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Topological properties and asymptotic behavior of generalized functions

by

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Preface

The present dissertation encompasses several results situated in the theory of ultradifferentiable functions and ultradistributions. More precisely, we characterize various topological properties for multiple classes of (generalized) functions and provide structural theorems for different types of asymptotic behavior of ultradistributions. The results discussed in the sequel are primarily based on the papers [34, 35, 36, 93, 94, 95] that I have coauthored. However, this text also contains many generalizations and, as of this writing, unpublished results; in particular, this is the case for Chapter 5. We also provide the reader with many open problems, whose solving we believe would lead to interesting new directions for the theory.

In Part I we are concerned with characterizing topological properties of ultradifferentiable function spaces with respect to their defining weight sequences and functions. Most notably, we fully characterize the nuclearity of several types of Gelfand-Shilov spaces, therefore settling an open problem that goes back to the 1960's in Mityagin's work. Another significant result we obtained is the first structural theorem for the space of ultradistributions vanishing at infinity we obtained, for which a completely novel approach was needed.

The asymptotic behavior of ultradistributions is considered in Part II. In particular, we provide complete structural theorems for three types of asymptotics related to translation and dilation, thus solving long standing open problems in the field. In addition to this we also extend the so-called general Tauberian theorem for the dilation group from distributions to ultradistributions.

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Part I

Topological properties of ultradifferentiable function spaces and their duals

Chapter 1

Introduction

Functional analysis is widely considered to be a vital part in the development of modern mathematical analysis. Starting from its roots in the study of function spaces and the linear operators between them, it has grown into an impressive and vibrant research area with an abundance of applications in fields such as partial differential equations, numerical analysis, approximation theory, and many more. One of its cornerstones is the theory of generalized functions, founded by L. Schwartz [125], providing powerful tools in various mathematical branches such as Fourier analysis, asymptotic analysis, and mathematical physics. A nice example is the Malgrange-Ehrenpreis theorem which states that any non-zero constant coefficient linear PDE admits a distributional fundamental solution. However for several natural problems the space of distributions is not a suitable setting, for instance Colombini and Spagnolo showed that there are Cauchy problems for weakly hyperbolic linear PDE's with smooth coefficients that are not well-posed in the space of distributions [31]. This motivated the search for linear spaces of generalized functions that are strictly larger than the space of distributions. Noteworthy examples are the spaces of ultradistributions [11, 21, 81, 121]. For instance, under suitable conditions the above Cauchy problems become well-posed in certain spaces of ultradistributions [29, 30, 60], whence the topological invariants of spaces of ultradifferentiable functions become of great interest.

In this part we will be mainly concerned with the so-called Gelfand-Shilov spaces. In [61, 62] Gelfand and Shilov introduced and systematically studied various kinds of spaces of smooth and ultradifferentiable functions satisfying global decay estimates. Such spaces, also known as spaces of type $\mathcal{S}$, were initially considered in the context of parabolic initial-value problems, and later turned out to be the right framework for the analysis of decay and regularity properties of global solutions to large classes of linear and semi-linear partial differential equations on $\mathbb{R}^d$. In this text, these spaces will allow us to circumvent the condition of non-quasianalyticity as would necessarily be the case if we were to consider compactly supported ultradifferentiable functions as our foundational space of test functions in view of the Denjoy-Carleman theorem. Our definition of the Gelfand-Shilov spaces is done using the notion of ultradifferentiability defined through weight matrices [119], called weight sequence systems in the present text. In particular, as explained in [119], this leads to a unified treatment of classes of ultradifferentiable functions defined via weight sequences [81] and via weight functions [21]. In the four main chapters of this first part, our primary interest will be the characterization of certain topological properties of either the Gelfand-Shilov spaces themselves, or spaces that contain them as a dense subspace.

The systematic approach of determining the topological invariants needed for the validity of many well-known theorems in mathematical analysis may be considered as one of the landmarks of the last century. One may just think of the open mapping theorem, the Banach-Steinhaus theorem, abstract Schwartz kernel theorems, and many more. This has led to a select list of topological notions, see e.g. [87, 92, 103, 107, 123, 131], whose specific characterizations are highly desirable. On an abstract level, the use of homological methods has been a remarkably fruitful way to obtain such results, see e.g. the monograph [151]. This paved the way for the characterization of topological properties of many well-known spaces, such as sequence spaces and spaces of continuous functions. In this text, we are primarily interested in the notions of nuclearity, (ultra-)bornologicity and barrelledness. In particular, we aim to characterize these for several spaces of ultradifferentiable functions and their duals by re-

ducing them to the simpler versions mentioned above. To do this, we will often employ techniques stemming from time-frequency analysis.

Time-frequency analysis [65], a modern branch of harmonic analysis, to this day presents itself as thoroughly studied yet vibrant area of research. Originating from the early development of quantum mechanics, it has matured into a formidable discipline with a plethora of applications ranging from fields such as signal processing, data compression, partial differential equations and many more. Lately it has also shown itself to be an invaluable tool in the theory of generalized functions, applicable in the context of regularity analysis [33, 71, 84, 85] but also for the study of intrinsic topological properties of function spaces [4, 44]. For this text in particular, we will often make use of the mapping properties of the short-time Fourier transform and Gabor frames to completely characterize the space at hand. It turns out that for spaces of ultradifferentiable functions studying decay in both time and frequency is an excellent way for grasping their essence, and has already been employed successfully before [42, 67].

The structure of Part I is as follows. We start in Chapter 2 with an overview of all notions and notations we will use in this part, as well as a recollection of several well-known results. We expect the reader to be mostly familiar with all that is written there, yet many references are provided that give great overviews for the theory at hand. After this, we formally introduce the Gelfand-Shilov spaces in Chapter 3 and discuss some of their basic properties. In particular we will study the invariance of their definition via L^q-norms, akin to the results made in [24], which will turn out to be equivalent to the spaces being nuclear. The chapter is then concluded with a time-frequency analysis of the Gelfand-Shilov spaces, specifically we discuss the continuity properties of the short-time Fourier transform and Gabor frames. The nuclearity of several variants of the Gelfand-Shilov spaces is characterized in Chapter 4. Though each space will require a different technique, the core idea will primarily be to embed a suitable space into the Gelfand-Shilov spaces or vice-versa. Here the complete characterization of the nuclearity of Köthe sequence

spaces will play a vital role. Continuing to Chapter 5, we consider the topological invariants of (PLB)-spaces of weighted ultradifferentiable functions, for example the multiplier space. In particular we characterize the ultrabornologicity and barrelledness using conditions similar to those of Vogt and Wagner for the splitting of short exact sequences of Fréchet spaces. Interestingly, the validity of our method for showing the necessity of these conditions will depend on the existence of Gabor frames whose windows have specified rapid decay in both time and frequency. In Chapter 6, the final chapter of Part I, we consider the space of bounded ultradistributions and the space of ultradistributions vanishing at infinity. In particular, we provide first structural theorems for both of them. These results will form the cornerstones of the theory build in Part II.

Chapter 2

Preliminaries

We fix in this chapter the notation and introduce several topological properties and spaces which we will use throughout this text.

2.1 Notation

For a topological space X, we denote for any subset $A \subseteq X$ by $\overline{A}$ its closure and by $\operatorname{int} A$ its interior. If we write $K \Subset X$, we mean K is a compact subset of $\operatorname{int} X$. We will always work with a Hausdorff topological space X, i.e. for any $x, y \in X$ such that $x \neq y$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

We always include 0 in the set $\mathbb{N}$ of all natural numbers, while we denote the set of all positive integers by $\mathbb{Z}_+$. By d we always mean an element in $\mathbb{Z}_+$ referring to the dimension. A multi-index is an element $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and we write $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$ for its length. We will also write $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ for the Euclidean norm of a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, but the distinction should always be clear from the context. For two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ we write $\alpha < \beta$ if $\alpha_j \leqslant \beta_j$ for all $j \in \{1, \ldots, d\}$ with at least one strict inequality, and $\alpha \leqslant \beta$ means that either $\alpha < \beta$ or $\alpha = \beta$. We employ the standard multi-index notation, namely, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$. The j-th partial derivative, with $j \in \{1, \ldots, d\}$, is denote by ∂_j and we write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. In the context of vectors, we write $e = (1, \ldots, 1)$.

For a function $f : \mathbb{R}^d \to \mathbb{C}$, we use the notation $M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t)$ and $T_x f(t) = f(t-x)$. For a subset $W \subseteq \mathbb{R}^d$ and $R > 0$ we denote $W_R := \{x \in \mathbb{R}^d \mid x = y + z \text{ with } y \in W, |z| < R\}$. We fix the constants in the Fourier transform as

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot \xi} dt, \qquad f \in L^1(\mathbb{R}^d).$$

We employ the following notation throughout this text.

Notation 2.1.1. We apply the brackets $(\,\cdot\,)$ for the *Beurling case* and the brackets $\{\,\cdot\,\}$ for the *Roumieu case*, which will have an explicit meaning in the specific context. Via the brackets $[\,\cdot\,]$ we consider both Beurling and Roumieu cases simultaneously, where we will often first give the statement for the Beurling case followed by the statement for the Roumieu case in parenthesis. If we use the brackets $\langle\,\cdot\,\rangle$ we mean the opposite case of $[\,\cdot\,]$, i.e. if $[\,\cdot\,] = (\,\cdot\,)$ then $\langle\,\cdot\,\rangle = \{\,\cdot\,\}$ and if $[\,\cdot\,] = \{\,\cdot\,\}$ then $\langle\,\cdot\,\rangle = (\,\cdot\,)$. In the literature one will often find the notation $*$ and $\dagger$ for $[\,\cdot\,]$, based on the notation first employed by Komatsu in [81].

2.2 Locally convex spaces

The topological spaces we will be concerned with are locally convex Hausdorff spaces (from now on abbreviated by lcHs). We refer the reader to [73, 87, 92, 131] for extensive overviews of the theory. For a lcHs E we denote the family of all continuous seminorms of E by $\mathrm{csn}(E)$. For two lcHs E, F we write $L(E, F)$ for the space of all continuous linear operators $E \to F$. By $L_b(E, F)$ we refer to $L(E, F)$ endowed with the topology of uniform convergence on the bounded sets of E. We write E' for the topological dual of E, i.e. $E' = L(E, \mathbb{C})$. Unless specified otherwise, we equip the dual E' with the strong topology, which we specifically write as E'_b. Some of the important topological properties on a lcHs E we will consider in this text are the following:

- E is *barrelled* (resp. *quasi-barrelled*) if the topology on E coincides with $b(E, E')$ (with the topology induced by E'').

- E is *Schwartz* if for each normed space F and any $A \in L(E, F)$ there exists a zero neighborhood U in E such that $A(U)$ is precompact in F.

- E is *Montel* if E is quasi-barrelled and every bounded set in E is relatively compact (called the *Heine-Borel property*).

- E is *(ultra-)bornological* if for any lcHs F and $A \in L(E, F)$, A is continuous if $A(B)$ is bounded for all bounded subsets (resp. bounded Banach disks) B in E.

A *Fréchet space* is a complete metrizable lcHs. A Fréchet space E is called *distinguished* if E' is bornological. A lcHs E is called a (DF)-*space* if it has a countable fundamental system of bounded sets and if every strongly bounded countable union of equicontinuous in E' is again equicontinuous. The dual of any Fréchet space is a (complete) (DF)-space, and conversely the dual of any (DF)-space is a Fréchet space. If a Fréchet space (resp. a (DF)-space) is Schwartz it is denoted as an (FS)-space (resp. a (DFS)-space), and if it is Montel it is denoted as an (FM)-space (resp. a (DFM)-space).

2.2.1 Inductive limits

An *inductive spectrum* $\mathcal{X}$ *of lcHs* is a sequence $(X_N)_{N \in \mathbb{N}}$ of *lcHs* such that $X_N \subseteq X_{N+1}$ with continuous inclusion for all $N \in \mathbb{N}$. The *inductive limit* of the spectrum $\mathcal{X}$, denoted by $X = \varinjlim_{N \in \mathbb{N}} X_N$, is given by the set $X = \bigcup_{N \in \mathbb{N}} X_N$ and endowed with the finest locally convex Hausdorff topology for which all inclusions $X_N \to X$ are continuous. In view of [92, Lemma 24.6, p. 280] and the inductive spectra considered in this text (whose algebraic dual always contains the translates of the Dirac delta function), such a topology will always exist. A lcHs X is called an (LB)-space if it can be written as the inductive limit of a spectrum consisting of Banach spaces. Similarly, X is called an (LF)-space (resp. (LFS)-space) if it can be written as the inductive limit of a spectrum consisting of Fréchet spaces (resp. (FS)-spaces).

For an inductive spectrum $\mathcal{X} = (X_N)_{N \in \mathbb{N}}$ of Fréchet spaces we consider the following two regularity conditions (cfr. [8, 143, 150, 151]):

- $\mathcal{X}$ is said to be *sequentially retractive* if for any null sequence in X there exists a $N \in \mathbb{N}$ such that the sequence is contained in X_N and converges to zero in X_N.

- $\mathcal{X}$ is said to be *regular* if for any bounded subset B of X there exists a $N \in \mathbb{N}$ such that B is contained and bounded in X_N.

Remark 2.2.1. Let $\mathcal{X} = (X_N)_{N \in \mathbb{N}}$ and $\mathcal{Y} = (Y_N)_{N \in \mathbb{N}}$ be two inductive spectra consisting of Fréchet spaces. Let (P) be any of the two conditions considered above. If $\varinjlim_{N \in \mathbb{N}} X_n \cong \varinjlim_{N \in \mathbb{N}} Y_N$ as locally convex spaces, then $\mathcal{X}$ satisfies (P) if and only if $\mathcal{Y}$ does so, as follows from Grothendieck's factorization theorem [92, Theorem 24.33, p. 290]. This justifies calling an (LF)-space sequentially retractive, respectively regular, if one (and hence all) of its defining inductive spectra has this property.

We have the following chain of implications (cfr. [151] and the references therein):

$$\text{sequentially retractive} \implies \text{(quasi-)complete} \implies \text{regular.} \quad (2.1)$$

In the special case where the spectrum consists of Fréchet-Montel spaces, these conditions become equivalent. We refer the reader to [150] for further information.

We end this section by considering the dual Mittag-Leffler theorem, which allows us to detect topological isomorphisms between inductive limits. We call an inductive spectrum $\mathcal{X} = (X_N)_{N \in \mathbb{N}}$ *compact* (resp. *weakly compact*) if each inclusion $\iota_N : X_N \to X_{N+1}$ is compact (resp. weakly compact), i.e. there is a neighborhood of zero in X_N that is relatively compact (resp. relatively weakly compact) in X_{N+1}.

Theorem 2.2.2 (Dual Mittag-Leffler theorem, [81, Lemma 1.4, p. 37]). *Let $\mathcal{X} = (X_N)_{N \in \mathbb{N}}$, $\mathcal{Y} = (Y_N)_{N \in \mathbb{N}}$ and $\mathcal{Z} = (Z_N)_{N \in \mathbb{N}}$ be inductive spectra of Banach spaces. Suppose that*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{\rho_1} & Y_1 & \xrightarrow{\pi_1} & Z_1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \xrightarrow{\rho_2} & Y_2 & \xrightarrow{\pi_2} & Z_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_N & \xrightarrow{\rho_N} & Y_N & \xrightarrow{\pi_N} & Z_N & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

is an inductive sequence of short topologically exact sequences. Set $\rho = \varinjlim_{N \in \mathbb{N}} \rho_N$. If $X = \varinjlim_{N \in \mathbb{N}} X_N$ is Montel, $\mathcal{Y}$ is regular and $\mathcal{Z}$ is weakly compact, then $X \cong \rho(X)$ as locally convex spaces.

2.2.2 Projective limits

A *projective spectrum $\mathcal{X}$ of lcHs* is a sequence $(X_N)_{N \in \mathbb{N}}$ of lcHs together with continuous linear linking mappings $\varrho_{N+1}^N : X_{N+1} \to X_N$ for all $N \in \mathbb{N}$. We write $\varrho_N^N = \mathrm{id}_{X_N}$ and $\varrho_M^N = \varrho_{N+1}^N \circ \cdots \circ \varrho_M^{M-1}$ for $N < M$. We set

$$
\mathrm{Proj}^0 \, \mathcal{X} = \varprojlim_{N \in \mathbb{N}} X_N
$$

$$
:= \{(x_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} X_N \mid \rho_{N+1}^N(x_{N+1}) = x_N \text{ for all } N \in \mathbb{N}\}
$$

and denote by $\varrho^M : \mathrm{Proj}^0 \, \mathcal{X} \to X_M : (x_N)_{N \in \mathbb{N}} \to x_M$ the projection on the Mth component. We call $\mathrm{Proj}^0 \, \mathcal{X}$ the *projective limit* of $\mathcal{X}$. We endow $\mathrm{Proj}^0 \, \mathcal{X}$ with the coarsest topology such that every projection ϱ^M is continuous. The projective spectrum $\mathcal{X}$ is called *reduced* if the image of $\varrho^M : \mathrm{Proj}^0 \, \mathcal{X} \to X_M$ is dense for every $M \in \mathbb{N}$. For such projective spectra, we have that $\mathcal{X}^* = (X_N')_{N \in \mathbb{N}}$ is an inductive

spectrum of lcHs, as the transposed of ρ_{N+1}^N may be considered as a continuous inclusion. We then have as sets [87]:

$$\left(\varprojlim_{N\in\mathbb{N}} X_N\right)' = \varinjlim_{N\in\mathbb{N}} X_N'. \tag{2.2}$$

Note that (2.2) holds topologically if $\mathcal{X}$ is *(weakly) compact* [80, 127], that is if the linking mappings ϱ_{N+1}^N are (weakly) compact. On the other hand, if $\mathcal{X} = (X_N)_{N\in\mathbb{N}}$ is an inductive spectrum, then $\mathcal{X}^* = (X_N', \varrho_{N+1}^N)$ is a projective spectrum where we put $\rho_{N+1}^N(x') = x'|_{X_N}$ for any $x' \in X_{N+1}'$. We have as sets [87]:

$$\left(\varinjlim_{N\in\mathbb{N}} X_N\right)' = \varprojlim_{N\in\mathbb{N}} X_N'. \tag{2.3}$$

If $\mathcal{X}$ is regular, (2.3) holds topologically.

Two projective spectra $\mathcal{X} = (X_N, \varrho_{N+1}^N)_{N\in\mathbb{N}}$ and $\mathcal{Y} = (Y_N, \sigma_{N+1}^N)$ are said to be *equivalent* if there exist increasing sequences $(k_N)_{N\in\mathbb{N}}$ and $(l_N)_{N\in\mathbb{N}}$ of natural numbers with $N \leqslant k_N \leqslant l_N \leqslant k_{N+1}$ and linear mappings $T_N : X_{l_N} \to Y_{k_N}$ and $S_N : Y_{k_N} \to X_{l_{N-1}}$ such that $S_N \circ T_N = \varrho_{l_N}^{l_{N-1}}$ and $T_N \circ S_{N+1} = \sigma_{k_{N+1}}^{k_N}$. Clearly, $\mathrm{Proj}^0\,\mathcal{X} \cong \mathrm{Proj}^0\,\mathcal{Y}$ as locally convex spaces if $\mathcal{X}$ and $\mathcal{Y}$ are equivalent projective spectra.

A lcHs is called a (PLB)-space (resp. a (PLS)-space) if it can be written as the projective limit of a reduced projective spectrum of (LB)-spaces (resp. (DFS)-spaces). We refer to [49] for more information on these spaces.

2.2.3 Nuclearity and topological tensor products

We recall some of the fundamentals of the theory of nuclear spaces and topological tensor products, introduced by Grothendieck in his doctoral thesis [69] for his study of the validity of abstract Schwartz kernel theorems.

A linear map $A : E \to F$ between two Banach spaces E, F is said to be *nuclear*, if there exist a sequence $(a_n)_{n\in\mathbb{N}}$ in E' and a sequence $(b_n)_{n\in\mathbb{N}}$ in F such that $\sum_{n\in\mathbb{N}} \|a_n\|_{E'} \|b_n\|_F < \infty$ and

$$A(x) = \sum_{n\in\mathbb{N}} \langle a_n, x \rangle b_n, \qquad x \in E.$$

For a lcHs E and a continuous seminorm $p \in \mathrm{csn}(E)$, we denote by E_p the local Banach space for the seminorm p, i.e. the completion of E/N_p w.r.t. the seminorm p where $N_p = \{x \in E \mid p(x) = 0\}$.

Definition 2.2.3. A lcHs E is called *nuclear* if for each $p \in \mathrm{csn}(E)$ there exists a $q \in \mathrm{csn}(E)$ with $q \geqslant p$ such that the canonical embedding $\iota : E_q \to E_p$ is nuclear.

We recall some results on nuclear spaces. Proofs for all statements may be found in [92], or follow easily therefrom.

Lemma 2.2.4. *Let E be a nuclear lcHs.*

(i) *E is Schwartz.*

(ii) *If E is complete and quasi-barrelled, then E is Montel.*

(iii) *Any subspace F of E is nuclear. If F is closed, then E/F is nuclear.*

(iv) *For any projective spectrum $\mathcal{X} = (X_N, \varrho_{N+1}^N)$ of nuclear spaces, the projective limit $\varprojlim_{N \in \mathbb{N}} X_N$ is nuclear.*

(v) *For any inductive spectrum $\mathcal{X} = (X_N)$ of nuclear spaces, the inductive limit $\varinjlim_{N \in \mathbb{N}} X_N$ is nuclear.*

We now consider the nuclearity of Fréchet and (DF)-spaces, which are then denoted as (FN)- and (DFN)-spaces. For such spaces Grothendieck provided a criterion for nuclearity in terms of summable sequences [69]. Let E be a lcHs. A sequence $(e_n)_{n \in \mathbb{N}}$ in E is called *weakly summable* if

$$\sum_{n \in \mathbb{N}} |\langle e', e_n \rangle| < \infty, \qquad \forall e' \in E'.$$

By Mackey's theorem, $(e_n)_{n \in \mathbb{N}}$ is weakly summable if and only if the set

$$\bigcup_{k \in \mathbb{N}} \left\{ \sum_{n=0}^{k} c_n e_n \; : \; |c_n| \leqslant 1, n = 0, \dots, k \right\}$$

is bounded in E. The sequence $(e_n)_{n \in \mathbb{N}}$ is called *absolutely summable* if

$$\sum_{n \in \mathbb{N}} p(e_n) < \infty, \qquad \forall p \in \operatorname{csn}(E).$$

Clearly, $(e_n)_{n \in \mathbb{N}}$ is absolutely summable if and only if $\sum_{n=0}^{\infty} p(e_n) < \infty$ for all p belonging to some fundamental system of continuous semi-norms on E. Moreover, if E is a Fréchet space or a (DF)-space, then $(e_n)_{n \in \mathbb{N}}$ is absolutely summable if and only if for every bounded set B in E it holds that $\sum_{n=0}^{\infty} p_B(e_n) < \infty$, where p_B is the gauge functional of B [107, Theorem 1.5.8].

Proposition 2.2.5 ([107, Theorem 4.2.5]). *Let E be a Fréchet space or a (DF)-space. Then, E is nuclear if and only if every weakly summable sequence in E is absolutely summable.*

We move on to topological tensor products. We only provide a brief discussion of the notations and results used in the sequel, for a more thorough overview we refer the reader to such works as [69, 83, 131]. For any two lcHs E and F we define the ε tensor product $E \varepsilon F$ as the space of all bilinear functionals on $E_c' \times F_c'$ which are hypocontinuous on the equicontinuous sets of E' and F'. We endow $E \varepsilon F$ with the topology of uniform convergence on the products of equicontinuous sets in E' and F'. The tensor product $E \otimes F$ is canonically imbedded in $E \varepsilon F$ under

$$(e \otimes f)(e', f') = \langle e', e \rangle \langle f', f \rangle.$$

We write $E \otimes_\varepsilon F$ if we equip $E \otimes F$ with the topology induced by $E \varepsilon F$, and write $E \widehat{\otimes}_\varepsilon F$ for its completion. For any two complete lcHS E and F, if either E or F is nuclear, then we have the following canonical isomorphisms as locally convex spaces:

$$E \varepsilon F \cong E \widehat{\otimes}_\varepsilon F.$$

The ε-tensor product behaves well under projective limits.

Lemma 2.2.6 ([83, Proposition 1.5]). *Let E be a lcHs and $\mathcal{X} = \{F_N, \varrho_{N+1}^N\}_{N \in \mathbb{N}}$ be a projective spectrum of lcHs. Then the following canonical isomorphism holds as locally convex spaces:*

$$E \widehat{\otimes}_\varepsilon (\varprojlim_{N \in \mathbb{N}} F_N) \cong \varprojlim_{N \in \mathbb{N}} E \widehat{\otimes}_\varepsilon F_N.$$

The *projective tensor product topology* π on $E \otimes F$ is the strongest locally convex topology such that the canonical bilinear mapping $E \times F \rightarrow E \otimes F$ is continuous, and we denote by $E \otimes_\pi F$ the tensor product $E \otimes F$ endowed with the topology π. Additionally, $E \widehat{\otimes}_\pi F$ is the completion of $E \otimes_\pi F$. If E and F are (DF)-spaces, then so are $E \otimes_\pi F$ and $E \widehat{\otimes}_\pi F$. Similarly, if E is an (FN)-space and F a Fréchet space, then $E \widehat{\otimes}_\pi F$ is a Fréchet space.

The topology of $E \otimes_\pi F$ is finer than that of $E \otimes_\varepsilon F$. However, if either one of the spaces is nuclear, then the topologies coincide (and in matter of fact this is an equivalent definition for nuclearity, see [131, Theorem 50.1, p. 511]). If such is the case, we simply write $E \otimes F = E \otimes_\varepsilon F = E \otimes_\pi F$ and $E \widehat{\otimes} F$ for its completion. We also note that if both E and F are nuclear, then so is $E \widehat{\otimes} F$ [131, Proposition 50.1, p. 514].

Suppose E_1, E_2, F_1 and F_2 are lcHs. If $E_1 \cong E_2$ and $F_1 \cong F_2$ as locally convex spaces, then also $E_1 \widehat{\otimes}_\varepsilon F_1 \cong E_2 \widehat{\otimes}_\varepsilon F_2$ [131, Proposition 43.7, p. 440]. Moreover, we will also need the following general fact.

Lemma 2.2.7. *Let E, E_0, F, F_0 be lcHs such that $E_0 \subseteq E$ and $F_0 \subseteq F$ with dense continuous inclusions. Then, $E_0 \widehat{\otimes}_\varepsilon F_0$ is dense in $E \widehat{\otimes}_\varepsilon F$.*

2.3 Classical locally convex spaces

We recall in this section several well-known locally convex spaces which we will frequently use throughout this text.

2.3.1 Köthe sequence spaces

Given a sequence $a = (a_j)_{j \in \mathbb{Z}^d}$ of positive numbers, we define $l^q(\mathbb{Z}^d, a) = l^q(a)$, $q \in [1, \infty]$, as the weighted Banach sequence space consisting of all $c = (c_j)_{j \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$ such that

$$\|c\|_{l^q(a)} = \left(\sum_{j \in \mathbb{Z}^d} (|c_j| a_j)^q \right)^{1/q} < \infty, \qquad q \in [1, \infty),$$

and

$$\|c\|_{l^\infty(a)} = \sup_{j \in \mathbb{Z}^d} |c_j| a_j < \infty.$$

Furthermore, we define $l^0(\mathbb{Z}^d, a)$ as the space consisting of all $(c_j)_{j \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$ such that

$$\lim_{j \in \mathbb{Z}^d} c_j a_j = 0$$

and endow it with the norm $\|\cdot\|_{l^\infty(a)}$.

A *Köthe set* is a family $A = \{a^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ of sequences a^λ of positive numbers such that $a_j^\lambda \leqslant a_j^\mu$ for all $j \in \mathbb{Z}^d$ and $\mu \leqslant \lambda$. We define the associated *Köthe sequence spaces* as

$$\lambda^q(A) = \varprojlim_{\lambda \to 0^+} l^q(a^\lambda), \qquad \lambda^q\{A\} = \varinjlim_{\lambda \to \infty} l^q(a^\lambda), \qquad q \in \{0\} \cup [1, \infty].$$

Note that $\lambda^q(A)$ is a Fréchet space, while $\lambda^q\{A\}$ is a regular (LB)-space, as follows from [8, Corollary 7, p. 80]. We denote by A° the Köthe set $A^\circ = \{1/a^{1/\lambda} \,|\, \lambda \in \mathbb{R}_+\}$.

The nuclearity of the spaces $\lambda^q[A]$ can be characterized in terms of the following conditions on the Köthe set A:

(N) $\forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ : a^\lambda/a^\mu \in l^1(\mathbb{Z}^d)$;

{N} $\forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ : a^\lambda/a^\mu \in l^1(\mathbb{Z}^d)$.

Proposition 2.3.1 ([8, Proposition 15, p. 75]). *Let A be a Köthe set. The following statements are equivalent:*

(i) *A satisfies* [N].

(ii) *$\lambda^q[A]$ is nuclear for all $q \in \{0\} \cup [1, \infty]$.*

(iii) *$\lambda^q[A]$ is nuclear for some $q \in \{0\} \cup [1, \infty]$.*

(iv) *$\lambda^q[A] = \lambda^r[A]$ as locally convex spaces for all $q, r \in \{0\} \cup [1, \infty]$.*

(v) *$\lambda^q[A] = \lambda^r[A]$ as sets for some $q, r \in \{0\} \cup [1, \infty]$ with $q \neq r$.*

2.3.2 Spaces of integrable and smooth functions

For any $q \in [1, \infty]$, we denote by $L^q(\mathbb{R}^d)$ the Banach space of all measurable functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|\varphi\|_{L^q} = \left(\int_{\mathbb{R}^d} |\varphi(x)|^q dx \right)^{1/q} < \infty, \qquad q \in [1, \infty),$$

and
$$\|\varphi\|_{L^\infty} = \sup_{x \in \mathbb{R}^d} |\varphi(x)| < \infty.$$

In the smooth case, we consider the Fréchet spaces $\mathcal{D}_{L^q}(\mathbb{R}^d)$, $q \in [1, \infty]$, of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{D}_{L^q}, k} = \max_{|\alpha| \leqslant k} \|\varphi^{(\alpha)}\|_{L^q} < \infty, \qquad \forall k \in \mathbb{N}.$$

In keeping with Schwartz, we will write $\mathcal{B}(\mathbb{R}^d)$ for the space $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$.

For a compact subset $K \Subset \mathbb{R}^d$, we by denote $\mathcal{D}_K$ the closed subspace of $\mathcal{B}(\mathbb{R}^d)$ of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi \subseteq K$. The space of all compactly supported smooth functions is then denoted by

$$\mathcal{D}(\mathbb{R}^d) = \varinjlim_{K \Subset \mathbb{R}^d} \mathcal{D}_K.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the Fréchet space of all smooth functions φ such that

$$\|\varphi\|_{\mathcal{S}, k} = \max_{|\alpha| \leqslant k} \sup_{x \in \mathbb{R}^d} (1 + |x|)^k |\varphi^{(\alpha)}(x)| < \infty, \qquad \forall k \in \mathbb{N}.$$

We note that $\mathcal{S}(\mathbb{R}^d)$ may be equivalently defined via any L^q-norm. Clearly the Fourier transform $\mathcal{F}$ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d)$.

2.3.3   Spaces of ultradifferentiable functions

Weight sequences

A multi-indexed sequence $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$ of positive numbers is called a *weight sequence* if $\lim_{\alpha \in \mathbb{N}^d} (M_\alpha / M_0)^{1/|\alpha|} = \infty$. We write e_j for the standard coordinate unit vectors in $\mathbb{R}^d$, $j = 1, \dots, d$. We consider the following conditions on a weight sequence M:

$(M.1)$ $M_{\alpha + e_j}^2 \leqslant M_\alpha M_{\alpha + 2e_j}$ for all $\alpha \in \mathbb{N}^d$ and $j \in \{1, \dots, d\}$;

$(M.2)'$ $M_{\alpha + e_j} \leqslant C_0 H^{|\alpha|} M_\alpha$ for all $\alpha \in \mathbb{N}^d$ and $j \in \{1, \dots, d\}$ and some $C_0, H > 0$;

$(M.2)$ $M_{\alpha+\beta} \leqslant C_0 H^{|\alpha+\beta|} M_\alpha M_\beta$ for all $\alpha, \beta \in \mathbb{N}^d$ and some $C_0, H > 0$.

