TRANSFER IMMITANCE SYNTHESIS

by

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INTRODUCTION

A general goal for analog computer simulation of transfer functions is the simulation of an arbitrary transfer function by a single operational amplifier; in this report special techniques limited to RC transfer impedances are developed.

Development of the state space approach to transfer function simulation is made and the number of amplifiers required is determined. The operational amplifier is characterized as a computing element, and a matrix approach is used to show the effects of utilizing two port networks as input and feedback elements. Having developed a general transfer impedance approach to the simulation of transfer functions, simulation of specific functions is attempted, but only achieved to the second order case.

Three appendices contain proofs, calculations, and an equivalent circuit development of single operational amplifier transfer function simulation which parallels the matrix approach given in the body of the report.

STATE SPACE APPROACH TO TRANSFER FUNCTION SIMULATION

The most widely used method of transfer function simulation is the classical state space approach. This technique has the advantage of being both systematic and simple to apply; however, implementation requires a rather large number of operational amplifiers for a given transfer function simulation.

Simulation of the transfer function

$$T(s) = \frac{\sum_{i=o}^{n} a_i s^i}{\sum_{j=o}^{n} b_j s^j} \qquad b_n = 1 \qquad (1)$$

will be shown to require in general $\left[(5+3n)/2\right]$ amplifiers for simulation by the state space method (1,3).*

A general state space development will now be made. Consider Fig. 1.

$$\bar{e}_o = \frac{1}{s} \left(\bar{e}_i - b_{n-1} \bar{e}_o\right) \qquad (2)$$

$$D_1(s) = \frac{\bar{e}_o}{\bar{e}_i} = \frac{1}{b_{n-1} + s} \qquad (3)$$

Figure 2 indicates the extension to the second order case.

$$\frac{\bar{e}_o}{\bar{e}_i} = \frac{\frac{1}{s} \frac{1}{(b_{n-1}+s)}}{1 + b_{n-2} \frac{1}{(s)} \frac{1}{(b_{n-1}+s)}} \qquad (4)$$

$$D_2(s) = \frac{1}{b_{n-2} + b_{n-1} s + s^2} \qquad (5)$$

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*Heavy brackets indicate the greatest integer function.

We are generating a denominator polynomial whose coefficients are determined by the feedback elements. A proof will now be given that this pattern continues in general.

Proof:

$$\text{Let } D_n(s) = \frac{1}{\sum_{i=o}^{n} a_i s^i}, \quad a_n = 1 \qquad (6)$$

Referring to Fig. 3 we see:

$$D_{n+1}(s) = \frac{\frac{1}{s} D_n(s)}{1 + \frac{k}{s} D_n(s)} = \frac{1}{k + s \sum_{i=o}^{n} a_i s^i} \qquad (7)$$

$$= \frac{1}{k + \sum_{i=o}^{n} a_i s^{(i+1)}} \qquad (8)$$

$$D_{n+1}(s) = \frac{1}{k + a_o s + a_1 s^1 + . \, . \, .+ a_n s^{n+1}} \qquad (9)$$

$$D_{n+1}(s) = \frac{1}{\sum_{j=o}^{n+1} b_j s^j} \qquad (10)$$

where: $b_o = k$
$b_j = a_{j-1} \quad j \geq 1$ Q.E.D.

We have shown that simulation of the polynomial

$$D_n(s) = \frac{1}{b_o + b_1 s^1 + . \, . \, .+ b_{n-1} s^{n-1} + s^n} \qquad (11)$$

is performed by the block diagram in Fig. 4. Such a transfer function can be generated by "patching" Fig. 4 on an analog computer.

Consider now the block diagram induced by an arbitrary numerator polynomial $N_n(s)$. Referring to Fig. 5 we see that:

$$\frac{\bar{e}_o}{\bar{e}_i} = \frac{a_n}{s} = \frac{N_o(s)}{s} \tag{12}$$

Fig. 6 indicates the extension to the first order case:

$$\frac{\bar{e}_o}{\bar{e}_i} = \frac{a_n}{s} + \frac{a_{n-1}}{s^2} = \frac{sa_n+a_{n-1}}{s^2} = \frac{N_1(s)}{s^2} \tag{13}$$

It is seen that an arbitrary numerator polynomial is being generated; its coefficients correspond to the feed forward elements. A proof will be given that this continues in general.

Proof: Let $B_n(s) = \dfrac{N_n(s)}{s^{n+1}} = \dfrac{\sum_{i=o}^{n} a_i s^i}{s^{n+1}}$ (14)

Generating $B_{n+1}(s)$ by the block diagram in Fig. 7.

$$B_{n+1}(s) = \frac{\bar{e}_o}{\bar{e}_i} = \frac{a_{n+1}}{s} + \frac{B_n(s)}{s}$$

$$= \frac{a_{n+1}s^{n+1} + \sum_{i=o}^{n} a_i s^i}{s^{n+2}} \tag{15}$$

$$B_{n+1}(s) = \frac{\sum_{i=o}^{n+1} a_i s^i}{s^{n+2}} = \frac{N_{n+1}(s)}{s^{n+2}} \qquad \text{Q.E.D.}$$

Considering the denominator and numerator generation procedures, and bearing in mind that an operational amplifier possesses the sign inverting property, we obtain the general computer diagram shown in Fig. 8. This diagram will generate the transfer function

$$T_n(s) = \frac{\sum_{i=o}^{n} a_i s^i}{\sum_{j=o}^{n} b_j s^j}, \quad b_n = 1 \tag{16}$$

Careful study of Fig. 8 shows that n integrators, n/2 inverters, and 2 summing amplifiers are required for n even; for n odd $\frac{n+1}{2}$ inverters are required with the same number of integrators and summing amplifiers. A tabulation of this for several values of n makes it apparent that in general $[(5+3n)/2]$ amplifiers are required for simulation of $T_n(s)$.

