2}L_1$ . If the values of the process mean which are unacceptable are  $m_a + \Delta$  and  $m_a - \Delta$ , then when the process mean equals  $m_a + \Delta$ ,  $L_2(m_a + \Delta)$  will usually be so large that  $L_0(m_a + \Delta) \doteq L_1(m_a + \Delta)$ . Similarly  $L_0(m_a - \Delta) \doteq L_2(m_a - \Delta)$ .

The following example is identical with the example given for the one-sided scheme, except that now deviations in both directions are considered. It is desired to design a scheme to control the mean at 4.00 and to detect changes in the mean of  $\pm 0.50$ . The standard deviation of the process is 1, and it is required to have  $L_0(m_a) = 500$  and  $L_0(m_a \pm 0.50) = 5.00$ . Choose a central reference value and use the results  $L_0(m_a) = \frac{1}{2}L_1(m_a)$ , and  $L_0(m_a + 0.50) \doteq L_1(m_a + 0.50)$ . The values of  $h$  and  $n$  are determined from the single sided scheme with  $L_a = 1000$  and  $L_r = 5.00$ . From Fig. 2,  $|k-m| / \sigma(x) = 0.8 = |m_a - m_r| \sqrt{n} / 2\sigma = (.5) \sqrt{n} / 2$ . Thus  $n = 10.34$ . Rounding this up, use  $n = 11$ . Recalculating  $|k-m| / \sigma(x) = 0.82$  and using this result along with  $L_0(m_a \pm 0.50) = 5$ ,  $L(m_a) > 1000/2$  and  $h \sqrt{n} / \sigma = 3.45$ . Thus  $h = 1.05$ . If it is desired to keep  $L(m_a) = 500$ , use  $|k-m| / \sigma(x) = 0.82$  and  $L(m_a) = 500$  to obtain  $L(m \pm 0.50) = 4.8$  and  $h \sqrt{n} / \sigma = 3.3$ . In this case,  $h = 1.00$ .

## 2.5 Fraction-defective Sampling Schemes

The distribution of defective items in a sample of size  $n$  has the binomial distribution. Usually the proportion defective is sufficiently

small and the Poisson approximation may be applied. Kemp (1962) constructed Table 3 which gives some values of  $m_a$ ,  $R$ ,  $h$ , and  $k$  for a variety of fraction defective sampling schemes. The values  $p_a$  and  $p_r$  represent the proportion of defective items at the acceptance and rejection quality levels respectively. The value  $R = m_r/m_a = N_r/N_p = p_r/p_a$ . The values chosen for inclusion in this table are those which will give sampling schemes requiring the smallest sample sizes.

R	$L_a$	$L_r$								
		5.00			7.50			10.00		
		$m_a$	k	h	$m_a$	k	h	$m_a$	k	h
2.50	500	1.18	2.00	5.00	0.64	1.20	3.75	0.50	0.90	3.75
	250	0.93	1.50	4.50	0.52	0.90	3.50	0.42	0.80	3.00
	125	0.71	1.20	3.75	0.47	0.70	3.25	0.32	0.60	2.25
3.00	500	0.66	1.20	4.00	0.46	0.90	3.50	0.32	0.70	3.00
	250	0.56	0.90	3.00	0.40	0.80	3.00	0.27	0.60	2.50
	125	0.48	0.80	3.00	0.31	0.60	3.00	0.15	0.30	2.00
3.50	500	0.54	1.20	3.00	0.35	0.80	3.00	0.24	0.60	2.75
	250	0.41	0.90	2.50	0.27	0.60	2.50	0.18	0.40	2.50
	125	0.34	0.70	2.25	0.18	0.40	2.00	0.13	0.30	1.75
4.00	500	0.38	0.90	2.75	0.24	0.60	2.75	0.16	0.40	2.50
	250	0.32	0.80	2.25	0.21	0.60	2.00	0.12	0.30	2.00
	125	0.28	0.70	1.75	0.16	0.40	1.75	0.07	0.20	1.50

Table 3. Values of  $m_a$ ,  $R$ ,  $h$ , and  $k$  for fraction-defective sampling schemes.