A well-known example is the *Gevrey sequence* $(\alpha!^s)_{\alpha \in \mathbb{N}^d}$ for some $s > 0$, which always satisfies $(M.1)$ and $(M.2)$.

A weight sequence M is called *isotropic* if $M_\alpha = M_\beta$ for any $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| = |\beta|$. We then also write $M = (M_p)_{p \in \mathbb{N}}$ where for any $p \in \mathbb{N}$ we set $M_p := M_\alpha$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = p$. To any isotropic weight sequence M we associate the sequence $m_p := M_p/M_{p-1}$ and note that m_p is a non-decreasing sequence if and only if M satisfies $(M.1)$.

For any two weight sequences M and N we write $M \leqslant N$ if $M_\alpha \leqslant C L^{|\alpha|} N_\alpha$ for any $\alpha \in \mathbb{N}^d$ and some $C, L > 0$. If $M \leqslant N$ and $N \leqslant M$, we write $M \approx N$. Note that when this holds, then, M satisfies $(M.2)'$ (resp. $(M.2)$) if and only if N satisfies the same condition. If for any $\varepsilon > 0$ there exists some $C = C_\varepsilon > 0$ such that $M_\alpha \leqslant C \varepsilon^{|\alpha|} N_\alpha$ for all $\alpha \in \mathbb{N}^d$, we write $M \prec N$.

The *associated function* of M is defined as

$$\omega_M(x) = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|x^\alpha| M_0}{M_\alpha}, \qquad |x| > 0,$$

and $\omega_M(0) = 0$. Then, ω_M vanishes in some neighborhood of the origin and increases faster than $\log|x|$ as $|x| \to \infty$ (cf. [81, p. 48]). Also observe that $\omega_M(x) = \omega_M(|x_1|, \ldots, |x_d|)$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

We define the tensor product of a finite number of weight sequences $M_j = (M_{j,\alpha})_{\alpha \in \mathbb{N}^{d_j}}$ on $\mathbb{N}^{d_j}$, with $j = 1, \ldots, k$, as the sequence $M_1 \otimes \cdots \otimes M_k = (M_{1,\alpha_1} \cdots M_{k,\alpha_k})_{(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^{d_1+\cdots+d_k}}$. Note that $M_1 \otimes \cdots \otimes M_k$ satisfies $(M.1)$ ($(M.2)'$ or $(M.2)$, respectively) if and only if this property holds for each M_j. Moreover,

$$\omega_{M_1 \otimes \cdots \otimes M_k}(x) = \sum_{j=1}^{k} \omega_{M_j}(x_j), \qquad x = (x_1, \ldots, x_k) \in \mathbb{R}^{d_1+\cdots+d_k}.$$

Let M be a weight sequence on $\mathbb{N}^d$. Given a permutation σ of the indices $\{1, \ldots, d\}$, we write $\sigma(M) = (M_{(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(d)})})_{\alpha_1, \ldots, \alpha_d \in \mathbb{N}^d}$. We call a weight sequence M *isotropically decomposable* if it can be written as a tensor product of k isotropic weight sequences, that is, if there is a permutation σ such that $\sigma(M) = M_1 \otimes \cdots \otimes M_k$ with

each M_j isotropic. Such weight sequences may be reconstructed from their associated function in case of logarithmic convexity.

Lemma 2.3.2. *Let M be an isotropically decomposable weight sequence. Then M satisfies $(M.1)$ if and only if*

$$M_\alpha = M_0 \sup_{x \in \mathbb{R}^d} \frac{|x^\alpha|}{\exp \omega_M(x)}. \tag{2.4}$$

If such is the case, then,

$$\omega_M(\sum_{j=1}^k x_j) \leqslant \sum_{j=1}^k \omega_M(kd^{1/2}x_j), \qquad x_1, \ldots, x_k \in \mathbb{R}^d, \tag{2.5}$$

for arbitrary $k \in \mathbb{Z}_+$.

Proof. It suffices to consider the isotropic case. It is a straightforward calculation to see that M satisfies $(M.1)$ if (2.4) holds. Conversely, if M satisfies $(M.1)$ then for any $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ such that $|\alpha| = p$, by [81, Proposition 3.2, p. 49],

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp \omega_M((t,0,\ldots,0))} \leqslant M_0 \sup_{x \in \mathbb{R}^d} \frac{|x^\alpha|}{\exp \omega_M(x)} \leqslant M_\alpha,$$

whence (2.4) holds.

Now suppose M satisfies $(M.1)$. Consider the function

$$\eta(t) = \sup_{p \in \mathbb{N}} \frac{t^p M_0}{M_p}, \qquad t \geqslant 0. \tag{2.6}$$

Note that

$$\eta(d^{-1/2}|x|) \leqslant \omega_M(x) \leqslant \eta(|x|), \qquad x \in \mathbb{R}^d. \tag{2.7}$$

As η is increasing, (2.7) implies that

$$\omega_M\left(\sum_{j=1}^k x_j\right) \leqslant \sum_{j=1}^k \omega_M(kd^{1/2}x_j)$$

for any $x_1, \ldots, x_k \in \mathbb{R}^d$. $\qquad\square$

We shall also consider the following two sets

$$(\mathfrak{R}) = \{(\ell_p)_{p\in\mathbb{Z}_+} : \ell_p = \ell \text{ for some } \ell > 0\},$$
$$\{\mathfrak{R}\} = \{(\ell_p)_{p\in\mathbb{Z}_+} : \ell_p \nearrow \infty \text{ and } \ell_p > 0, \forall p \in \mathbb{N}\},$$

and use $[\mathfrak{R}]$ as a common notation. For any $(\ell_p) \in [\mathfrak{R}]$, we associate to it the isotropic weight sequence $L = (L_p)_{p\in\mathbb{N}}$ with $L_p = \prod_{j=1}^{p} \ell_p$ for $p \geqslant 1$ and $L_0 = 1$. Then, for any isotropic weight sequence M we consider the isotropic weight sequence $M_{\ell_p} = (M_p L_p)_{p\in\mathbb{N}}$. Whenever M satisfies $(M.1)$ then the ensuing useful assertions [40, Lemma 4.5, p. 417] hold on the growth of a function $g : [0, \infty) \to [0, \infty)$:

$$\forall q > 0 : g(t) = O\left(e^{\omega_M(qt)}\right)$$
$$\Longleftrightarrow \exists(\ell_p) \in \{\mathfrak{R}\} : g(t) = O\left(e^{\omega_{M_{\ell_p}}(t)}\right) \quad (2.8)$$

and

$$\forall(\ell_p) \in \{\mathfrak{R}\} : g(t) = O\left(e^{-\omega_{M_{\ell_p}}(t)}\right)$$
$$\Longleftrightarrow \exists q > 0 : g(t) = O\left(e^{-\omega_M(qt)}\right). \quad (2.9)$$

It is important to point out that if M satisfies $(M.2)$ or $(M.2)'$, then for any given $(\ell_p) \in [\mathfrak{R}]$ one can always find a $(k_p) \in [\mathfrak{R}]$ such that $k_p \leqslant \ell_p$, $\forall p \in \mathbb{Z}_+$, and M_{k_p} satisfies the same conditions as M. For the $(\mathfrak{R})$-case this is trivial, whereas the assertion for the $\{\mathfrak{R}\}$-case directly follows from [117, Lemma 2.3].

Ultradifferentiable functions

Given a weight sequence M, a compact subset $K \Subset \mathbb{R}^d$ and $\ell > 0$, we denote by $\mathcal{E}^{M,\ell}(K)$ the space of all smooth functions $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{E}^{M,\ell}(K)} = \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|\varphi^{(\alpha)}(x)|}{\ell^{|\alpha|} M_\alpha} < \infty.$$

Then we consider the *space of all ultradifferentiable functions w.r.t. M of Beurling type, resp. Roumieu type*, on $\mathbb{R}^d$:

$$\mathcal{E}^{(M)}(\mathbb{R}^d) = \varprojlim_{K\Subset\mathbb{R}^d} \varprojlim_{\ell\to 0^+} \mathcal{E}^{M,\ell}(K), \qquad \mathcal{E}^{\{M\}}(\mathbb{R}^d) = \varprojlim_{K\Subset\mathbb{R}^d} \varinjlim_{\ell\to\infty} \mathcal{E}^{M,\ell}(K).$$

Note that the condition $(M.1)$ implies that $\mathcal{E}^{[M]}(\mathbb{R}^d)$ is closed under multiplication, while $(M.2)'$ guarantees that $\mathcal{E}^{[M]}(\mathbb{R}^d)$ is closed under differentiation.

In this part, we will primarily be concerned with the so-called *Gelfand-Shilov spaces* [24, 61]. Given two weight sequences M and N and $\ell, q > 0$, we denote by $\mathcal{S}_{N,q}^{M,\ell}(\mathbb{R}^d)$ the Banach space of all $\varphi \in C^{\infty}(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{S}_{N,q}^{M,\ell}} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{e^{\omega_N(qx)}|\varphi^{(\alpha)}(x)|}{\ell^{|\alpha|}M_\alpha} < \infty.$$

Then we consider the spaces

$$\mathcal{S}_{(N)}^{(M)}(\mathbb{R}^d) = \varprojlim_{q\to\infty}\varprojlim_{\ell\to 0^+} \mathcal{S}_{N,q}^{M,\ell}(\mathbb{R}^d), \qquad \mathcal{S}_{\{N\}}^{\{M\}}(\mathbb{R}^d) = \varprojlim_{q\to 0^+}\varinjlim_{\ell\to\infty} \mathcal{S}_{N,q}^{M,\ell}(\mathbb{R}^d).$$

In Chapter 3 we will extend the definition of the Gelfand-Shilov spaces using so-called weight function and weight sequence systems.

Chapter 3

Gelfand-Shilov spaces

3.1   Introduction

The aim of this chapter is to introduce and demonstrate several general properties of the so-called Gelfand-Shilov spaces of ultradifferentiable functions, both of Beurling and Roumieu type. Such spaces, also known as spaces of type $\mathcal{S}$, were first considered by Gelfand and Shilov in the context of parabolic initial-value problems [62] and systematically studied in [61]. Thereafter the Gelfand-Shilov spaces turned out to be the right framework for the analysis of decay and regularity properties of global solutions to large classes of linear and semi-linear partial differential equations on $\mathbb{R}^d$. We refer to the monograph [96] and the survey article [64] for accounts on applications of Gelfand-Shilov spaces; see also [23, 118] for global pseudo-differential calculus in this setting. For our purposes, the Gelfand-Shilov spaces and their duals will serve as the fundamental spaces on which we will build our theory in the chapters to come. In particular this allows us to circumvent the condition of non-quasianalyticity as would be necessarily the case if we were to consider compactly supported ultradifferentiable functions as our foundational space of test functions in view of the Denjoy-Carleman theorem.

In this chapter, and those remaining in Part I, we will work with the notion of ultradifferentiability defined through weight matrices [119], called weight sequence systems in the present text. In particular, as explained in [119], this leads to a unified treatment of classes

of ultradifferentiable functions defined via weight sequences [81] and via weight functions [21]. Whence Section 3.2 is devoted to a general discussion of weight function and sequence systems, where we establish several properties which will be employed throughout this text. Moreover, we further extend the considerations from [119] to multi-indexed weight sequence systems in order to cover the anisotropic case as well.

We introduce the type of Gelfand-Shilov spaces employed throughout this text, denoted by $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$, in Section 3.3. The index $q \in [1, \infty]$ refers to the ultradifferentiability of the smooth test functions with respect to the weight sequence system $\mathfrak{M}$ of their L^q-norm. Aside of establishing some basic topological properties, one of the main results we will obtain is the independence of q under certain conditions for $\mathfrak{M}$ and $\mathscr{W}$, which later we will show is exactly the case when the spaces are nuclear, see Chapter 4.

Finally, in Section 3.4, we consider time-frequency analysis in the framework of the Gelfand-Shilov spaces. The results obtained there will prove to be invaluable tools in several of our proofs to come, see in particular Chapters 4, 5, 6 and 11. In recent times the field of time-frequency analysis has been employed successfully for the study of functions and generalized functions, applicable in the context of regularity analysis but also for the study of intrinsic topological properties of function spaces, see e.g. [4, 42, 44, 59, 67, 84]. A comprehensive overview of the field may be found in the monograph [65]. For our purposes, we will specifically discuss continuity properties of the short-time Fourier transform and Gabor frames.

3.2 Weight function and sequence systems

In this section, we define and study weight sequence systems (introduced in [119] under the name weight matrices) and weight function systems.

3.2.1 Weight function systems

Let X be a topological space. A continuous function $w : X \to \mathbb{R}_+$ is called a *weight function* on X. A *weight function system* on X is a family $\mathscr{W} = \{w^\lambda \mid \lambda \in \mathbb{R}_+\}$ of weight functions w^λ on X such that $w^\lambda(x) \leqslant w^\mu(x)$ for all $x \in X$ and $\mu \leqslant \lambda$. If X is a locally compact topological vector space, we consider the following conditions on a weight function system $\mathscr{W}$:

(wM) $\forall K \Subset X \; \forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ \; \exists C > 0 \; \forall x \in X \; : \; \sup_{y \in K} w^\lambda(x+y) \leqslant Cw^\mu(x)$;

{wM} $\forall K \Subset X \; \forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; \exists C > 0 \; \forall x \in X \; : \; \sup_{y \in K} w^\lambda(x+y) \leqslant Cw^\mu(x)$;

(M) $\forall \lambda \in \mathbb{R}_+ \; \exists \mu, \eta \in \mathbb{R}_+ \; \exists C > 0 \; \forall x, y \in X \; : \; w^\lambda(x + y) \leqslant Cw^\mu(x)w^\eta(y)$;

{M} $\forall \mu, \eta \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; \exists C > 0 \; \forall x, y \in X \; : \; w^\lambda(x + y) \leqslant Cw^\mu(x)w^\eta(y)$.

Clearly, [M] implies [wM]. We say $\mathscr{W}$ is *non-degenerate* if $\inf_{x \in X} w^\lambda(x) > 0$ for any $\lambda \in \mathbb{R}_+$. Moreover, $\mathscr{W}$ is called *symmetric* if $w^\lambda(x) = w^\lambda(-x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in X$.

In the case where $X = \mathbb{R}^d$, we also consider the conditions

(N) $\forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ \; : \; w^\lambda/w^\mu \in L^1(\mathbb{R}^d)$;

{N} $\forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; : \; w^\lambda/w^\mu \in L^1(\mathbb{R}^d)$.

In the sequel, if we do not specify X we always mean $X = \mathbb{R}^d$.

The following result is easy to verify.

Lemma 3.2.1. *Let $\mathscr{W}$ be a weight function system on X satisfying* [wM]. *Then*

$$\forall K \Subset X \; \forall \lambda \in \mathbb{R}_+ \; \exists \lambda' \in \mathbb{R}_+ \; \forall \mu' \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+$$
$$(\forall K \Subset X \; \forall \lambda' \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; \forall \mu \in \mathbb{R}_+ \; \exists \mu' \in \mathbb{R}_+)$$
$$\exists C > 0 \; \forall x \in X \; : \; \sup_{y \in K} \frac{w^\lambda(x + y)}{w^\mu(x + y)} \leqslant C \frac{w^{\lambda'}(x)}{w^{\mu'}(x)}.$$

Two weight function systems $\mathscr{W}$ and $\mathscr{V}$ on X may be compared in the following ways:

$$\mathscr{W} \,(\leqslant)\, \mathscr{V} \quad \Leftrightarrow \quad \forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ \; \exists C > 0 \; \forall x \in X : \omega^\lambda(x) \leqslant C v^\mu(x);$$
$$\mathscr{W} \,\{\leqslant\}\, \mathscr{V} \quad \Leftrightarrow \quad \forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; \exists C > 0 \; \forall x \in X : \omega^\lambda(x) \leqslant C v^\mu(x);$$
$$\mathscr{W} \,[\approx]\, \mathscr{V} \quad \Leftrightarrow \quad \mathscr{W} \,[\leqslant]\, \mathscr{V} \text{ and } \mathscr{V} \,[\leqslant]\, \mathscr{W}.$$

Clearly, if $\mathscr{W}[\approx]\mathscr{V}$ then $\mathscr{W}$ satisfies [wM] ([M] or [N], respectively) if and only if $\mathscr{V}$ does so.

We define the tensor product of a finite number of weight function systems $\mathscr{W}_j = \{w_j^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ on X_j, $j = 1, \ldots, k$, as

$$\mathscr{W}_1 \otimes \cdots \otimes \mathscr{W}_k = \{w_1^\lambda \otimes \cdots \otimes w_k^\lambda \,|\, \lambda \in \mathbb{R}_+\},$$

where $w_1^\lambda \otimes \cdots \otimes w_k^\lambda(x) = w_1^\lambda(x_1) \cdots w_k^\lambda(x_k)$ for $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$. Note that $\mathscr{W}_1 \otimes \cdots \otimes \mathscr{W}_k$ satisfies [wM] (resp. [M]) if and only if each $\mathscr{W}_j$ does so.

We end with some considerations on the condition [N]. The following will be a useful result in the sequel. As is standard, $C_0(\mathbb{R}^d)$ denotes the space of continuous functions vanishing at infinity equipped with the L^∞-norm.

Lemma 3.2.2. *Let $\mathscr{W}$ be a weight function system on $\mathbb{R}^d$ satisfying* [wM] *and* [N]. *Then,*

$$\forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ \; (\forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+) \; : \; w^\lambda / w^\mu \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d).$$

Proof. This is a consequence of Lemma 3.2.1. $\qquad\square$

Given a weight function system $\mathscr{W}$ on $\mathbb{Z}^d$, we associate to it the Köthe set

$$A_{\mathscr{W}} = \{(w^\lambda(j))_{j \in \mathbb{Z}^d} \,|\, \lambda \in \mathbb{R}_+\}.$$

The next result shows that the notion [N] is unambiguous.

Lemma 3.2.3. *Let $\mathscr{W}$ be a weight function system satisfying* [wM]. *Then, $\mathscr{W}$ satisfies* [N] *if and only if $A_{\mathscr{W}}$ satisfies* [N].

Proof. This again follows from Lemma 3.2.1. $\qquad\square$

3.2.2 Weight sequence systems

A *weight sequence system* on $\mathbb{R}^d$ is a family $\mathfrak{M} = \{M^\lambda \mid \lambda \in \mathbb{R}_+\}$ of weight sequences M^λ on $\mathbb{N}^d$ satisfying $(M.1)$ such that $M_\alpha^\mu \leqslant M_\alpha^\lambda$ for all $\alpha \in \mathbb{N}^d$ and $\mu \leqslant \lambda$. We will often work with some of the following conditions on a weight sequence system $\mathfrak{M}$:

(L) $\forall R > 0 \ \forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C_0 > 0 \ \forall \alpha \in \mathbb{N}^d : R^{|\alpha|} M_\alpha^\mu \leqslant C_0 M_\alpha^\lambda;$

{L} $\forall R > 0 \ \forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C_0 > 0 \ \forall \alpha \in \mathbb{N}^d : R^{|\alpha|} M_\alpha^\mu \leqslant C_0 M_\alpha^\lambda;$

$(\mathfrak{M}.2)'$ $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C_0, H > 0 \ \forall \alpha \in \mathbb{N}^d \ \forall j \in \{1, \ldots, d\} : M_{\alpha+e_j}^\mu \leqslant C_0 H^{|\alpha|} M_\alpha^\lambda;$

$\{\mathfrak{M}.2\}'$ $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C_0, H > 0 \ \forall \alpha \in \mathbb{N}^d \ \forall j \in \{1, \ldots, d\} : M_{\alpha+e_j}^\mu \leqslant C_0 H^{|\alpha|} M_\alpha^\lambda;$

$(\mathfrak{M}.2)$ $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C_0, H > 0 \ \forall \alpha, \beta \in \mathbb{N}^d : M_{\alpha+\beta}^\mu \leqslant C_0 H^{|\alpha+\beta|} M_\alpha^\lambda M_\beta^\lambda;$

$\{\mathfrak{M}.2\}$ $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C_0, H > 0 \ \forall \alpha, \beta \in \mathbb{N}^d : M_{\alpha+\beta}^\mu \leqslant C_0 H^{|\alpha+\beta|} M_\alpha^\lambda M_\beta^\lambda.$

Furthermore, $\mathfrak{M}$ is called *accelerating* if $M_{\alpha+e_j}^\mu / M_\alpha^\mu \leqslant M_{\alpha+e_j}^\lambda / M_\alpha^\lambda$ for all $\alpha \in \mathbb{N}^d$, $j \in \{1, \ldots, d\}$, and $\mu \leqslant \lambda$.

Two weight sequence systems $\mathfrak{M}$ and $\mathfrak{N}$ may be compared in the following ways:

$$\mathfrak{M} \ (\leqslant) \ \mathfrak{N} \quad \Leftrightarrow \quad \forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ : M^\mu \leqslant N^\lambda;$$
$$\mathfrak{M} \ \{\leqslant\} \ \mathfrak{N} \quad \Leftrightarrow \quad \forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ : M^\mu \leqslant N^\lambda;$$
$$\mathfrak{M} \ [\approx] \ \mathfrak{N} \quad \Leftrightarrow \quad \mathfrak{M} \ [\leqslant] \ \mathfrak{N} \text{ and } \mathfrak{N} \ [\leqslant] \ \mathfrak{M}.$$

Clearly, if $\mathfrak{M}[\approx]\mathfrak{N}$ then $\mathfrak{M}$ satisfies [L] ($[\mathfrak{M}.2]'$ or $[\mathfrak{M}.2]$, respectively) if and only if $\mathfrak{N}$ does so.

We define the tensor product of a finite number of weight sequence systems $\mathfrak{M}_j = \{M_j^\lambda \mid \lambda \in \mathbb{R}_+\}$ on $\mathbb{N}^{d_j}$, $j = 1, \ldots, k$, as

$$\mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_k = \{M_1^\lambda \otimes \cdots \otimes M_k^\lambda \mid \lambda \in \mathbb{R}_+\}.$$

Clearly, $\mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_k$ satisfies [L] ($[\mathfrak{M}.2]'$ or $[\mathfrak{M}.2]$, respectively) if and only if each $\mathfrak{M}_j$ does so.

A weight sequence system $\mathfrak{M} = \{M^\lambda \mid \lambda \in \mathbb{R}_+\}$ is called *isotropic* if M^λ is isotropic for each $\lambda \in \mathbb{R}_+$. Given a permutation σ of the indices $\{1, \ldots, d\}$, we write $\sigma(\mathfrak{M}) = \{\sigma(M^\lambda) \mid \lambda \in \mathbb{R}_+\}$. We call $\mathfrak{M}$ *isotropically decomposable* if it can be written as a tensor product of isotropic weight sequence systems, that is, if there is a permutation σ such that $\sigma(\mathfrak{M}) = \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_k$ with each $\mathfrak{M}_j$ isotropic.

Given a weight sequence system $\mathfrak{M}$, we associate to it the non-degenerate symmetric weight function system on $\mathbb{R}^d$

$$\mathscr{W}_{\mathfrak{M}} = \{e^{\omega_{M^\lambda}} \mid \lambda \in \mathbb{R}_+\}.$$

If $\mathfrak{M}$ is isotropically decomposable, the conditions on $\mathfrak{M}$ may be characterized by $\mathscr{W}_{\mathfrak{M}}$ as follows.

Lemma 3.2.4. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system.*

(i) *$\mathfrak{M}$ satisfies* [L] *if and only if*

$$\forall R > 0 \ \forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C > 0$$
$$(\forall R > 0 \ \forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C > 0)$$
$$\omega_{M^\lambda}(Rx) \leqslant \omega_{M^\mu}(x) + \log C.$$

(ii) *$\mathfrak{M}$ satisfies* $[\mathfrak{M}.2]'$ *if and only if*

$$\forall N \in \mathbb{N} \ \forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C, H > 0$$
$$(\forall N \in \mathbb{N} \ \forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C, H > 0)$$
$$\omega_{M^\lambda}(x) + N \log |x| \leqslant \omega_{M^\mu}(Hx) + \log C.$$

(iii) *$\mathfrak{M}$ satisfies* $[\mathfrak{M}.2]$ *if and only if*

$$\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+ \ \exists C, H \geqslant 0$$
$$(\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+ \ \exists C, H \geqslant 0)$$
$$2\omega_{M^\lambda}(x) \leqslant \omega_{M^\mu}(Hx) + \log C.$$

Proof. (i) Suppose that $R^{|\alpha|} M_\alpha^\mu \leqslant C_0 M_\alpha^\lambda$ for all $\alpha \in \mathbb{N}^d$. Then

$$\omega_{M^\lambda}(Rx) = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|(Rx)^\alpha| M_0^\lambda}{M_\alpha^\lambda} \leqslant \sup_{\alpha \in \mathbb{N}^d} \log C_0 \frac{|x^\alpha| M_0^\lambda}{M_\alpha^\mu}$$
$$= \omega_{M^\mu}(x) + \log \frac{C_0 M_0^\lambda}{M_0^\mu},$$

whence the first implication. Conversely, suppose the inequality holds, then for any $\alpha \in \mathbb{N}^d$, by (2.4),

$$R^{|\alpha|} M_\alpha^\mu = M_0^\mu \sup_{x \in \mathbb{R}^d} \frac{|(Rx)^\alpha|}{e^{\omega_{M^\mu}(x)}} \leqslant C M_0^\mu \sup_{x \in \mathbb{R}^d} \frac{|(Rx)^\alpha|}{e^{\omega_{M^\lambda}(Rx)}} = \frac{C M_0^\mu}{M_0^\lambda} M_\alpha^\lambda.$$

(ii) Suppose that $M_{\alpha+e_j}^\mu \leqslant C_0 H^{|\alpha|} M_\alpha^\lambda$ for all $\alpha \in \mathbb{N}^d$ and $1 \leqslant j \leqslant d$. Then for any $j \in \{1, \ldots, d\}$:

$$\omega_{M^\lambda}(x) + \log|x_j| = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|x^{\alpha+e_j}| M_0^\lambda}{M_\alpha^\lambda} \leqslant \sup_{\alpha \in \mathbb{N}} \log \frac{C_0}{H} \frac{|(Hx)^{\alpha+e_j}| M_0^\lambda}{M_{\alpha+e_j}^\mu}$$

$$\leqslant \omega_{M^\mu}(Hx) + \log \frac{C_0 M_0^\lambda}{H M_0^\mu}.$$

As $\log|x| \leqslant \max_{1 \leqslant j \leqslant d} \log|x_j| + \log \sqrt{d}$, we find

$$\omega_{M^\lambda}(x) + \log|x| \leqslant \omega_{M^\mu}(Hx) + \log C_1$$

for some $C_1 > 0$. For any $N \in \mathbb{Z}_+$, iterating N-times then gives the result. Next, assume the inequality holds and set $N = 1$. Then for any $\alpha \in \mathbb{N}^d$ and $j \in \{1, \ldots, d\}$, by (2.4),

$$M_{\alpha+e_j}^\mu = M_0^\mu \sup_{x \in \mathbb{R}^d} \frac{|x^{\alpha+e_j}|}{e^{\omega_{M^\mu}(x)}} \leqslant \frac{M_0^\mu}{M_0^\lambda} H^{|\alpha|} M_\alpha^\lambda \sup_{x \in \mathbb{R}^d} |x_j| e^{\omega_{M^\lambda}(H^{-1}x) - \omega_{M^\mu}(x)}$$

$$\leqslant C \frac{M_0^\mu}{M_0^\lambda} H^{|\alpha|} M_\alpha^\lambda.$$

(iii) Suppose that $M_{\alpha+\beta}^\mu \leqslant C_0 H^{|\alpha+\beta|} M_\alpha^\lambda M_\beta^\lambda$ for all $\alpha, \beta \in \mathbb{N}^d$. Then

$$2\omega_{M^\lambda}(x) = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|x^{2\alpha}| (M_0^\lambda)^2}{M_\alpha^\lambda M_\alpha^\lambda} \leqslant \sup_{\alpha \in \mathbb{N}^d} \log C_0 \frac{|(Hx)^{2\alpha}| (M_0^\lambda)^2}{M_{2\alpha}^\mu}$$

$$\leqslant \omega_{M^\mu}(Hx) + \log \frac{C_0 (M_0^\lambda)^2}{M_0^\mu}.$$

If the inequality holds, then for any $\alpha, \beta \in \mathbb{N}^d$, by (2.4),

$$M_{\alpha+\beta}^\mu = M_0^\mu \sup_{x \in \mathbb{R}^d} \frac{|x^{\alpha+\beta}|}{e^{\omega_{M^\mu}(x)}} \leqslant C M_0^\mu \sup_{x \in \mathbb{R}^d} \left(\frac{|x^\alpha|}{e^{\omega_{M^\lambda}(H^{-1}x)}} \right) \left(\frac{|x^\beta|}{e^{\omega_{M^\lambda}(H^{-1}x)}} \right)$$

$$\leqslant \frac{C M_0^\mu}{(M_0^\lambda)^2} H^{|\alpha+\beta|} M_\alpha^\lambda M_\beta^\mu.$$

$\square$

The conditions on $\mathfrak{M}$ and $\mathscr{W}_{\mathfrak{M}}$ are related as follows.

Lemma 3.2.5. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system satisfying* [L].

(a) *$\mathscr{W}_{\mathfrak{M}}$ satisfies* [M].

(b) *Consider the following statements:*

(i) *$\mathfrak{M}$ satisfies* [$\mathfrak{M}.2$]′.

(ii) *$A_{\mathscr{W}_{\mathfrak{M}}}$ satisfies* [N].

(iii) *$\mathscr{W}_{\mathfrak{M}}$ satisfies* [N].

Then, $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. If in addition $\mathfrak{M}$ is accelerating, then $(iii) \Rightarrow (i)$.

Proof. We may assume that $\mathfrak{M}$ is isotropic.
  (a) This follows from (2.5) and Lemma 3.2.4(i).
  (b) $(i) \Rightarrow (ii)$ Is a direct consequence of Lemma 3.2.4(ii).
  $(ii) \Leftrightarrow (iii)$ In view of (a), Lemma 3.2.3 yields the result.
  $(iii) \Rightarrow (i)$ *(if $\mathfrak{M}$ is accelerating)* Let $m^\lambda(t) = \sum_{m_p^\lambda \leqslant t} 1$ for $t \geqslant 0$.
Let η^λ be as in (2.6) and set $w^\lambda(x) = e^{\eta^\lambda(|x|)}$ for $x \in \mathbb{R}^d$. Note that the weight function system $\{w^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ also satisfies [N] because of (2.7) and Lemma 3.2.4(i). It is well-known that [81, Equation (3.11), p. 50]

$$w^\lambda(x) = \exp\left(\int_0^{|x|} \frac{m^\lambda(u)}{u} du\right), \qquad x \in \mathbb{R}^d.$$

Let $\lambda > 0$ $(\mu > 0)$ be arbitrary and choose $\mu > 0$ $(\lambda > 0)$ such that $w^\lambda/w^\mu \in L^1(\mathbb{R}^d)$. In particular, $\mu \leqslant \lambda$. Since $\mathfrak{M}$ is accelerating, we have that $m_p^\mu \leqslant m_p^\lambda$ for all $p \geqslant 1$ and thus $m^\lambda(t) \leqslant m^\mu(t)$ for all $t \geqslant 0$. Hence,

$$\int_{t_1}^{t_2} \frac{m^\lambda(u) - m^\mu(u)}{u} du \leqslant 0$$

for all $t_2 \geqslant t_1 \geqslant 0$, which implies that $w^\lambda(x)/w^\mu(x)$ is non-increasing in $|x|$. Therefore,

$$|y|^d \frac{w^\lambda(y)}{w^\mu(y)} \leqslant \frac{1}{|B(0,1)|} \int_{B(0,|y|)} \frac{w^\lambda(x)}{w^\mu(x)} dx \leqslant \frac{1}{|B(0,1)|} \int_{\mathbb{R}^d} \frac{w^\lambda(x)}{w^\mu(x)} dx < \infty$$

for all $y \in \mathbb{R}^d$. This implies

$$\forall \lambda \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+ \; (\forall \mu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+) \; \exists C', H' > 0 \; \forall t \geqslant 0 \; :$$
$$\eta^\lambda(t) + d \log t \leqslant \eta^\mu(t) + \log C.$$

Then, by iteration and applying (2.7), we infer from Lemma 3.2.4(*i*) and (*ii*) that $\mathfrak{M}$ satisfies [$\mathfrak{M}$.2]′. $\qquad\square$

3.2.3 Two examples

We present two examples of important instances of classes of weight sequence systems and weight function systems, which we will use throughout this text.