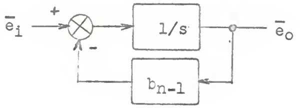

Fig. 1. Basic building block for state space denominator generation.

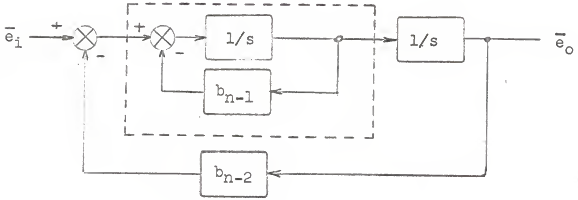

Fig. 2. Block diagram for generation of second order denominator by the state space approach.

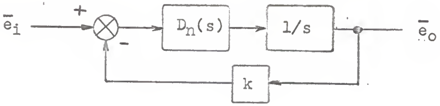

Fig. 3. Block diagram to generate n+1 order denominator by the state space approach.

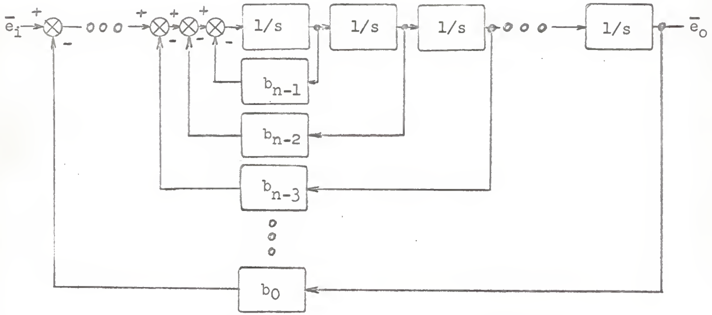

Fig. 4. Block diagram for generation of denominator of order n by the state space approach.

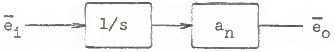

Fig. 5. Basic building block for state space numerator generation.

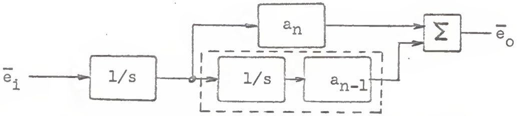

Fig. 6. Block diagram for generation of second order numerator by the state space approach.

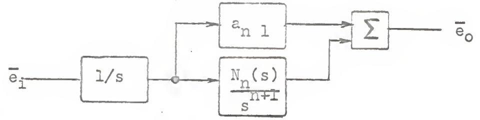

Fig. 7. Block diagram for generation of n+1 order numerator polynomial.

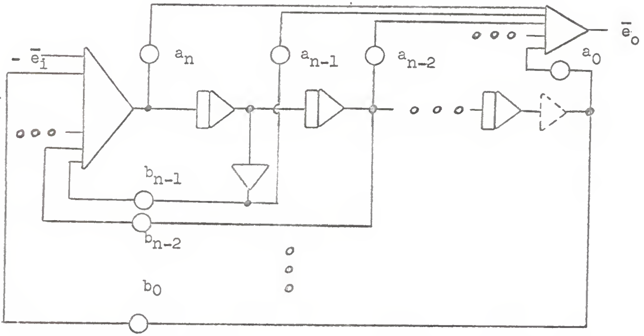

Fig. 8. Computer diagram for simulation of general transfer function of order n.

ONE AMPLIFIER REALIZATION OF TRANSFER FUNCTIONS

Transfer function simulation by the state space approach uses simple input and feedback impedances, but requires a large number of amplifiers.

Recalling that $\bar{e}_o/\bar{e}_i = -Z_{FB}/Z_i$ it would appear that a more economical alternative would be to utilize more complicated input and feedback impedances and fewer amplifiers. For purposes of analog computation, however, the only building blocks available are resistors and capacitors (Recalling that a 1 per unit (pu) resistance = 1 megohm, a 1 pu capacitance = 1 microfarad, and a 1 pu inductance = 1 megahenry, it becomes apparent why such a limitation must be imposed).

A characteristic of RC driving point impedances is that all poles and zeros must fall on the negative real axis (4); this places severe restrictions on the class of transfer functions which can be simulated. In this report an investigation is made of the feasibility of utilizing RC transfer impedances as the input and feedback impedances of an operational amplifier. The goal, which was only partially achieved, is simulation of a prescribed transfer function utilizing only a single operational amplifier.

OPERATIONAL AMPLIFIER CHARACTERISTICS

Consider the operational amplifier given in Fig. 9. The equivalent circuit for this amplifier is given in Fig. 10 for the following ideal amplifier assumptions:

1) The ideal amplifier's input impedance (Z_{in}) is infinite.

2) There is no coupling impedance between output and input of the ideal amplifier. This is the isolation property.

3) The output impedance is $1/Y_o$ and the linearily dependent generator is proportional to μe_g.

