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ON POSITIVE REAL FUNCTIONS

by

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MASTER OF SCIENCE

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INTRODUCTION

The purpose of this report is to investigate some interesting aspects of positive real functions. These have been uncovered by preliminary investigation of A. Lee (6). This particular investigation was easily executed after A. Talbot (7) published a fundamental paper on proper positive real functions.

Aspects of interest essentially concern addition of a positive real function with a non-positive real function (loosely speaking) to obtain a positive real function result.

TERMINOLOGY

Definition: The right half s-plane exclusive of the j-axis will be denoted as the region R.

Definition: A rational function of s is called a "positive real function" (abbreviated p.r.f.) if it satisfies the following conditions:

$\mathrm{Re}\, f(s) \geq 0$ when $\mathrm{Re}\, s \geq 0$

$f(s) = \text{real}$ when $s = \text{real}$

where $s = \sigma + j\omega$ is a complex number.

Definition: If $\mathrm{Re}\, f(s) > 0$ in the definition of a p.r.f., then f(s) is called a "proper positive real function".

Foster reactances are a class of improper positive real functions. A more subtle example of an improper p.r.f. is $(1+s^2)/(1+2s+s^2)$.

Geometrically speaking, a p.r.f. simply maps the entire right half of s-plane into the entire right half of f(s)-plane; furthermore, it transforms the real axis of s-plane onto the real axis of the f(s)-plane.

Definition: An "integral function" has no finite singularities.

Every such function of finite order η may be written as:

$$C\ s^d \exp(Q) \prod_{r=1}^{n} (1-s_r) \exp\left\{\frac{s}{s} + \frac{1}{2}\left[\frac{s}{s_r}\right]^2 + \cdots + \frac{1}{p}\left[\frac{s}{s_r}\right]^p\right\}$$

where $d = 0, 1, 2, \cdots$, $Q(s)$ is a polynomial of degree $q \leq \eta$, $p \leq \eta$.

We shall say that an integral function is of type (h, 1) if:

$$p \leq h, \quad q \leq 1.$$

Definition: Polynomials having no poles or zeros in R are called "Hurwitz polynomials".

A p.r.f. is necessarily a ratio of Hurwitz polynomials but the reverse implication is not true. The reader will find more detailed discussion about this fact in this report. As an example, the integral function will be "Hurwitz" if Re $s_r \leq 0$ for every r.

BASIC THEOREMS ON POSITIVE REAL FUNCTIONS

With these definitions in mind, it is not difficult to note the important tautology namely "A p.r.f. of a p.r.f. is also a p.r.f.". This statement will be dignified as a theorem.

Theorem 1: If the functions $f(s)$ and $g(s)$ are p.r.f., then $f(g(s))$ is also a p.r.f..

Proof: Observe that the right half of the s-plane is mapped into the right half of g(s)-plane which, in turn, maps into the right half of f(g(s))-plane by the function f. This completes the proof.

Theorem 2: If $f(s)$ is a p.r.f., then the reciprocal of $f(s)$ is also a p.r.f..

Proof: Let $f(s) = \mathrm{Re}(f) + j\,\mathrm{Im}(f)$,

Then we have:

$$g(s) = \frac{1}{f} = \frac{1}{\mathrm{Re}(f) + j\,\mathrm{Im}(f)}$$

$$= \frac{\mathrm{Re}(f)}{\mathrm{Re}^2(f) + \mathrm{Im}^2(f)} - \frac{j\,\mathrm{Im}(f)}{\mathrm{Re}^2(f) + \mathrm{Im}^2(f)} .$$

Hence, $\mathrm{Re}(g) \geq 0$ if $\mathrm{Re}(f) \geq 0$.

Since $f(s)$ is a p.r.f., it is apparent that $\mathrm{Re}(g(s)) \geq 0$ for $\mathrm{Re}\ s \geq 0$.

Through the study of the theory of functions of a complex variable, we may delve further into detailed properties of a p.r.f.. We do this by showing that the real part of a function changes its sign at least twice as we travel around a pole's immediate vicinity. Therefore, a p.r.f. cannot have any poles or zeros in the right half plane since we restrict the real part of a p.r.f. to be non-negative throughout the region and do not allow any change of sign. Detailed analysis will be presented in the following theorem's proof.