First, given a single weight sequence M satisfying $(M.1)$, we set $\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha \in \mathbb{N}^d} \mid \lambda \in \mathbb{R}_+\}$ and $\mathscr{W}_M = \mathscr{W}_{\mathfrak{M}_M} = \{e^{\omega_M(\cdot/\lambda)} \mid \lambda \in \mathbb{R}_+\}$.

Lemma 3.2.6. *Let M be an isotropically decomposable weight sequence satisfying $(M.1)$.*

(a) *$\mathfrak{M}_M$ is accelerating and satisfies* [L].

(b) *$\mathscr{W}_M$ satisfies* [M].

(c) *M satisfies $(M.2)$′ if and only if $\mathfrak{M}_M$ satisfies* [$\mathfrak{M}$.2]′ *if and only if $\mathscr{W}_M$ satisfies* [N].

Proof. Part (*a*) is obvious, while (*b*) and (*c*) have been established in Lemma 3.2.5. $\qquad\square$

As a second example, following [119, Section 5], we can also introduce weight sequence systems and weight function systems generated by a weight function in the sense of [21]. We consider the following conditions on a non-negative non-decreasing continuous function ω on $[0, \infty)$:

(α) $\omega(2t) = O(\omega(t))$;

(γ) $\log t = O(\omega(t))$;

(γ_0) $\log t = o(\omega(t))$;

(δ) $\phi : [0, \infty) \to [0, \infty)$, $\phi(x) = \omega(e^x)$, is convex.

We call ω a *Braun-Meise-Taylor weight function (BMT weight function)* if $\omega_{|[0,1]} \equiv 0$ and ω satisfies (α), (γ_0) and (δ). In such a case, we define the *Young conjugate ϕ^* of ϕ* as

$$\phi^* : [0, \infty) \to [0, \infty), \qquad \phi^*(y) = \sup_{x \geq 0}(xy - \phi(x)).$$

Note that ϕ^* is convex and $y = o(\phi^*(y))$. We define $\mathfrak{M}_\omega = \{M_\omega^\lambda \mid \lambda \in \mathbb{R}_+\}$, where $M_\omega^\lambda = \left(\exp\left(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)\right)\right)_{\alpha \in \mathbb{N}^d}$; the above stated properties of ϕ^* imply that M_ω^λ is an isotropic weight sequence satisfying $(M.1)$. Furthermore, we set $\mathscr{W}_\omega = \{e^{\frac{1}{\lambda}\omega(|\cdot|)} \mid \lambda \in \mathbb{R}^+\}$ (for general ω).

Lemma 3.2.7. *Let ω be a non-negative non-decreasing continuous function on $[0, \infty)$.*

(a) *If ω is a BMT weight function, then $\mathfrak{M}_\omega$ satisfies [L] and [$\mathfrak{M}$.2].*

(b) *If ω satisfies (α), then $\mathscr{W}_\omega$ satisfies [M].*

(c) *ω satisfies (γ) ((γ_0)) if and only if $\mathscr{W}_\omega$ satisfies (N) ({N}).*

Proof. (a) This is shown in [119, Corollary 5.15].
(b) This follows from the fact that ω is non-decreasing.
(c) As ω is non-decreasing, this can be shown by using a similar argument as in the proof of implication $(iii) \Rightarrow (i)$ in Lemma 3.2.5(b). $\square$

3.3 The Gelfand-Shilov spaces

We now introduce the Gelfand-Shilov spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$. Let $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$ be a sequence of positive numbers and let w be a non-negative function on $\mathbb{R}^d$. We define $\mathcal{S}_{w,q}^M = \mathcal{S}_{w,q}^M(\mathbb{R}^d)$, $q \in [1, \infty]$, as the seminormed space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{S}_{w,q}^M} = \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M_\alpha} \left(\int_{\mathbb{R}^d} (|\varphi^{(\alpha)}(x)|w(x))^q dx\right)^{1/q} < \infty, \qquad q \in [1, \infty),$$

and

$$\|\varphi\|_{\mathcal{S}_{w,\infty}^M} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|w(x)}{M_\alpha} < \infty.$$

If w is positive and continuous, then $\mathcal{S}_{w,q}^M$ is a Banach space. Given a weight sequence system $\mathfrak{M}$ and weight function system $\mathscr{W}$, we define the *Gelfand-Shilov spaces (of Beurling and Roumieu type)*

$$\mathcal{S}_{(\mathscr{W}),q}^{(\mathfrak{M})} = \varprojlim_{\lambda \to 0^+} \mathcal{S}_{w^\lambda,q}^{M^\lambda}, \qquad \mathcal{S}_{\{\mathscr{W}\},q}^{\{\mathfrak{M}\}} = \varinjlim_{\lambda \to \infty} \mathcal{S}_{w^\lambda,q}^{M^\lambda}, \qquad q \in [1, \infty].$$

Whenever $\mathfrak{M}_1[\approx]\mathfrak{M}_2$ and $\mathscr{W}_1[\approx]\mathscr{W}_2$, then clearly $\mathcal{S}_{[\mathscr{W}_1],q}^{[\mathfrak{M}_1]} = \mathcal{S}_{[\mathscr{W}_2],q}^{[\mathfrak{M}_2]}$ as locally convex spaces.

Notation 3.3.1. Should $\mathfrak{M}[\approx]\mathfrak{M}_M$ or $\mathscr{W}[\approx]\mathscr{W}_M$ for some weight sequence M we shall simply write $[M]$ instead, i.e. $\mathcal{S}_{[\mathscr{W}],q}^{[M]} = \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}_M]}$ and $\mathcal{S}_{[M],q}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}_M],q}^{[\mathfrak{M}]}$. Similarly, if $\mathfrak{M}[\approx]\mathfrak{M}_\omega$ or $\mathscr{W}[\approx]\mathscr{W}_\omega$ for some BMT weight function ω we will write $[\omega]$ instead, i.e. $\mathcal{S}_{[\mathscr{W}],q}^{[\omega]} = \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}_\omega]}$ and $\mathcal{S}_{[\omega],q}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}_\omega],q}^{[\mathfrak{M}]}$.

Note that $\mathcal{S}_{(\mathscr{W}),q}^{(\mathfrak{M})}$ is a Fréchet space, while $\mathcal{S}_{\{\mathscr{W}\},q}^{\{\mathfrak{M}\}}$ is an (LB)-space. Also, if $\mathscr{W}$ satisfies [wM], then $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ is translation-invariant. In the Romieu case, we may further characterize the topology.

Lemma 3.3.2. *Let $\mathfrak{M}$ be a weight sequence system, let $\mathscr{W}$ be a nondegenerate weight function system and let $q \in [1, \infty]$. Then, the (LB)-space $\mathcal{S}_{\{\mathscr{W}\},q}^{\{\mathfrak{M}\}}$ is regular.*

Proof. By [8, Corollary 7, p. 80], it suffices to show that, for each $\lambda > 0$, the closed unit ball B_λ in $\mathcal{S}_{w^\lambda,q}^{M^\lambda}$ is closed in $\mathcal{S}_{\{\mathscr{W}\},q}^{\{\mathfrak{M}\}}$. Note that $\mathcal{S}_{\{\mathscr{W}\},q}^{\{\mathfrak{M}\}} \subset \mathcal{D}_{L^q}(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ with continuous inclusion; the first inclusion is a consequence of the fact that $1 \leqslant w^\lambda$ for all $\lambda > 0$ and the second one is a classical result of Schwartz [125]. Therefore it is enough to show that B_λ is closed in $\mathcal{B}(\mathbb{R}^d)$. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in B_λ and $\varphi \in \mathcal{B}(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ in $\mathcal{B}(\mathbb{R}^d)$. In particular, $\varphi_n^{(\alpha)}(x) \to \varphi^{(\alpha)}(x)$ for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. Hence, we obtain that

$$\|\varphi^{(\alpha)}\|_{L^q} \leqslant \liminf_{n \to \infty} \|\varphi_n^{(\alpha)}\|_{L^q} \leqslant M_\alpha^\lambda$$

for all $\alpha \in \mathbb{N}^d$, where we have used Fatou's lemma for $q < \infty$. This shows that $\varphi \in B_\lambda$ and the proof is complete. $\qquad\square$

We now discuss the role of q. We first consider how the space may possibly become enlarged as q increases.

Lemma 3.3.3. *Let* $\mathfrak{M}$ *be a weight sequence system satisfying* [L] *and* [$\mathfrak{M}.2$]′ *and* $\mathscr{W}$ *be a weight function system satisfying* [wM]. *Then, for any* $1 \leqslant q \leqslant r \leqslant \infty$, *we have the following continuous inclusions*

$$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],1} \subseteq \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \subseteq \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],r} \subseteq \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}.$$

Proof. Since $\|f\|_{L^r} \leqslant \|f\|_{L^\infty}^{(r-q)/r} \|f\|_{L^q}^{q/r}$ for all $f \in L^\infty(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, it is enough to consider the case $r = \infty$. Denote by H the characteristic function of the orthonant $[0, \infty)^d$ and let $e = (1, \dots, 1)$. Then $\partial^e H = \delta$. Choose $\psi \in \mathcal{D}_{[-1/2,1/2]^d}$ such that $\psi \equiv 1$ on a neighborhood of 0. Then, $\partial^e(H\psi) - \delta = \chi \in L^\infty(\mathbb{R}^d)$ has support in $[-1/2, 1/2]^d$. Hence, $\varphi = (\partial^e \varphi) * (H\psi) - \varphi * \chi$ for all $\varphi \in C^\infty(\mathbb{R}^d)$. By [wM], [$\mathfrak{M}.2$]′ and [L], we find that for each $\lambda > 0$ there are $\mu > 0$ and $C, C' > 0$ (for each $\mu > 0$ there are $\lambda > 0$ and $C, C' > 0$) such that $w^\lambda(x + t) \leqslant C w^\mu(x)$ for all $x \in \mathbb{R}^d$ and $t \in [-1/2, 1/2]^d$ and $M^\mu_{\alpha+e} \leqslant C' M^\lambda_\alpha$ for all $\alpha \in \mathbb{N}^d$. We may assume that $\mu \leqslant \lambda$. Hence, by Jensen's inequality,

$$
\begin{aligned}
\|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda,\infty}} &= \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)| w^\lambda(x)}{M^\lambda_\alpha} \\
&\leqslant C \|\psi\|_{L^\infty} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{1}{M^\lambda_\alpha} \int_{x+[-1/2,1/2]^d} |\partial^e \varphi^{(\alpha)}(t)| w^\mu(t) dt \\
&\quad + C \|\chi\|_{L^\infty} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{1}{M^\lambda_\alpha} \int_{x+[-1/2,1/2]^d} |\varphi^{(\alpha)}(t)| w^\mu(t) dt \\
&\leqslant CC' \|\psi\|_{L^\infty} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{1}{M^\mu_{\alpha+e}} \left(\int_{\mathbb{R}^d} (|\varphi^{(\alpha+e)}(t)| w^\mu(t))^q dt \right)^{1/q} \\
&\quad + C \|\chi\|_{L^\infty} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{1}{M^\mu_\alpha} \left(\int_{\mathbb{R}^d} (|\varphi^{(\alpha)}(t)| w^\mu(t))^q dt \right)^{1/q} \\
&\leqslant C'' \|\varphi\|_{\mathcal{S}^{M^\mu}_{w^\mu,q}},
\end{aligned}
$$

for any $\varphi \in \mathcal{S}^{M^\mu}_{w^\mu,q}$, where $C'' = C(C' \|\psi\|_{L^\infty} + \|\chi\|_{L^\infty})$. $\qquad\square$

Next, we examine when the definition of the Gelfand-Shilov spaces are q-invariant. Of course, such considerations are only relevant if the spaces are non-trivial. A direct consequence of Lemma 3.3.3 is that every $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ is non-trivial, $q \in [1, \infty]$, should $\mathcal{S}_{[\mathscr{W}],1}^{[\mathfrak{M}]} \neq \{0\}$. However, we will now show that it suffices to prove non-triviality for *some* $q \in [1, \infty]$ in order to conclude non-triviality for *all* $q \in [1, \infty]$. To this purpose, given a weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$, we introduce the auxiliary spaces

$$\widetilde{\mathcal{S}}_{(\mathscr{W})}^{(\mathfrak{M})} = \bigcap_{\lambda > 0} \bigcap_{k \in \mathbb{N}} \mathcal{S}_{(1+|\cdot|)^k w^\lambda, \infty}^{M^\lambda}, \qquad \widetilde{\mathcal{S}}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}} = \bigcup_{\lambda > 0} \bigcap_{k \in \mathbb{N}} \mathcal{S}_{(1+|\cdot|)^k w^\lambda, \infty}^{M^\lambda}.$$

Lemma 3.3.4. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and let $\mathscr{W}$ be a weight function system satisfying* [wM]. *The following statements are equivalent:*

(i) $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \neq \{0\}$ *for all $q \in [1, \infty]$.*

(ii) $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \neq \{0\}$ *for some $q \in [1, \infty]$.*

(iii) $\widetilde{\mathcal{S}}_{[\mathscr{W}]}^{[\mathfrak{M}]} \neq \{0\}$.

Proof. $(i) \Rightarrow (ii)$ Trivial.

$(ii) \Rightarrow (iii)$ Let $\varphi \in \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ be such that $\varphi(0) = 1$. Choose $\psi \in \mathcal{D}(\mathbb{R}^d)$ so that $\int_{\mathbb{R}^d} \varphi(x)\psi(-x)dx = 1$. Next, pick $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \chi(x)dx = 1$. Then, $\varphi_0 = (\varphi * \psi)\widehat{\chi} \in \widetilde{\mathcal{S}}_{[\mathscr{W}]}^{[\mathfrak{M}]}$ and $\varphi_0 \not\equiv 0$ (as $\varphi_0(0) = 1$).

$(iii) \Rightarrow (i)$ This follows from the fact that $\widetilde{\mathcal{S}}_{[\mathscr{W}]}^{[\mathfrak{M}]} \subseteq \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ for all $q \in [1, \infty]$. $\qquad \square$

Remark 3.3.5. The non-triviality of Gelfand-Shilov spaces is an interesting problem in and of itself. For weight sequences, a classical result by Gelfand and Shilov states that $\mathcal{S}_{[p!^\tau],q}^{[p!^\sigma]}$ is non-trivial if $\sigma + \tau > 1$ (if and only if $\sigma + \tau \geqslant 1$) [61, p. 235]. Other non-triviality conditions can be found in [41].

We will now prove that the spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ coincide if and only if $\mathscr{W}$ satisfies [N]. Moreover, in Chapter 4 we shall see that in the non-degenerate case this is exactly the case whenever $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ is nuclear. The sufficiency of [N] is an easy result.

Lemma 3.3.6. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system satisfying* [wM] *and* [N]. *Then,* $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \subseteq \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],r}$ *with continuous inclusion for all $q, r \in [1, \infty]$ with $q \geqslant r$.*

Proof. This follows from Hölder's inequality and Lemma 3.2.2. $\square$

Next, we establish an important connection between the spaces $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ and $\lambda^q[A_{\mathscr{W}}]$.

Proposition 3.3.7. *Let $\mathfrak{M}$ be a weight sequence system, let $\mathscr{W}$ be a weight function system satisfying* [wM], *and let $q \in [1, \infty]$. The mapping*

$$S_q = S : \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \to \lambda^q[A_{\mathscr{W}}], \qquad S(\varphi) = (\varphi(j))_{j \in \mathbb{Z}^d},$$

is continuous.

Proof. For $q = \infty$ this is obvious. Assume now that $q < \infty$. We again denote by H the characteristic function of the orthonant $[0, \infty)^d$ and $e = (1, \ldots, 1)$. As in the proof of Lemma 3.3.3 there exist $\psi \in \mathcal{D}_{[-1/2,1/2]^d}$ and $\chi \in L^\infty(\mathbb{R}^d)$ with support in $[-1/2, 1/2]^d$ such that $\varphi = (\partial^e \varphi) * (H\psi) - \varphi * \chi$ for all $\varphi \in C^\infty(\mathbb{R}^d)$. For each $\lambda > 0$ there are $\mu > 0$ and $C > 0$ (for each $\mu > 0$ there are $\lambda > 0$ and $C > 0$) such that $w^\lambda(x + t) \leqslant Cw^\mu(x)$ for all $x \in \mathbb{R}^d$ and $t \in [-\frac{1}{2}, \frac{1}{2}]^d$. We obtain that

$$|\varphi(x)w^\lambda(x)|$$

$$\leqslant C \left(\|\psi\|_{L^\infty} \int_{x+[-\frac{1}{2},\frac{1}{2}]^d} |\partial^e \varphi(t)|w^\mu(t)dt \right.$$

$$\left. + \|\chi\|_{L^\infty} \int_{x+[-\frac{1}{2},\frac{1}{2}]^d} |\varphi(t)|w^\mu(t)dt \right)$$

for all $x \in \mathbb{R}^d$ and $\varphi \in C^\infty(\mathbb{R}^d)$. By Jensen's inequality, the latter inequality implies that

$$\left\| (\varphi(j)w^\lambda(j))_{j \in \mathbb{Z}^d} \right\|_{l^q} \leqslant C(\|\psi\|_{L^\infty} \|\varphi^{(e)} w^\mu\|_{L^q} + \|\chi\|_{L^\infty} \|\varphi w^\mu\|_{L^q})$$

for all $\varphi \in \mathcal{S}^{M^\mu}_{w^\mu, q}$, from which the result follows. $\square$

Proposition 3.3.8. *Let $\mathfrak{M}$ be a weight sequence system, let $\mathscr{W}$ be a weight function system satisfying* [M], *and let $q \in [1, \infty]$. For each $\psi \in \widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$, the mapping*

$$T_{\psi,q} = T_\psi = T : \lambda^q[A_\mathscr{W}] \to \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}, \qquad T((c_j)_{j \in \mathbb{Z}^d}) = \sum_{j \in \mathbb{Z}^d} c_j \psi(\cdot - j),$$

is continuous.

Proof. We only show the result for $q \in (1, \infty)$; the proofs for $q = 1$ and $q = \infty$ are similar and in fact less involved. Let $\nu > 0$ be such that $\psi \in \bigcap_{k \in \mathbb{N}} \mathcal{S}^{M^\nu}_{(1 + |\cdot|)^k w^\nu, \infty}$; this means that ν is fixed in the Roumieu case but can be taken as large as needed in the Beurling case. For each $\lambda > 0$ there are $\mu, \nu > 0$ and $C > 0$ (for each $\mu, \nu > 0$ there are $\lambda > 0$ and $C > 0$) such that $w^\lambda(x + y) \leqslant C w^\mu(x) w^\nu(y)$ for all $x, y \in \mathbb{R}^d$. We may assume that $\nu \leqslant \lambda$. Let $q' = q/(q - 1)$ be the conjugate exponent of q. By Hölder's inequality, we have that, for all $(c_j)_{j \in \mathbb{Z}^d} \in l^q((w^\mu(j))_{j \in \mathbb{Z}^d})$,

$$\sum_{j \in \mathbb{Z}^d} |c_j| |\psi^{(\alpha)}(x - j)| w^\lambda(x)$$

$$\leqslant C \sum_{j \in \mathbb{Z}^d} \frac{|c_j| w^\mu(j)}{(1 + |x - j|)^{(d+1)/q}} |\psi^{(\alpha)}(x - j)| w^\nu(x - j)(1 + |x - j|)^{(d+1)/q}$$

$$\leqslant C \left(\sum_{j \in \mathbb{Z}^d} \frac{(|c_j| w^\mu(j))^q}{(1 + |x - j|)^{d+1}} \right)^{1/q} \times$$

$$\left(\sum_{j \in \mathbb{Z}^d} \left(|\psi^{(\alpha)}(x - j)| w^\nu(x - j)(1 + |x - j|)^{(d+1)/q} \right)^{q'} \right)^{1/q'}$$

$$\leqslant C' \left\| \psi^{(\alpha)} (1 + |\cdot|)^{d+1} w^\nu \right\|_{L^\infty} \left(\sum_{j \in \mathbb{Z}^d} \frac{(|c_j| w^\mu(j))^q}{(1 + |x - j|)^{d+1}} \right)^{1/q}$$

for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, where $C' = 2^{\frac{d+1}{q'}} C \left(\sum_{j \in \mathbb{Z}^d} (1 + |j|)^{-d-1} \right)^{1/q'}$.

Hence,

$$\| \sum_{j\in\mathbb{Z}^d} c_j\psi(\,\cdot\,-j)\|_{\mathcal{S}^{M^\lambda}_{w^\lambda,q}} \leqslant \sup_{\alpha\in\mathbb{N}^d} \frac{1}{M^\nu_\alpha}\| \sum_{j\in\mathbb{Z}^d} c_j\psi^{(\alpha)}(\,\cdot\,-j)w^\lambda\|_{L^q}$$
$$\leqslant C'' \,\|\psi\|_{\mathcal{S}^{M^\nu}_{(1+|\,\cdot\,|)^{d+1}w^\nu,\infty}} \,\big\|(c_jw^\mu(j))_{j\in\mathbb{Z}^d}\big\|_{l^q}\,,$$

with $C'' = C'\left(\int_{x\in\mathbb{R}^d}(1+|x|)^{-d-1}dx\right)^{1/q}$. $\qquad\qquad\square$

Lemma 3.3.9. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and let $\mathscr{W}$ be a weight function system satisfying* [wM]. *Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1,\infty]$. Then there exists a $\psi \in \widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ such that $\psi(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$.*

Proof. By Lemma 3.3.4, there exist $\varphi \in \widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ such that $\varphi(0) = 1$. Set

$$\chi(\xi) = \mathcal{F}(1_{[-\frac12,\frac12]^d})(\xi) = \int_{[-\frac12,\frac12]^d} e^{-2\pi i\xi\cdot x}dx, \qquad \xi \in \mathbb{R}^d.$$

Then, $\chi(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Hence, $\psi = \varphi\chi$ satisfies all requirements. $\qquad\square$

We obtain the following useful corollary.

Corollary 3.3.10. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L]*, let $\mathscr{W}$ be a weight function system satisfying* [M]*, and let $q \in [1,\infty]$. Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$. Then, $\lambda^q[A_{\mathscr{W}}]$ is isomorphic to a complemented subspace of $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$.*

Proof. Choose ψ as in Lemma 3.3.9. Consider the continuous linear mappings $S : \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \to \lambda^q[A_{\mathscr{W}}]$ and $T_\psi : \lambda^q[A_{\mathscr{W}}] \to \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ from Proposition 3.3.7 and Proposition 3.3.8, respectively, and note that $S \circ T_\psi = \mathrm{id}_{\lambda^q[A_{\mathscr{W}}]}$. $\qquad\square$

We are now ready to characterize exactly when the Gelfand-Shilov spaces coincide.

Theorem 3.3.11. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and* [$\mathfrak{M}.2$]′ *and let $\mathscr{W}$ be a weight function system satisfying* [wM]. *Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1,\infty]$. Consider the following statements:*

(i) $\mathscr{W}$ satisfies [N].

(ii) $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}],r}^{[\mathfrak{M}]}$ as locally convex spaces for all $q, r \in [1, \infty]$.

(iii) $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}],r}^{[\mathfrak{M}]}$ as sets for some $q, r \in [1, \infty]$ with $q \neq r$.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. If in addition $\mathscr{W}$ satisfies [M], then also $(iii) \Rightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$ Follows by Lemma 3.3.3 and 3.3.6.

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (i)$ *(if $\mathscr{W}$ satisfies [M])* Suppose that $q < r$. Choose ψ as in Lemma 3.3.9. Consider the mappings $S_q : \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \to \lambda^q[A_{\mathscr{W}}]$ and $T_{\psi,r} : \lambda^r[A_{\mathscr{W}}] \to \mathcal{S}_{[\mathscr{W}],r}^{[\mathfrak{M}]}$ from Proposition 3.3.7 and Proposition 3.3.8, respectively. Note that $c = S_q(T_{\psi,r}(c)) \in \lambda^q[A_{\mathscr{W}}]$ for all $c \in \lambda^r[A_{\mathscr{W}}]$, that is, $\lambda^r[A_{\mathscr{W}}] \subseteq \lambda^q[A_{\mathscr{W}}]$. Since $\lambda^q[A_{\mathscr{W}}] \subseteq \lambda^r[A_{\mathscr{W}}]$ always holds true, we have that $\lambda^r[A_{\mathscr{W}}] = \lambda^q[A_{\mathscr{W}}]$ as sets. The result now follows from Proposition 2.3.1 and Lemma 3.2.3. $\qquad\square$

Notation 3.3.12. In the sequel, we shall often drop the index q in the notation $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ if $\mathfrak{M}$ is a weight sequence system satisfying [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}$ is a weight function system satisfying [wM] and [N]. This is justified by Theorem 3.3.11.

3.4 Time-frequency analysis

We now consider time-frequency analysis in the framework of Gelfand-Shilov spaces. In particular, we will be interested in continuity results for the short-time Fourier transform and Gabor frames. For a general overview of the research area we refer to [65].

3.4.1 The short-time Fourier transform

The *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window $\psi \in L^2(\mathbb{R}^d)$ is defined as

$$V_\psi f(x, \xi) = (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t)\overline{\psi(t-x)}e^{-2\pi i \xi \cdot t}dt, \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Now $\|V_\psi f\|_{L^2} = \|\psi\|_{L^2} \|f\|_{L^2}$, so that in particular $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous. The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int \int_{\mathbb{R}^{2d}} F(x, \xi) M_\xi T_x \psi dx d\xi, \qquad F \in L^2(\mathbb{R}^{2d}).$$

If $\psi \neq 0$ then $\gamma \in L^2(\mathbb{R}^d)$ is called a *synthesis window* for ψ if $(\gamma, \psi)_{L^2} \neq 0$. One shows [65, Corollary 3.2.3, p. 44]

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{L^2(\mathbb{R}^d)} . \tag{3.1}$$

We now wish to consider the continuity properties of the (adjoint) STFT on the spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$. To do this, we will need adequate decay properties on the window ψ. To this purpose, we introduce the notion of admissibility for weight function systems. Let $\mathscr{W}$ and $\mathscr{V}$ be two weight function systems, then $\mathscr{W}$ is said to be $(\mathscr{V})$-admissible if

$$\forall \lambda \in \mathbb{R}_+ \; \exists \mu, \nu \in \mathbb{R}_+ \; \exists C > 0 \; \forall x, y \in \mathbb{R}^d : w^\lambda(x + y) \leqslant C w^\mu(x) v^\nu(y),$$

while $\mathscr{W}$ is said to be $\{\mathscr{V}\}$-admissible if

$$\forall \mu, \nu \in \mathbb{R}_+ \; \exists \lambda \in \mathbb{R}_+ \; \exists C > 0 \; \forall x, y \in \mathbb{R}^d : w^\lambda(x + y) \leqslant C w^\mu(x) v^\nu(y).$$

Note that in particular $\mathscr{W}$ is $[\mathscr{W}]$-admissible if and only if $\mathscr{W}$ satisfies [M]. Also, if $\mathscr{W}$ is $[\mathscr{V}]$-admissible, then $\mathscr{W}$ satisfies [wM]. Finally, if $\mathscr{W}$ is non-degenerate then so is $\mathscr{V}$.

We study the continuity of the (adjoint) STFT on $\mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}$. As we are interested in preserving the reconstruction formula (3.1), we need an adequate space on which the STFT maps into. For a weight function w we denote by $C_w(\mathbb{R}^d)$ the Banach space of all continuous functions $\varphi \in C(\mathbb{R}^d)$ such that $\|\varphi \cdot w\|_{L^\infty} < \infty$. Then for a weight function system $\mathscr{W}$ we consider the spaces

$$C_{(\mathscr{W})}(\mathbb{R}^d) = \varprojlim_{\lambda \to 0^+} C_{w^\lambda}(\mathbb{R}^d), \qquad C_{\{\mathscr{W}\}}(\mathbb{R}^d) = \varinjlim_{\lambda \to \infty} C_{w^\lambda}(\mathbb{R}^d).$$

We find the following result.

Proposition 3.4.1. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system satisfying $[\mathrm{L}]$ and $[\mathfrak{M}.2]'$ and let $\mathscr{W}$ be a weight function system which is $[\mathscr{V}]$-admissible for some weight function system $\mathscr{V}$. For any $\psi \in \mathcal{S}_{[\mathscr{V}],1}^{[\mathfrak{M}]}$, the mappings*

$$V_\psi : \mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]} \to C_{[\mathscr{W} \otimes \mathscr{w}_{\mathfrak{M}}]}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$$

and

$$V_\psi^* : C_{[\mathscr{W} \otimes \mathscr{w}_{\mathfrak{M}}]}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d) \to \mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}$$

are well-defined and continuous. Moreover, when $\mathscr{V}$ is symmetric, if $\psi \in \widetilde{\mathcal{S}}_{[\mathscr{V}]}^{[\mathfrak{M}]} \neq 0$ and $\gamma \in \widetilde{\mathcal{S}}_{[\mathscr{V}]}^{[\mathfrak{M}]}$ is a synthesis window for ψ, then the reconstruction formula

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}} \tag{3.2}$$

holds.

Proof. Throughout the proof we have that $\psi \in \mathcal{S}_{v^{\mu_3},1}^{M^{\lambda_3}}$ for any (resp. for some) $\lambda_3, \mu_3 \in \mathbb{R}_+$.

We first show the continuity of V_ψ. It suffices to show

$$\forall \lambda_1, \mu_1 \in \mathbb{R}_+ \; \exists \lambda_2, \mu_2 \in \mathbb{R}_+ \; (\forall \lambda_2, \mu_2 \in \mathbb{R}_+ \; \exists \lambda_1, \mu_1 \in \mathbb{R}_+) :$$

$V_\psi : \mathcal{S}_{w^{\mu_2}}^{M^{\lambda_2}} \to C_{w^{\mu_1} \otimes \exp \omega_{M^{\lambda_1}}}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ is well-defined and continuous.

For any $\lambda_1 \in \mathbb{R}_+$ we choose $\lambda_1 = \lambda_2 = \lambda_3$ (for every $\lambda_2 \in \mathbb{R}_+$ we choose $\lambda_1 = \max(\lambda_2, \lambda_3)$). Next, for any $\mu_1 \in \mathbb{R}_+$ there exist $\mu_2, \mu_3 \in \mathbb{R}_+$ (for any $\mu_2 \in \mathbb{R}_+$ and fixed μ_3 there exists a $\mu_1 \in \mathbb{R}_+$) such that $w^{\mu_1}(x+y) \leqslant C_1 w^{\mu_2}(x) v^{\mu_3}(y)$ for some $C_1 > 0$ and any $x, y \in \mathbb{R}^d$. We see that for $\varphi \in \mathcal{S}_{w^{\mu_2},\infty}^{M^{\lambda_2}}$ and $\gamma \in \mathbb{N}^d$

$$w^{\mu_1}(x) \left| \frac{\xi^\gamma}{M_\gamma^{\lambda_1}} V_\psi \varphi(x, \xi) \right|$$

$$\leqslant C_1 M_0^{\lambda_1} (2\pi)^{-|\gamma|}$$

$$\sum_{\gamma' \leqslant \gamma} \binom{\gamma}{\gamma'} \int_{\mathbb{R}^d} \frac{|\varphi^{(\gamma')}(t)| w^{\mu_2}(t)}{M_{\gamma'}^{\lambda_2}} \cdot \frac{|\psi^{(\gamma-\gamma')}(x-t)| v^{\mu_3}(x-t)}{M_{\gamma-\gamma'}^{\lambda_3}} dt$$

$$\leqslant C_1 M_0^{\lambda_1} \|\varphi\|_{\mathcal{S}_{w^{\mu_2},\infty}^{M^{\lambda_2}}} \|\psi\|_{\mathcal{S}_{v^{\mu_3},1}^{M^{\lambda_3}}} \, .$$

It follows that

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} e^{\omega_{M^{\lambda_1}}(\xi)} w^{\mu_1}(x)|V_\psi\varphi(x,y)| \leqslant C \left\|\varphi\right\|_{\mathcal{S}^{M^{\lambda_2}}_{w^{\mu_2},\infty}}$$

for $C = C_1(M_0^{\lambda_1})^2 \left\|\psi\right\|_{\mathcal{S}^{M^{\lambda_3}}_{v^{\mu_3},1}}$. The continuity of V_ψ now follows.