Straightforward, but tedious, calculation (see appendices I and II) yields the following equations:

$$\bar{e}_g = \bar{e}_i \frac{Y_1Y_2 + Y_oY_1}{Y + \mu Y_2Y_o} \tag{17}$$

$$\bar{e}_o = \bar{e}_i \frac{Y_1 \ (Y_2-\mu Y_o)}{Y + \mu Y_oY_2} \tag{18}$$

$$Z_{out} = \frac{Y_1 + Y_2}{Y + \mu Y_2Y_o} \tag{19}$$

where: $Y_1 = 1/Z_i$, $Y_2 = 1/Z_{FB}$, $Y = Y_1Y_2+Y_oY_1+Y_2Y_o$

Note that:

$$\lim_{\mu \to \infty} \bar{e}_g = 0 \tag{17a}$$

$$\lim_{\mu \to \infty} \frac{\bar{e}_o}{\bar{e}_i} = - \frac{Y_1}{Y_2} = - \frac{Z_{FB}}{Z_i} \tag{18a}$$

$$\lim_{\mu \to \infty} Z_{out} = 0 \tag{19a}$$

It is worthy of mention that a parallel development may be made in which the ideal amplifier is represented by a voltage controlled current generator ($-ge_g$) in parallel with an impedance (Z_o). It seems obvious that in the limit as $\mu \to \infty$, g and Z_o would likewise become infinite. This mode of thinking will be useful in a later section where a matrix derivation of transfer characteristics will be made.

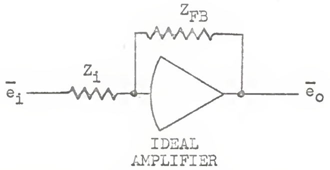

Fig. 9. Operational amplifier as used for computation.

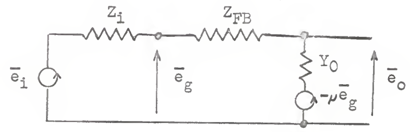

Fig. 10. Equivalent circuit for Fig. 9.

ANALOG COMPUTATION AND THE TRANSFER MATRIX

Consider the transfer matrix of the two port network shown in Fig. 11 with given voltage and current conventions. The linear dependence is given by

$$E_1 = AE_2 + BI_2 \tag{20}$$

$$I_1 = CE_2 + DI_2 \tag{21}$$

or in matrix form it is

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \tag{22}$$

Utilizing two port networks as input and feedback impedances for an operational amplifier, a simple relationship will be established between the transfer matrices of the input and feedback networks, and the transfer function generated. While a purely matrix approach will be used in this section, identical results can be generated by considering equivalent circuits. This is done in Appendix III. The matrix approach is more rigorous; the equivalent circuit approach is included for those who desire a physical "feel" for the problem.

Figure 12 indicates the situation under consideration, and under the normal ideal amplifier assumptions this reduces to Fig. 13. For Fig. 13 it is seen that:

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ ge_1 + i_2 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} o & o \\ g & o \end{pmatrix} \begin{pmatrix} e_1 \\ i_1 \end{pmatrix} \qquad (24)$$

$$\begin{pmatrix} 1-bg & 0 \\ -dg & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} \qquad (25)$$

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \frac{1}{1-bg} \begin{pmatrix} 1 & 0 \\ dg & 1-bg \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} \qquad (26)$$

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \frac{1}{1-bg} \begin{pmatrix} a & b \\ g+c & d \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} \qquad (27)$$

We now cascade an input network onto the left to obtain

$$\begin{pmatrix} e_i \\ i_i \end{pmatrix} = \frac{1}{1-bg} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ g+c & d \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} \qquad (28)$$

Setting $i_2=o$ yields the open circuit voltage

$$e_2 = \frac{1-bg}{a\alpha + \beta(g+c)} e_i \qquad (29)$$

Since ad-bc = 1 in a passive network, the above equation becomes

$$\frac{e_2}{e_i} = \frac{ad - b(g+c)}{a\alpha + \beta(g+c)} \qquad (30)$$

If $ad \ll b(g+c)$ and $a\alpha \ll \beta(g+c)$ then

$$\frac{e_2}{e_i} = -\frac{b}{\beta} \qquad (31)*$$

This last result is analogous to the well known result (Fig. 9)

$$\frac{e_o}{e_i} = -\frac{Z_{FB}}{Z_i}$$

but we are not now working with driving point RC impedances. We have acquired a richer source for generation of transfer functions.

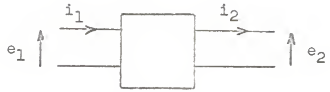

Fig. 11. A linear two port network.

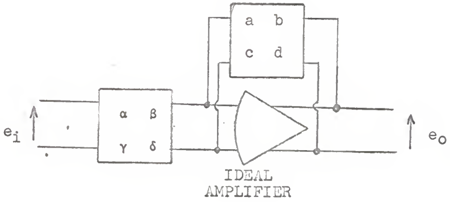

Fig. 12. Operational amplifier with two port networks as input and feedback impedances.

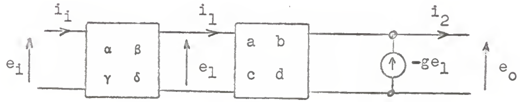

Fig. 13. Equivalent circuit for Fig. 12.