In order to design a scheme when  $p_a$  and  $p_r$  have specified values, calculate  $R$  and find from Table 3 the values of  $m_a$ ,  $k$ , and  $h$  which correspond to the values of  $L_a$  and  $L_r$  that the scheme is to have. The quantities  $h$  and

$k$  are the actual decision interval and reference value respectively, and  $m_a$  represents the mean number of defective items per sample at acceptance quality level for the scheme. If  $\underline{n}$  is the number of items tested per sample,  $m_a = np_a$ , and hence the required value of  $\underline{n}$  can be found.

To illustrate, suppose a process is such that  $p_a = 0.01$  and  $p_r = 0.03$ . It is desired to have  $L_a = 500$  and  $L_r = 7.50$ . Then calculate  $R = .03 / .01 = 3$ , and use Table 3. It is found that  $m_a = 0.46$ ,  $k = 0.90$ , and  $h = 3.50$ . Since  $m_a = np_a$ ,  $n = 46$ . Hence the scheme consists of taking samples of 46 items, counting the number of defective items, subtracting 0.90 from this figure, and plotting the results cumulatively using a value of 3.50 for the decision interval.

Tables 1 and 2 may also be used for fraction defective schemes. As an example, suppose the proportion of defectives that can be tolerated at acceptance quality level is 0.005 and the proportion at rejection quality level is 0.015. Suppose also that the average run length is to be 500 at the acceptance quality level, and 7 at the rejection quality level. The value nearest to  $R = 0.015 / 0.005 = 3$ , in Table 3, is  $R = 3.04$ . For this  $R$  value,  $k = 1$ ,  $h = 4$ , and  $m_a = 0.52$ . Then  $n = m_a / p_a = 0.52 / 0.005 = 104$ . Therefore the scheme would consist of taking samples of size 104 at regular intervals, counting the number of defective items, subtracting  $k = 1$  from this figure, and plotting the results cumulatively, using a value of 4 for the decision interval.

## 2.6 Two-sided V-mask Scheme

This scheme was devised by Barnard (1959) and is much like the two-sided decision interval scheme. First the origin of measurement is shifted



to the target value by letting the reference value be  $\mu$ . The cumulative sums are then  $S_r = \sum_{i=1}^r (x_i - \mu)$ . As long as the process mean remains near the target value, the graph of the cumulative sums does not deviate too much from the horizontal. To check if the process is on target, a V-shaped mask is superimposed on the cumulative sum chart with the vertex of the V pointing forward and at a distance  $d$  ahead of the most recent point on the chart. This is illustrated in Fig. 3, where  $\theta$  is the angle between each of the limbs of the V-mask and the horizontal.

If all of the curve is visible, it is assumed that the process is in statistical control. If the cumulative sum path cuts one of the limbs of the V-mask, then the decision is that the process mean has changed. When the upper limb is crossed, a decrease in the process mean is indicated, while if the lower limb is crossed an increase in the process mean is indicated. The V-mask is moved along the chart as each new cumulative sum is plotted.

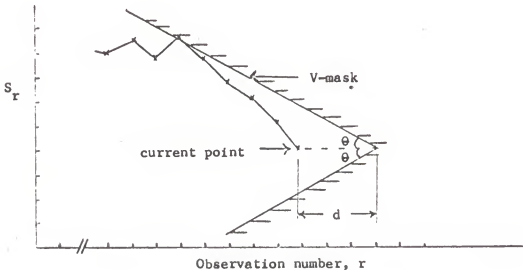


Fig. 3. V-mask superimposed on a cumulative sum chart.

The properties of the V-mask scheme are determined by the two parameters  $\underline{d}$  and  $\underline{\theta}$ . It is easily seen that the larger the lead distance and the angle of the V, the fewer will be decisions which state that the process mean has changed. The form of the V should be such that no decision is reached if the process is operating in a satisfactory manner, but a real change should be detected as quickly as possible.

One way of selecting the proper parameter values for a particular application is to try out a variety of masks on historical records. Since this method of selection is subjective, Barnard (1959) developed a method using the ARL, as was done with decision interval schemes.

Goldsmith and Whitfield (1961) have evaluated several average run lengths for V-mask quality control schemes where the results are mutually independent and come from a Normal distribution with variance  $\underline{\sigma}^2$ . For standardization purposes it was assumed that the plotting interval on the horizontal axis was equal to a  $2\sigma$  unit on the vertical axis. With this scale, when the process mean shifts  $2\sigma$  from the target, the mean path of the cumulative graph makes an angle of  $45^\circ$  with the horizontal. This arrangement permits a rapid visual picture of the behavior of the process. When the plotting interval on the horizontal axis is equal to  $\nu\sigma$  ( $\nu \neq 2$ ) on the vertical axis, the values of  $\tan \theta$  used should be multiplied by  $2/\nu$ .