Next we treat the continuity of the adjoint mapping V_ψ^*. Here, it suffices to show

$$\forall\lambda_1,\mu_1\in\mathbb{R}_+ \ \exists\lambda_2,\mu_2\in\mathbb{R}_+ \ (\forall\lambda_2,\mu_2\in\mathbb{R}_+ \ \exists\lambda_1,\mu_1\in\mathbb{R}_+):$$

$V_\psi^*: C_{w^{\mu_2}\otimes\exp\omega_{M^{\lambda_2}}}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)\to\mathcal{S}^{M^{\lambda_1}}_{w^{\mu_1}}$ is well-defined and continuous.

By Lemma 3.2.4(i) and (ii), for any $\lambda_1\in\mathbb{R}_+$ there exists a $\lambda_2\leqslant\lambda_1$ (for any $\lambda_2\in\mathbb{R}_+$ there exists a $\lambda_1\geqslant\lambda_2$) such that $\exp[\omega_{M^{\lambda_1}}(4\pi\cdot)-\omega_{M^{\lambda_2}}(\cdot)]\in L^1(\mathbb{R}^d)$. Additionally, by [L] there is some $\lambda_3\in\mathbb{R}_+$ (we may possibly enlarge λ_1) such that $2^{|\alpha|}M^{\lambda_3}\leqslant C_0 M_\alpha^{\lambda_1}$ for all $\alpha\in\mathbb{N}^d$ and some $C_0>0$. Again, for any $\mu_1\in\mathbb{R}_+$ there exist $\mu_2,\mu_3\in\mathbb{R}_+$ (for any $\mu_2\in\mathbb{R}_+$ and fixed μ_3 there exists a $\mu_1\in\mathbb{R}_+$) such that $w^{\mu_1}(x+y)\leqslant C_1 w^{\mu_2}(x)v^{\mu_3}(y)$ for some $C_1>0$ and any $x,y\in\mathbb{R}^d$. For $\Phi\in C_{w^{\mu_2}\otimes\exp[\omega_{M^{\lambda_2}}]}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)$ and $\alpha\in\mathbb{N}^d$ we have,

$$w^{\lambda_1}(t)\frac{|\partial^\alpha V_\psi^*\Phi(t)|}{M_\alpha^{\lambda_1}} \leqslant C_1 M_0^{\lambda_1}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}$$

$$\int\int_{\mathbb{R}^{2d}}\frac{|\Phi(x,\xi)||2\pi\xi|^{|\beta|}w^{\mu_2}(x)}{M_\beta^{\lambda_2}}\frac{|\psi^{(\alpha-\beta)}(t-x)|v^{\mu_3}(t-x)}{M_{\alpha-\beta}^{\lambda_1}}dxd\xi$$

$$\leqslant C_0 C_1\frac{M_0^{\lambda_1}}{M_0^{\lambda_2}}\left\|\psi\right\|_{\mathcal{S}^{M^{\lambda_3}}_{v^{\mu_3},1}}\left\|\Phi\right\|_{C_{w^{\mu_2}\otimes\exp[\omega_{M^{\lambda_2}}]}}$$

$$2^{-|\alpha|}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^d}e^{\omega_{M^{\lambda_1}}(4\pi\xi)-\omega_{M^{\lambda_2}}(\xi)}d\xi$$

$$\leqslant C\left\|\Phi\right\|_{C_{w^{\mu_2}\otimes e[\omega_{M^{\lambda_2}}]}},$$

whence the continuity of V_ψ^*.

Finally, suppose $\mathcal{V}$ is symmetric, $\psi\in\widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathcal{V}]}\neq 0$ and $\gamma\in\widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathcal{V}]}$ is a synthesis window for ψ. For any $\varphi\in\mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}],\infty}$ we have $V_\psi\varphi(x,\xi)=\mathcal{F}_t(\varphi T_x\overline\psi)(\xi)$. As $\mathcal{W}$ is $[\mathcal{V}]$-admissible and $\mathcal{V}$ is symmetric, it is clear

that $\varphi T_x\overline{\psi} \in L^1(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, whence $\varphi T_x\overline{\psi} = \mathcal{F}^{-1}(V_\psi\varphi(x,\cdot))$. By our previous calculations, it then follows from Fubini's Theorem that

$$
\begin{aligned}
\varphi(t) &= \frac{1}{(\gamma,\psi)_{L^2}} \int_{\mathbb{R}^d} (\varphi(t)T_x\overline{\psi}(t))T_x\gamma(t)dx \\
&= \frac{1}{(\gamma,\psi)_{L^2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} V_\psi\varphi(x,\xi)e^{2\pi i\xi\cdot t}d\xi \right) T_x\gamma(t)dx \\
&= \frac{1}{(\gamma,\psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi\varphi(x,\xi)M_\xi T_x\gamma(t)dxd\xi.
\end{aligned}
$$

$\square$

We may now impose the following conditions on $\mathfrak{M}$ and $\mathscr{W}$:

Assumption 3.4.2. $\mathfrak{M}$ *is isotropically decomposable satisfying* [L] *and* $[\mathfrak{M}.2]'$ *and* $\mathscr{W}$ *is* $[\mathscr{V}]$-*admissible for some symmetric weight function system* $\mathscr{V}$ *such that* $\widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{V}]} \neq \{0\}$.

Corollary 3.4.3. *Let* $\mathfrak{M}$ *be a weight sequence system and* $\mathscr{W}$ *be a weight function system for which Assumption 3.4.2 holds. Then* $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}$ *is isomorphic to a complemented subspace of* $C_{[\mathscr{W}\otimes\mathscr{W}_\mathfrak{M}]}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$.

3.4.2   Gabor frames

We start with a brief discussion on the theory of Gabor frames in $L^2(\mathbb{R}^d)$. See [65, Chaps. 5-8] for a complete account on the subject. Given a non-zero window function $\psi \in L^2(\mathbb{R}^d)$ and lattice parameters $a, b > 0$, the set of times frequency shifts

$$
G(\psi, a, b) = \{M_{bn}T_{ak}\psi : k, n \in \mathbb{Z}^d\}
$$

is called a *Gabor frame* for $L^2(\mathbb{R}^d)$ if there exist $A, B > 0$ (frame bounds) such that

$$
A\|f\|^2_{L^2(\mathbb{R}^d)} \leqslant \sum_{k,n\in\mathbb{Z}^d} |V_\psi f(ak,bn)|^2 \leqslant B\|f\|^2_{L^2(\mathbb{R}^d)}, \qquad (3.3)
$$

for all $f \in L^2(\mathbb{R}^d)$. The Gabor frame operator

$$Sf = S^{a,b}_{\psi,\psi} f = \sum_{k,n \in \mathbb{Z}^d} V_\psi f(ak, bn) M_{bn} T_{ak} \psi$$

is then bounded, positive and invertible on $L^2(\mathbb{R}^d)$. Additionally, we consider the *Gabor coefficient operator* and *Gabor synthesis operator* for a window $\psi \in L^2(\mathbb{R}^d)$ and lattice parameters $a, b > 0$

$$C^{a,b}_\psi : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d}) : \quad f \mapsto (V_\psi f(ak, bn))_{(k,n) \in \mathbb{Z}^{2d}}, \quad (3.4)$$

$$D^{a,b}_\psi : \ell^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d) : \quad (\{c_{k,n}\}) \mapsto \sum_{(k,n) \in \mathbb{Z}^{2d}} c_{k,n} M_{bn} T_{ak} \psi, \quad (3.5)$$

or in any space where the mappings make sense. In particular, we have

$$S = D^{a,b}_\psi \circ C^{a,b}_\psi.$$

The *canonical dual frame* of $G(\psi, a, b)$ is the Gabor frame $G(\gamma^\circ, a, b)$ where the canonical dual window is given by $\gamma^\circ = S^{-1}\psi \in L^2(\mathbb{R}^d)$. Every $f \in L^2(\mathbb{R}^d)$ then possesses the Gabor frame series expansion

$$f = \sum_{k,n \in \mathbb{Z}^d} V_\psi f(ak, bn) M_{bn} T_{ak} \gamma^\circ = \sum_{k,n \mathbb{Z}^d} V_{\gamma^\circ} f(ak, bn) M_{bn} T_{ak} \psi \quad (3.6)$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. The choice of γ° for the validity of (3.6) is however not unique. Any $\gamma \in L^2(\mathbb{R}^d)$ is called a dual window for $\mathcal{G}(\psi, a, b)$ if

$$S_{\psi,\gamma} = D^{a,b}_\gamma \circ C^{a,b}_\psi = \mathrm{id}_{L^2(\mathbb{R}^d)}.$$

The duals of a window ψ are characterized via the Wexler-Raz biorthogonality relations.

Lemma 3.4.4 ([65, Theorem 7.3.1, p. 133]). *Let $a, b > 0$. For $\psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$ the following statements are equivalent:*

(i) $\mathcal{G}(\psi, a, b)$ *is a Gabor frame and γ is a dual window for ψ;*

(ii) $(\psi, M_{\frac{l}{a}} T_{\frac{k}{b}} \gamma)_{L^2} = (ab)^d \delta_{k,0} \delta_{l,0}$ *for all $k, l \in \mathbb{Z}^d$.*

For any lattice parameters $a, b > 0$ we consider the Köthe set

$$A_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}}^{a,b} = \{w^\lambda(ak)e^{\omega_{M^\lambda}(bn)} \mid \lambda \in \mathbb{R}_+\}.$$

We may now describe the continuity of the Gabor coefficient operator and Gabor synthesis operator on the Gelfand-Shilov spaces as follows.

Proposition 3.4.5. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system for which Assumption 3.4.2 holds for some weight function system $\mathscr{V}$ and $\mathscr{W}$ satisfies [N]. For any $\psi \in \mathcal{S}_{[\mathscr{V}],1}^{[\mathfrak{M}]} \cap \mathcal{S}_{[\mathscr{V}],\infty}^{[\mathfrak{M}]}$, the mappings*

$$C_\psi^{a,b} : \mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]} \to \lambda^\infty[A_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}}^{a,b}]$$

and

$$D_\psi^{a,b} : \lambda^\infty[A_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}}^{a,b}] \to \mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}.$$

are well-defined and continuous. Moreover, the series (3.5) is absolutely summable in $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$.

Proof. The case of $C_\psi^{a,b}$ follows directly from Proposition 3.4.1. For $D_\psi^{a,b}$ it suffices to show

$$\forall \lambda_1, \mu_1 \in \mathbb{R}_+ \ \exists \lambda_2, \mu_2 \in \mathbb{R}_+ \ (\forall \lambda_2, \mu_2 \in \mathbb{R}_+ \ \exists \lambda_1, \mu_1) :$$

$$l^\infty \left(w^{\mu_2}(a \cdot) \otimes e^{\omega_{M^{\lambda_2}}(b \cdot)} \right) \to \mathcal{S}_{w^{\mu_1},\infty}^{M^{\lambda_1}} \text{ is well-defined and continuous.}$$

We have that $\psi \in \mathcal{S}_{v^{\mu_3},\infty}^{M^{\lambda_3}}$ for all (resp. for some) $\lambda_3, \mu_3 \in \mathbb{R}_+$. For every $\mu_1 \in \mathbb{R}_+$ there exists $\mu_3, \mu_4 \in \mathbb{R}_+$ (for every $\mu_4 \in \mathbb{R}_+$ and fixed μ_3 there exists a $\mu_1 \in \mathbb{R}_+$) such that $w^{\mu_1}(x + y) \leqslant C_0 w^{\mu_4}(x)v^{\mu_3}(y)$ for all $x, y \in \mathbb{R}^d$ and some $C_0 > 0$. By Lemma 3.2.3 there exists a $\mu_2 \in \mathbb{R}_+$ (for every $\mu_2 \in \mathbb{R}_+$ there exists a $\mu_4 \in \mathbb{R}_+$) such that

$$\sum_{k \in \mathbb{Z}^d} \frac{w^{\mu_4}(ak)}{w^{\mu_2}(ak)} < \infty.$$

Using condition [L] we find for every $\lambda_1 \in \mathbb{R}_+$ a $\lambda_4 \in \mathbb{R}_+$ (for every $\lambda_4 \in \mathbb{R}_+$ a $\lambda_1 \in \mathbb{R}_+$) such that $(4\pi)^{|\alpha|}M_\alpha^{\lambda_4} \leqslant C_1 M_\alpha^{\lambda_1}$ for all $\alpha \in \mathbb{N}^d$ and some $C_1 > 0$. Then, by application of Lemma 3.2.5 and Lemma 3.2.3,

there exists a $\lambda_2 \in \mathbb{R}_+$ (for every $\lambda_2 \in \mathbb{R}_+$ there exists a $\lambda_4 \in \mathbb{R}_+$) such that

$$\sum_{n \in \mathbb{Z}^d} \frac{e^{\omega_{M^{\lambda_4}}(bn)}}{e^{\omega_{M^{\lambda_2}}(bn)}} < \infty.$$

Additionally, by condition [L], for every $\lambda_1 \in \mathbb{R}_+$ there is some $\lambda_3 \in \mathbb{R}_+$ (for every $\lambda_3 \in \mathbb{R}_+$ there is some $\lambda_1 \in \mathbb{R}_+$) such that $2^{|\alpha|} M_\alpha^{\lambda_3} \leqslant C_2 M_\alpha^{\lambda_1}$ for all $\alpha \in \mathbb{N}^d$ and some $C_2 > 0$. We now have for any $(c_{k,n})_{k,n \in \mathbb{Z}^d} \in l^\infty \left(w^{\mu_2}(a \cdot) \otimes e^{\omega_{M^{\lambda_2}}(b \cdot)} \right)$

$$\left\| D_\psi^{a,b}((c_{k,n})_{k,n \in \mathbb{Z}^d}) \right\|_{\mathcal{S}_{w^{\mu_1},\infty}^{M^{\lambda_1}}} \leqslant \sum_{(k,n) \in \mathbb{Z}^{2d}} \left\| c_{k,n} M_{bn} T_{ak} \psi \right\|_{\mathcal{S}_{w^{\mu_1},\infty}^{M^{\lambda_1}}}$$

$$\leqslant M_0^{\lambda_1} C_0 \sum_{k,n \in \mathbb{Z}^d} |c_{k,n}| w^{\mu_4}(ak)$$

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{t \in \mathbb{R}^d} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{(2\pi bn)^{|\beta|}}{M_\beta^{\lambda_1}} \frac{|\psi^{(\alpha-\beta)}(t-ak)| v^{\mu_3}(t-ak)}{M_{\alpha-\beta}^{\lambda_1}}$$

$$\leqslant C \left\| (c_{k,n})_{k,n \in \mathbb{Z}^d} \right\|_{l^\infty \left(w^{\mu_2}(a \cdot) \otimes e^{\omega_{M^{\lambda_2}}(b \cdot)} \right)}$$

$$\cdot \left(\sum_{k \in \mathbb{Z}^d} \frac{w^{\mu_4}(ak)}{w^{\mu_2}(ak)} \right) \left(\sum_{n \in \mathbb{Z}^d} \frac{e^{\omega_{M^{\lambda_4}}(bn)}}{e^{\omega_{M^{\lambda_2}}(bn)}} \right).$$

for some $C > 0$. From here we may conclude the continuity of $D_\psi^{a,b} : \lambda^\infty[A_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}}^{a,b}] \to \mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$ as well as the absolute summability in $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$. □

Corollary 3.4.6. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system for which Assumption 3.4.2 holds for some weight function system $\mathscr{V}$ and $\mathscr{W}$ satisfies [N]. Take $\psi \in \mathcal{S}_{[\mathscr{V}],1}^{[\mathfrak{M}]} \cap \mathcal{S}_{[\mathscr{V}],\infty}^{[\mathfrak{M}]}$ such that $\mathcal{G}(\psi, a, b)$ is a Gabor frame for certain $a, b > 0$ and let γ be a dual window for ψ. If $\gamma \in \mathcal{S}_{[\mathscr{V}],1}^{[\mathfrak{M}]} \cap \mathcal{S}_{[\mathscr{V}],\infty}^{[\mathfrak{M}]}$, then $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$ is isomorphic to a complemented subspace of $\lambda^\infty[A_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}}^{a,b}]$.*

The previous corollary carries with it an interesting problem: when may the rapid decay in time and frequency of the window of a Gabor frame be carried over to a dual window? We introduce the following notion.

Definition 3.4.7. A lcHs $\mathcal{X}$ of functions on which the translation and modulation operators work continuously is called *Gabor accessible* if there exist $\psi, \gamma \in \mathcal{X}$ and $a, b > 0$ such that $\mathcal{G}(\psi, a, b)$ is a Gabor frame and γ is a dual window for it.

For the remainder of this section we will discuss the problem of Gabor accessibility for the Gelfand-Shilov spaces of Roumieu type associated to the Gevrey sequences, i.e. $\mathcal{S}_r^s(\mathbb{R}^d) = \mathcal{S}_{\{p!^r\}}^{\{p!^s\}}(\mathbb{R}^d)$. To this purpose, we introduce the following notion.

Definition 3.4.8. A pair (r, s) of positive real numbers is called a *Gabor couple (for dimension d)* if there exists some $\psi \in \mathcal{S}_r^s(\mathbb{R}^d)$ for which $\mathcal{G}(\psi, a, b)$ is a Gabor frame for some lattice parameters $a, b > 0$ and such that there exists a dual window γ for ψ in $\mathcal{S}_r^s(\mathbb{R}^d)$.

A classical result by Walnut [148] (see also [65, Theorem 6.5.1, p. 121]) implies that for any $\psi \in \mathcal{S}(\mathbb{R}^d)$ there exists $a, b > 0$ such that $\mathcal{G}(\psi, a, b)$ is a Gabor frame. Then Janssen showed [76] that in this case the canonical dual window γ° of ψ belongs to $\mathcal{S}(\mathbb{R}^d)$ as well (see also [65, Theorem 13.5.4, p. 296]). This statement does not, in general, have an ultradifferentiable analog. Take for instance, for $d = 1$, the special case where $\psi(x) = e^{-\pi x^2}$ is the Gaussian, so that $\psi \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R})$, then $\mathcal{G}(\psi, a, b)$ is a Gabor frame if $ab < 1$ but its canonical dual window does not have Gaussian decay in time and frequency [77]. However, by considering other dual windows for ψ, we may provide valuable sufficient conditions for pairs to be a Gabor couple. We first start with the following observations, which state that Gabor couples on the real line may be extended to arbitrary dimension and are always symmetrical.

Lemma 3.4.9. *For any $(r, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ the following statements hold:*

(a) *If (r, s) is a Gabor couple for dimension 1, then (r, s) is a Gabor couple for any dimension $d \in \mathbb{Z}_+$.*

(b) *If (r, s) is a Gabor couple for dimension d, then so is (s, r).*

Proof. (a) Let (r, s) be a Gabor couple for dimension 1. Then take ψ_1 and γ_1 in $\mathcal{S}_r^s(\mathbb{R})$ and $a, b > 0$ such that $\mathcal{G}(\psi_1, a, b)$ is a Gabor frame for

$L^2(\mathbb{R})$ and γ_1 is a dual window for ψ_1. For arbitrary dimension $d \geqslant 2$ we define $\psi_d(x) := \psi_1(x_1) \cdots \psi_1(x_d)$ and $\gamma_d(x) = \gamma_1(x_1) \cdots \gamma_1(x_d)$, then one easily sees that both ψ_d and γ_d lie in $\mathcal{S}_r^s(\mathbb{R}^d)$. By Lemma 3.4.4 it follows that

$$(\psi_d, M_{\frac{l}{a}} T_{\frac{k}{b}} \gamma_d)_{L^2} = \prod_{j=1}^{d} (\psi_1, M_{\frac{l_j}{a}} T_{\frac{k_j}{b}} \gamma_1)_{L^2} = \prod_{j=1}^{d} ab\delta_{k_j,0}\delta_{l_j,0}$$
$$= (ab)^d \delta_{k,0}\delta_{l,0}$$

for any $k = (k_1, \ldots, k_d)$ and $l = (l_1, \ldots, l_d)$. Another application of Lemma 3.4.4 shows that $\mathcal{G}(\psi_d, a, b)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ and γ_d is a dual window for ψ_d.

(b) Suppose (r, s) is a Gabor couple for dimension d, and let $\psi, \gamma \in \mathcal{S}_r^s(\mathbb{R}^d)$ and $a, b > 0$ be such that $\mathcal{G}(\psi, a, b)$ is a Gabor frame and γ is a dual window for ψ. By [27, Corollary 2.5] both $\widehat{\psi}$ and $\widehat{\gamma}$ lie in $\mathcal{S}_s^r(\mathbb{R}^d)$. From the Plancherel theorem and Lemma 3.4.4 it then follows

$$(\widehat{\gamma}, M_{\frac{k}{b}} T_{\frac{l}{a}} \widehat{\psi})_{L^2} = \overline{(\psi, M_{-\frac{l}{a}} T_{\frac{k}{b}} \gamma)}_{L^2} = (ab)^d \delta_{l,0}\delta_{k,0}.$$

Whence, by Lemma 3.4.4, $\mathcal{G}(\widehat{\gamma}, b, a)$ is a Gabor frame and $\widehat{\psi}$ is a dual window for $\widehat{\gamma}$. $\qquad \square$

Corollary 3.4.10. *Let $(r, s) \in \mathbb{R}_+^2$. Then, (r, s) is Gabor couple, for any dimension d, if one of the following conditions is satisfied:*

(i) $\min(r, s) \geqslant 1/2$;

(ii) $\max(r, s) > 1$.

Proof. In virtue of Lemma 3.4.9(a) it suffices to show this for $d = 1$.

(i) It is shown in [17], by use of the Bargmann transform [75], that for the Gaussian $\psi(x) = e^{-\pi x^2}$ there exists a dual window of Gaussian decay in both time and frequency.

(ii) By Lemma 3.4.9(b), we may suppose w.l.o.g. that $s > 1$. In [17], those Gabor frames whose window and canonical dual window have compact support are characterised. Each one of these form an example. Simple examples may also be constructed using [26, Theorem 2.2]. $\qquad \square$

Open problem 3.4.11. Determine exactly which pairs $(r, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ form a Gabor couple in dimension d. As $\mathcal{S}_r^s(\mathbb{R}^d)$ is trivial if and only if $r + s < 1$ [61, p. 235], it follows from Lemma 3.4.9(b) and Corollary 3.4.10 that only the cases $0 < r < 1/2$ and $1 - r \leqslant s \leqslant 1$ are open. If one can show, for $d = 1$, that any point on the line $r + s = 1$ is a Gabor couple, then the problem would effectively be solved for arbitrary dimension in view of Lemma 3.4.9(a). Should this not be the case, it would be interesting to study whether or not the dimension has an influence on the characterization of Gabor couples.

Chapter 4

Characterizations of nuclearity

4.1 Introduction

Nuclear spaces play a major role in functional analysis. One of their key features is the validity of abstract Schwartz kernel theorems, which often allows for the representation and study of important classes of continuous linear mappings via kernels. Therefore, establishing whether a given function space is nuclear becomes a central question from the point of view of both applications and understanding its locally convex structure.

In the case of weighted Fréchet spaces of smooth functions on $\mathbb{R}^d$, the nuclearity question has been completely settled. Let $\mathscr{W}$ be a weight function system and consider the associated Gelfand-Shilov spaces of smooth functions

$$\mathcal{S}_{(\mathscr{W}),q} = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \max_{|\alpha| \leqslant n} \|\varphi^{(\alpha)} w_n\|_{L^q} < \infty \quad \forall n \in \mathbb{N}\}, \quad q \in [1, \infty],$$

endowed with their natural Fréchet space topologies. If $\mathscr{W}$ satisfies (wM), then $\mathcal{S}_{(\mathscr{W}),q}$ is nuclear if and only if $\mathscr{W}$ satisfies (N). In fact, this result follows from Vogt's sequence space representation of $\mathcal{S}_{(\mathscr{W}),q}$ [144, Theorem 3.1] and the well-known corresponding characterization of nuclearity for Köthe sequence spaces, i.e. Proposition 2.3.1. Condition (N) appears already in the work of Gelfand and Shilov,

who proved the nuclearity of $\mathcal{S}_{(\mathscr{W}),\infty}$ under it and some extra regularity assumptions in a direct fashion [62, p. 181].

The aim of this chapter is to discuss several results centred around the characterization of the nuclearity of Gelfand-Shilov type spaces. In Section 4.2 we consider the Gelfand-Shilov spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ introduced in Chapter 3. We give sufficient conditions for $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ to be nuclear in terms of $\mathfrak{M}$ and $\mathscr{W}$; see Theorem 4.2.1 . Actually, we show that for an important class of weight sequence systems our hypotheses also become necessary, providing a full characterization of nuclearity in such a case; see Theorem 4.2.6. Moreover, nuclearity is related to the identity $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}],r}^{[\mathfrak{M}]}$, $q \neq r$, see also Theorem 3.3.11. A useful feature of our approach is that our considerations are stable under tensor products. We shall exploit this fact to derive new kernel theorems for Gelfand-Shilov spaces in Section 4.2.3. Note that these kernel theorems are 'global' counterparts of Petzsche's results from [104].

Secondly, in Section 4.3, we consider the nuclearity of the so-called Beurling-Björck spaces $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$. In recent works Boiti et al. [14, 15, 16] have investigated the nuclearity of the Beurling-Björck space $\mathcal{S}_{(\omega)}^{(\omega)}(\mathbb{R}^d)$ (in our notation). Their most general result [16, Theorem 3.3] establishes the nuclearity of this Fréchet space when ω is a Braun-Meise-Taylor type weight function [21] (where non-quasianalyticity is replaced by $\omega(t) = o(t)$ and the condition $\log(t) = o(\omega(t))$ from [21] is relaxed to $\log t = O(\omega(t))$). Our aim is to improve and generalize [16, Theorem 3.3] by considerably weakening the set of hypotheses on the weight functions, providing a complete characterization of the nuclearity of these spaces (for radially increasing weight functions), and considering anisotropic spaces and the Roumieu case as well. Particularly, we shall show that the conditions (β) and (δ) from [16, Definition 2.1] play no role in deducing nuclearity. Furthermore, we discuss the equivalence of the various definitions of Beurling-Björck type spaces given in the literature [16, 28, 67] but considered here under milder assumptions. In particular, we show that, if ω satisfies (α) and (γ), our definition of $\mathcal{S}_{(\omega)}^{(\omega)}(\mathbb{R}^d)$ coincides with the one employed in [16].

4.2 The Gelfand-Shilov spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$

4.2.1 Nuclearity

In this section, we characterize the nuclearity of the Gelfand-Shilov spaces $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ in terms of $\mathfrak{M}$ and $\mathscr{W}$. We start by providing sufficient conditions, whose prove is based on Grothendieck's criterion for nuclearity in terms of summable sequences [69].

Theorem 4.2.1. *Let $\mathfrak{M}$ be a weight sequence system satisfying [L] and [$\mathfrak{M}$.2]$'$ and let $\mathscr{W}$ be a non-degenerate weight function system satisfying [wM] and [N]. Then, $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$ is nuclear.*

Proof. We shall show that $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]} = \mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}$ is nuclear. To this end, we employ Proposition 2.2.5 with $E = \mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}$. Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}_{[\mathscr{W}],\infty}^{[\mathfrak{M}]}$ be a weakly summable sequence. This means that for all $\lambda > 0$ (for some $\lambda > 0$) there is $C > 0$ such that

$$\| \sum_{n=0}^{k} c_n\varphi_n \|_{\mathcal{S}_{w^\lambda,\infty}^{M^\lambda}} \leqslant C$$

for all $k \in \mathbb{N}$ and $|c_n| \leqslant 1$, $n = 0, \ldots, k$, where we have used Lemma 3.3.2 in the Roumieu case. We claim that

$$\sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{1}{M_\alpha^\lambda} \sum_{n=0}^{\infty} |\varphi_n^{(\alpha)}(x)| w^\lambda(x) \leqslant C. \tag{4.1}$$

Fix arbitrary $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. Let $k \in \mathbb{N}$. Choose $|c_n(\alpha, x)| \leqslant 1$ such that $c_n(\alpha, x)\varphi_n^{(\alpha)}(x) = |\varphi_n^{(\alpha)}(x)|$. Then,

$$\frac{1}{M_\alpha^\lambda} \sum_{n=0}^{k} |\varphi_n^{(\alpha)}(x)| w^\lambda(x) = \frac{1}{M_\alpha^\lambda} \left| \sum_{n=0}^{k} c_n(\alpha, x)\varphi_n^{(\alpha)}(x) \right| w^\lambda(x) \leqslant C,$$

whence the claim follows by letting $k \to \infty$. We now employ (4.1) to show that $(\varphi_n)_{n\in\mathbb{N}}$ is absolutely summable. By Theorem 3.3.11, it is enough to prove that

$$\sum_{n=0}^{\infty} \|\varphi_n\|_{\mathcal{S}_{w^\mu,1}^{M^\mu}} < \infty$$

for all $\mu > 0$ (for some $\mu > 0$). Let $\mu > 0$ be arbitrary (let $\lambda > 0$ be such that (4.1) holds). Conditions [L] and [N] imply that there is $\lambda > 0$ (there is $\mu > 0$) such that $2^{|\alpha|}M_\alpha^\lambda \leqslant C'M_\alpha^\mu$ for all $\alpha \in \mathbb{N}^d$ and some $C' > 0$ and $w^\mu/w^\lambda \in L^1(\mathbb{R}^d)$. Hence,

$$\sum_{n=0}^{\infty} \|\varphi_n\|_{\mathcal{S}_{w^\mu,1}^{M^\mu}} = \sum_{n=0}^{\infty} \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M_\alpha^\mu} \int_{\mathbb{R}^d} |\varphi_n^{(\alpha)}(x)|w^\mu(x)dx$$

$$\leqslant C' \sum_{\alpha \in \mathbb{N}^d} \frac{1}{2^{|\alpha|}} \int_{\mathbb{R}^d} \frac{1}{M_\alpha^\lambda} \sum_{n=0}^{\infty} |\varphi_n^{(\alpha)}(x)|w^\lambda(x)\frac{w^\mu(x)}{w^\lambda(x)}dx$$

$$\leqslant 2^d CC' \left\|w^\mu/w^\lambda\right\|_{L^1}.$$

$\square$

Our next goal is to discuss the necessity of the conditions $[\mathfrak{M}.2]'$ and [N] for $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}(\mathbb{R}^d)$ to be nuclear.

Proposition 4.2.2. *Let $\mathfrak{M}$ be a weight sequence system satisfying [L], let $\mathscr{W}$ be a weight function system satisfying [M], and let $q \in [1,\infty]$. Suppose that $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}(\mathbb{R}^d)$ is non-trivial and nuclear. Then, $\mathscr{W}$ satisfies [N].*

Proof. Since nuclearity is inherited to subspaces, Corollary 3.3.10 implies that $\lambda^q[A_\mathscr{W}]$ is nuclear. The result therefore follows from Proposition 2.3.1 and Lemma 3.2.3. $\square$

Proposition 4.2.3. *Let $\mathfrak{M}$ be a weight sequence system satisfying [L], let $\mathscr{W}$ be a non-degenerate weight function system satisfying [M], and let $q \in [1,\infty]$. Suppose that $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}(\mathbb{R}^d)$ is non-trivial and nuclear. Then, $A_{\mathscr{W}_\mathfrak{M}}$ satisfies [N].*

We shall make use of the ensuing result due to Petzsche [104] in order to show Proposition 4.2.3.