ANALOG COMPUTATION AND THE AIDENTITY

Use of transfer impedances enriches the number of transfer functions which can be constructed on a single operational amplifier. An attempt will now be made to apply this method to specific cases.

Of considerable interest in the study of servo systems is the class of functions known as approximate identities (2), hereafter referred to as aidentities. An approximate identity of order p, $(\hat{1}_p)$, is defined as a ratio of polynomials A(s) such that:

$$A(s) = \frac{\sum_{i=o}^{n} a_i s^i}{\sum_{i=o}^{m} b_i s^i} \qquad (32)$$

where $a_i = b_i$ for $i<p<n+1\leq m+1$. That is, the first p successive pairs of coefficients of s^i are equal. (33)

We shall now develop a network for the analog computer simulation of aidentities by transfer impedances.

Lemma:

For the RC ladder network in Fig. 14 the components of the transfer matrix

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \qquad (34)$$

are polynomials in 1/s.

An induction proof will be made. The assertion is obviously true for n = 1 (Fig. 15). Assume α_n, β_n, γ_n, δ_n, are polynomials in 1/s; cascade on an RC "L" section (Fig. 15).

$$\begin{pmatrix} \alpha_{n+1} & \beta_{n+1} \\ \gamma_{n+1} & \delta_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_n(1/s) & \beta_n(1/s) \\ \gamma_n(1/s) & \delta_n(1/s) \end{pmatrix} \begin{pmatrix} 1 & 1/sC \\ 1/R & 1+1/sC \end{pmatrix} \quad (35)$$

$$\alpha_{n+1} = \alpha_n(1/s) + (1/R) \quad \beta_n(1/s) = \alpha_{n+1}(1/s) \quad (36)$$

$$\beta_{n+1} = (1/sC)\alpha_n(1/s) + (1 + 1/sC) \quad \beta_n(1/s) = \beta_{n+1}(1/s) \quad (37)$$

$$\gamma_{n+1} = \gamma_n + (1/R) \quad \delta_n = \gamma_{n+1}(1/s) \quad (38)$$

$$\delta_{n+1} = (1/sC) \quad \gamma_n(1/s) + (1 + 1/sRC) \quad \beta_n(1/s) = \delta_{n+1}(1/s) \quad (39)$$

Q.E.D.

The element of primary interest is β_n. It can be shown

$$\beta_n = \sum_{i=o}^{n} a_i s^{-i} \quad (40)$$

where n is the number of capacitors present in the network of Fig. 14.

Let us now place a unit resistor in parallel with β_n (The equivalent circuit approach is the best viewpoint to take here; refer to Appendix III). This physically corresponds to the network of Fig. 16.

$$Z_{FB} = \frac{1(\beta_n)}{1 + \beta_n} \quad (41)$$

$$Z_{FB} = \frac{\sum_{i=o}^{n} a_{n-i} s^i}{\sum_{i=o}^{n-1} a_{n-i} s^i + (1+a_o)\ s^n} \quad (42)$$

This is an aidentity of order n.

It is now obvious that the network of Fig. 16 when placed in the feedback loop of an operational amplifier having $Z_i = 1$ will generate aidentity transfer functions. For practical application we seek a simple relationship between the component values of Fig. 16 and the coefficients of equation (42); with this as a goal we investigate a specific aidentity.

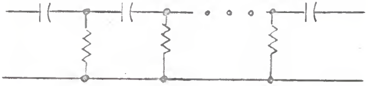

Fig. 14. Network for P(1/s) generation.

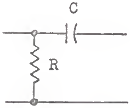

Fig. 15. Basic component of Fig. 14.

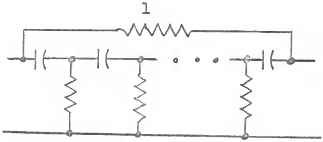

Fig. 16. Network for aidentity generation.

THE RAULT FUNCTION

There is a theorem developed by Reza which states that if A is a positive real function, and B is defined by

$$\frac{1 - B}{1 + B} = \sum_{k=o}^{n} p_k \left(\frac{1 - A}{1 + A}\right)^k \tag{43}$$

where

$$o \le p_k \le 1, \qquad \sum_{k=o}^{n} p_k = 1 \tag{44}$$

then B is a positive real function. The proof is elementary and will not be presented.

Consider a special case of this theorem where:

$$p_i = \frac{1}{2^n} \frac{n!}{i!(n-i)!} = \frac{1}{2^n} \binom{n}{i} \tag{45}$$

Therefore:

$$\frac{1 - B}{1 + B} = \frac{1}{2^n} \left(1 - \frac{1 - A}{1 + A}\right)^n \tag{46}$$

$$= \left(\frac{1}{1 + A}\right)^n \tag{47}$$

Solving for B yields

$$B = \frac{1 - \left(\frac{1}{1 + A}\right)^n}{1 + \left(\frac{1}{1 + A}\right)^n} \tag{48}$$

The simplest non-trivial positive real function possible for A is

$$A = 1/s \tag{49}$$

This implies that

$$B_n(s) = \frac{(s+1)^n - s^n}{(s+1)^n + s^n} \qquad (50)$$

For the remainder of this report this function shall be referred to as the Rault function of order n (2). It is obvious that since the numerator and denominator of $B_n(s)$ differ only in the s^n position, $B_n(s)$ is an aidentity of order n. By the binomial expansion we may write $B_n(s)$ as

$$B_n(s) = \frac{\sum_{i=o}^{n-1} \binom{n}{i} s^i}{\sum_{i=o}^{n-1} \binom{n}{i} s^i + 2s^n} \qquad (51)$$

Let us investigate the pole and zero locations of this function.