Theorem 3: A p.r.f. $f(s)$ can have neither poles nor zeros in R. Poles of $f(s)$ and $\frac{1}{f}$ on the imaginary axis must be simple with real positive residues.

Proof: Suppose $f(s)$ has a pole of order n at s_o. Its Laurent series expansion about s_o is:

$$f(s) = \frac{C_{-n}}{(s-s_o)^n} + \frac{C_{-n+1}}{(s-s_o)^{n-1}} + \cdots + \frac{C_{-1}}{s-s_o} + C_o + C_1(s-s_o)$$

$$+ \cdots\cdots$$

In the immediate vicinity of s_o, this series may be approximated by its dominant term, $C_{-n}/(s-s_o)^n$. This dominant term can be written in polar form as:

$$C_{-n} = C\,e^{j\psi},$$

$$s-s_o = r\,e^{j\theta}.$$

Since $f(s) \doteq C\, e^{j\psi} \cdot \frac{1}{r^n} \cdot e^{-jn\theta}$, we obtain

$$\text{Re } f(s) \doteq \frac{C}{r^n} \cdot \cos(n\theta - \psi),$$

and ψ is a constant angle. It is obvious that the Re f(s) changes sign 2n times as θ varies from zero to 2π, which is a journey once around the pole as shown in Fig. 1.

For a p.r.f., Re f(s) will not be allowed to change sign in R. These investigations explain why no poles can be found in R for a p.r.f..

However on the j-axis, a boundary of R, only two sectors are possible. The real part of the function must be positive on one side of the boundary and negative on the other. That is to say if any pole is found on the j-axis, we must have $\psi = 0$, $n = 1$ which means that poles must be simple with positive and real residues.

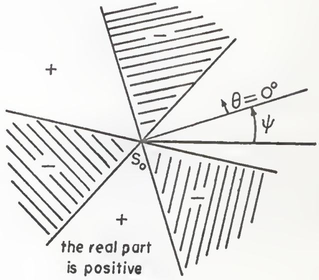


Fig. 1. Showing the immediate vicinity of a pole of order 3 in the f(s)-plane.

We have so far discussed only the property of poles. The argument extends immediately to zeros because of Theorem 2.

We have gathered sufficient information to give a more restricted definition of a p.r.f.. To have clearer impression as to how a p.r.f. behaves on the boundary and in R, we modify the definition as follows:

$\operatorname{Re} f(s) > 0$	when	$\operatorname{Re}(s) > 0$, i.e. s in R.
$\operatorname{Re} f(s) \geq 0$	when	$\operatorname{Re}(s) = 0$, i.e. s on j-axis
f(s) is real	when	s is real.

Through the first definition, we can easily see that every p.r.f. is "Hurwitz". Yet it is possible for a Hurwitz polynomial to have zeros of higher order than unity on the j-axis which is not true for a p.r.f..

Corollary: If $\frac{f(s)}{g(s)}$ is a p.r.f., then the degree of f(s) and g(s) should not differ by more than unity.

Proof: The points $s = 0$, and $s = \infty$ lie on the boundary of R. Consequently, at these points only simple poles and zeros can exist. Consider the point at infinity:

$$\lim_{s \to \infty} \frac{f(s)}{g(s)} = k \frac{s^n}{s^m} = k\, s^{n-m}$$

where n, m are degrees of f(s) and g(s) respectively. If $n-m>1$, the function has non-simple zeros at ∞. If $n-m<1$, the function has non-simple poles at ∞. Both cases are not permissible for a p.r.f..

NEW THEOREMS ON POSITIVE REAL FUNCTIONS

Theorems stated in the previous section have been known for some time. This section will present some new theorems on positive real functions. As troubles always occur on the boundary of the right half s-plane, this section will be divided into two parts. Starting out with theorems valid for both proper and improper positive real functions, this investigation proceeds further into a theorem on proper positive real functions.

Theorems for Both Proper and Improper Positive Real Functions

Theorem 4 (Talbot): a). If $\frac{f(s)}{g(s)}$ is a p.r.f., both f and g have no common zeros, then $xf(s) + jyg(s)$ is "Hurwitz" for all real constant x, y not both zero.

b). If f(s) and g(s) are holomorphic in R, $xf + jyg$ is "Hurwitz" for all real non-zero, x, y, then $\pm \frac{f(s)}{g(s)}$ is a p.r.f..