Lemma 4.2.4 ([104, Satz 3.5 and Satz 3.6])**.** *Let A be a Köthe set and let E be a lcHs.*

(a) *Suppose that E is nuclear and that there are continuous linear mappings $T : \lambda^1(A) \to E$ and $S : E \to \lambda^\infty(A)$ such that $S \circ T = \iota$, where $\iota : \lambda^1(A) \to \lambda^\infty(A)$ denotes the natural embedding. Then, $\lambda^1(A)$ is nuclear.*

(b) *Suppose that E'_b is nuclear and that there are continuous linear mappings $T : \lambda^1\{A\} \to E$ and $S : E \to \lambda^\infty\{A\}$ such that $S \circ T = \iota$, where $\iota : \lambda^1\{A\} \to \lambda^\infty\{A\}$ denotes the natural embedding. Then, $\lambda^1\{A\}$ is nuclear.*

Proof. This is essentially shown in [104, Satz 3.5 and Satz 3.6] but we repeat the argument here for the sake of completeness and because our assumptions are slightly more general.

(a) Since nuclearity is inherited to subspaces, it suffices to show that T is a topological isomorphism onto its image. We write $e_i = (\delta_{j,i})_{j \in \mathbb{Z}^d}$ for $i \in \mathbb{Z}^d$. Then, $(e_i)_{i \in \mathbb{Z}^d}$ is a Schauder basis for $\lambda^1(A)$ with coefficient functionals

$$\xi_i : \lambda^1(A) \to \mathbb{C}, \quad \left\langle \xi_i, (c_j)_{j \in \mathbb{Z}^d} \right\rangle = c_i, \qquad i \in \mathbb{Z}^d.$$

Since T is continuous and $S \circ T = \iota$, $(T(e_i))_{i \in \mathbb{Z}^d}$ is a Schauder basis for $T(\lambda^1(A))$ with coefficient functionals $\eta_i = \xi_i \circ T^{-1} = \xi_i \circ S$ for $i \in \mathbb{Z}^d$. We claim that the Schauder basis $(T(e_i))_{i \in \mathbb{Z}^d}$ is equicontinuous, that is,

$$\forall p \in \mathrm{csn}(E) \, \exists q \in \mathrm{csn}(E) \, \forall x \in T(\lambda^1(A)) : \sup_{i \in \mathbb{Z}^d} |\left\langle \eta_i, x \right\rangle | p(T(e_i)) \leqslant q(x).$$

Let $p \in \mathrm{csn}(E)$ be arbitrary. As T is continuous, there is $\lambda > 0$ such that

$$\begin{aligned}
|\left\langle \eta_i, x \right\rangle | p(T(e_i)) &\leqslant |\left\langle \xi_i, S(x) \right\rangle | \, \|e_i\|_{l^1(a^\lambda)} \\
&= |\left\langle \xi_i, S(x) \right\rangle | a_i^\lambda \\
&\leqslant \|S(x)\|_{l^\infty(a^\lambda)}
\end{aligned}$$

for all $x \in E$ and $i \in \mathbb{Z}^d$. The claim now follows from the continuity of S. Since $T(\lambda^1(A))$ is nuclear (as a subspace of the nuclear space E), the Dymin-Mityagin basis theorem [107, Theorem 10.2.1] yields that $(T(e_i))_{i \in \mathbb{Z}^d}$ is an absolute Schauder basis for $T(\lambda^1(A))$, that is,

$$\forall p \in \mathrm{csn}(E) \, \exists q \in \mathrm{csn}(E) \, \forall x \in T(\lambda^1(A)) :$$
$$\sum_{i \in \mathbb{Z}^d} |\left\langle \eta_i, x \right\rangle | p(T(e_i)) \leqslant q(x). \tag{4.2}$$

We now show that $T^{-1} : T(\lambda^1(A)) \to \lambda^1(A)$ is continuous. Let $\lambda > 0$ be arbitrary. Since S is continuous, there is $p \in \mathrm{csn}(E)$ such that

$$
\begin{aligned}
\|(c_i)_{i \in \mathbb{Z}^d}\|_{l^1(a^\lambda)} &= \sum_{i \in \mathbb{Z}^d} |c_i| \|e_i\|_{l^1(a^\lambda)} \\
&= \sum_{i \in \mathbb{Z}^d} |\langle \eta_i, T((c_i)_{i \in \mathbb{Z}^d}) \rangle| \, \|S(T(e_i))\|_{l^\infty(a^\lambda)} \\
&\leqslant \sum_{i \in \mathbb{Z}^d} |\langle \eta_i, T((c_i)_{i \in \mathbb{Z}^d}) \rangle| \, p(T(e_i))
\end{aligned}
$$

for all $(c_i)_{i \in \mathbb{Z}^d} \in \lambda^1(A)$, whence the continuity of T^{-1} follows from (4.2).

(b) By transposing, we obtain continuous linear mappings $T^t : E'_b \to (\lambda^1\{A\})'_b$ and $S : (\lambda^\infty\{A\})'_b \to E'_b$ such that $T^t \circ S^t = \iota^t$. Consider the natural continuous embeddings $\iota_1 : \lambda^1(A^\circ) \to (\lambda^\infty\{A\})'_b$ and $\iota_2 : (\lambda^1\{A\})'_b \to \lambda^\infty(A^\circ)$. Note that $(\iota_2 \circ T^t) \circ (S^t \circ \iota_1) = \tau$, where $\tau : \lambda^1(A^\circ) \to \lambda^\infty(A^\circ)$ denotes the natural embedding. Hence, part (a) yields that $\lambda^1(A^\circ)$ is nuclear, which is equivalent to the nuclearity of $\lambda^1\{A\}$ by Proposition 2.3.1. $\qquad \square$

We also need the existence of a specific element in $\widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$.

Lemma 4.2.5. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and $\mathscr{W}$ a weight function system satisfying* [wM]. *Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1, \infty]$. Then there exists $\psi \in \widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ such that $\sum_{j \in \mathbb{Z}^d} \psi(\cdot - j) \equiv 1$.*

Proof. By Lemma 3.3.4, there is $\varphi \in \widetilde{\mathcal{S}}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ such that $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Then,

$$
\psi(x) = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \varphi(x - t) dt, \qquad x \in \mathbb{R}^d,
$$

satisfies all requirements. $\qquad \square$

Proof of Proposition 4.2.3. By Proposition 2.3.1, it suffices to show that $\lambda^1[A_{\mathscr{W}_{\mathfrak{M}}}]$ is nuclear. To this end, we use Lemma 4.2.4 with $A = A_{\mathscr{W}_{\mathfrak{M}}}$ and $E = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ (in the Roumieu case, E'_b is nuclear as the strong dual of a nuclear (DF)-space). For $r = 1$ or $r = \infty$ we define

$\mathcal{E}_{\mathrm{per},r}^{[\mathfrak{M}]}$ as the space consisting of all $\mathbb{Z}^d$-periodic functions $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \frac{1}{M_\alpha^\lambda} \left\| \varphi^{(\alpha)} \right\|_{L^r\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)} < \infty$$

for all $\lambda > 0$ (for some $\lambda > 0$). We endow $\mathcal{E}_{\mathrm{per},r}^{[\mathfrak{M}]}$ with its natural Fréchet space topology ((LB)-space topology). The mappings

$$T_0 : \lambda^1[A_{\mathscr{W}_\mathfrak{M}}] \to \mathcal{E}_{\mathrm{per},\infty}^{[\mathfrak{M}]}, \quad T_0((c_j)_{j \in \mathbb{Z}^d}) = \left[\xi \to \sum_{j \in \mathbb{Z}^d} c_j e^{-2\pi i j \cdot \xi} \right]$$

and

$$S_0 : \mathcal{E}_{\mathrm{per},1}^{[\mathfrak{M}]} \to \lambda^\infty[A_{\mathscr{W}_\mathfrak{M}}], \qquad S_0(\varphi) = \left(\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \varphi(\xi) e^{2\pi i j \cdot \xi} d\xi \right)_{j \in \mathbb{Z}^d}$$

are continuous. Next, choose ψ as in Lemma 4.2.5 and consider the continuous linear mapping

$$T_1 : \mathcal{E}_{\mathrm{per},\infty}^{[\mathfrak{M}]} \to \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}, \quad T_1(\varphi) = \psi\varphi.$$

Note that $\mathscr{W}$ satisfies [N] by Proposition 4.2.2. Hence, Lemma 3.3.6 yields that the mapping

$$S_1 : \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \to \mathcal{E}_{\mathrm{per},1}^{[\mathfrak{M}]}, \qquad S_1(\varphi) = \sum_{j \in \mathbb{Z}^d} \varphi(\,\cdot\, - j)$$

is continuous. Finally, we define the continuous linear mappings $T = T_1 \circ T_0 : \lambda^1[A_{\mathscr{W}_\mathfrak{M}}] \to \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]}$ and $S = S_0 \circ S_1 : \mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \to \lambda^\infty[A_{\mathscr{W}_\mathfrak{M}}]$. The choice of ψ implies that $S \circ T = \iota$. □

We obtain the following two important results.

Theorem 4.2.6. *Let $\mathfrak{M}$ be an isotropically decomposable accelerating weight sequence system satisfying* [L] *and let $\mathscr{W}$ be a non-degenerate weight function system satisfying* [M]. *Suppose that $\mathcal{S}_{[\mathscr{W}],q}^{[\mathfrak{M}]} \neq \{0\}$ for some $q \in [1, \infty]$. Then, the following statements are equivalent:*

(i) $\mathfrak{M}$ satisfies [$\mathfrak{M}.2$]$'$ *and $\mathscr{W}$ satisfies* [N].

(ii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ is nuclear for all $q \in [1, \infty]$.

(iii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ is nuclear for some $q \in [1, \infty]$.

Proof. $(i) \Rightarrow (ii)$ This has been shown in Theorem 4.2.1.

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (i)$ In view of Lemma 3.3.4, $\mathscr{W}$ satisfies [N] by Proposition 4.2.2, while $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ by Proposition 4.2.3 and Lemma 3.2.5. $\square$

Theorem 4.2.7. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and* $[\mathfrak{M}.2]'$. *Let $\mathscr{W}$ be a non-degenerate weight function system satisfying* [M]. *Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1, \infty]$. Then, the following statements are equivalent:*

(i) $\mathscr{W}$ *satisfies* [N].

(ii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ *is nuclear for all $q \in [1, \infty]$.*

(iii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ *is nuclear for some $q \in [1, \infty]$.*

(iv) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],r}$ *as locally convex spaces for all $q, r \in [1, \infty]$.*

(v) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],r}$ *as sets for some $q, r \in [1, \infty]$ with $q \neq r$.*

Proof. In view of Lemma 3.3.4, this follows from Theorem 3.3.11, Theorem 4.2.1 and Proposition 4.2.2. $\square$

In the specific case of Gelfand-Shilov spaces defined by weight sequences we obtain the following useful characterizations.

Theorem 4.2.8. *Let M and A be two isotropically decomposable weight sequences satisfying* $(M.1)$. *Suppose that $\mathcal{S}^{[M]}_{[A],q} \neq \{0\}$ for some $q \in [1, \infty]$. Then, the following statements are equivalent:*

(i) M *and A both satisfy* $(M.2)'$.

(ii) $\mathcal{S}^{[M]}_{[A],q}$ *is nuclear for all $q \in [1, \infty]$.*

(iii) $\mathcal{S}^{[M]}_{[A],q}$ *is nuclear for some $q \in [1, \infty]$.*

Proof. By Lemma 3.2.6 and Theorem 4.2.6. □

Theorem 4.2.9. *Let M be an isotropically decomposable weight sequence satisfying $(M.1)$ and $(M.2)'$. Let A be an isotropically decomposable weight sequence satisfying $(M.1)$. Suppose that $\mathcal{S}^{[M]}_{[A],q} \neq \{0\}$ for some $q \in [1, \infty]$. Then, the following statements are equivalent:*

(i) *A satisfies $(M.2)'$.*

(ii) *$\mathcal{S}^{[M]}_{[A],q}$ is nuclear for all $q \in [1, \infty]$.*

(iii) *$\mathcal{S}^{[M]}_{[A],q}$ is nuclear for some $q \in [1, \infty]$.*

(iv) *$\mathcal{S}^{[M]}_{[A],q} = \mathcal{S}^{[M]}_{[A],r}$ as locally convex spaces for all $q, r \in [1, \infty]$.*

(v) *$\mathcal{S}^{[M]}_{[A],q} = \mathcal{S}^{[M]}_{[A],r}$ as sets for some $q, r \in [1, \infty]$ with $q \neq r$.*

Proof. By Lemma 3.2.6 and Theorem 4.2.7. □

In case of Gelfand-Shilov spaces defined by BMT weight functions we get the following result.

Theorem 4.2.10. *Let ω be a BMT weight function and let η be a non-negative non-decreasing continuous function on $[0, \infty)$ satisfying (α). Suppose that $\mathcal{S}^{[\omega]}_{[\eta],q} \neq \{0\}$ for some $q \in [1, \infty]$. Then, the following statements are equivalent:*

(i) *η satisfies (γ) $((\gamma_0))$.*

(ii) *$\mathcal{S}^{[\omega]}_{[\eta],q}$ is nuclear for all $q \in [1, \infty]$.*

(iii) *$\mathcal{S}^{[\omega]}_{[\eta],q}$ is nuclear for some $q \in [1, \infty]$.*

(iv) *$\mathcal{S}^{[\omega]}_{[\eta],q} = \mathcal{S}^{[\omega]}_{[\eta],r}$ as locally convex spaces for all $q, r \in [1, \infty]$.*

(v) *$\mathcal{S}^{[\omega]}_{[\eta],q} = \mathcal{S}^{[\omega]}_{[\eta],r}$ as sets for some $q, r \in [1, \infty]$ with $q \neq r$.*

Proof. By combining Theorem 4.2.9 with Lemma 3.2.7. □

In Section 4.3 we will further refine Theorem 4.2.10 by dropping the necessity of ω being a BMT weight function (cfr. Theorem 4.3.4).

4.2.2 Projective description

In this auxiliary section, we provide a projective description of the Gelfand-Shilov spaces $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}$. This result will be used in the next section to prove kernel theorems.

We start by recalling some basic results about the projective description of weighted (LB)-spaces of continuous functions [8]. Let X be a completely regular Hausdorff space. Given a non-negative function v on X, we write $Cv(X)$ for the seminormed space consisting of all $f \in C(X)$ such that $\|f\|_v = \sup_{x \in X} |f(x)|v(x) < \infty$. If v is positive and continuous, then $Cv(X)$ is a Banach space. A family $\mathcal{V} = \{v^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ consisting of positive continuous functions v^λ on X such that $v^\lambda(x) \leqslant v^\mu(x)$ for all $x \in X$ and $\mu \leqslant \lambda$ is said to be a *Nachbin family on X* . We define the associated (LB)-space

$$\mathcal{V}C(X) = \varinjlim_{\lambda \to \infty} Cv^\lambda(X).$$

The *maximal Nachbin family associated with $\mathcal{V}$* , denoted by $\overline{V} = \overline{V}(\mathcal{V})$, is given by the space consisting of all non-negative upper semi-continuous functions v on X such that $\sup_{x \in X} v(x)/v^\lambda(x) < \infty$ for all $\lambda \in \mathbb{R}_+$. The *projective hull of $\mathcal{V}C(X)$* is defined as the space $C\overline{V}(X)$ consisting of all $f \in C(X)$ such that $\|f\|_v < \infty$ for all $v \in \overline{V}$. We endow $C\overline{V}(X)$ with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_v \,|\, v \in \overline{V}\}$. The spaces $\mathcal{V}C(X)$ and $C\overline{V}(X)$ are always equal as sets. If $\mathcal{V}$ satisfies the condition [8, p. 94]

(S) $\forall \lambda \in \mathbb{R}_+ \, \exists \mu \in \mathbb{R}_+ : v^\mu/v^\lambda$ vanishes at infinity,

then these spaces also coincide topologically [8, Corollary 5, p. 116].

Let X_j be a completely regular Hausdorff space and let $\mathcal{V}_j = \{v_j^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ be a Nachbin family on X_j for $j = 1, 2$. We denote by $\mathcal{V}_1 \otimes \mathcal{V}_2$ the Nachbin family $\{v_1^\lambda \otimes v_2^\lambda \,|\, \lambda \in \mathbb{R}_+\}$ on $X_1 \times X_2$, where $v_1^\lambda \otimes v_2^\lambda(x_1, x_2) = v_1^\lambda(x_1)v_2^\lambda(x_2)$, $x_1 \in X_1, x_2 \in X_2$. Note that $\mathcal{V}_1 \otimes \mathcal{V}_2$ satisfies (S) if and only if both $\mathcal{V}_1$ and $\mathcal{V}_2$ do so. Moreover, $\overline{V}(\mathcal{V}_1) \otimes \overline{V}(\mathcal{V}_2)$ is upwards dense in $\overline{V}(\mathcal{V}_1 \otimes \mathcal{V}_2)$, that is, for every $v \in \overline{V}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ there are $v_j \in \overline{V}(\mathcal{V}_j)$, $j = 1, 2$, such that $v(x_1, x_2) \leqslant v_1 \otimes v_2(x_1, x_2)$ for all $x_1 \in X_1, x_2 \in X_2$.

Note that every weight function system $\mathscr{W}$ on $\mathbb{R}^d$ is a Nachbin family on $X = \mathbb{R}^d$. Lemma 3.2.2 implies that $\mathscr{W}$ satisfies (S) if $\{N\}$

and {wM} hold for $\mathscr{W}$. Likewise, a weight sequence system $\mathfrak{M}$ on $\mathbb{N}^d$ defines a Nachbin family on $X = \mathbb{N}^d$ via

$$\mathfrak{M}^\circ = \{1/M^\lambda \,|\, \lambda \in \mathbb{R}_+\}.$$

If $\mathfrak{M}$ satisfies {L}, then $\mathfrak{M}^\circ$ satisfies (S). We define $\overline{V}(\mathfrak{M})$ as the family consisting of all sequences M of positive numbers such that $1/M \in \overline{V}(\mathfrak{M}^\circ)$. More concretely, $\overline{V}(\mathfrak{M})$ consists of all sequences M of positive numbers such that $\sup_{\alpha \in \mathbb{N}^d} M_\alpha^\lambda / M_\alpha < \infty$ for all $\lambda \in \mathbb{R}_+$. Note that each element in $\overline{V}(\mathfrak{M})$ is automatically a weight sequence.

Remark 4.2.11. Let M be a weight sequence. Then, the set

$$\{(M_\alpha \prod_{j=0}^{|\alpha|} r_j)_{\alpha \in \mathbb{N}^d} \,|\, (r_j)_{j \in \mathbb{N}} \in \{\mathfrak{R}\}\} \tag{4.3}$$

is downwards dense in $\overline{V}(\mathfrak{M}_M)$ (cf. [83, Lemma 3.4]). The family $\{\mathfrak{R}\}$ was introduced by Komatsu to obtain a projective description of the space $\mathcal{E}^{\{M\}}(\Omega)$ of ultradifferentiable functions of Roumieu type [83, Proposition 3.5]. Later on, this family was also used by Pilipović to give a projective description of the Gelfand-Shilov spaces $\mathcal{S}_{\{A\}}^{\{M\}}$ [108, Lemma 4]. For general weight sequence systems $\mathfrak{M}$, the family $\overline{V}(\mathfrak{M})$ is the natural generalization of the family in (4.3).

The following observations will be of great use in the sequel.

Lemma 4.2.12. *Let $(a_\alpha)_{\alpha \in \mathbb{N}^d}$ be a sequence of positive numbers.*

(i) *We have $\sup_{\alpha \in \mathbb{N}^d} a_\alpha / M_\alpha^\lambda < \infty$ for some $\lambda \in \mathbb{R}_+$ if and only if $\sup_{\alpha \in \mathbb{N}^d} a_\alpha / M_\alpha < \infty$ for all $M \in \overline{V}(\mathfrak{M})$.*

(ii) *We have $\sup_{\alpha \in \mathbb{N}^d} a_\alpha M_\alpha < \infty$ for some $M \in \overline{V}(\mathfrak{M})$ if and only if $\sup_{\alpha \in \mathbb{N}^d} a_\alpha M_\alpha^\lambda < \infty$ for all $\lambda \in \mathbb{R}_+$.*

Proof. The direct implications are clear, we now show the reverse implications.

(i) Suppose that $\sup_{\alpha \in \mathbb{N}^d} a_\alpha / M_\alpha < \infty$ for all $M \in \overline{V}(\mathfrak{M})$, but $\sup_{\alpha \in \mathbb{N}^d} a_\alpha / M_\alpha^\lambda = \infty$ for any $\lambda > 0$. We construct a sequence $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$ inductively. There is a $\alpha_1 \in \mathbb{N}^d$ such that $a_{\alpha_1} / M_{\alpha_1}^1 > 1$,

and we set $M_\alpha = M_\alpha^1$ for all $|\alpha| \leqslant |\alpha_1|$. Suppose we have arrived at step $n + 1$. There exists a $\alpha_{n+1} \in \mathbb{N}^d$ such that $|\alpha_{n+1}| > |\alpha_n|$ and $a_{\alpha_{n+1}}/M_{\alpha_{n+1}}^{n+1} > n + 1$. Then we set $M_\alpha = M_\alpha^{n+1}$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha_n| < |\alpha| \leqslant |\alpha_{n+1}|$. It is clear that $M \in \overline{V}(\mathfrak{M})$ but $\sup_{\alpha \in \mathbb{N}^d} a_\alpha/M_\alpha = \infty$, giving a contradiction. Whence the existence of some $\lambda > 0$ such that $\sup_{\alpha \in \mathbb{N}^d} a_\alpha/M_\alpha^\lambda < \infty$.

(ii) Suppose that $C_\lambda = \sup_{\alpha \in \mathbb{N}^d} a_\alpha M_\alpha^\lambda < \infty$ for all $\lambda \in \mathbb{R}_+$. We define $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$ as $M_\alpha := \sup_{\lambda \in \mathbb{R}_+} M_\alpha^\lambda / C_\lambda$. Then clearly $M \in \overline{V}(\mathfrak{M})$ and $\sup_{\alpha \in \mathbb{N}^d} a_\alpha M_\alpha \leqslant 1$. $\qquad\square$

Corollary 4.2.13. *Let $\mathfrak{M}$ be a weight sequence system.*

(i) *If $\mathfrak{M}$ satisfies $\{L\}$ then for any $R > 0$ and $M \in \overline{V}(\mathfrak{M})$ there exists a $N \in \overline{V}(\mathfrak{M})$ and $C_0 > 0$ such that $R^{|\alpha|} N_\alpha \leqslant C_0 M_\alpha$ for any $\alpha \in \mathbb{N}^d$.*

(ii) *Suppose $\mathfrak{M}$ is isotropically decomposable satisfying $\{L\}$. If $\mathfrak{M}$ satisfies $\{\mathfrak{M}.2\}'$ then for any $M \in \overline{V}(\mathfrak{M})$ there exists a $N \in \overline{V}(\mathfrak{M})$ and $C_0 > 0$ such that $N_{\alpha+e_j} \leqslant C_0 M_\alpha$ for any $\alpha \in \mathbb{N}^d$ and $j \in \{1, \ldots, d\}$.*

Proof. (i) Suppose $\mathfrak{M}$ satisfies $\{L\}$. Take arbitrary $R > 0$ and $M \in \overline{V}(\mathfrak{M})$ and consider the sequence $a_\alpha = R^{|\alpha|}/M_\alpha$. For any $\mu > 0$ there exists a $\lambda > 0$ and $C_0' > 0$ such that $R^{|\alpha|} M_\alpha^\mu \leqslant C_0' M_\alpha^\lambda$. Then $\sup_{\alpha \in \mathbb{N}^d} a_\alpha M_\alpha^\mu \leqslant C_0' \sup_{\alpha \in \mathbb{N}^d} M_\alpha^\lambda / M_\alpha < \infty$. Whence by Lemma 4.2.12(ii) there exists a $N \in \overline{V}(\mathfrak{M})$ such that $\sup_{\alpha \in \mathbb{N}^d} a_\alpha N_\alpha < \infty$ which is equivalent to $R^{|\alpha|} N_\alpha \leqslant C_0 M_\alpha$ for some $C_0 > 0$ and any $\alpha \in \mathbb{N}^d$.

(ii) We may assume $\mathfrak{M}$ is isotropic and $d = 1$. Suppose $\mathfrak{M}$ satisfies $\{\mathfrak{M}.2\}'$. Take any $M \in \overline{V}(\mathfrak{M})$. Let $\mu > 0$ be arbitrary and $\lambda > 0$ be such that $M_{p+1}^\mu \leqslant C_0' M_p^\lambda$ for some $C_0' > 0$ and any $p \in \mathbb{N}$. Consider the sequence $a_0 = 1$ and $a_n = 1/M_{n-1}$ for $n \geqslant 1$. Then $\sup_{n \geqslant 1} a_n M_n^\mu \leqslant C_0' M_{n-1}^\lambda / M_{n-1} < \infty$. Then by Lemma 4.2.12($ii$) there is some $N \in \overline{V}(\mathfrak{M})$ such that $N_{n+1} \leqslant C_0 M_n$ for some $C_0 > 0$ and all $n \geqslant 0$. $\qquad\square$

We are ready to state and prove the main result of this section.

Theorem 4.2.14. *Let $\mathfrak{M}$ be a weight sequence system satisfying $\{L\}$ and $\{\mathfrak{M}.2\}'$, and let $\mathscr{W}$ be a non-degenerate weight function system satisfying $\{wM\}$ and $\{N\}$. Then, $\varphi \in C^{\infty}(\mathbb{R}^d)$ belongs to $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}$ if and only if $\|\varphi\|_{\mathcal{S}_{w,\infty}^M} < \infty$ for all $M \in \overline{V}(\mathfrak{M})$ and $w \in \overline{V}(\mathscr{W})$. Moreover, the topology of $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}$ is generated by the system of seminorms $\{\|\cdot\|_{\mathcal{S}_{w,\infty}^M} \mid M \in \overline{V}(\mathfrak{M}), w \in \overline{V}(\mathscr{W})\}$.*

We define

$$\iota : C^{\infty}(\mathbb{R}^d) \to C(\mathbb{N}^d \times \mathbb{R}^d), \qquad \iota(\varphi) = [(\alpha, x) \mapsto \varphi^{(\alpha)}(x)].$$

The proof of Theorem 4.2.14 is based on the ensuing lemma.

Lemma 4.2.15. *Let $\mathfrak{M}$ be a weight sequence system satisfying $\{L\}$ and $\{\mathfrak{M}.2\}'$, and let $\mathscr{W}$ be a non-degenerate weight function system satisfying $\{wM\}$ and $\{N\}$. Then, $\varphi \in C^{\infty}(\mathbb{R}^d)$ belongs to $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}$ if and only if $\iota(\varphi) \in (\mathfrak{M}^{\circ} \otimes \mathscr{W})C(\mathbb{N}^d \times \mathbb{R}^d)$. Moreover,*

$$\iota : \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}} \to (\mathfrak{M}^{\circ} \otimes \mathscr{W})C(\mathbb{N}^d \times \mathbb{R}^d)$$

is a topological embedding.

Proof. The first part and the fact that ι is continuous are obvious. We now show that ι is a topological embedding. Fix an arbitrary $q \in (1, \infty)$. For $n \in \mathbb{Z}_+$ we write X_n for the Banach space consisting of all $\varphi \in C^{\infty}(\mathbb{R}^d)$ such that

$$\|\varphi\|_{X_n} = \left(\sum_{\alpha \in \mathbb{N}^d} \left(\frac{\left\| \varphi^{(\alpha)} w^n \right\|_{L^q}}{M_{\alpha}^n} \right)^q \right)^{1/q} < \infty$$

and Y_n for the Banach space consisting of all sequences $(\varphi_{\alpha})_{\alpha \in \mathbb{N}^d}$ of measurable functions such that

$$\|(\varphi_{\alpha})_{\alpha \in \mathbb{N}^d}\|_{Y_n} = \left(\sum_{\alpha \in \mathbb{N}^d} \left(\frac{\left\| \varphi_{\alpha} w^n \right\|_{L^q}}{M_{\alpha}^n} \right)^q \right)^{1/q} < \infty.$$

Note that both X_n and Y_n are reflexive. The mapping $\rho_n : X_n \to Y_n, \varphi \mapsto (\varphi^{(\alpha)})_{\alpha \in \mathbb{N}^d}$ is a topological embedding. Set $X = \varinjlim_{n \in \mathbb{Z}_+} X_n$

and $Y = \varinjlim_{n \in \mathbb{Z}_+} Y_n$. Condition {L} implies that $X = \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\},q} = \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ as locally convex spaces. We claim that $\rho = \varinjlim_{n \in \mathbb{Z}_+} \rho_n : X = \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} \to Y$ is a topological embedding. Before we prove the claim, let us show how it entails the result. Condition {L} and Lemma 3.2.2 imply that the mapping

$$\tau : (\mathfrak{M}^\circ \otimes \mathscr{W}) C(\mathbb{N}^d \times \mathbb{R}^d) \to Y, \qquad f \mapsto (f(\alpha, \, \cdot \,))_{\alpha \in \mathbb{N}^d}$$

is well-defined and continuous. Note that $\rho = \tau \circ \iota$. Hence, ι is a topological embedding because ρ is so. We now show the claim with the aid of the dual Mittag-Leffler theorem 2.2.2. For $n \in \mathbb{Z}_+$ we set $Z_n = Y_n / \rho_n(X_n)$. Hence, Z_n is a reflexive Banach space. We denote by $\pi_n : Y_n \to Z_n$ the quotient mapping. The natural linking mappings $Z_n \to Z_{n+1}$ are injective since $\rho(X_{n+1}) \cap Y_n = \rho(X_n)$. Consider the following injective inductive sequence of short topologically exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{\rho_1} & Y_1 & \xrightarrow{\pi_1} & Z_1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \xrightarrow{\rho_2} & Y_2 & \xrightarrow{\pi_2} & Z_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

The linking mappings of the inductive spectra $(X_n)_{n \in \mathbb{Z}_+}$, $(Y_n)_{n \in \mathbb{Z}_+}$ and $(Z_n)_{n \in \mathbb{Z}_+}$ are weakly compact as continuous linear mappings between reflexive Banach spaces. In particular, these inductive spectra are regular [80, Lemma 3]. Furthermore, $\varinjlim_{n \in \mathbb{Z}_+} X_n = X = \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ is Montel since it is a nuclear (DF)-space. Therefore, the dual Mittag-Leffler theorem 2.2.2 yields that $\rho = \varinjlim_{n \in \mathbb{Z}_+} \rho_n$ is a topological embedding. $\qquad \square$

Proof of Theorem 4.2.14. We write E for the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathcal{S}^M_{w,\infty}} < \infty$ for all $M \in \overline{V}(\mathfrak{M})$ and $w \in \overline{V}(\mathscr{W})$ endowed with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_{\mathcal{S}^M_{w,\infty}} \mid M \in \overline{V}(\mathfrak{M}), w \in \overline{V}(\mathscr{W})\}$. We need to show that

$\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ and E coincide as locally convex spaces. Since $\overline{V}(\mathfrak{M}^\circ)\otimes\overline{V}(\mathscr{W})$ is upward dense in $\overline{V}(\mathfrak{M}^\circ\otimes\mathscr{W})$, we have that $\varphi\in C^\infty(\mathbb{R}^d)$ belongs to E if and only if $\iota(\varphi)\in C\overline{V}(\mathfrak{M}^\circ\otimes\mathscr{W})(\mathbb{R}^d)$ and that $\iota : E \to C\overline{V}(\mathfrak{M}^\circ\otimes\mathscr{W})(\mathbb{R}^d)$ is a topological embedding. As both the Nachbin families $\mathfrak{M}^\circ$ and $\mathscr{W}$ satisfy (S), $\mathfrak{M}^\circ\otimes\mathscr{W}$ does so as well. Hence, $(\mathfrak{M}^\circ\otimes\mathscr{W})C(\mathbb{N}^d\times\mathbb{R}^d) = C\overline{V}(\mathfrak{M}^\circ\otimes\mathscr{W})(\mathbb{R}^d)$ as locally convex spaces. The result now follows from Lemma 4.2.15. $\qquad\square$

4.2.3 Kernel theorems

We prove kernel theorems for the spaces $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ in this section. To this end, we introduce vector-valued versions of $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ and give a tensor product representation for them.