Pole locations:

$$(s+1)^n = -s^n \qquad (52)$$

$$(s+1)^n = e^{j(2k-1)\pi} s^n, \; k = 1,2,3. \; . \; .$$

Therefore: $$(s+1) = e^{\frac{j(2k-1)\pi}{n}} s, \; k = 1,2,3. \; . \; . \qquad (53)$$

Let $$\theta = \frac{(2k-1)\pi}{n}, \; k = 1,2,3. \; . \; . \qquad (54)$$

$$s_k = \frac{1}{1 - e^{j\theta}} \qquad (55)$$

$$= \frac{1}{(1-\cos\theta) - j \sin \theta} \qquad (56)$$

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^2 + \sin^2\theta} - j\,\frac{\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta} \qquad (57)$$

But:

$$(1-\cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta \qquad (58)$$

Therefore:

$$s_k = -\frac{1}{2} + j\,\frac{1}{2}\,\frac{\sin\theta}{(\cos\theta)-1} \qquad (59)$$

A similar development shows that the zeros are:

$$s_q = -\frac{1}{2} + j\,\frac{1}{2}\,\frac{\sin\phi}{(\cos\phi)-1} \qquad (60)$$

Where

$$\phi = \frac{2q\pi}{n}, \quad q = 1,2,3 \ldots \qquad (61)$$

The Rault function is unusual in that all its poles and zeros fall on a line parallel to the imaginary axis and intersecting the negative real axis at $-1/2$.

SIMULATION OF THE RAULT FUNCTION

As a specific case of single operational amplifier simulation of transfer functions the Rault function will be simulated.

The Rault function of order one is:

$$B_1(s) = \frac{1}{1 + 2s} \tag{62}$$

Consider the network of Fig. 17 (a degenerate case of Fig. 16). Using this as a feedback network for an operational amplifier with $Z_i = 1$ (Fig. 9).

$$\frac{\bar{e}_o}{\bar{e}_i} = - \beta(s) = - \frac{1}{1 + sC_1} \tag{63}$$

Setting $C_1 = 2$ the Rault function of order one is obtained up to a sign by a single amplifier.

The Rault function of order two is

$$B_2(s) = \frac{1 + 2s}{1 + 2s + 2s^2} \tag{64}$$

The feedback network of Fig. 18 possesses the following transfer matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1/sC_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/R & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/sC_2 \\ 0 & 1 \end{pmatrix} \tag{65}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{sRC_1} & \frac{1}{sC_2}\left[1 + \frac{1}{sRC_1}\right] + \frac{1}{sC_1} \\ 1/R & \frac{1}{sC_2} + 1 \end{pmatrix} \tag{66}$$

Therefore:

$$\beta_2(s) = \left[\frac{1}{C_1} + \frac{1}{C_2}\right] s^{-1} + \frac{1}{RC_1C_2} s^{-2} \tag{67}$$

Placing a unit resistance in shunt with the top of the Tee network in Fig. 18 (see Appendix III) gives the following result:

$$\beta_2(s) = -\frac{\frac{1}{R_1C_1C_2} + \left[\frac{1}{C_1} + \frac{1}{C_2}\right]s}{\frac{1}{R_1C_1C_2} + \left[\frac{1}{C_1} + \frac{1}{C_2}\right]s + s^2} \tag{68}$$

$$= -\frac{\frac{2}{R_1C_1C_2} + 2\left[\frac{1}{C_1} + \frac{1}{C_2}\right]s}{\frac{2}{R_1C_1C_2} + 2\left[\frac{1}{C_1} + \frac{1}{C_2}\right]s + 2s^2} \tag{69}$$

Setting

$$\frac{2}{R_1C_1C_2} = 1 \tag{70}$$

and

$$\frac{1}{C_1} + \frac{1}{C_2} = 1 \tag{71}$$

yields the Rault function of order 2 (up to a sign). Equations (70) and (71) are satisfied if:

$$C_1 = C_2 = 2 \tag{72}$$

$$R_1 = 1/2 \tag{73}$$

The appropriate computer diagram is shown in Fig. 19.

The Rault function of order three is:

$$B_3(s) = \frac{1 + 3s + 3s^2}{1 + 3s + 3s^2 + 2s^3} \tag{74}$$

Computation similar to that shown for the second order case, when performed on the network of Fig. 20 yields

$$B_3(s) = -\frac{\frac{2}{R_1R_2C_1C_2C_3} + 2\left[\frac{1}{R_2C_3}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)+\frac{1}{R_1C_1}\left(\frac{1}{C_2}+\frac{1}{C_3}\right)\right]s + 2\ \frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\ s^2}{\frac{2}{R_1R_2C_1C_2C_3} + 2\left[\frac{1}{R_2C_3}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)+\frac{1}{R_1C_1}\left(\frac{1}{C_2}+\frac{1}{C_3}\right)\right]s^2 + 2s^3} \tag{75}$$

We agree to neglect the sign and now equate coefficients of equation (75) to those of (74).

$$\frac{1}{2} = \frac{1}{R_1R_2C_1C_2C_3} \tag{76}$$

$$\frac{3}{2} = \frac{1}{R_2C_3}\ \ \frac{1}{C_1} + \frac{1}{C_2}\ \ + \frac{1}{R_1C_1}\ \ \frac{1}{C_2} + \frac{1}{C_3} \tag{77}$$

$$\frac{3}{2} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{78}$$

Since we have three equations in five unknowns it would seem that a solution for three elements in terms of the other two would be a simple matter; a demonstration will be made that this is not the case.