The graphs of Goldsmith and Whitfield give average run lengths with lead distances,  $d$ , of 1, 2, 5, and 8 horizontal axis units, and  $\tan \theta$  values to give a range of  $L_0$  from 20 to 1500 samples. Fig. 4 illustrates one of the graphs. The average run length evaluation was carried out by Monte Carlo simulation on a Ferranti "Mercury" digital computer. A sequence of

Normal deviates was generated by a fast table look-up procedure. Each calculated ARL had a coefficient of variation of less than 10 per cent, which was concluded to be adequate enough for practical purposes. Kemp (1961) stated that these average run lengths agree with those obtained by Ewan and Kemp to within the degree claimed.

Two empirical formulas which can be used to obtain these results were given by Goldsmith and Whitfield. They are:

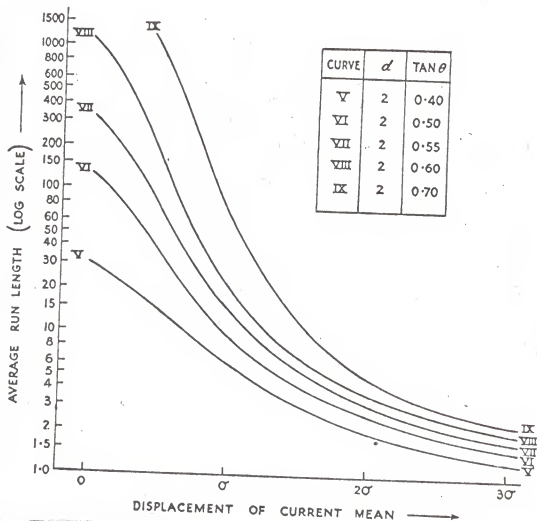


Fig. 4. Average run lengths of current mean for symmetric V-mask with  $d=2$ .

$$\log_{10} \log_{10} L_0 = -0.5244 + 0.0398d + 1.1687 \tan \theta + 1.2641 \tan \theta \times \log_{10} d .$$

$$L_1 = (2d \tan \theta) / (q - 2 \tan \theta) + 2/3 .$$

The first of these equations provided a good approximation for  $L_0$  at all values of  $d$  and  $\theta$  investigated, and the second provided a good approximation when  $1.5 < q < 4$ , where  $q$  is the amount the process is off target.

As an example, suppose it is required to have  $L_0 = 500$  and  $L_1 = 15$  with  $q = 1$ . Then by interpolation from Fig. 4, the mask with parameters  $d = 2$ , and  $\tan \theta = 0.565$  is satisfactory.

Barnard suggested that a parabolic mask may be more appropriate than a V-mask. He arrived at this suggestion by applying several different V-masks to the same set of results. Other than Barnard's work however, there seems to have been little done with this type of mask.

## 2.7 Equivalence of V-mask and Two-sided Decision Interval Scheme

The equivalence between the V-mask and two-sided decision interval scheme can be demonstrated by reference to Fig. 5. The cumulative sum at  $\underline{A}$ , which is the last plotted point, is  $S_n$ , and at  $\underline{C}$  is  $S_{n-r}$ . The V-mask has a lead distance =  $d$  horizontal plotting intervals and an angle  $\theta$  between its limbs and the horizontal.  $\underline{B}$  is the intersection of a vertical line from  $\underline{C}$  and a horizontal line from  $\underline{A}$ . Therefore,

$$BC = S_n - S_{n-r} .$$

The path of the cumulative sum will cross the lower limb of the V when

$$BC \geq BD , \text{ or}$$

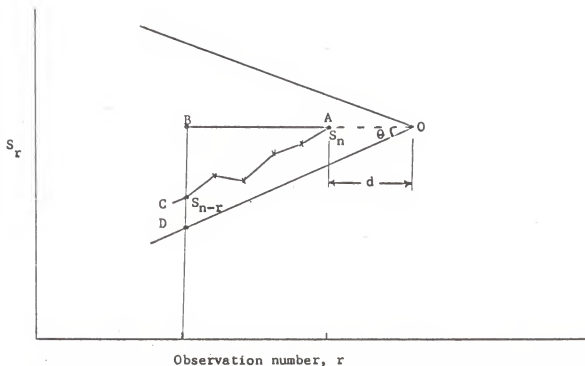


Fig. 5. Cumulative sum of  $(\bar{x} - \mu)$  plotted against number of samples.