Proof: a). f and g are "Hurwitz".

f/g has neither poles nor zeros in R because of the positive realness. By the assumption that f and g has no common zeros, we conclude that f and g can have no zeros in R. So that f and g must be Hurwitz polynomials.

If $xf(s_o) + jyg(s_o) = 0$, s_o in R with x and y not both zero, then both must be non-zero; otherwise this contradicts the fact that f and y are Hurwitz polynomial.

We thus have $\frac{f(s_o)}{g(s_o)} = j \frac{y}{x}$ or $\mathrm{Re}\ \frac{f}{g} = 0$ at s_o in R.

But this is a contradiction to f/g is a p.r.f. (by Theorem 3).

So we conclude that xf + jyg is a Hurwitz polynomial.

b). f and g are Hurwitz polynomials., i.e. has no zeros in R.

Suppose $g(s_o) = 0$, s_o in R, we can put g(s) in the form:

$$g(s) = C_o(s-s_o)^n + C_1(s-s_o)^{n-1} + \cdots\cdots + C_{n-1}(s-s_o).$$

For the immediate vicinity of s_o, that is for $|s-s_o| = r < \delta \ll 1$:

$$g(s) \doteq C_{n-1}(s-s_o) \doteq r\, e^{j\theta}$$

$$jyg(s) \doteq y\, e^{j\pi/2} \cdot r\, e^{j\theta} \doteq y\, r\, e^{j(\theta + \frac{\pi}{2})},\quad 0 \le \theta \le 2\pi.$$

Now we determine how f(s) behaves in this vicinity:

$$f(s_o) = |f_o| e^{j\phi_o},$$

$$f(s) = |f_o + \xi_1| e^{j(\phi_o + \xi_2)} \text{ where } \xi_1,\ \xi_2 \to 0 \text{ as } \delta \to 0$$

due to continuity. If we allow y to be large enough so that:

$$y\, r = |f_o|$$

and θ takes on the value $\phi_o - \frac{3}{2}\pi$, then as $\delta \to 0$, we have:

$$f(s) + jyg(s) = |f_o| e^{j\phi_o} + y r e^{j(\phi_o - \pi)}$$

$$= |f_o| e^{j\phi_o} - |f_o| e^{j\phi_o} = 0.$$

This contradicts the assumption that $xf + jyg$ is Hurwitz. So we conclude that g is Hurwitz. Thus $\frac{f}{g}$ is holomorphic in R because f and g are holomorphic and $g \neq 0$ in R.

$\text{Re}\,\frac{f}{g} \neq 0$ (given $xf + jyg \neq 0$), together with the property of continuity (f/g is holomorphic) in R imply $\pm \frac{f}{g}$ is a p.r.f..

The inequality, $xf(s) + jyg(s) \neq 0$, s in R, which Talbot used in his theorem needs closer investigation. Detailed discussion will be presented in the following corollary.

Corollary: If $f(s)$ and $g(s)$ are Hurwitz polynomials, and $\frac{f(s)}{g(s)}$ is not a p.r.f., then $xf(s) + jyg(s) = 0$ for certain s in R, where x, y are real numbers not both zero.

Proof: Given that $\frac{f(s)}{g(s)}$ is not a p.r.f., then $\text{Re}\,\frac{f(s)}{g(s)}$ achieves positive as well as negative values in R. Since $f(s)$ and $g(s)$ are Hurwitz polynomials, $\frac{f(s)}{g(s)}$ is holomorphic in R. By continuity, it is known that $\text{Re}\,\frac{f(s)}{g(s)} = 0$ for certain s in R. $\frac{f(s)}{g(s)} = -j\frac{y}{x}$ for these values of s.

Two necessary consequences of this corollary will be emphasized and stated separately for future reference, namely:

1. If $f(s)$ and $g(s)$ are Hurwitz polynomials and the degrees of $f(s)$ and $g(s)$ differ by more than unity, then $xf(s) + jyg(s) = 0$ for certain s in R, where x, y are real numbers not both zero;
2. If $f(s)$ and $g(s)$ are Hurwitz polynomials having zeros of higher order than unity on the j-axis, then $xf(s) + jyg(s) = 0$ for certain s in R, where x, y are real numbers not both zero.

This corollary and the above statements bring out certain possibilities for $xf(s) + jyg(s)$ going to zero in R, which were not obvious when Talbot presented his theorem.