Pick $\quad R_1 = 1/2 \qquad C_1 = 2 \qquad (79)$

We thus obtain:

$$1/2 = \frac{1}{R_2C_2C_3} \tag{80}$$

$$\frac{3}{2} = \frac{1}{2R_2C_3} + \frac{1}{R_2C_2C_3} + \frac{1}{C_2} + \frac{1}{C_3} \tag{81}$$

$$1 = \frac{1}{C_2} + \frac{1}{C_3} \tag{82}$$

Substituting equations (80) and (82) into (81) yields

$$\frac{3}{2} = \frac{1}{2R_2C_3} + \frac{3}{2} \tag{83}$$

which is a contradiction.

Similar contradictions were obtained for a large number of trial values for R_1 and C_1. While it has not been proven that no solutions exist to equations (76), (77) and (78), the existence of such solutions seems unlikely. Our method of simulation has failed for the third order Rault function. It was also established that the method fails for higher order Rault functions.

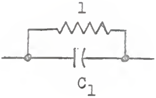

Fig. 17. Network for $\hat{1}_1$ generation.

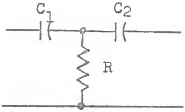

Fig. 18. Partial network for $\hat{1}_2$ generation.

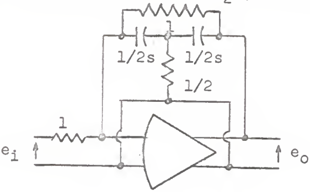

Fig. 19. Computer diagram for Rault function of order two.

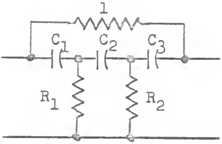

Fig. 20. Network for $\hat{1}_3$ generation.

A PATTERN SEARCH

Simulation of the Rault aidentity was attained up to the second order case by appropriate networks of the type given in Fig. 16 having $R_i = 1/2$ and all $C_i = 2$ for all i. Third order aidentity generation with the same values yields

$$\frac{\bar{e}_o}{\bar{e}_i} = \frac{1 + 4s + 3s^2}{1 + 4s + 3s^2 + 2s^3} \qquad (84)$$

which differs from the Rault function by only one coefficient. Pole-zero locations of transfer functions generated by semi-ladders of the type shown in Fig. 16 with $R_i = 1/2$, $C_i = 2$ for all i were investigated for up to the ninth order case by use of a digital computer. Partial results are given in Table I. All zeros lie on the negative real axis, and higher order networks merely place more and more poles on the negative real axis. It is suggested that an extensive investigation of RC transfer impedances is required in order to determine limitations on pole zero locations.

Table 1. Pole-Zero Locations

$$\frac{.5+2.0s+1.5s^2}{.5+2.0s+1.5s^2+s^3}$$

Zeros	Poles
s= -.333	s = -.306
s= -.999	s = $-.597\pm j1.131$

$$\frac{.5+3.0s+5.0s^2+2.0s^3}{.5+3.0s+5.0s^2+2.0s^3+s^4}$$

Zeros	Poles
s = -.293	s = -.310
s = -.500	s = -.407
s = -1.707	s = $-.641\pm j1.882$

$$\frac{.5+4.0s+10.5s^2+10.0s^3+s.5s^4}{.5+4.0s+10.5s^2+10.0s^3+2.5s^4+s^5}$$

Zeros	Poles
s = -.276	s = -.271
s = -.381	s = -.464
s = -.724	s = -.499
s = -2.618	s = $-.632\pm j2.747$

$$\frac{.5+5.0s+18.0s^2+28.0s^3+17.5s^4+3.0s^5}{.5+5.0s+18.0s^2+28.0s^3+17.5s^4+3.0s^5+s^6}$$

Zeros	Poles
s = -.268	s = -.272
s = -.333	s = -.317
s = -.499	s = $-.572\pm j3.716$
s = -.3.732	s = $-.633\pm j.097$

INTENTIONAL FEEDBACK

Use of an RC feedback network and a single amplifier has failed to produce the desired results. The configuration we have been considering is shown in Fig. 21, and gives the following transfer functions.

$$e_o = -\lambda e_i \tag{85}$$

Consider the network of Fig. 22 which yields the following transfer function.

$$e_o = -(\kappa e_i + \mu\nu) \tag{86}$$

Two obvious cases are:

1. $\nu = e_o$ (Feedback)
2. $\nu = e_i$ (Feedthrough)

The second case is discarded as representing merely a modification of the input network. Consider case one:

$$e_o(1 + \mu) = -\kappa e_i$$

$$\frac{e_o}{e_i} = -\frac{\kappa}{1 + \mu}$$

For this to give us an aidentity κ and μ must be identical polynomials in 1/s. It is apparent this is what we have been doing all along by the inclusion of the unity feedback resistor; nothing is gained by this approach.