$$S_n - S_{n-r} \geq BO \tan \theta .$$

But  $BO = w(r + d)$ , where  $w$  = the vertical scale distance per horizontal plotting interval. Therefore,

$$S_n - S_{n-r} \geq w(r + d) \tan \theta ,$$

$$S_n - S_{n-r} - (rw \tan \theta) \geq wd \tan \theta ,$$

$$\sum_{i=n-r+1}^n (x_i - \mu - w \tan \theta) \geq wd \tan \theta .$$

This is equivalent to accumulating the deviations of  $x_i$  from a reference value,  $k_1 = \mu + w \tan \theta$ , and using a decision interval  $h = wd \tan \theta$ .

A similar argument shows that the upper limb is crossed when

$$\sum_{i=n-r+1}^n (x_i - \mu + w \tan \theta) \leq -wd \tan \theta .$$

This is equivalent to the decision procedure with reference value  $k_2 = \mu - w \tan \theta$ , and  $h = -wd \tan \theta$ .

There are situations when one method of presentation is favored over the other. If it is expected that the process level will fluctuate in a random fashion about the target value  $\mu$ , and if the past history of these deviations is useful for technical investigation as well as control purposes, the combined use of the V-mask and cumulative sum chart is a very powerful technique. On the other hand, if the only information required is whether the process is running at an acceptable level, the two-sided decision interval may be better.

## 2.8 Gauging in Cumulative Sum Schemes

Cumulative sum schemes using gauging are advantageous if it is possible to build equipment to automatically do the testing. Page (1962) developed schemes for controlling the mean and standard deviation of a process using gauging. He was mainly concerned with sampling, gauging, and recording one observation at a time, since this is the most convenient arrangement to perform automatically.

With this method, observations from the process are assumed to be normal and independent with variance  $\sigma^2$ . The gauges are set at  $\mu \pm G$ . If  $A$ ,  $B$ , and  $C$  represent the number of articles that fall below  $\mu - G$ , that fall between the gauges, and that exceed  $\mu + G\sigma$  respectively, then  $C-A$  is sensitive to changes in the mean and  $C+A$  is sensitive to changes in the standard deviation.

In order to control the mean, a cumulative sum of the  $(C - A)$  for samples of size one is plotted and a V-mask is applied to the chart to test for a change. Page presented and compared several gauging schemes with Shewhart schemes and cumulative sum schemes. In general, the gauging schemes are less sensitive, than regular cumulative sum schemes, to moderate departures in the mean, but are more sensitive than Shewhart schemes. For large departures, both Shewhart and cumulative sum schemes act much faster than the gauging schemes.

In order to control the standard deviation, the cumulative sum  $\sum_i (C_i + A_i - k)$  is plotted. A change in the standard deviation is assumed to have occurred if the path rises a height  $h$  above the minimum. Several one-sided gauging schemes for single observations for controlling the variance of a population were presented. Some of these schemes were compared with Shewhart range schemes. The comparison suggested that it was possible to find a cumulative sum gauging scheme which could be carried out automatically and which would give a similar performance to a Shewhart range scheme.

## 2.9 Comparison of Cumulative Sum and Shewhart Schemes

Several different methods have been used to compare the average run lengths for Shewhart and cumulative sum schemes. Goldsmith and Whitfield (1961) selected the two schemes so that values of the average run length at the acceptance quality level were approximately equal. The results of this comparison are illustrated in Tables 4a and 4b.