There is doubt whether Talbot's theorem applies to improper p.r.f.. Since his proof does not imply the p.r.f. to be a proper one, it should and, in fact it does, apply to improper p.r.f.. An example in the following section will demonstrate how well Talbot's theorem applies to a Foster reactance function.

Theorem 5 (Talbot-Lucas): If $f(s)$ is a Hurwitz polynomial or integral function of type (0, 2) with $Q(s) = as^2 + bs$, $a \geq 0$, $Re(b) \geq 0$, then $f'(s) = \frac{df}{ds}$ is also a Hurwitz polynomial.

Proof: Let $f(s) = s^p \exp(as^2 + bs) \prod_{r=1}^{n} (s - s_r)$, with $Re(s_r) \leq 0$.

From

$$\log f(s) = p \log s + as^2 + bs + \sum_{r=1}^{n} \log (s-s_r)$$

one can deduce that

$$\frac{f'(s)}{f(s)} = \frac{p}{s} + 2\,as + b + \sum_{r=1}^{n} \frac{1}{s-s_r} \quad \text{-------------(*)}$$

Every term of f'/f is positive real and non-zero in R; one can then state that

$$f'(s) = f(s).\ \frac{f'(s)}{f(s)} \text{ is Hurwitz polynomial.}$$

Notice that the degree of polynomial Q is limited to 2; otherwise, terms of s^q, where q is an integer greater than unity, will be introduced into f'/f and thus f'/f ceases to be positive real.

Corollary: If $f(s)$ and $g(s)$ are Hurwitz polynomials or integral functions of type (0, 2) as in the above theorem, and $xf(s) + jyg(s) \neq 0$ in R, then $xf' + jyg' \neq 0$ in R for every pair of real non-zero numbers x, y.

The proof of this corollary is a straight forward application of Theorem 5.

Theorem 6: If $\frac{f(s)}{g(s)}$ is a p.r.f. (including the Foster reactance functions),

then $\frac{\psi'(s)}{\gamma'(s)} = \frac{f' + b(1+bs)f}{g' + b(1+bs)g}$ is also a p.r.f..

Here, b is any non-negative real number including zero and the monomial (1+bs) cannot extend to $1+bs+(bs)^2+(bs)^3 + \cdots\cdots + (bs)^p$.

Proof: Let $\psi(s) = f(s) \cdot \exp\left[bs+(bs)^2\right]$, and let

$$\gamma(s) = g(s) \cdot \exp\left[bs+(bs)^2\right],$$

these are integral functions of type (0, 2).

Since it is given that $\frac{\psi(s)}{\gamma(s)}$ is a p.r.f., by Theorem 4 $x\psi + jy\gamma \neq 0$ in R for all real non-zero x and y.

The corollary of the previous theorem is applicable and $x\psi' + jy\gamma' \neq 0$ in R for every real non-zero x and y. So that $\frac{\psi'}{\gamma'}$ is a p.r.f..

The limitation on the monomial (1+bs) lies in the proof of Theorem 5.

This is a powerful theorem which enables us to generate a p.r.f. of higher or lower degree from a given p.r.f.. It also enables us to convert a Foster reactance function into a proper positive real function!

A Theorem for Proper Positive Real Functions

Theorem 7: If $\frac{f(s)}{g(s)}$ is a proper p.r.f., then $\frac{f(s)}{g(s)} + \left[\frac{f(s)}{g(s)}\right]^n$ is also a proper p.r.f..

Proof: Given that $\frac{f(s)}{g(s)}$ is a p.r.f., we assert that there exists non-zero real numbers x, y such that $xf(s) + jyg(s) \neq 0$, s in R.

If $x\left[f(g)^{n-1} + (f)^n\right] + jy(g)^n = 0$,

then $(g)^{n-1}\cdot\left[xf + jyg\right] + x(f)^n = 0$,

and $xf + jyg = -x \frac{(f)^n}{(g)^{n-1}}$.

Now, $x(f)^n/(g)^{n-1}$ goes to zero as $x = 0$, $y \neq 0$ and contradicts the fact that g(s) is "Hurwitz" (Theorem 4).

So we conclude that:

$x\left[f(g)^{n-1} + (f)^n\right] + jy(g)^n \neq 0$ in R,

and $\frac{f(s)}{g(s)} + \left[\frac{f(s)}{g(s)}\right]^n$ is a p.r.f..