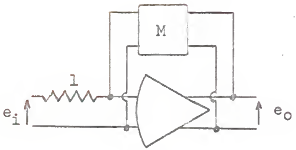

Fig. 21. Operational amplifier with conventional transfer impedance feedback.

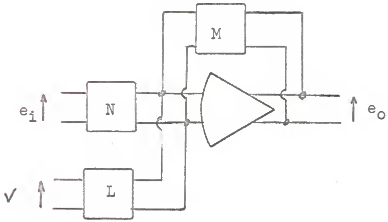

Fig. 22. Two input system for investigation of intentional feedback.

CONCLUSION

The state space approach to transfer function simulation is summarized and the operational amplifier characterized as a computational device in this report. A method has been developed for single amplifier simulation of high order transfer functions by use of the transfer impedance of a two port network.

The specific case of a single operational amplifier simulation of the Rault aidentity of arbitrary order is attempted, but is only achieved up to the second order case.

It is felt that more extensive investigation of the transfer impedance, β, is required to determine what constraints must be placed upon it when considering only RC networks. The problem of transfer function simulation on a single amplifier is reduced to the synthesis of the transfer impedance β, but little can be said in general about transfer impedance synthesis.

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APPENDIX I

Millman's Theorem

A useful tool for the investigation of operational amplifier characteristics is Millman's Theorem:

If the circuit of Fig. 1A is given then

$$E_{out} = \frac{\sum_{i=1}^{n} e_i Y_i}{\sum_{i=1}^{n} Y_i} \qquad (1A)$$

Proof:

Assume E_{out} to be true for a given n. Consider Fig. 2A which is equivalent to Fig. 1A with one more branch added. Applying Norton's Theorem to Fig. 2A, Fig. 3A is generated. Parallel admittances and current generators are now combined in the normal fashion, and after taking a Thevenin equivalent we arrive at the network shown in Fig. 4A. We see that e_{out} is given by

$$e_{out} = \frac{\sum_{i=1}^{n} e_i Y_i}{\sum_{i=1}^{n} Y_i} \qquad (2A)$$

and the theorem has been proven.

Fig. 1A. Network for Millman's theorem.

$$e=\frac{\sum_{i=1}^{n} e_i Y_i}{\sum_{i=1}^{n} Y_i}$$

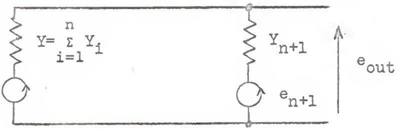

Fig. 2A. Extention of Fig. 1A. to n+1 case.

$$i_n=\sum_{i=1}^{n} e_i Y_i$$

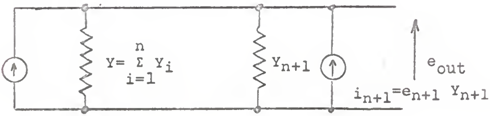

Fig. 3A. Norton equivalent of Fig. 2A.

$$e=\frac{\sum_{i=1}^{n} e_i Y_i}{\sum_{i=1}^{n} Y_i}$$

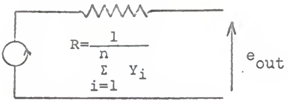

Fig. 4A. Thevenin equivalent of Fig. 3A.

APPENDIX II

Derivation of Ideal Operational Amplifier Characteristics

Consider Fig. 10 and apply Millman's Theorem (see Appendix I) :

$$e_g = \frac{e_i Y_1 + (-\mu e_g) \dfrac{Y_2 Y_o}{Y_2 + Y_o}}{Y_1 + \dfrac{Y_2 Y_o}{Y_2 + Y_o}} = \frac{e_i Y_1 (Y_2 + Y_o) - \mu e_g Y_2 Y_o}{Y_1 (Y_2 + Y_o) + Y_2 Y_o} \quad (3A)$$

where: $Y_1 = 1/Z_i$, $Y_2 = 1/Z_{FB}$, $Y = Y_1 Y_2 + Y_o Y_1 + Y_2 Y_o$

$$e_o = \frac{e_i \left(\dfrac{Y_1 Y_2}{Y_1 + Y_2}\right) - \mu e_g Y_o}{\dfrac{Y_1 Y_2}{Y_1 + Y_2} + Y_o} = \frac{e_i (Y_1 Y_2) - \mu e_g Y_o (Y_1 + Y_2)}{Y_1 Y_2 + Y_o (Y_1 + Y_2)} \quad (4A)$$

Therefore:

$$e_g = \frac{e_i (Y_1 Y_2 + Y_o Y_1) - \mu e_g Y_2 Y_o}{Y} \quad (5A)$$

$$e_o = \frac{e_i (Y_1 Y_2) - \mu e_g (Y_o Y_1) - \mu e_g Y_o Y_2}{Y} \quad (6A)$$

Solving for e_g:

$$e_g \left(1 + \frac{\mu Y_2 Y_0}{Y}\right) = e_i \frac{(Y_1 Y_2 + Y_o Y_1)}{Y} \quad (7A)$$

$$e_g = \frac{e_i (Y_1 Y_2 + Y_o Y_1)}{Y + \mu Y_2 Y_o} \quad (17)$$

Also:

$$e_o = e_i \left[\frac{Y_1 Y_2}{Y} - \frac{\mu \, (Y_o Y_1 + Y_o Y_2) \; (Y_1 Y_2 + Y_o Y_1)}{Y \; (Y + \mu Y_2 Y_o)} \right] \qquad (8A)$$

$$e_o = e_i \left[\frac{Y_1 Y_2}{Y} - \frac{\mu \; Y_o Y_1 \; (Y_1 + Y_2) \; (Y_o + Y_2)}{Y \; (Y + \mu \; Y_2 Y_o)} \right] \qquad (9A)$$