Table 4a.  $d = 5$ ,  $\tan \theta = 0.35$ , equivalent control lines at  $\pm 2.96$ 

Average run length with	Deviation of current mean						
	0	$\frac{1}{2}\sigma$	$\sigma$	$2\sigma$	$3\sigma$	$4\sigma$	$5\sigma$
Cumulative scheme	319	42.3	9.23	3.50	2.09	1.59	1.21
Shewhart scheme	320	137	39.5	5.89	1.92	1.17	1.02

Table 4b.  $d = 2$ ,  $\tan \theta = 0.40$ , equivalent control lines at  $\pm 2.14$ 

Average run length with	Deviation of current mean						
	0	$\frac{1}{2}\sigma$	$\sigma$	$2\sigma$	$3\sigma$	$4\sigma$	$5\sigma$
Cumulative scheme	30.7	13.9	5.84	2.04	1.30	1.05	1.005
Shewhart scheme	30.5	18.1	7.75	2.24	1.24	1.03	1.002

Table 4. Comparisons between cumulative sum and Shewhart schemes with approximately equal average run length at acceptance quality level.

The value of the cumulative sum scheme is most evident for moderate deviations from the current mean, but exceptionally large changes are picked up a little more rapidly with the Shewhart chart. If it is allowable to have slack in the process, then it may be better to use Shewhart charts, as cumulative sum charts will cause interruptions when the quality of production is acceptable much more frequently than Shewhart charts.

Table 5 compares the required sample sizes for a Shewhart scheme and a cumulative sum scheme to have equivalent run lengths. It may be seen that for the higher values of  $L_p$ , more than 100% additional testing is required for a Shewhart chart. There are circumstances when this additional sampling is regarded as trivial, and in these cases the Shewhart chart could be used.



$L_r$	$L_a$		
	250	500	1000
2.50	1.3	1.3	1.3
5.00	1.8	1.9	2.0
7.50	2.2	2.4	2.5
10.00	2.5	2.8	3.0

Table 5. Ratio of the sample size required for a Shewhart scheme to that required for a cumulative sum scheme with equivalent run lengths at acceptance and rejection quality level.

Several other comparisons of cumulative sum and Shewhart schemes were found in the literature. Kemp (1961) gave the values of  $L_r$  for cumulative sum charts and for single-sided Shewhart charts with the same sample size and average run length at acceptance quality level. Kemp also gave comparisons between cumulative sum charts and more efficient Shewhart charts. Ewan (1963) presented a comparison with the same  $L_a$ , but with a different sample size and interval.

### 3. ESTIMATION OF THE CURRENT PROCESS MEAN

The current process mean may be estimated from a cumulative sum chart by superimposing a parabola-shaped cursor on the chart, as illustrated by Fig. 6. This is done by placing the vertex of the cursor directly over the current point on the graph of the cumulative sum, with the reference value equal to the target value, and rotating the cursor so that it includes

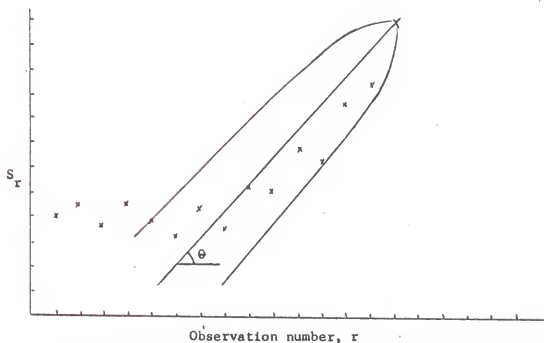


Fig. 6. Use of parabolic cursor to estimate the current process mean.

between its limbs the greatest possible number of consecutive points, counting backwards from the current point. When a maximum number,  $\underline{m}$ , of such points are included, the cursor should be rotated so that one limb is caused to pass through the  $\underline{m}$  th point. Then the slope of the axis of symmetry corresponds to the change in the process average. For example, if  $\underline{w}$  is the vertical scale distance per horizontal plotting interval, then the estimated change in the process mean will be  $w \tan \theta$ . The estimate of the process mean is  $k + w \tan \theta$ , where  $\underline{k}$  is the reference value.

A parabola need not always be used for the cursor, as a quartic or a rectangular cursor may be better for certain sets of data. Whichever cursor is chosen however, the shape should be symmetric and should exclude points on the cumulative sum chart which relate to a considerably earlier period.

Before a particular shape is adopted, a "dry running" procedure should be applied to past records. For a numerical comparison of different shapes

and sizes of parabolas, a measure of the effectiveness of a given shape is obtained by computing the mean square difference between the estimated current mean and the actual value of the next observation. The shape which gives the smallest mean square difference on a series of past observations should be chosen. Although this method of obtaining the correct shape for the cursor is quite lengthy, the operation of the estimation procedure is straightforward.