This theorem does not extend to improper positive real functions; for if g(s) has a simple zero on the j-axis, $\left[g(s)\right]^n$ will then have a non-simple zero on the j-axis. Thus statement No. 2 (p. 11) under the corollary of Talbot's theorem applies.

EXAMPLES

In this section some numerical examples will be presented to give a brief idea of applications of theorems discussed in the previous section.

An Example for Theorem 4.

An example of the Talbot theorem's applicability to the Foster reactance function $\frac{s}{s^2 + 1}$ (where $s = \sigma + j\omega$ with $\sigma \geq 0$) is now given.

Calculation shows that,

$$xf + jyg = xs + jy(s^2 + 1)$$

$$= x(\sigma + j\omega) + jy\left[(\sigma + j\omega)^2 + 1\right]$$

$$= (x\sigma - 2y\sigma\omega) + j(x\omega + y\sigma^2 + y - y\omega^2).$$

The real part goes to zero when

$$x = 2y\omega .$$

The imaginary part goes to zero when

$$x = \frac{y(\omega^2 - \sigma^2 - 1)}{\omega} .$$

Both parts must be zero in order to make $xf + jyg = 0$, which is the case when

$$2\omega^2 = \omega^2 - \sigma^2 - 1,$$

i.e. when

$$\sigma^2 + \omega^2 + 1 = 0.$$

For $\sigma \geq 0$, no real ω will satisfy this equation; this simple means no points in R will make $xf + jyg$ go to zero as Talbot's theorem predicts.

An Example for Theorem 6.

This problem is to convert a Foster reactance function into a proper positive real function. Given the Foster reactance $\frac{1}{s}$, choose b=1, and execute the procedure described in the theorem:

$$\frac{f(s)}{g(s)} = \frac{1}{s}$$

$$\frac{\psi(s)}{\gamma(s)} = \frac{\exp\left(s + \frac{1}{2}s^2\right)}{s \cdot \exp\left(s + \frac{1}{2}s^2\right)}$$

$$\frac{\psi'(s)}{\gamma'(s)} = \frac{1+s}{s(1+s)+1} = \frac{1+s}{1+s+s^2} .$$

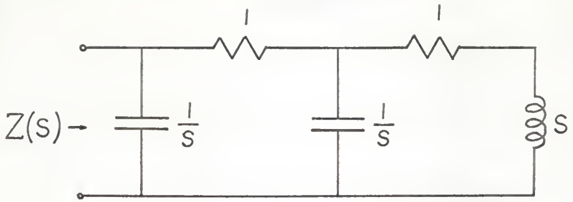
and the result is a proper positive real function.

An Example for Theorem 7.

Suppose that a series resistor and inductor load must be protected from a sharp pulse voltage. This can be interpreted as the need for a driving point impedance,

$$Z(s) = \frac{1}{1+s} + \frac{1}{(1+s)^3} .$$

Straight-forward continued fraction expansion yields the following ladder network:



This driving point impedance will act as a pulse stretcher and will reduce the magnitude of the input spike voltage at the load impedance (1 + s).

SUMMARY

This report is concerned with the truth value in the reverse implication of the true statement:

If f(s) and g(s) are positive real functions, then f(s) + g(s) is a positive real function.

On the other hand, if f(s) + g(s) is a given positive real function, this does not imply that both f(s) and g(s) are positive real functions.

After reviewing well-known theorems on positive real functions, this report presents some theorems on the nature of a non-positive real function that can be added to a positive real function to yield a positive real function. Three applications are presented.

ACKNOWLEDGMENT

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Brune (2) has shown in the early 1930's that the driving point impedance of every finite passive network is a positive real function; and conversely, it is always possible to construct a finite passive network for each of these functions. This fact explains why positive real functions play such an important role in the study of network synthesis.

The main purpose of this report is to present certain procedures (which are described in the form of theorems) of generating another positive real function from a given one. These procedures are concerned with non-positive real functions which can be added to the given positive real function to form a new positive real function.

The development begins by constructing the frame work from definitions of a positive real function. Some important fundamental theorems follow. A complete proof of Talbot's theorem (7) is presented. With the help of this important theorem, several new theorems on both proper and improper positive real functions are introduced and followed by a theorem valid for proper positive real functions.

In the last section, some numerical examples are worked out to give applications of these new theorems.