But:

$$(Y_1 + Y_2) \; (Y_o + Y_2) = Y_1 Y_2 + Y_o Y_2 + Y_1 Y_o + Y_2^{\;2} = Y + Y_2^{\;2} \qquad (10A)$$

Therefore:

$$e_o = e_i \left[\frac{Y_1 Y_2}{Y} - \frac{\mu \; Y_o Y_1 \; (Y + Y_2^{\;2})}{Y \; (Y + \mu \; Y_o Y_2)} \right] \qquad (11A)$$

$$e_o = e_i \left[\frac{Y_1 \; (Y_2 - \mu \; Y_o)}{Y + \mu \; Y_o Y_2} \right] \qquad (18)$$

Using the same model (Fig. 10) and assumptions we shall calculate the output impedance of the ideal amplifier. Placing a voltage e across the output as shown in Fig. 5A it is seen that:

$$e_g = \left[\frac{Y_2}{Y_1 + Y_2} \right] e \qquad (12A)$$

$$i = e \left(\frac{Y_1 Y_2}{Y_1 + Y_2} \right) + (e + \mu e_g) \; Y_o$$

$$= e \left[\frac{Y_1 Y_2}{Y_1 + Y_2} + Y_o + \frac{\mu \; Y_2 Y_o}{Y_1 + Y_2} \right] \qquad (13A)$$

$$Z_{out} = \frac{e}{i} = \frac{1}{\dfrac{Y_1 Y_2}{Y_1 + Y_2} + Y_o + \dfrac{\mu \; Y_2 Y_o}{Y_1 + Y_2}} \qquad (14A)$$

$$Z_{out} = \frac{Y_1 + Y_2}{Y_1Y_2 + Y_oY_1 + Y_oY_1 + \mu Y_2Y_o} \qquad (15A)$$

$$Z_{out} = \frac{Y_1 + Y_2}{Y + \mu Y_2Y_o} \qquad (19)$$

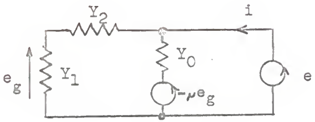

Fig. 5A. Model for output impedance calculations.

APPENDIX III

An Equivalent Circuit Approach to Analog Transfer Functions

Any two port network may be represented mathematically as a T or pi equivalent, although, of course, the Tee and Pi elements are not necessarily physically realizable.

The shunt network in Fig. 6A possesses the transfer matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \qquad (16A)$$

For the series network in Fig. 7A:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \qquad (17A)$$

Straightforward cascade operations on these basic building blocks yeild transfer matrices for the Tee and Pi equivalents. For the Pi in Fig. 8A it can be shown that

$$B = Z_2 \qquad (18A)$$

Consider again the transfer function generated by Fig. 12, and think of the pi equivalent of the two networks (Fig. 9A). If μ is large equations (17a), (18a), and (19a) apply. e_i is thought of as an ideal voltage source (or take a Thevenin equivalent of the actual source shunted by Z, and let $\bar{\beta} = \beta + Z_{THEV}$), therefore Z_1 may be neglected (or included in $\bar{\beta}$). Z_3 and Z_a may be neglected for large μ as $e_g \to 0$, Z_b appears across an ideal voltage source (μ large implies $Z_{out} \to 0$) and may also be neglected. Therefore, one can conclude that

$$\frac{e_o}{e_i} = -\frac{Z_{FB}}{Z_i} = -\frac{b}{\beta} \qquad (31)$$

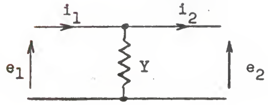

Fig. 6A. Basic parallel network.

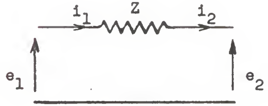

Fig. 7A. Basic series network.

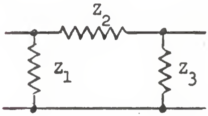

Fig. 8A. A pi network.

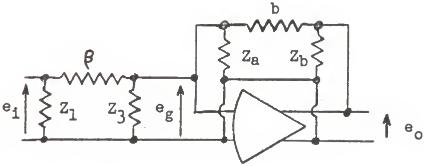

Fig. 9A. Pi networks used as input and feedback elements of an operational amplifier.

TRANSFER IMMITANCE SYNTHESIS

by

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OF TECHNOLOGY, 1964

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

The goal of this report is the simulation of a specific transfer function class by a single operational amplifier with two port networks as input and feedback impedances.

The report gives a detailed development of the state space approach to analog computer transfer function simulation with attention given to the number of amplifiers required. From a straightforward analysis of an operational amplifier equations are developed which define the amplifier as a computing element. A matrix approach is used to derive a relationship between the transfer impedances of the input and feedback networks and the transfer function thus generated. Turning to the specific case, a network is developed to simulate the general class of transfer functions known as approximate identities. The Rault function of arbitrary order is derived and simulation of this Rault function is attempted. This simulation is only successful up to the second order case. The report indicates that further investigation is required to determine restrictions to be placed on transfer impedances when only RC networks are permitted.