Let $\mathfrak{M}$ be a weight sequence system, let $\mathscr{W}$ be a weight function system and let E be a lcHs. We define $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}(\mathbb{R}^d;E) = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d;E)$ as the space consisting of all $\boldsymbol{\varphi}\in C^\infty(\mathbb{R}^d;E)$ such that for all $p\in\mathrm{csn}(E)$ and $\lambda\in\mathbb{R}_+$ (for all $p\in\mathrm{csn}(E)$, $M\in\overline{V}(\mathfrak{M})$ and $w\in\overline{V}(\mathscr{W})$)

$$p_\lambda(\boldsymbol{\varphi}) = \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d}\frac{p(\boldsymbol{\varphi}^{(\alpha)}(x))w^\lambda(x)}{M_\alpha^\lambda} < \infty$$
$$\left(p_{M,w}(\boldsymbol{\varphi}) = \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d}\frac{p(\boldsymbol{\varphi}^{(\alpha)}(x))w(x)}{M_\alpha} < \infty\right).$$

We endow $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d;E)$ with the locally convex topology generated by the system of seminorms $\{p_\lambda\,|\,p\in\mathrm{csn}(E),\lambda\in\mathbb{R}_+\}$ ($\{p_{M,w}\,|\,p\in\mathrm{csn}(E), M\in\overline{V}(\mathfrak{M}), w\in\overline{V}(\mathscr{W})\}$).

Proposition 4.2.16. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and* [$\mathfrak{M}$.2]′, *let $\mathscr{W}$ be a non-degenerate weight function system satisfying* [wM] *and* [N], *and let E be a complete lcHs. Then, the following canonical isomorphisms of locally convex spaces hold*

$$\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d;E) \cong \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)\varepsilon E \cong \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}(\mathbb{R}^d)\widehat{\otimes}E.$$

We will make use of the ensuing result of Komatsu [83] to show Proposition 4.2.16.

Lemma 4.2.17 ([83, Lemma 1.12]). *Let G be a semi-Montel lcHs such that G is continuously included in $C(\mathbb{R}^d)$ and let E be a complete lcHs. Then, every function $\boldsymbol{\varphi} \in C(\mathbb{R}^d; E)$ satisfying*

$$\langle e', \boldsymbol{\varphi} \rangle : \mathbb{R}^d \to \mathbb{C}, \ x \mapsto \langle e', \boldsymbol{\varphi}(x) \rangle \ \text{belongs to } G \text{ for all } e' \in E' \quad (4.4)$$

defines an element of $G \varepsilon E$ via $E'_c \to G$, $e' \mapsto \langle e', \boldsymbol{\varphi} \rangle$. Conversely, for every $T \in G \varepsilon E$ there is a unique $\boldsymbol{\varphi} \in C(\mathbb{R}^d; E)$ satisfying (4.4) such that $T(e') = \langle e', \boldsymbol{\varphi} \rangle$ for all $e' \in E'$.

Proof of Proposition 4.2.16. We only show the Roumieu case as the Beurling case is similar. The second isomorphism follows from the fact that $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d)$ is complete and nuclear (recall that every nuclear (DF)-space is complete). We now show the first isomorphism. This amounts to showing that the mapping

$$\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d; E) \to \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d) \varepsilon E, \quad \boldsymbol{\varphi} \mapsto [e' \mapsto \langle e', \boldsymbol{\varphi} \rangle] \quad (4.5)$$

is a topological isomorphism. We first show that it is a well-defined bijective mapping. By Lemma 4.2.17 with $G = \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d)$ (G is semi-Montel because it is nuclear), it suffices to show that a function $\boldsymbol{\varphi} \in C(\mathbb{R}^d; E)$ belongs to $\mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d; E)$ if and only if $\langle e', \boldsymbol{\varphi} \rangle \in \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d)$ for all $e' \in E'$. The direct implication is obvious. Conversely, let $\boldsymbol{\varphi} \in C(\mathbb{R}^d; E)$ be such that $\langle e', \boldsymbol{\varphi} \rangle \in \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d)$ for all $e' \in E'$. In particular, $\langle e', \boldsymbol{\varphi} \rangle \in C^\infty(\mathbb{R}^d)$ for all $e' \in E'$. By [126, Appendice Lemme II], we have that $\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^d; E)$ and

$$\langle e', \boldsymbol{\varphi} \rangle^{(\alpha)} = \left\langle e', \boldsymbol{\varphi}^{(\alpha)} \right\rangle, \qquad e' \in E', \alpha \in \mathbb{N}^d.$$

Theorem 4.2.14 implies that for all $M \in \overline{V}(\mathfrak{M})$ and $w \in \overline{V}(\mathscr{W})$ the set

$$\left\{ \frac{\boldsymbol{\varphi}^{(\alpha)}(x) w(x)}{M_\alpha} \,|\, x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d \right\}$$

is weakly bounded in E. Hence, this set is bounded in E by Mackey's theorem. This means that $\boldsymbol{\varphi} \in \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d; E)$. Next, we show that the isomorphism in (4.5) holds topologically. Let $M \in \overline{V}(\mathfrak{M})$, $w \in \overline{V}(\mathscr{W})$

and $p \in \operatorname{csn}(E)$ be arbitrary. We denote by B the polar set of the p-unit ball in E. The bipolar theorem yields that

$$\sup_{e' \in B} \| \langle e', \boldsymbol{\varphi} \rangle \|_{\mathcal{S}_{w,\infty}^M}$$

$$= \sup \left\{ \frac{| \langle e', \boldsymbol{\varphi}^{(\alpha)}(x) \rangle | w(x)}{M_\alpha} \mid x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d, e' \in B \right\}$$

$$= p_{M,w}(\boldsymbol{\varphi})$$

for all $\boldsymbol{\varphi} \in \mathcal{S}_{\{\mathscr{W}\}}^{\{\mathfrak{M}\}}(\mathbb{R}^d; E)$. The result now follows from Theorem 4.2.14. $\qquad\square$

We are ready to prove the kernel theorems.

Theorem 4.2.18. *Let $\mathfrak{M}_j$ be a weight sequence system on $\mathbb{N}^{d_j}$ satisfying* [L] *and* [$\mathfrak{M}$.2]$'$*, and let $\mathscr{W}_j$ be a non-degenerate weight function system on $\mathbb{R}^{d_j}$ satisfying* [wM] *and* [N] *for $j = 1, 2$. The following canonical isomorphisms of locally convex spaces hold*

$$\mathcal{S}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}(\mathbb{R}^{d_1 + d_2}) \cong \mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})$$

$$\cong L_b(\mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1})_b', \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})) \qquad (4.6)$$

and

$$\mathcal{S}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}(\mathbb{R}^{d_1 + d_2})_b' \cong \mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1})_b' \widehat{\otimes} \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})_b'$$

$$\cong L_b(\mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1}), \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})_b'). \qquad (4.7)$$

Proof. The isomorphisms in (4.7) follow from those in (4.6) and the general theory of nuclear Fréchet and (DF)-spaces, see e.g. [83, Theorem 2.2]. We now show the isomorphisms in (4.6). By Proposition 4.2.16 and the fact that $\mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1})$ is Montel (as it is nuclear and barreled), it is enough to show that the following canonical isomorphism of locally convex spaces holds

$$\mathcal{S}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}(\mathbb{R}^{d_1 + d_2}) \cong \mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1}; \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})).$$

This amounts to verify that the mappings

$$\mathcal{S}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}(\mathbb{R}^{d_1 + d_2}) \to \mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1}; \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})) : \varphi \mapsto [x_1 \mapsto \varphi(x_1, \cdot)]$$

and

$$\mathcal{S}_{[\mathscr{W}_1]}^{[\mathfrak{M}_1]}(\mathbb{R}^{d_1}; \mathcal{S}_{[\mathscr{W}_2]}^{[\mathfrak{M}_2]}(\mathbb{R}^{d_2})) \to \mathcal{S}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}(\mathbb{R}^{d_1 + d_2}) :$$
$$\boldsymbol{\varphi} \mapsto [(x_1, x_2) \mapsto \boldsymbol{\varphi}(x_1)(x_2)],$$

who are each others inverses, are well-defined and continuous. But the proofs of these facts are standard and therefore omitted (we only remark that in the Roumieu case one needs to use Theorem 4.2.14).
□

4.3 The Beurling-Björck spaces

4.3.1 The space $\mathcal{S}_{[\eta]}^{[\omega]}$

In this section, by a *weight function* on $\mathbb{R}^d$ we simply mean a non-negative, measurable, and locally bounded function. We consider the following standard conditions [11, 21] (see also Section 3.2.3):

(α) There are $L, C > 0$ such that $\omega(x+y) \leqslant L(\omega(x)+\omega(y))+\log C$, for all $x, y \in \mathbb{R}^d$.

(γ) There are $A, B > 0$ such that $A \log(1 + |x|) \leqslant \omega(x) + \log B$, for all $x \in \mathbb{R}^d$.

(γ_0) $\displaystyle\lim_{|x| \to \infty} \frac{\omega(x)}{\log |x|} = \infty.$

A weight function ω is called *radially increasing* if $\omega(x) \leqslant \omega(y)$ whenever $|x| \leqslant |y|$.

Given a weight function ω and a parameter $\lambda > 0$, we introduce the family of norms

$$\|\varphi\|_{\omega,\lambda} = \sup_{x \in \mathbb{R}^d} |\varphi(x)| e^{\lambda \omega(x)}.$$

If η is another weight function, we consider the Banach space $\mathcal{S}_{\eta,\omega}^{\lambda}(\mathbb{R}^d)$ consisting of all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathcal{S}_{\eta,\omega}^{\lambda}} := \|\varphi\|_{\eta,\lambda} + \|\widehat{\varphi}\|_{\omega,\lambda} <$

∞. Finally, we define the *Beurling-Björck spaces* (of Beurling and Roumieu type) as

$$\mathcal{S}_{(\eta)}^{(\omega)}(\mathbb{R}^d) = \varprojlim_{\lambda\to\infty} \mathcal{S}_{\eta,\omega}^{\lambda}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}_{\{\eta\}}^{\{\omega\}}(\mathbb{R}^d) = \varinjlim_{\lambda\to 0^+} \mathcal{S}_{\eta,\omega}^{\lambda}(\mathbb{R}^d).$$

In literature, various definitions of the Beurling-Björck spaces have been given [16, 28, 67]. In this section, we study the connection between the conditions (γ) and (γ_0) and the equivalence of said alternate definitions of Beurling-Björck type spaces. Let ω and η be two weight functions. Given parameters $k, l \in \mathbb{N}$ and $\lambda > 0$, we introduce the family of norms

$$\|\varphi\|_{\omega,k,l,\lambda} = \max_{|\alpha|\leqslant k} \max_{|\beta|\leqslant l} \sup_{x\in\mathbb{R}^d} |x^\beta \varphi^{(\alpha)}(x) e^{\lambda\omega(x)}|.$$

We define $\widetilde{\mathcal{S}}_{\eta,\omega}^{\lambda}(\mathbb{R}^d)$ as the Fréchet space consisting of all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\widetilde{\mathcal{S}}_{\eta,\omega}^{k,\lambda}} := \|\varphi\|_{\eta,k,k,\lambda} + \|\widehat{\varphi}\|_{\omega,k,k,\lambda} < \infty, \qquad \forall k \in \mathbb{N}.$$

We set

$$\widetilde{\mathcal{S}}_{(\eta)}^{(\omega)}(\mathbb{R}^d) = \varprojlim_{\lambda\to\infty} \widetilde{\mathcal{S}}_{\eta,\omega}^{\lambda}(\mathbb{R}^d) \quad \text{and} \quad \widetilde{\mathcal{S}}_{\{\eta\}}^{\{\omega\}}(\mathbb{R}^d) = \varinjlim_{\lambda\to 0^+} \widetilde{\mathcal{S}}_{\eta,\omega}^{\lambda}(\mathbb{R}^d).$$

The following result is a generalization of [28, Theorem 3.3] and [67, Corollary 2.9] (see also [16, Theorem 2.3]).

Theorem 4.3.1. *Let ω and η be two weight functions satisfying (α). Suppose that $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \neq \{0\}$. The following statements are equivalent:*

(i) ω *and* η *satisfy* (γ) *((γ_0) in the Roumieu case).*

(ii) $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) = \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ *as locally convex spaces.*

(iii) $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) = \{\varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \forall \lambda > 0 \ (\exists \lambda > 0) \ \forall \alpha \in \mathbb{N}^d :$
$$\sup_{x\in\mathbb{R}^d} |x^\alpha \varphi(x)| e^{\lambda\eta(x)} < \infty \quad and \quad \sup_{\xi\in\mathbb{R}^d} |\xi^\alpha \widehat{\varphi}(\xi)| e^{\lambda\omega(\xi)} < \infty\}.$$

(iv) $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) = \{\varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \forall \lambda > 0 \ (\exists \lambda > 0) \ \forall \alpha \in \mathbb{N}^d :$

$$\int_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)| e^{\lambda \eta(x)} dx < \infty \ \text{ and } \ \int_{\xi \in \mathbb{R}^d} |\widehat{\varphi}^{(\alpha)}(\xi)| e^{\lambda \omega(\xi)} d\xi < \infty \}.$$

(v) $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d).$

Following [67], our proof of Theorem 4.3.1 is based on the mapping properties of the short-time Fourier transform (see Section 3.4.1). To this purpose, we introduce two additional function spaces. Given a parameter $\lambda > 0$, we define $\mathcal{S}_{\omega}^{\lambda}(\mathbb{R}^d)$ as the Fréchet space consisting of all $\varphi \in C^{\infty}(\mathbb{R}^d)$ such that $\|\varphi\|_{\omega,k,\lambda} := \|\varphi\|_{\omega,k,0,\lambda} < \infty$ for all $k \in \mathbb{N}$ and set

$$\mathcal{S}_{(\omega)}(\mathbb{R}^d) = \varprojlim_{\lambda \to \infty} \mathcal{S}_{\omega}^{\lambda}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}_{\{\omega\}}(\mathbb{R}^d) = \varinjlim_{\lambda \to 0^+} \mathcal{S}_{\omega}^{\lambda}(\mathbb{R}^d).$$

Given a parameter $\lambda > 0$, we define $C_{\omega}^{\lambda}(\mathbb{R}^d)$ as the Banach space consisting of all $\varphi \in C(\mathbb{R}^d)$ such that $\|\varphi\|_{\omega,\lambda} < \infty$ and set

$$C_{(\omega)}(\mathbb{R}^d) = \varprojlim_{\lambda \to \infty} C_{\omega}^{\lambda}(\mathbb{R}^d) \quad \text{and} \quad C_{\{\omega\}}(\mathbb{R}^d) = \varinjlim_{\lambda \to 0^+} C_{\omega}^{\lambda}(\mathbb{R}^d).$$

We need the following extension of [67, Theorem 2.7].

Proposition 4.3.2. *Let ω and η be weight functions satisfying (α) and (γ) $((\gamma_0)$ in the Roumieu case). Define the weight $\eta \oplus \omega(x,\xi) := \eta(x) + \omega(\xi)$ for $(x,\xi) \in \mathbb{R}^{2d}$. Fix a window $\psi \in \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$.*

(a) *The linear mappings*

$$V_{\check{\psi}} : \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \ \text{ and } \ V_{\psi}^* : C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$$

are continuous.

(b) *The linear mappings*

$$V_{\check{\psi}} : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to \mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \ \text{ and } \ V_{\psi}^* : \mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$$

are continuous.

Proof. It suffices to show that $V_{\check{\psi}} : \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to \mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d})$, $V_{\check{\psi}} :$ $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d})$, and $V_{\psi}^* : C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ are continuous. Indeed, the continuity of $V_{\psi}^* : \mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ and $V_{\check{\psi}} : \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d})$ would be immediate consequences, whereas, in view of (3.1), we could then always factor $V_{\check{\psi}}$ on $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ as a composition of continuous mappings,

$$V_{\check{\psi}} : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \xrightarrow{V_{\check{\psi}}} C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \xrightarrow{V_{\chi}^*} \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \xrightarrow{V_{\check{\psi}}} \mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}), \quad (4.8)$$

where, when $\psi \neq 0$, the window χ is chosen such that $\chi \in \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ and $(\psi, \check{\chi})_{L^2} = 1$. (The relation (4.8) actually yields $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) = \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$.)

Suppose that $\psi \in \widetilde{\mathcal{S}}_{\eta,\omega}^{\lambda_0}(\mathbb{R}^d)$, so that $\lambda_0 > 0$ is fixed in the Roumieu case but can be taken as large as needed in the Beurling case. Let A and $B = B_A$ be the constants occurring in (γ) (in the Roumieu case, A can be taken as large as needed due to (γ_0)). Furthermore, we assume that all constants occurring in (α) and (γ) (resp. (γ_0)) are the same for both ω and η. We may also assume that $\lambda_0 - k/A > 0$. We first consider $V_{\check{\psi}}$. Let $\lambda < (\lambda_0 - k/A)/L$ be arbitrary. For all $k \in \mathbb{N}$ and $\varphi \in \mathcal{S}_{\eta,\omega}^{\lambda L + \frac{k}{A}}(\mathbb{R}^d)$, it holds that

$$\max_{|\alpha+\beta| \leq k} \sup_{(x,\xi) \in \mathbb{R}^{2d}} |\partial_\xi^\beta \partial_x^\alpha V_{\check{\psi}} \varphi(x,\xi)| e^{\lambda \eta(x)}$$

$$\leq (2\pi)^k \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} e^{\lambda \eta(x)} \int_{\mathbb{R}^d} |\varphi(t)|(1+|t|)^k |\psi^{(\alpha)}(x-t)| dt$$

$$\leq (2\pi)^k \|\psi\|_{\eta,k,\lambda_0} \|\varphi\|_{\eta,\lambda L + \frac{k}{A}}$$

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{\lambda(\eta(x) - L\eta(t))} (1+|t|)^k e^{-\frac{k}{A}\eta(t)} e^{-\lambda_0 \eta(x-t)} dt$$

$$\leq (2\pi)^k B^{\frac{k}{A}} C^\lambda \|\psi\|_{\eta,k,\lambda_0} \|\varphi\|_{\eta,\lambda L + \frac{k}{A}} \int_{\mathbb{R}^d} e^{-(\lambda_0 - \lambda L)\eta(y)} dy$$

and

$$
\max_{|\alpha+\beta|\leqslant k} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_\xi^\beta \partial_x^\alpha V_{\check{\psi}}\varphi(x,\xi)|e^{\lambda\omega(\xi)}
$$

$$
= \max_{|\alpha+\beta|\leqslant k} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_\xi^\beta \partial_x^\alpha V_{\mathcal{F}(\check{\varphi})}\widehat{\psi}(\xi,-x)|e^{\lambda\omega(\xi)}
$$

$$
\leqslant (2\pi)^k \max_{|\beta|\leqslant k} \sup_{\xi\in\mathbb{R}^d} e^{\lambda\omega(\xi)} \int_{\mathbb{R}^d} |\widehat{\psi}(t)|(1+|t|)^k|\widehat{\varphi}^{(\beta)}(\xi-t)|dt
$$

$$
\leqslant (2\pi)^k B^{\frac{k}{A}} C^\lambda \|\widehat{\varphi}\|_{\omega,k,L\lambda}\|\widehat{\psi}\|_{\omega,\lambda_0} \int_{\mathbb{R}^d} e^{-(\lambda_0-\lambda L-k/A)\omega(t)}dt.
$$

These inequalities imply the continuity of $V_{\check{\psi}} : \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to \mathcal{S}_{[\eta\oplus\omega]}(\mathbb{R}^{2d})$. Taking $k = 0$ in the above norm bounds, we also obtain that $V_{\check{\psi}} : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta\oplus\omega]}(\mathbb{R}^{2d})$ is continuous. Next, we treat V_ψ^*. Let $\lambda < \lambda_0/L$ be arbitrary. For all $k \in \mathbb{N}$ and $\Phi \in C_{\eta\oplus\omega}^{\lambda L+\frac{k}{A}}(\mathbb{R}^{2d})$ it holds that

$$
\|V_\psi^*\Phi\|_{\eta,k,\lambda}
$$

$$
\leqslant (2\pi)^k \max_{|\alpha|\leqslant k} \sup_{t\in\mathbb{R}^d} e^{\lambda\eta(t)}
$$

$$
\sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \iint_{\mathbb{R}^{2d}} |\Phi(x,\xi)|(1+|\xi|)^k|\psi^{(\beta)}(t-x)|dxd\xi
$$

$$
\leqslant (4\pi)^k \|\psi\|_{\eta,k,\lambda_0}\|\Phi\|_{\eta\oplus\omega,\lambda L+\frac{k}{A}}
$$

$$
\iint_{\mathbb{R}^{2d}} (1+|\xi|)^k e^{-\left(\frac{k}{A}+\lambda L\right)\omega(\xi)} e^{\lambda(\eta(t)-L\eta(x))} e^{-\lambda_0\eta(t-x)}dxd\xi
$$

$$
\leqslant (4\pi)^k B^{\frac{k}{A}} C^\lambda \|\psi\|_{\eta,k,\lambda_0}\|\Phi\|_{\eta\oplus\omega,\lambda L+\frac{k}{A}} \iint_{\mathbb{R}^{2d}} e^{-\lambda L\omega(\xi)-(\lambda_0-\lambda L)\eta(y)}dyd\xi
$$

and

$$
\|\mathcal{F}(V_\psi^*\Phi)\|_{\omega,k,\lambda}
$$

$$
= \max_{|\alpha|\leqslant k} \sup_{t\in\mathbb{R}^d} e^{\lambda\omega(t)} \left|\partial_t^\alpha \iint_{\mathbb{R}^d} \Phi(x,\xi)e^{2\pi i\xi\cdot x}e^{-2\pi it\cdot x}\widehat{\psi}(t-\xi)dxd\xi\right|
$$

$$\leqslant (4\pi)^k B^{\frac{k}{A}} C^\lambda \|\widehat{\psi}\|_{\omega,k,\lambda_0} \|\Phi\|_{\eta\oplus\omega,\lambda L+\frac{k}{A}} \iint\limits_{\mathbb{R}^{2d}} e^{-\lambda L\eta(x)-(\lambda_0-\lambda L)\omega(\xi)} dx d\xi.$$

Since $\| \cdot \|_{\eta,k,k,\lambda} \leqslant B^{\frac{k}{A}} \| \cdot \|_{\eta,k,\lambda+\frac{k}{A}}$ and $\| \cdot \|_{\omega,k,k,\lambda} \leqslant B^{\frac{k}{A}} \| \cdot \|_{\omega,k,\lambda+\frac{k}{A}}$ for all $\lambda > 0$ and $k \in \mathbb{N}$, the above inequalities show the continuity of V_ψ^*. $\qquad\square$

In order to be able to apply Proposition 4.3.2, we show the ensuing simple lemma.

Lemma 4.3.3. *Let ω and η be weight functions satisfying (α). If $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \neq \{0\}$, then also $\widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \neq \{0\}$.*

Proof. Let $\varphi \in \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)\backslash\{0\}$. Pick $\psi, \chi \in \mathcal{D}(\mathbb{R}^d)$ for which we have $\int_{\mathbb{R}^d} \varphi(x)\psi(-x)dx = 1$ and $\int_{\mathbb{R}^d} \chi(x)dx = 1$. Then, $\varphi_0 = (\varphi * \chi)\mathcal{F}^{-1}(\psi) \in \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ and $\varphi_0 \not\equiv 0$ (as $\varphi_0(0) = 1$). $\qquad\square$

Proof of Theorem 4.3.1. $(i) \Rightarrow (ii)$ In view of Lemma 4.3.3, this follows from Proposition 4.3.2 and the reconstruction formula (3.1).

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (v)$ *and* $(iv) \Rightarrow (v)$ These implications follow from the fact that $\mathcal{S}(\mathbb{R}^d)$ consists precisely of all those $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \varphi(x)| < \infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^d} |\xi^\alpha \widehat{\varphi}(\xi)| < \infty$$

for all $\alpha \in \mathbb{N}^d$ (see e.g. [28, Corollary 2.2]).

$(v) \Rightarrow (i)$ Since the Fourier transform is an isomorphism from $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ onto $\mathcal{S}_{[\omega]}^{[\eta]}(\mathbb{R}^d)$ and from $\mathcal{S}(\mathbb{R}^d)$ onto itself, it is enough to show that η satisfies (γ) $((\gamma_0)$ in the Roumieu case). We start by constructing $\varphi_0 \in \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ such that $\varphi(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$, similar as in Lemma 3.3.9. Choose $\psi \in \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ such that $\psi(0) = 1$. Set

$$\chi(x) = \int_{[-\frac{1}{2},\frac{1}{2}]^d} e^{-2\pi i x \cdot t} dt, \qquad x \in \mathbb{R}^d.$$

Then, $\chi(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Hence, $\varphi_0 = \psi\chi$ satisfies all requirements. Let $(\lambda_j)_{j\in\mathbb{Z}^d}$ be an arbitrary multi-indexed sequence of

positive numbers such that $\lambda_j \to \infty$ as $|j| \to \infty$ $((\lambda_j)_{j \in \mathbb{Z}^d} = (\lambda)_{j \in \mathbb{Z}^d}$ for $\lambda > 0$ in the Roumieu case). Consider

$$\varphi = \sum_{j \in \mathbb{Z}^d} \frac{e^{-\lambda_j \eta(j)}}{(1+|j|)^{d+1}} \varphi_0(\cdot - j) \in \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d).$$

Since $\mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$, there is $C > 0$ such that

$$\frac{e^{-\lambda_j \eta(j)}}{(1+|j|)^{d+1}} = |\varphi(j)| \leqslant \frac{C}{(1+|j|)^{d+2}}$$

for all $j \in \mathbb{Z}^d$. Hence,

$$\log(1+|j|) \leqslant \lambda_j \eta(j) + \log C$$

for all $j \in \mathbb{Z}^d$. As η satisfies (α) and $(\lambda_j)_{j \in \mathbb{Z}^d}$ is arbitrary, the latter inequality is equivalent to (γ) $((\gamma_0)$ in the Roumieu case).

$(i) \Rightarrow (iv)$ Let us denote the space in the right-hand side of (iv) by $\mathcal{S}^{[\omega],1}_{[\eta],1}(\mathbb{R}^d)$. Since we already showed that $(i) \Rightarrow (ii)$ and we have that $\tilde{\mathcal{S}}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \subseteq \mathcal{S}^{[\omega],1}_{[\eta],1}(\mathbb{R}^d)$, it suffices to show that $\mathcal{S}^{[\omega],1}_{[\eta],1}(\mathbb{R}^d) \subseteq \tilde{\mathcal{S}}^{[\omega]}_{[\eta]}(\mathbb{R}^d)$. By Proposition 4.3.2$(a)$, Lemma 4.3.3 and the reconstruction formula (3.1), it suffices to show that $V_{\check\psi}(\varphi) \in C_{[\eta \oplus \omega]}(\mathbb{R}^{2d})$ for all $\varphi \in \mathcal{S}^{[\omega],1}_{[\eta],1}(\mathbb{R}^d)$, where $\psi \in \tilde{\mathcal{S}}^{[\omega]}_{[\eta]}(\mathbb{R}^d)$ is a fixed window. But the latter can be shown by using the same method employed in the first part of the proof of Proposition 4.3.2. $\qquad\square$

4.3.2   Nuclearity

We characterize in this section the nuclearity of the Beurling-Björck spaces. In particular, our goal is to show the following result.

Theorem 4.3.4. *Let ω and η be weight functions satisfying (α). If ω and η satisfy (γ) $((\gamma_0)$ in the Roumieu case), then $\mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d)$ is nuclear. Conversely, if in addition ω and γ are radially increasing, then the nuclearity of $\mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d)$ implies that ω and η satisfy (γ) $((\gamma_0)$ in the Roumieu case), provided that $\mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \neq \{0\}$.*

Our proof of Theorem 4.3.4 is based on Proposition 4.3.2(*b*), Lemma 4.2.4 and the next auxiliary result.

Proposition 4.3.5. *Let η be a weight function satisfying (α) and (γ) $((\gamma_0)$ in the Roumieu case). Then, $\mathcal{S}_{[\eta]}(\mathbb{R}^d)$ is nuclear.*

Proof. We present two different proofs:

(i) The first one is based on a classical result of Gelfand and Shilov [62, p. 181]. The nuclearity of $\mathcal{S}_{(\eta)}(\mathbb{R}^d)$ is a particular case of this result, as the increasing sequence of weight functions $(e^{n\eta})_{n\in\mathbb{N}}$ satisfies the so-called (P) and (N) conditions because of (γ). For the Roumieu case, note that

$$\mathcal{S}_{\{\eta\}}(\mathbb{R}^d) = \varinjlim_{n\in\mathbb{Z}_+} \varprojlim_{k\geqslant n} \mathcal{S}_\eta^{\frac{1}{n}-\frac{1}{k}}(\mathbb{R}^d)$$

as locally convex spaces. The above mentioned result implies that, for each $n \in \mathbb{Z}_+$, the Fréchet space $\varprojlim_{k\geqslant n} \mathcal{S}_\eta^{\frac{1}{n}-\frac{1}{k}}(\mathbb{R}^d)$ is nuclear, as the increasing sequence of weight functions $(e^{\left(\frac{1}{n}-\frac{1}{k}\right)\eta})_{k\geqslant n}$ satisfies the conditions (P) and (N) because of (γ_0). The result now follows from the fact that the inductive limit of a countable spectrum of nuclear spaces is again nuclear.