#### 4. CONCLUSION

This report has covered the work in the literature concerning cumulative sum techniques. It was seen that most of the applications deal with controlling the mean of some industrial process, but there are many other applications of cumulative sums possible. Goldsmith and Whitfield (1964) briefly discussed some of these further applications in the fields of sales forecasting, accounting, economics, and job categorisation. Overall, cumulative sum techniques are useful whenever a series of results has been produced at regular intervals of time.

## ACKNOWLEDGMENTS

I am very much indebted to Dr. A. M. Feyerherm for his guidance in preparing this report. I would also like to thank Miss Anita Schroeder for her help in editing and typing my notes.

## BIBLIOGRAPHY

- Barnard, G. A. (1959). Control charts and stochastic processes. Journal Royal Statistical Society, 21, 239-271.
- Beattie, D. W. (1962). A continuous acceptance sampling procedure based upon a cumulative sum chart for the number of defectives. Applied Statistics, 11, 137-147.
- Duncan, A. J. (1965). Quality Control and Industrial Statistics. Homewood, Illinois: Richard D. Irwin, Inc.
- Ewan, W. D. and Kemp, K. W. (1960). Sampling inspection of continuous processes with no autocorrelation between successive results. Biometrika, 47, 363-380.
- Ewan, W. D. (1963). When and how to use cusum charts. Technometrics, 5, 1-22.
- Goldsmith, P. L. and Whitfield, H. (1961). Average run lengths in cumulative chart quality control schemes. Technometrics, 3, 11-22.
- Goldsmith, P. L. and Whitfield, H. (1964). Cumulative Sum Techniques. I.C.I. Monograph, Oliver & Boyd.
- Kemp, K. W. (1958). Formulae for calculating the operating characteristics and the average sample number of some sequential tests. Journal Royal Statistical Society, 20B, 379-386.
- Kemp, K. W. (1961). The average run length of the cumulative sum chart when a V-mask is used. Journal Royal Statistical Society, 23B, 149-153.
- Kemp, K. W. (1962). The use of cumulative sums for sampling inspection schemes. Applied Statistics, 11, 16-31.
- Page, E. S. (1954). Continuous inspection schemes. Biometrika, 41, 100-115.
- Page, E. S. (1961). Cumulative sum charts. Technometrics, 3, 1-9.
- Page, E. S. (1962). Cumulative sum schemes using gauging. Technometrics, 4, 197-209.
- Roberts, S. W. (1966). A comparison of some control chart procedures. Technometrics, 8, 411-430.
- Truax, H. M. (1960). Report on cumulative sum charts, Atlas Powder Co.

CUMULATIVE SUM TECHNIQUES

by

DAVID DONALD KRUEGER

B. S., Wisconsin State University, Oshkosh, 1965

---

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY  
Manhattan, Kansas

1967

Although cumulative sums have been used for a long time, the techniques for their use have been developed only recently. The principal application of cumulative sums has been its use as a quality control technique for controlling the mean of an industrial process. In this case, the sums,  $S_n = \sum_{i=1}^n (x_i - k)$ , are plotted on a chart against  $n$ , the number of observations. One of the advantages which this representation of a series of results has over the usual Shewhart control chart is the ease with which a change in quality and the point of change in quality can be seen. Improvement in efficiency for changes in the process mean between  $0.5$  and  $2.0\sigma$  is another.

The first development of the techniques of cumulative sums was done in 1954 by E. S. Page, who devised one-sided decision interval schemes for controlling the mean of a process. These techniques were extended to two-sided decision interval schemes by Ewan and Kemp. The average run length is usually used as the criteria for the choice of a scheme. It is usually defined as the average number of samples taken before making the decision that a change in the process mean has occurred.

G. A. Barnard introduced the method of superimposing a V-mask on cumulative sum charts in order to control quantitative variables. It was later shown that the V-mask schemes and the two-sided decision interval schemes were equivalent.

E. S. Page has evaluated the characteristics of cumulative sum schemes designed to control the mean and standard deviation of a Normal distribution, where the results are given a score according to the zone in which they fall. These gauging schemes are useful if equipment is available to do the recording automatically.

Cumulative sum charts are also useful for estimating the process mean. This is done by superimposing a parabola on the cumulative sum path so that the slope of the axis of symmetry will measure the change in the process mean.