(ii) Next, we give a proof that only makes use of the fact that $\mathcal{S}(\mathbb{R}^d)$ is nuclear. Our argument adapts an idea of Hasumi [70]. Fix a non-negative function $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \chi(y)dy = 1$ and for each $\lambda > 0$ let

$$\Psi_\lambda(x) = \exp\left(\lambda L \int_{\mathbb{R}^d} \chi(y)\eta(x + y)dy\right).$$

It is clear from the assumption (α) that η should have at most polynomial growth. So, we fix $q > 0$ such that $(1+|x|)^{-q}\eta(x)$ is bounded. We obtain that there are positive constants c_λ, C_λ, $C_{\lambda,\beta}$, and $C_{\lambda_1,\lambda_2,\beta}$ such that

$$c_\lambda \exp\left(\lambda\eta(x)\right) \leqslant \Psi_\lambda(x) \leqslant C_\lambda \exp(L^2\lambda\eta(x)), \qquad (4.9)$$

$$|\Psi_\lambda^{(\beta)}(x)| \leqslant C_{\lambda,\beta}(1 + |x|)^{q|\beta|}\Psi_\lambda(x), \qquad (4.10)$$

and

$$\left| \left(\frac{\Psi_{\lambda_1}}{\Psi_{\lambda_2}} \right)^{(\beta)} (x) \right| \leqslant C_{\lambda_1,\lambda_2,\beta} (1 + |x|)^{q|\beta|}, \qquad (4.11)$$

for each $\beta \in \mathbb{N}^d$, and $\lambda_1 \leqslant \lambda_2$. Let $X_\lambda = \Psi_\lambda^{-1} \mathcal{S}(\mathbb{R}^d)$ and topologize each of these spaces in such a way that the multiplier mappings $M_{\Psi_\lambda} : X_\lambda \to \mathcal{S}(\mathbb{R}^d) : \varphi \mapsto \Psi_\lambda \cdot \varphi$ are isomorphisms. The bounds (4.11) guarantee that the inclusion mappings $X_{\lambda_2} \to X_{\lambda_1}$ are continuous whenever $\lambda_1 \leqslant \lambda_2$. If A is a constant such that (γ) holds for η, then the inequalities (4.9) and (4.10) clearly yield

$$\max_{|\beta| \leqslant k} \sup_{x \in \mathbb{R}^d} (1 + |x|)^k |(\Psi_\lambda \varphi)^{(\beta)}(x)| \leqslant B_{k,\lambda,A} \|\varphi\|_{\eta,k,\lambda L^2 + (1+q)k/A}$$

and

$$
\begin{aligned}
\|\varphi\|_{\eta,k,\lambda} &\leqslant \frac{1}{c_\lambda} \max_{|\beta| \leqslant k} \left\| \Psi_\lambda \varphi^{(\beta)} \right\|_{L^\infty} \\
&\leqslant \frac{1}{c_\lambda} \max_{|\beta| \leqslant k} \left(\left\| (\Psi_\lambda \varphi)^{(\beta)} \right\|_{L^\infty} + \sum_{\nu < \beta} \binom{\beta}{\nu} \left\| \Psi_\lambda^{(\beta - \nu)} \varphi^{(\nu)} \right\|_{L^\infty} \right) \\
&\leqslant b'_{k,\lambda} \max_{|\beta| \leqslant k} \left\| (\Psi_\lambda \varphi)^{(\beta)} \right\|_{L^\infty} + \max_{|\beta| \leqslant k-1} \left\| (1 + |\cdot|)^{qk} \Psi_\lambda \varphi^{(\beta)} \right\|_{L^\infty} \\
&\leqslant b_{k,\lambda} \max_{|\beta| \leqslant k} \left\| (1 + |\cdot|)^{qk(k+1)/2} (\Psi_\lambda \varphi)^{(\beta)} \right\|_{L^\infty},
\end{aligned}
$$

for some positive constants $B_{k,\lambda,A}, b'_{k,\lambda}$ and $b_{k,\lambda}$. This gives, as locally convex spaces,

$$\mathcal{S}_{(\eta)}(\mathbb{R}^d) = \varprojlim_{n \in \mathbb{Z}_+} X_n$$

and the continuity of the inclusion $X_\lambda \to \mathcal{S}_\eta^\lambda(\mathbb{R}^d)$. If in addition (γ_0) holds, we can choose A arbitrarily large above. Consequently, the inclusion $\mathcal{S}_\eta^{L^2 \lambda + \varepsilon}(\mathbb{R}^d) \to X_\lambda$ is continuous as well for any arbitrary $\varepsilon > 0$, whence we infer the topological equality

$$\mathcal{S}_{\{\eta\}}(\mathbb{R}^d) = \varinjlim_{n \in \mathbb{Z}_+} X_{1/n}.$$

The claimed nuclearity of $\mathcal{S}_{[\eta]}(\mathbb{R}^d)$ therefore follows from that of $\mathcal{S}(\mathbb{R}^d)$ and the well-known stability of this property under projective and (countable) inductive limits [131, Proposition 50.1, p. 514]. $\qquad \square$

Proof of Theorem 4.3.4. We first suppose that ω and η satisfy (γ) ((γ_0) in the Roumieu case). W.l.o.g. we may assume that $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \neq \{0\}$. In view of Lemma 4.3.3, Proposition 4.3.2(b) and the reconstruction formula (3.1) imply that $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ is isomorphic to a (complemented) subspace of $\mathcal{S}_{[\eta \oplus \omega]}(\mathbb{R}^{2d})$. The latter space is nuclear by Proposition 4.3.5. The result now follows from the fact that nuclearity is inherited to subspaces.

Next, we suppose that ω and η are radially increasing and that $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ is nuclear and non-trivial. Since the Fourier transform is a topological isomorphism from $\mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ onto $\mathcal{S}_{[\omega]}^{[\eta]}(\mathbb{R}^d)$, it is enough to show that η satisfies (γ) ((γ_0) in the Roumieu case). Consider the Köthe set $A_\eta = \{(e^{\lambda \eta(j)})_{j \in \mathbb{Z}^d} \mid \lambda \in \mathbb{R}_+\}$. Note that, by Proposition 2.3.1, $\lambda^1[A_\eta]$ is nuclear if and only if

$$\exists \lambda > 0 \ (\forall \lambda > 0) : \sum_{j \in \mathbb{Z}^d} e^{-\lambda \eta(j)} < \infty.$$

As η is radially increasing and satisfies (α), the above condition is equivalent to (γ) ((γ_0) in the Roumieu case). Hence, it suffices to show that $\lambda^1[A_\eta]$ is nuclear. To this end, we use Lemma 4.2.4 with $A = A_\eta$ and $E = \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$. We start by constructing $\varphi_0 \in \mathcal{S}_\eta^\omega(\mathbb{R}^d)$ such that

$$\int_{[0,\frac{1}{2}]^d} \varphi_0(j + x)dx = \delta_{j,0}, \qquad j \in \mathbb{Z}^d. \tag{4.12}$$

By Lemma 4.3.3, there is a $\varphi \in \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ such that $\varphi(0) = 1$. Set

$$\chi(x) = \frac{1}{2^d} \int_{[-1,1]^d} e^{-2\pi i x \cdot t} dt, \qquad x \in \mathbb{R}^d.$$

Then, $\chi(j/2) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Hence, $\psi = \varphi \chi \in \widetilde{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ and $\psi(j/2) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Then, $\varphi_0 = (-1)^d \partial_d \cdots \partial_1 \psi$ satisfies all requirements. The linear mappings

$$T : \lambda^1[A_\eta] \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d), \quad T((c_j)_{j \in \mathbb{Z}^d}) = \sum_{j \in \mathbb{Z}^d} c_j \varphi_0(\cdot - j)$$

and

$$S : \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \to \lambda^\infty[A_\eta], \quad S(\varphi) = \left(\int_{[0,\frac{1}{2}]^d} \varphi(x+j) dx \right)_{j \in \mathbb{Z}^d}$$

are continuous. Moreover, by (4.12), we have that $S \circ T = \iota$. $\qquad \square$

Chapter 5

(PLB)-spaces of weighted ultradifferentiable functions

5.1 Introduction

(PLB)-spaces, i.e. countable projective limits of countable inductive limits of Banach spaces, often arise naturally in the context of generalized functions. For instance, the space of distributions, the space of real analytic functionals or the multiplier space $\mathcal{O}_M(\mathbb{R}^d)$ are all classical examples which exhibit this topological structure. Moreover, due to the elevated complexity ultradifferentiability imposes on the topology of the spaces associated to it, (PLB)-spaces appear frequently in the theory of ultradistributions. Thus the determination of the topological properties of such spaces becomes an important problem in functional analysis. We refer to the survey article [49] for applications, examples, and further references on the subject.

In general, the projective limit of bornological spaces is not again bornological, and the same is true for barrelledness. As such topological properties are highly desirable, characterizing when they hold poses an interesting problem. One of the first and arguably most famous results in this direction was Grothendieck's proof for the barrelledness of the multiplier space $\mathcal{O}_M(\mathbb{R}^d)$ [69], which he did by showing it is isomorphic to a complemented subspace of $s \widehat{\otimes} s'$ (later Valdivia showed that actually the tensor representation $\mathcal{O}_M(\mathbb{R}^d) \cong s \widehat{\otimes} s'$ holds [132]). More recently, in [1] (see also [149]) the locally convex

properties of weighted (PLB)-spaces of continuous functions were described in terms of their defining double sequence of weights. Similarly as in Chapter 4, these results allow us to characterize the topological invariants for vastly more complex spaces of test functions.

In this chapter we consider certain variants of the Gelfand-Shilov spaces, more precisely

$$\mathcal{S}_{\{\mathscr{W}\}}^{(\mathfrak{M})} = \varprojlim_{\lambda \to 0^+} \varinjlim_{\mu \to \infty} \mathcal{S}_{w^\mu,\infty}^{M^\lambda} \quad \text{and} \quad \mathcal{S}_{(\mathscr{W})}^{\{\mathfrak{M}\}} = \varprojlim_{\mu \to 0^+} \varinjlim_{\lambda \to \infty} \mathcal{S}_{w^\mu,\infty}^{M^\lambda}$$

for a certain weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$, where we will employ the notation $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]} = \mathcal{S}_{\langle\mathscr{W}\rangle}^{[\mathfrak{M}]}$. A particularly interesting example is the space of multipliers for the Gelfand-Shilov spaces $\mathcal{O}_M^{[\mathfrak{M}],[\mathscr{W}]} = \mathcal{Z}_{\mathscr{W}^\circ}^{[\mathfrak{M}]}$, where $\mathscr{W}^\circ = \{1/w^{1/\lambda} \,|\, \lambda \in \mathbb{R}_+\}$, see [47]. The goal is to characterize the ultrabornologicity and barrelledness of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ through conditions on $\mathscr{W}$ closely related to the topological invariants (Ω) and (DN) for Fréchet spaces of Vogt and Wagner [92]. In the case of $\mathcal{O}_M^{[\mathfrak{M}],[\mathscr{W}]}$, we will see a clear distinction between the Beurling and Roumieu case, where the former will in most cases posses these properties while the latter often does not. Interestingly, our method for showing the necessity of these conditions will require the Gabor accessibility of the Gelfand-Shilov spaces.

The chapter is organized as follows. In Section 5.2 we formally introduce the conditions (Ω) and $\{DN\}$ (and two variants) on a weight function system $\mathscr{W}$. Moreover, we link some of these conditions on $\mathscr{W}_{\mathfrak{M}}$, for a weight sequence system $\mathfrak{M}$, to a condition $[\mathfrak{M}.2]^*$ on $\mathfrak{M}$ itself, and they are equivalent under the right circumstances. The condition $[\mathfrak{M}.2]^*$ is actually a generalization of the condition $(M.2)^*$ on weight sequences [89], which determines when the associated function is a BMT weight function [20]. Then, in Section 5.3, we characterize the topological properties of (PLB)-spaces of weighted continuous functions in our framework. After this, the strategy will be to either embed $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ as a complemented subspace in such a space or to do the reverse, so that the locally convex property in question is inherited by the subspace and may then be determined. Our main result, Theorem 5.4.3, characterizing the ultrabornologicity and barrelledness of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ is then shown in Section 5.4, where we also discuss some specific examples.

5.2   The conditions (Ω) and $\{DN\}$

In this section we introduce and study several conditions for a weight function system $\mathscr{W}$ on a topological space X which will be intrinsically linked to the topological properties of the spaces considered in this chapter. To this regard, we introduce the following notions.

Definition 5.2.1. A weight function system $\mathscr{W}$ on X is said to satisfy condition (Ω) if

$$\forall \lambda \in \mathbb{R}_+ \; \exists \mu < \lambda \; \forall \eta < \mu \; \exists \theta \in (0,1) \; \exists C > 0 \; \forall x \in X :$$

$$\frac{1}{w^\mu(x)} \leqslant C \left(\frac{1}{w^\lambda(x)} \right)^\theta \left(\frac{1}{w^\eta(x)} \right)^{1-\theta} .$$

If "$\exists \theta \in (0,1)$" is replaced by "$\forall \theta \in (0,1)$", then $\mathscr{W}$ is said to satisfy $(\overline{\overline{\Omega}})$.

Definition 5.2.2. A weight function system $\mathscr{W}$ on X is said to satisfy condition $\{DN\}$ if

$$\exists \lambda \in \mathbb{R}_+ \; \forall \mu > \lambda \; \forall \theta \in (0,1) \; \exists \eta > \mu \; \exists C > 0 \; \forall x \in X :$$

$$\frac{1}{w^\mu(x)} \leqslant C \left(\frac{1}{w^\lambda(x)} \right)^\theta \left(\frac{1}{w^\eta(x)} \right)^{1-\theta} .$$

If "$\forall \theta \in (0,1)$" is replaced by "$\exists \theta \in (0,1)$", then $\mathscr{W}$ is said to satisfy $\{\underline{DN}\}$.

Remark 5.2.3. The previous conditions are inspired by and closely related to the topological linear invariants (DN), $(\underline{DN})$, (Ω) and $(\overline{\overline{\Omega}})$ for Fréchet spaces of Vogt and Wagner [92, 142, 146]; see [149] for more information.

The following observation, a direct consequence of Lemma 3.2.1, will prove itself useful in the sequel.

Lemma 5.2.4. *Let $\mathscr{W}$ be a weight function system on $\mathbb{R}^d$ that satisfies* [wM]. *Then $\mathscr{W}$ satisfies (Ω) or $(\overline{\overline{\Omega}})$ (resp. $\{DN\}$ or $\{\underline{DN}\}$) if and only if $\mathscr{W}_{|\mathbb{Z}^d}$ does.*

Our first goal is to relate for any weight sequence system $\mathfrak{M}$ the conditions (Ω), resp. $\{\underline{DN}\}$, on $\mathscr{W}_{\mathfrak{M}}$ to conditions on the weight sequence system itself. To this purpose, we say that $\mathfrak{M}$ satisfies $(\mathfrak{M}.2)^*$ if

$$\exists \lambda \in \mathbb{R}_+ \ \forall \mu \leqslant \lambda \ \exists Q \in \mathbb{Z}_+ \ \exists C_0 > 0 \ \forall \alpha \in \mathbb{N}^d : (M_\alpha^\lambda)^Q \leqslant C_0 M_{Q\alpha}^\mu$$

and $\mathfrak{M}$ satisfies $\{\mathfrak{M}.2\}^*$ if

$$\exists \mu \in \mathbb{R}_+ \ \forall \lambda \geqslant \mu \ \exists Q \in \mathbb{Z}_+ \ \exists C_0 > 0 \ \forall \alpha \in \mathbb{N}^d : (M_\alpha^\lambda)^Q \leqslant C_0 M_{Q\alpha}^\mu.$$

Note that if $(M_\alpha^\lambda)^Q \leqslant C_0 M_{Q\alpha}^\mu$, $\mu \leqslant \lambda$, for some $Q \in \mathbb{Z}_+$, then also $(M_\alpha^\lambda)^{NQ} \leqslant (C_0 M_0^\mu)^N M_{NQ\alpha}^\mu$ for any $N \in \mathbb{Z}_+$. As a consequence, any isotropically decomposable weight sequence system $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]^*$ if and only if every isotropic weight sequence system in its decomposition satisfies $[\mathfrak{M}.2]^*$.

We may characterize $[\mathfrak{M}.2]^*$ via $\mathscr{W}_{\mathfrak{M}}$ in the following way.

Lemma 5.2.5. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system. Then $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]^*$ if and only if*

$$\exists \lambda \in \mathbb{R}_+ \ \forall \mu \leqslant \lambda \ (\exists \mu \in \mathbb{R}_+ \ \forall \lambda \geqslant \mu) \ \exists R > 1 \ \exists C > 0 \ :$$
$$\omega_{M^\mu}(t) \leqslant R \omega_{M^\lambda}(t) + \log C.$$

Proof. Suppose $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]^*$. Let $\lambda \in \mathbb{R}_+$ be such that for any $\mu \leqslant \lambda$ ($\mu \in \mathbb{R}_+$ be such that for any $\lambda \geqslant \mu$) we have $(M_\alpha^\lambda)^Q \leqslant C_0 M_{Q\alpha}^\mu$ for some $Q \in \mathbb{Z}_+$ and $C_0 > 0$. For any $\beta < (Q, \dots, Q)$ we then have

$$\sup_{\alpha \in \mathbb{N}^d} \log \frac{|t^{Q\alpha+\beta}| M_0^\mu}{M_{Q\alpha+\beta}^\mu}$$

$$\leqslant \sup_{\alpha \in \mathbb{N}^d} \log \frac{|t^{Q\alpha}| M_0^\mu}{M_{Q\alpha}^\mu} \cdot \frac{|t^\beta| M_0^\mu}{M_\beta^\mu}$$

$$\leqslant \sup_{\alpha \in \mathbb{N}^d} \log \frac{|t^{Q\alpha}| (M_0^\lambda)^Q}{(M_\alpha^\lambda)^Q} + \log |t^\beta| + \log \frac{C_0 M_0^\mu (M_0^\lambda)^{1-Q}}{\inf_{\beta < Qe} M_\beta^\mu}$$

$$\leqslant (Q+1) \omega_{M^\lambda}(t) + \log C$$

for some $C > 0$, from which we may infer

$$\omega_{M^\mu}(t) = \max_{\beta < Qe} \sup_{\alpha \in \mathbb{N}^d} \log \frac{t^{Q\alpha+\beta} M_0^\mu}{M_{Q\alpha+\beta}^\mu} \leqslant (Q+1) \omega_{M^\lambda}(t) + \log C.$$

Conversely, suppose the inequality holds for some $\lambda \in \mathbb{R}_+$ and any $\mu \leqslant \lambda$ (for some $\mu \in \mathbb{R}_+$ and any $\lambda \geqslant \mu$). We may suppose $R \in \mathbb{Z}_+$. For any $\alpha \in \mathbb{N}^d$ we get

$$M_\alpha^\lambda = M_0^\lambda \sup_{t \in \mathbb{R}^d} |t^\alpha| e^{-\omega_{M^\lambda}(t)}$$

$$\leqslant C M_0^\lambda \sup_{t \in \mathbb{R}^d} |t^\alpha| e^{-\omega_{M^\mu}(t)/R} = \frac{C M_0^\lambda}{(M_0^\mu)^{1/R}} (M_{R\alpha}^\mu)^{1/R}.$$

$\square$

We get the following relation between the conditions $[\mathfrak{M}.2]^*$ on $\mathfrak{M}$ and (Ω) and $\{\underline{DN}\}$ on $\mathscr{W}_{\mathfrak{M}}$.

Proposition 5.2.6. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system. Consider the statements*

(i) $\mathscr{W}_{\mathfrak{M}}$ satisfies (Ω) (resp. $\{\underline{DN}\}$).

(ii) $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]^*$.

Then, (i) $\Rightarrow$ (ii). If $\mathfrak{M}$ satisfies $[L]$ and $[\mathfrak{M}.2]$, then, also (ii) $\Rightarrow$ (i).

Proof. We only show the Beurling case, the Roumieu case may be done similarly. Suppose $\mathscr{W}_{\mathfrak{M}}$ satisfies (Ω). There exists some $\lambda \leqslant 1$ such that for any $\mu \leqslant \lambda$ there is some $\theta \in (0, 1)$ and $C > 0$ for which

$$(1 - \theta)\omega_{M^\mu}(t) \leqslant \theta\omega_{M^1}(t) + (1 - \theta)\omega_{M^\mu}(t) \leqslant \omega_{M^\lambda}(t) + \log C.$$

Putting $R = (1 - \theta)^{-1}$, it follows from Lemma 5.2.5 that $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]^*$.

Suppose now $\mathfrak{M}$ satisfies $[L]$, $[\mathfrak{M}.2]$ and $[\mathfrak{M}.2]^*$. Take any $\lambda \in \mathbb{R}_+$ and let $\lambda_0 \leqslant \lambda$ be such that the inequality in Lemma 5.2.5 holds for any $\mu \leqslant \lambda_0$. By Lemma 3.2.4(i) and (iii) there exists a $\mu \leqslant \lambda_0$ such that $2\omega_{M^{\lambda_0}}(t) \leqslant \omega_{M^\mu}(t) + \log C_1$ for some $C_1 > 0$. By putting $\theta = (R - 1)/R$ we get for any $\eta \leqslant \mu$:

$$\theta\omega_{M^\lambda}(t) + (1-\theta)\omega_{M^\eta}(t) \leqslant (1+\theta)\omega_{M^{\lambda_0}}(t) + \log C \leqslant \omega_{M^\mu}(t) + \log C_1 C.$$

We may conclude $\mathscr{W}_{\mathfrak{M}}$ satisfies (Ω). $\square$

Our notation of the condition $[\mathfrak{M}.2]^*$ is based on the condition $(M.2)^*$ on weight sequences [20, 89]. An isotropic weight sequence M is said to satisfy $(M.2)^*$ if and only if

$$\exists Q \in \mathbb{Z}_+ : \liminf_{p \to \infty} m_{Qp}/m_p > 1, \qquad p \geqslant 1.$$

We mention that $(M.1)$ and $(M.3)$ imply $(M.2)^*$ [105, Proposition 1.1]. The condition is intrinsically linked to the BMT weight functions, which we explore in the following result.

Proposition 5.2.7. *Let M be an isotropic weight sequence satisfying $(M.1)$ and $(M.2)$. Then, the following statements are equivalent:*

  (i) M satisfies $(M.2)^*$.

 (ii) ω_M is a BMT weight function.

(iii) $\mathfrak{M}_M$ satisfies $[\mathfrak{M}.2]^*$.

(iv) $\mathfrak{M}_M$ satisfies (Ω).

 (v) $\mathfrak{M}_M$ satisfies $\{\underline{DN}\}$.

If these statements are satisfied, we have $\mathfrak{M}_M \, [\approx] \, \mathfrak{M}_{\omega_M}$ and $\mathscr{W}_M [\approx] \mathscr{W}_{\omega_M}$.

Proof. $(i) \Leftrightarrow (ii)$ is shown in [20, Proposition 13], while $(iii) \Leftrightarrow (iv)$ and $(iii) \Leftrightarrow (v)$ follow from Proposition 5.2.6. We are left with $(i) \Leftrightarrow (iii)$. By [20, Proposition 13] we have that M satisfies $(M.2)^*$ if and only if

$$\omega_M(2t) \leqslant H\omega_M(t) + \log C, \qquad t \geqslant 0, \tag{5.1}$$

for some $C, H \geqslant 1$. By Lemma 5.2.5 this is equivalent to $\mathfrak{M}$ satisfying $[\mathfrak{M}.2]^*$.

One may now easily verify the last part of the proposition. $\qquad\square$

As a matter of fact, any weight sequence associated to a BMT weight function satisfies (Ω) and $\{\underline{DN}\}$.

Proposition 5.2.8. *Let ω be a BMT weight function. Then, $\mathfrak{M}_\omega$ satisfies $[\mathfrak{M}.2]^*$. In particular, $\mathfrak{M}_\omega$ satisfies (Ω) and $\{\underline{DN}\}$.*

Proof. By using the fact that for the Young conjugate ϕ^* the function $y \mapsto \phi^*(y)/y$ is increasing on $[0, \infty)$, it follows that for any $\lambda \in \mathbb{R}_+$, $Q \in \mathbb{Z}_+$ and $\alpha \in \mathbb{N}^d$:

$$(M_{\omega,\alpha}^\lambda)^Q = e^{\frac{Q}{\lambda}\phi^*(\lambda|\alpha|)} \leqslant e^{\frac{1}{\lambda}\phi^*(\lambda Q|\alpha|)} = M_{\omega,Q\alpha}^\lambda.$$

We may conclude $\mathfrak{M}_\omega$ satisfies $[\mathfrak{M}.2]^*$. As $\mathfrak{M}_\omega$ satisfies $[L]$ and $[\mathfrak{M}.2]$ by Lemma 3.2.7, it follows from Proposition 5.2.6 that $\mathfrak{M}_\omega$ satisfies (Ω) and $\{\underline{DN}\}$. □

We now end with a discussion on the conditions $(\overline{\overline{\Omega}})$ and $\{DN\}$. As we will see in Theorem 5.4.3, these will determine whether or not our spaces in question are ultrabornological and barrelled. Therefore it is of great interest to us to determine if the weight function system $\mathscr{W}$, or $\mathscr{W}^\circ = \{1/w^{1/\lambda} \,|\, \lambda \in \mathbb{R}_+\}$ when considering multiplier spaces, satisfies the necessary conditions. Our primary interest are the weight function systems that arise from a weight sequence M or a weight function ω as in Section 3.2.3. For $\mathscr{W}_M$ and $\mathscr{W}_\omega$, we will show that the conditions $(\overline{\overline{\Omega}})$ and $\{DN\}$ are rarely met. For $\mathscr{W}_M^\circ$ and $\mathscr{W}_\omega^\circ$, we will see a clear distinction between the Beurling and Roumieu case. We first look at $\mathscr{W}_\omega$.

Proposition 5.2.9. *Let ω be a non-negative non-decreasing continuous function on $[0, \infty)$ going to infinity. The following statements hold.*

  (i) $\mathscr{W}_\omega$ does not satisfy $\{DN\}$ and $(\overline{\overline{\Omega}})$.

  (ii) $\mathscr{W}_\omega^\circ$ satisfies $\{DN\}$, but not $(\overline{\overline{\Omega}})$.

Proof. That $\mathscr{W}_\omega$ and $\mathscr{W}_\omega^\circ$ do not satisfy $(\overline{\overline{\Omega}})$ follows easily from the fact that for any $\eta < \mu < \lambda$, the inequalities

$$\frac{\theta}{\lambda} + \frac{1-\theta}{\eta} \leqslant \frac{1}{\mu} \quad \text{and} \quad \mu \leqslant \theta\lambda + (1-\theta)\eta$$

do not hold for all $\theta \in (0,1)$. That $\mathscr{W}_\omega$ also does not satisfy $\{DN\}$ follows from the fact that for any $\mu > 1$ and $\theta \in (\frac{1}{\mu}, 1)$, the inequality

$$\theta + \frac{1-\theta}{\eta} \leqslant \frac{1}{\mu}$$

cannot hold for any $\eta > \mu$. Finally, as for any $\mu > 1$ and $\theta \in (0, 1)$ one may always find an $\eta > \mu$ such that

$$\mu \leqslant 1 + (1 - \theta)\eta,$$

we infer that $\mathscr{W}_\omega^\circ$ satisfies $\{DN\}$. $\square$

Next, we consider $\mathscr{W}_M$. We first show the following general result.

Lemma 5.2.10. *Let* $\mathfrak{M}$ *be an isotropically decomposable weight sequence system satisfying* (L) *and* $(\mathfrak{M}.2)$. *Then,* $\mathscr{W}_\mathfrak{M}^\circ$ *satisfies* $\{DN\}$.

Proof. It suffices to show that for any $\mu > 0$ and $\theta \in (0, 1)$ there exists a $\eta < \mu$ such that

$$\omega_{M^\mu}(x) \leqslant (1 - \theta)\omega_{M^\eta}(x) + \log C,$$

for some $C > 0$. This follows easily from Lemma 3.2.4 (i) and (iii). $\square$

Corollary 5.2.11. *Let* M *be an isotropically decomposable weight sequence satisfying* $(M.1)$ *and* $(M.2)$. *The following statements hold.*

(i) $\mathscr{W}_M$ *does not satisfy* $\{DN\}$ *and* $(\overline{\overline{\Omega}})$.

(ii) $\mathscr{W}_M^\circ$ *satisfies* $\{DN\}$. *If* M *satisfies* $(M.2)^*$, *then,* $\mathscr{W}_M^\circ$ *does not satisfy* $(\overline{\overline{\Omega}})$.

Proof. (i) If $\mathscr{W}_M$ satisfies $(\overline{\overline{\Omega}})$, respectively $\{DN\}$, then in particular each isotropic sequence in its decomposition does so. Hence these satisfy (Ω), respectively $\{\underline{DN}\}$, so that by Proposition 5.2.7 the weight function system arises from a BMT weight function. We now get a contradiction from Proposition 5.2.9(i).

(ii) By Lemma 5.2.10 we see that $\mathscr{W}_M^\circ$ satisfies $\{DN\}$. If M satisfies $(M.2)^*$, then by a similar argument as in (i) one shows that $\mathscr{W}_M^\circ$ cannot satisfy $(\overline{\overline{\Omega}})$ by virtue of Proposition 5.2.9(ii). $\square$

5.3 (PLB)-spaces of weighted continuous functions

Let X be a topological space. A family $\mathcal{A} = \{a^{\lambda,\mu} \mid \lambda, \mu \in \mathbb{R}_+\}$ of continuous functions $X \to \mathbb{R}_+$ is called a *weight function grid on X* if $a^{\lambda_1,\mu_1}(x) \leqslant a^{\lambda_2,\mu_2}(x)$ for every $x \in X$ when $\lambda_2 \leqslant \lambda_1$ and $\mu_2 \leqslant \mu_1$. The following two conditions will be crucial for our analysis.

Definition 5.3.1. The weight function grid $\mathcal{A}$ on X is said to satisfy $\mathcal{Q}$ if

$$\forall \lambda_1 \in \mathbb{R}_+ \ \exists \lambda_2 \leqslant \lambda_1 \ \exists \mu_1 \in \mathbb{R}_+ \ \forall \lambda_3 \leqslant \lambda_2$$
$$\forall \mu_2 \geqslant \mu_1 \ \forall \varepsilon > 0 \ \exists \mu_3 \geqslant \mu_2 \ \exists C > 0:$$
$$\frac{1}{a^{\lambda_2,\mu_2}(x)} \leqslant \frac{\varepsilon}{a^{\lambda_1,\mu_1}(x)} + \frac{C}{a^{\lambda_3,\mu_3}(x)}, \qquad \forall x \in X.$$

If "$\forall \varepsilon > 0$" is replaced by "$\exists \varepsilon > 0$", then $\mathcal{A}$ is said to satisfy $w\mathcal{Q}$.

Let X be a locally compact σ-compact topological vector space. Given a weight function grid $\mathcal{A}$ on X, we define the space

$$\mathcal{A}C(X) = \varprojlim_{\lambda \to 0^+} \varinjlim_{\mu \to \infty} C_{a^{\lambda,\mu}}(X).$$

Then, $\mathcal{A}C(X)$ is a (PLB)-space. We now recall two important results from [1] concerning the linear topological properties of the spaces $\mathcal{A}C(X)$ that will be used later on.

Theorem 5.3.2. *Let $\mathcal{A}$ be a weight function grid on X. If $\mathcal{A}$ satisfies $\mathcal{Q}$, then $\mathcal{A}C(X)$ is ultrabornological.*

Proof. In view of [151, Theorem 3.3.5, p. 41], this was shown in [1, Theorem 3.5]. $\square$

Theorem 5.3.3 ([1, Theorem 3.8(2)])**.** *Let $\mathcal{A}$ be a weight function grid on X. If $\mathcal{A}C(X)$ is barrelled, then $\mathcal{A}$ satisfies $w\mathcal{Q}$.*

Consider two topological spaces X and Y. Let $\mathscr{W}$ be a weight function system on X and $\mathscr{V}$ be a weight function system on Y. We will primarily be interested in weight function grids of the form $\mathcal{A} = \mathcal{A}_{\mathscr{W} \otimes \mathscr{V}} = \mathscr{W} \otimes \mathscr{V}$ on $X \times Y$, where we put $a^{\lambda,\mu} = w^\lambda \otimes v^\mu$. Our first concern is to determine when $\mathcal{A}_{\mathscr{W} \otimes \mathscr{V}}$ satisfies $\mathcal{Q}$. We find the following result.

Proposition 5.3.4. *Let $\mathscr{W}$ and $\mathscr{V}$ be weight function systems on X and Y, respectively. Suppose that one of the following statements holds:*

(i) $\mathscr{W}$ satisfies (Ω) and $\mathscr{V}$ satisfies $\{DN\}$;

(ii) $\mathscr{W}$ satisfies $(\overline{\overline{\Omega}})$ and $\mathscr{V}$ satisfies $\{\underline{DN}\}$.

Then, $\mathcal{A}_{\mathscr{W}\otimes\mathscr{V}}$ satisfies $\mathcal{Q}$.

Proof. Suppose (i) holds. Take any $\lambda_1 \in \mathbb{R}_+$ and take some $\lambda_2 \leqslant \lambda_1$ as in (Ω). Pick $\mu_1 \in \mathbb{R}_+$ as in $\{DN\}$ and any $\lambda_3 \leqslant \lambda_2$ and fix a corresponding $\theta \in (0,1)$ as implied by (Ω). By $\{DN\}$, for any $\mu_2 \geqslant \mu_1$ there exists some $\mu_3 \geqslant \mu_2$ and $C > 0$ such that for any $\varepsilon > 0$ and $(x,y) \in X \times Y$:

$$
\begin{aligned}
\frac{1}{w^{\lambda_2}(x)v^{\mu_2}(y)} &\leqslant C^2 \left(\frac{1}{w^{\lambda_1}(x)v^{\mu_1}(y)} \right)^{\theta} \left(\frac{1}{w^{\lambda_3}(x)v^{\mu_3}(y)} \right)^{1-\theta} \\
&= \left(\frac{\varepsilon}{w^{\lambda_1}(x)v^{\mu_1}(y)} \right)^{\theta} \left(\frac{\varepsilon^{-\frac{\theta}{1-\theta}}C^{\frac{2}{1-\theta}}}{w^{\lambda_3}(x)v^{\mu_3}(y)} \right)^{1-\theta} \\
&\leqslant \max\left\{ \frac{\varepsilon}{w^{\lambda_1}(x)v^{\mu_1}(y)}, \frac{\varepsilon^{-\frac{\theta}{1-\theta}}C^{\frac{2}{1-\theta}}}{w^{\lambda_3}(x)v^{\mu_3}(y)} \right\} \\
&\leqslant \frac{\varepsilon}{w^{\lambda_1}(x)v^{\mu_1}(y)} + \frac{\varepsilon^{-\frac{\theta}{1-\theta}}C^{\frac{2}{1-\theta}}}{w^{\lambda_3}(x)v^{\mu_3}(y)}.
\end{aligned}
$$

Consequently, $\mathcal{A}_{\mathscr{W}\otimes\mathscr{V}}$ satisfies $\mathcal{Q}$.

If (ii) holds, then one may analogously conclude $\mathcal{Q}$, where θ is fixed by $\mathscr{V}$ instead of $\mathscr{W}$. $\qquad\square$

Next, suppose either $\mathscr{W}$ or $\mathscr{V}$ is of the form $\mathscr{W}_{\mathfrak{M}}$ for some weight sequence system $\mathfrak{M}$. If we combine the observations made in Proposition 5.2.6 with those of Proposition 5.3.4 we obtain the following equivalencies, which will form a vital argument in showing necessary conditions for ultrabornologicity and barrelledness in the sequel.

Lemma 5.3.5. *Take a topological space X. Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system. For a weight function system $\mathscr{W}$ on X the following hold true.*

(a) *If* $\mathfrak{M}$ *satisfies* (L), $(\mathfrak{M}.2)$, *and* $(\mathfrak{M}.2)^*$, *the following statements are equivalent:*

   (i) $\mathcal{A}_{\mathscr{W}_{\mathfrak{M}} \otimes \mathscr{W}}$ *on* $\mathbb{Z}^d \times X$ *satisfies* $\mathcal{Q}$.

   (ii) $\mathcal{A}_{\mathscr{W}_{\mathfrak{M}} \otimes \mathscr{W}}$ *on* $\mathbb{Z}^d \times X$ *satisfies* $w\mathcal{Q}$.

   (iii) $\mathscr{W}$ *satisfies* $\{DN\}$.

(b) *If* $\mathfrak{M}$ *satisfies* $\{L\}$, $\{\mathfrak{M}.2\}$, *and* $\{\mathfrak{M}.2\}^*$, *the following statements are equivalent:*

   (i) $\mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}}$ *on* $X \times \mathbb{Z}^d$ *satisfies* $\mathcal{Q}$.

   (ii) $\mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}}$ *on* $X \times \mathbb{Z}^d$ *satisfies* $w\mathcal{Q}$.

   (iii) $\mathscr{W}$ *satisfies* $(\overline{\overline{\Omega}})$.

Proof. (a) (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Condition $w\mathcal{Q}$ implies that

$$\forall \lambda_1 \in \mathbb{R}_+ \; \exists \lambda_2 \leqslant \lambda_1 \; \exists \mu_1 \in \mathbb{R}_+ \; \forall \lambda_3 \leqslant \lambda_2 \; \forall \mu_2 \geqslant \mu_1 \; \exists \mu_3 \geqslant \mu_2 \; \exists C > 0 :$$

$$\frac{1}{w^{\mu_2}(x)} \leqslant C \left(\frac{e^{\omega_{M^{\lambda_2}}(j) - \omega_{M^{\lambda_1}}(j)}}{w^{\mu_1}(x)} + \frac{e^{\omega_{M^{\lambda_2}}(j) - \omega_{M^{\lambda_3}}(j)}}{w^{\mu_3}(x)} \right).$$

for every $x \in X$ and $j \in \mathbb{Z}^d$. Using Lemma 5.2.5, let $\lambda_1 \in \mathbb{R}_+$ be such that for any $\lambda \leqslant \lambda_1$ we have $\omega_{M^\lambda}(j) \leqslant R\omega_{M^{\lambda_1}}(j) + \log C_1$ for some $R > 1$ and $C_1 > 0$. We then fix λ_2 and μ_1 as in $w\mathcal{Q}$. In particular we have

$$\omega_{M^{\lambda_2}}(j) - \omega_{M^{\lambda_1}}(j) \leqslant \frac{R-1}{R}\omega_{M^{\lambda_2}}(j) + \log C_1.$$

Fix some $\nu > 0$. By Lemma 3.2.4(i) and (iii) there is a $\eta \leqslant \lambda_2$ such that

$$\frac{R-1}{R}\nu\omega_{M^{\lambda_2}}(2j) \leqslant \omega_{M^\eta}(j) + \log C_2$$

for some $C_2 \geqslant 1$. Next, again by Lemma 3.2.4(i) and (iii), there exists a $\lambda_3 \leqslant \eta$ so that $2\omega_{M^\eta}(j) \leqslant \omega_{M^{\lambda_3}}(j) + \log C_3$ for some $C_3 > 0$. Specifically, we have

$$\omega_{M^{\lambda_2}}(j) - \omega_{M^{\lambda_3}}(j) \leqslant -\omega_{M^\eta}(j) + \log C_3.$$

Set $C' = \max(C_1, C_3)$, we get from wQ

$$\exists \mu_1 \in \mathbb{R}_+ \; \forall \mu_2 \geqslant \mu_1 \; \exists \mu_3 \geqslant \mu_2 \; \exists C > 0 \; \forall x \in X \; \forall j \in \mathbb{Z}^d :$$

$$\frac{1}{w^{\mu_2}(x)} \leqslant CC' \left(\frac{e^{\frac{R-1}{R} \omega_{M^{\lambda_2}}(j)}}{w^{\mu_1}(x)} + \frac{e^{-\omega_{M^\eta}(j)}}{w^{\mu_3}(x)} \right).$$

Consider the sequence $j_k = (k, 0, \ldots, 0)$ for $k \in \mathbb{Z}_+$, then note that $e^{\omega_{M^{\lambda_2}}(j_k)}$ is a non-decreasing sequence such that $\lim_{k \to \infty} e^{\omega_{M^{\lambda_2}}(j_k)} = \infty$. Also note that $e^{\omega_{M^{\lambda_2}}(j_{k+1})} \leqslant C_2 e^{\frac{1}{\nu} \frac{R}{R-1} \omega_{M^\eta}(j_k)}$. For any $r \geqslant e^{\omega_{M^{\lambda_2}}(j_1)}$ let $k = k_r \geqslant 1$ be such that

$$e^{\omega_{M^{\lambda_2}}(j_k)} \leqslant r < e^{\omega_{M^{\lambda_2}}(j_{k+1})} \leqslant C_2 e^{\frac{1}{\nu} \frac{R}{R-1} \omega_{M^\eta}(j_k)}.$$

We now see that

$$\exists \mu_1 \in \mathbb{R}_+ \; \forall \mu_2 \geqslant \mu_1 \; \forall \nu > 0 \; \exists \mu_3 \geqslant \mu_2 \; \exists C > 0 \; \forall x \in X \; \forall r > 0 :$$

$$\frac{1}{w^{\mu_2}(x)} \leqslant CC'C_2 \left(\frac{r}{w^{\mu_1}(x)} + \frac{r^{-\nu}}{w^{\mu_3}(x)} \right).$$

The result then follows by calculating the minimum (with respect to $r > 0$) of the right-hand side of the above inequality.

$(iii) \Rightarrow (i)$ This follows from Proposition 5.3.4 as $\mathscr{W}_{\mathfrak{M}}$ satisfies (Ω) by Proposition 5.2.6.

(b) One may show the equivalencies similar as in part (a). $\qquad \square$

5.4   The space $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$

We now define the weighted (PLB)-spaces of ultradifferentiable functions that we will be concerned with in the remainder of this chapter. Let M be a weight sequence and let w be a continuous positive function on $\mathbb{R}^d$. We write $\mathcal{Z}_w^M(\mathbb{R}^d) = \mathcal{Z}_w^M$ for the space $\mathcal{S}_{w,\infty}^M$, i.e. for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{Z}_w^M} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{w(x)|\varphi^{(\alpha)}(x)|}{M_\alpha} < \infty.$$

Then given a weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$, we define the spaces

$$\mathcal{Z}_{\mathscr{W}}^{(\mathfrak{M})} = \varprojlim_{\lambda \to 0^+} \varinjlim_{\mu \to \infty} \mathcal{Z}_{w^\mu}^{M^\lambda}, \qquad \mathcal{Z}_{\mathscr{W}}^{\{\mathfrak{M}\}} = \varprojlim_{\mu \to 0^+} \varinjlim_{\lambda \to \infty} \mathcal{Z}_{w^\mu}^{M^\lambda}.$$

Then, $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ is a (PLB)-space. We will also write $\mathcal{O}_{M}^{[\mathfrak{M}],[\mathscr{W}]}$ for the space $\mathcal{Z}_{\mathscr{W}^{\circ}}^{[\mathfrak{M}]}$, due to its motivation as being the multiplier space associated to $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$. Note however that it is far from trivial whether or not $\mathcal{O}_{M}^{[\mathfrak{M}],[\mathscr{W}]}$ is *actually* the multiplier space of $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$.

Open problem 5.4.1. Determine when $\mathcal{O}_{M}^{[\mathfrak{M}],[\mathscr{W}]}$ is the multiplier space of $\mathcal{S}_{[\mathscr{W}]}^{[\mathfrak{M}]}$. In [47] this is shown to be the case for $\mathcal{S}_{[M]}^{[M]}$, where M is an isotropic weight sequence satisfying $(M.1)$, $(M.2)$ and $(M.3)$.

The goal of this chapter is characterize the topological properties of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ via the conditions $(\overline{\overline{\Omega}})$ and $\{DN\}$ on $\mathscr{W}$. To this purpose, we introduce the following assumption on $\mathfrak{M}$ and $\mathscr{W}$ under which we will characterize the ultrabornologicity and barrelledness of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$.

Assumption 5.4.2. $\mathfrak{M}$ *is an isotropically decomposable weight sequence system satisfying* [L] *and* $[\mathfrak{M}.2]'$ *such that* $\mathscr{W}_{\mathfrak{M}}$ *satisfies* (Ω) *(resp.* $\{\underline{DN}\}$*). Additionally,* $\mathscr{W}$ *is a weight function system satisfying* [wM] *such that there exists a symmetric weight function system* $\mathscr{V}$ *satisfying* [wM] *and* [N] *for which* $\mathscr{W}^{\circ}$ *is* $[\mathscr{V}]$-*admissible, i.e.*

$$\forall \mu \in \mathbb{R}_+ \; \exists \lambda, \eta \in \mathbb{R}_+ \; (\forall \lambda, \eta \in \mathbb{R}_+ \; \exists \mu \in \mathbb{R}_+) \; \exists C > 0 \; \forall x, y \in \mathbb{R}^d :$$
$$w^{\lambda}(x + y) \leqslant C w^{\mu}(x) v^{\eta}(y) \tag{5.2}$$

and $\mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]} \neq \{0\}$.

We are now ready to formulate the main result of this chapter.

Theorem 5.4.3. *Let* $\mathfrak{M}$ *be a weight sequence system and* $\mathscr{W}$ *be a weight function system such that Assumption 5.4.2 holds for some weight function sequence* $\mathscr{V}$. *Consider the statements:*

  (i) $\mathscr{W}$ *satisfies* $\{DN\}$ *(resp.* $(\overline{\overline{\Omega}})$*).*

  (ii) $\mathcal{Z}_{\mathscr{W}}^{(\mathfrak{M})}$ *(resp.* $\mathcal{Z}_{\mathscr{W}}^{\{\mathfrak{M}\}}$*) is ultrabornological.*

  (iii) $\mathcal{Z}_{\mathscr{W}}^{(\mathfrak{M})}$ *(resp.* $\mathcal{Z}_{\mathscr{W}}^{\{\mathfrak{M}\}}$*) is barrelled.*

Then the following are true:

(A) *The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold.*

(B) *If $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]$ and $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{V}]}$ is Gabor accessible, then, $(iii) \Rightarrow (i)$.*

Sufficient conditions for a Gelfand-Shilov space to be Gabor accessible were explored in Section 3.4.2. In particular, if $\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{V}\}} = \mathcal{S}^{\{p!^s\}}_{\{p!^r\}}(\mathbb{R}^d)$ this is exactly the case when (r, s) is a Gabor couple, so that by Corollary 3.4.10 it suffices that either $\min(r, s) \geqslant 1/2$ or $\max(r, s) > 1$.

To demonstrate the applicability of Theorem 5.4.3, let us consider the multiplier spaces associated to Gelfand-Shilov spaces defined by weight functions or weight sequences as in Section 3.2.3. In both examples we will see a clear distinction between the Beurling and Roumieu case.

Theorem 5.4.4. *Let $\omega, \eta : [0, \infty) \rightarrow \mathbb{R}_+$ be continuous and non-decreasing going to infinity such that ω is a BMT weight function and η satisfies (α) and (γ) (resp. (γ_0)). Suppose that $\mathcal{S}^{[\omega]}_{[\eta]} \neq \{0\}$. Then, the following statements hold.*

(i) *$\mathcal{O}^{(\omega),(\eta)}_M$ is ultrabornological and barrelled.*

(ii) *If $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$ is Gabor accessible, then $\mathcal{O}^{\{\omega\},\{\eta\}}_M$ is not ultrabornological or barrelled.*

Proof. We start by verifying that Assumption 5.4.2 is met for $\mathfrak{M} = \mathfrak{M}_\omega$, $\mathscr{W} = \mathscr{W}^\circ_\eta$ and $\mathscr{V} = \mathscr{W}_\eta$. For $\mathfrak{M}$, this follows directly from Lemma 3.2.7 and Proposition 5.2.8. Also, by Lemma 3.2.7 it follows that $\mathscr{W}$ satisfies [wM] while $\mathscr{V}$ satisfies [wM] and [N]. Finally, as η satisfies (α), we have

$$\eta(x) \leqslant L\eta(x + y) + L\eta(y) + \log C,$$

for some $C > 0$, from which we may conclude (5.2) holds.

The result now follows directly from Theorem 5.4.3 and Proposition 5.2.9(ii). $\qquad \square$

Theorem 5.4.5. *Let M and A be two isotropically decomposable weight sequences satisfying $(M.1)$ and $(M.2)$ and let M satisfy $(M.2)^*$. Suppose that $\mathcal{S}^{[M]}_{[A]} \neq \{0\}$. Then, the following statements hold.*

(i) $\mathcal{O}_M^{(M),(A)}$ is ultrabornological and barrelled.

(ii) If $\mathcal{S}_{\{A\}}^{\{M\}}$ is Gabor accessible and A satisfies $(M.2)^*$, then $\mathcal{O}_M^{\{M\},\{A\}}$ is not ultrabornological or barrelled.

Proof. As a direct consequence of Lemma 2.3.2, Lemma 3.2.6 and Proposition 5.2.7, we see that Assumption 5.4.2 holds for $\mathfrak{M} = \mathfrak{M}_M$, $\mathscr{W} = \mathscr{W}_A^\circ$ and $\mathscr{V} = \mathscr{W}_A$. Whence the result follows directly from Theorem 5.4.3 and Corollary 5.2.11. □

In particular, we get the following result for Gevrey sequences.

Corollary 5.4.6. *For any* $(r, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ *such that* $r + s > 1$ $(r + s \geqslant 1)$ *we have that:*

(i) $\mathcal{O}_M^{(p!^s),(p!^r)}$ *is ultrabornological and barrelled;*

(ii) *If* (r, s) *is a Gabor couple, then,* $\mathcal{O}_M^{\{p!^s\},\{p!^r\}}$ *is not ultrabornological or barrelled.*

Proof. For any $s > 0$ and $M_p = p!^s$ we have that $m_{2p}/m_p = 2^s > 1$, so that in particular M satisfies $(M.2)^*$. The proof is now completed by Theorem 5.4.5. □

Before we move on to the proof of Theorem 5.4.3, we first pose the following interesting open problem.

Open problem 5.4.7. Consider a continuous function $\rho : \mathbb{N}^d \times \mathbb{R}^d \to (0, \infty)$ and a weight sequence M. We may consider the Banach space of all smooth functions $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|\rho(\alpha, x)}{M_\alpha} < \infty.$$

Then, using a weight sequence system $\mathfrak{M}$ and an alternate weight function system $\mathcal{P} = \{\rho^\mu \mid \mu \in \mathbb{R}_+\}$ with continuous functions $\rho^\mu : \mathbb{N}^d \times \mathbb{R}^d \to (0, \infty)$, one may define several variants of Gelfand-Shilov type spaces as we did before. The problem now becomes the determination of the topological invariants of these spaces. Note that many of these test function spaces exhibit interference in their

time-frequency decay, so that the techniques employed throughout this text are often not directly applicable. A particularly interesting example is where $\rho^\mu(\alpha, x) = p_\alpha(x)w^\mu(x)$ for a weight function system $\mathscr{W}$ and a family $\{p_\alpha \mid \alpha \in \mathbb{N}^d\}$ of continuous functions $p_\alpha : \mathbb{R}^d \to (0, \infty)$. Such spaces arise naturally in the context of asymptotic behavior for ultradistributions, see Chapters 9 and 10.

5.4.1   The proof of Part (A)

The proof of Part (A) employs the short-time Fourier transform in order to embed $\mathcal{Z}_\mathscr{W}^{[\mathfrak{M}]}$ into the space $\mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{R}^{2d})$ (resp. $\mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(\mathbb{R}^{2d})$), after which the statement follows from Proposition 5.3.4 and Theorem 5.3.2. The following result considers the mapping properties of the (adjoint) STFT on $\mathcal{Z}_\mathscr{W}^{[\mathfrak{M}]}$.

Proposition 5.4.8. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ a weight function system such that Assumption 5.4.2 holds for some weight function system $\mathscr{V}$. For any $\psi \in \mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]}$, the mappings*

$$V_\psi : \mathcal{Z}_\mathscr{W}^{(\mathfrak{M})} \to \mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{R}_{\xi, x}^{2d}), \qquad V_\psi : \mathcal{Z}_\mathscr{W}^{\{\mathfrak{M}\}} \to \mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(\mathbb{R}_{x, \xi}^{2d}),$$

and

$$V_\psi^* : \mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{R}_{\xi, x}^{2d}) \to \mathcal{Z}_\mathscr{W}^{(\mathfrak{M})}, \qquad V_\psi^* : \mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(\mathbb{R}_{x, \xi}^{2d}) \to \mathcal{Z}_\mathscr{W}^{\{\mathfrak{M}\}},$$

are continuous. In particular, $\mathcal{Z}_\mathscr{W}^{[\mathfrak{M}]}$ is isomorphic to a complemented subspace of $\mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{R}^{2d})$ (of $\mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(\mathbb{R}^{2d})$).

Proof. We have that $\psi \in \mathcal{S}_{v^{\mu_3}, 1}^{M^{\lambda_3}}$ for any (resp. for some) $\lambda_3, \mu_3 \in \mathbb{R}_+$ by Theorem 3.3.11.

We first consider V_ψ. It suffices to show

$$\forall \lambda_1 \in \mathbb{R}_+ \exists \lambda_2 \in \mathbb{R}_+ \ \forall \mu_2 \in \mathbb{R}_+ \ \exists \mu_1 \in \mathbb{R}_+$$
$$(\forall \mu_1 \in \mathbb{R}_+ \ \exists \mu_2 \in \mathbb{R}_+ \ \forall \lambda_2 \in \mathbb{R}_+ \ \exists \lambda_1 \in \mathbb{R}_+)$$
$$V_\psi : \mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}} \to C_{\exp \omega_{M^{\lambda_1}} \otimes w^{\mu_1}}(\mathbb{R}_{\xi, x}^{2d}) \text{ is well-defined and continuous.}$$

In the Beurling case we put $\lambda_1 = \lambda_2 = \lambda_3$, while in the Roumieu case we put $\lambda_1 = \max(\lambda_2, \lambda_3)$. For any $\mu_2 \in \mathbb{R}_+$ we choose $\mu_1, \mu_3 \in \mathbb{R}_+$

(for any $\mu_1 \in \mathbb{R}_+$ and fixed $\mu_3 \in \mathbb{R}_+$ we choose $\mu_2 \in \mathbb{R}_+$) such that $w^{\mu_1}(x + y) \leqslant C_1 w^{\mu_2}(x) v^{\mu_3}(y)$ for some $C_1 > 0$ and any $x, y \in \mathbb{R}^d$. Then for $\varphi \in \mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}}$ and $\alpha \in \mathbb{N}^d$ we get

$$
\frac{|\xi^\alpha V_\psi \varphi(x, \xi)| w^{\mu_1}(x)}{M_\alpha^{\lambda_1}}
$$
$$
\leqslant \frac{C_1 M_0^{\lambda_1}}{(2\pi)^{|\alpha|}} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \frac{|\varphi^{(\beta)}(t)| w^{\mu_2}(t)}{M_\beta^{\lambda_2}} \cdot \frac{|\psi^{(\alpha-\beta)}(x - t)| v^{\mu_3}(x - t)}{M_{\alpha-\beta}^{\lambda_3}} dt
$$
$$
\leqslant \left(C_1 M_0^{\lambda_1} \|\psi\|_{\mathcal{S}_{v^{\mu_3},1}^{M^{\lambda_3}}} \right) \|\varphi\|_{\mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}}},
$$

whence

$$
\sup_{(x,\xi) \in \mathbb{R}^{2d}} |V_\psi \varphi(x, \xi)| w^{\mu_1}(x) e^{\omega_{M^{\lambda_1}}(\xi)} \leqslant C \|\varphi\|_{\mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}}}
$$

for some $C > 0$. The continuity of V_ψ follows.

We now show the continuity of V_ψ^*. It suffices to show

$$
\forall \lambda_1 \in \mathbb{R}_+ \; \exists \lambda_2 \in \mathbb{R}_+ \; \forall \mu_2 \in \mathbb{R}_+ \; \exists \mu_1 \in \mathbb{R}_+
$$
$$
(\forall \mu_1 \in \mathbb{R}_+ \; \exists \mu_2 \in \mathbb{R}_+ \; \forall \lambda_2 \in \mathbb{R}_+ \; \exists \lambda_1 \in \mathbb{R}_+)
$$
$$
V_\psi : C_{\exp \omega_{M^{\lambda_2}} \otimes w^{\mu_2}}(\mathbb{R}_{\xi,x}^{2d}) \to \mathcal{Z}_{w^{\mu_1}}^{M^{\lambda_1}} \text{ is well-defined and continuous.}
$$

Again, for any $\mu_2 \in \mathbb{R}_+$ we choose $\mu_1, \mu_3 \in \mathbb{R}_+$ (for any $\mu_1 \in \mathbb{R}_+$ and fixed $\mu_3 \in \mathbb{R}_+$ we choose $\mu_2 \in \mathbb{R}_+$) such that $w^{\mu_1}(x + y) \leqslant C_1 w^{\mu_2}(x) v^{\mu_3}(y)$ for some $C_1 > 0$ and any $x, y \in \mathbb{R}^d$. Using condition [L] and $[\mathfrak{M}.2]'$ and Lemma 3.2.5, we find that for any $\lambda_1, \lambda_3 \in \mathbb{R}_+$ there exists $\lambda_2, \lambda_4 \in \mathbb{R}_+$ (for any $\lambda_2 \in \mathbb{R}_+$ and fixed $\lambda_3 \in \mathbb{R}_+$ there exists $\lambda_1, \lambda_4 \in \mathbb{R}_+$) such that $(4\pi)^{|\alpha|} \max(M_\alpha^{\lambda_3}, M_\alpha^{\lambda_4}) \leqslant C_2 M_\alpha^{\lambda_1}$ for some $C_2 > 0$ and $e^{\omega_{M^{\lambda_4}}(\cdot)} / e^{\omega_{M^{\lambda_2}}(\cdot)} \in L^1(\mathbb{R}^d)$. Take any $\Phi \in C_{w^{\mu_2} \otimes \exp[\omega_{M^{\lambda_2}}]}(\mathbb{R}_{x,\xi}^{2d})$, then for any $\alpha \in \mathbb{N}^d$,

$$
\left| \partial_t^\alpha \int \int_{\mathbb{R}^{2d}} \Phi(x, \xi) e^{2\pi i \xi \cdot t} \psi(t - x) dx d\xi \right| \frac{w^{\mu_1}(t)}{M_\alpha^{\lambda_1}}
$$

$$\leqslant \frac{C_1 C_2^2 M_0^{\lambda_1}}{2^{|\alpha|}} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta}$$

$$\iint_{\mathbb{R}^{2d}} \frac{|t^\beta||\Phi(x,\xi)|w^{\mu_2}(x)}{M_\beta^{\lambda_4}} \frac{|\psi^{(\alpha-\beta)}(t-x)|v^{\mu_3}(t-x)}{M_{\alpha-\beta}^{\lambda_3}} dx d\xi$$

$$\leqslant \frac{C_1 C_2^2 M_0^{\lambda_1}}{M_0^{\lambda_4}} \|e^{\omega_{M^{\lambda_4}}(\cdot)}/e^{\omega_{M^{\lambda_2}}(\cdot)}\|_{L^1} \|\psi\|_{\mathcal{S}_{v^{\mu_3},1}^{M^{\lambda_3}}} \|\Phi\|_{C_{w^{\mu_2} \otimes \exp[\omega_{M^{\lambda_2}}]}}.$$

We may conclude the continuity of V_ψ^*.

The final statement will follow if we show (3.1) holds over $\mathcal{Z}_{\mathcal{W}}^{[\mathfrak{M}]}$. We choose some $\psi \in \tilde{\mathcal{S}}_{[\mathcal{V}]}^{[\mathfrak{M}]}$ with synthesis window $\gamma \in \tilde{\mathcal{S}}_{[\mathcal{V}]}^{[\mathfrak{M}]}$, where we used Lemma 3.3.4. Next, take any $\varphi \in \mathcal{Z}_{\mathcal{W}}^{[\mathfrak{M}]}$, then $V_\psi \varphi(x,\xi) = \mathcal{F}_t(\varphi T_x \overline{\psi})(\xi)$. One may easily deduce from (5.2) that $\varphi T_x \overline{\psi} \in L^1(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, whence $\varphi T_x \overline{\psi} = \mathcal{F}^{-1}(V_\psi \varphi(x,\cdot))$. By our previous calculations, it then follows from Fubini's Theorem that

$$\varphi(t) = \frac{1}{(\gamma,\psi)_{L^2}} \int_{\mathbb{R}^d} (\varphi(t) T_x \overline{\psi}(t)) T_x \gamma(t) dx$$

$$= \frac{1}{(\gamma,\psi)_{L^2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} V_\psi \varphi(x,\xi) e^{2\pi i \xi \cdot t} d\xi \right) T_x \gamma(t) dx$$

$$= \frac{1}{(\gamma,\psi)_{L^2}} \iint_{\mathbb{R}^{2d}} V_\psi \varphi(x,\xi) M_\xi T_x \gamma(t) dx d\xi.$$

We have thus shown that $\mathcal{Z}_{\mathcal{W}}^{[\mathfrak{M}]}$ is isomorphic to a complemented subspace of $\mathcal{A}_{\mathcal{W}_{\mathfrak{M}} \otimes \mathcal{W}} C(\mathbb{R}^{2d})$ (of $\mathcal{A}_{\mathcal{W} \otimes \mathcal{W}_{\mathfrak{M}}} C(\mathbb{R}^{2d})$). $\qquad \square$

Proof of Theorem 5.4.3 Part A. $(i) \Rightarrow (ii)$ By Assumption 5.4.2 and Proposition 5.3.4 it follows that $\mathcal{A}_{\mathcal{W}_{\mathfrak{M}} \otimes \mathcal{W}}$ (resp. $\mathcal{A}_{\mathcal{W} \otimes \mathcal{W}_{\mathfrak{M}}}$) satisfies $\mathcal{Q}$. Then $\mathcal{Z}_{\mathcal{W}}^{[\mathfrak{M}]}$ is ultrabornological by Theorem 5.3.2 and Proposition 5.4.8.

$(ii) \Rightarrow (iii)$ This holds for any lcHs [103, Observation 6.1.2, p. 167]. $\qquad \square$

5.4.2 The proof of Part (B)

We now move on to prove Part (B). We need some preparation. Let M be a weight sequence and $a > 0$. By $\mathcal{E}_{p,a}^M(\mathbb{R}^d) = \mathcal{E}_{p,a}^M$ we denote the

space of all $a\mathbb{Z}^d$-periodic functions $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{E}_{p,a}^M} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|}{M_\alpha} < \infty.$$

For a weight sequence system $\mathfrak{M}$ we then consider the spaces

$$\mathcal{E}_{p,a}^{(\mathfrak{M})}(\mathbb{R}^d) = \varprojlim_{\lambda \to 0^+} \mathcal{E}_{p,a}^{M^\lambda}(\mathbb{R}^d), \qquad \mathcal{E}_{p,a}^{\{\mathfrak{M}\}}(\mathbb{R}^d) = \varinjlim_{\lambda \to \infty} \mathcal{E}_{p,a}^{M^\lambda}(\mathbb{R}^d).$$

Let $b > 0$ and w be a positive function on $b\mathbb{Z}^d$. For a weight sequence M we write $Cw(b\mathbb{Z}^d; \mathcal{E}_{p,a}^M(\mathbb{R}^d))$ for the space consisting of all $(\varphi_\beta)_{\beta \in b\mathbb{Z}^d} \in (\mathcal{E}_{p,a}^M(\mathbb{R}^d))^{b\mathbb{Z}^d}$ such that

$$\left\|(\varphi_\beta)_{\beta \in b\mathbb{Z}^d}\right\|_{w,\mathcal{E}_{p,a}^M} = \sup_{\beta \in b\mathbb{Z}^d} w(\beta) \|\varphi_\beta\|_{\mathcal{E}_{p,a}^M} < \infty.$$

Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system on $b\mathbb{Z}^d$. We introduce the spaces

$$(\mathscr{W})C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{\{\mathfrak{M}\}}(\mathbb{R}^d)) = \varprojlim_{\mu \to 0^+} \varinjlim_{\lambda \to \infty} Cw^\mu(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^\lambda}(\mathbb{R}^d))$$

and

$$\{\mathscr{W}\}C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{(\mathfrak{M})}(\mathbb{R}^d)) = \varinjlim_{\lambda \to 0^+} \varinjlim_{\mu \to \infty} Cw^\mu(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^\lambda}(\mathbb{R}^d)).$$

Then $\langle\mathscr{W}\rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d))$ is a (PLB)-space.

Lemma 5.4.9. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system satisfying [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}$ be a weight function system on $b\mathbb{Z}^d$. Then,*

$$\{\mathscr{W}\}C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{(\mathfrak{M})}(\mathbb{R}^d)) \cong \mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}}C(\mathbb{Z}^d \times b\mathbb{Z}^d),$$
$$(\mathscr{W})C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{\{\mathfrak{M}\}}(\mathbb{R}^d)) \cong \mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}}C(b\mathbb{Z}^d \times b\mathbb{Z}^d),$$

as locally convex spaces.

Proof. Take $\lambda, \mu \in \mathbb{R}_+$ and any $(\varphi_\beta)_{\beta \in b\mathbb{Z}^d} \in Cw^\mu(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^\lambda}(\mathbb{R}^d))$. For any $m \in \mathbb{Z}^d$, $\beta \in b\mathbb{Z}^d$ and $\gamma \in \mathbb{N}^d$ we have

$$\frac{w^\mu(\beta)}{M_\gamma^\lambda} \left| \left(\frac{-i2\pi m}{a}\right)^\gamma a^{-d} \int_{[0,a]^d} \varphi_\beta(x) e^{-i\frac{2\pi m \cdot x}{a}} dx \right|$$

$$\leqslant a^{-d} \int_{[0,a]^d} \frac{w^\mu(\beta)|\varphi_\beta^{(\gamma)}(x)|}{M_\beta^\lambda} \leqslant \left\| (\varphi_\beta)_{\beta \in b\mathbb{Z}^d} \right\|_{w, \mathcal{E}_{p,a}^{M^\lambda}}.$$

From here we may conclude that the injective mapping

$$S : \{\mathscr{W}\}C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{(\mathfrak{M})}(\mathbb{R}^d)) \to \mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{Z}^d \times b\mathbb{Z}^d)$$

$$(\text{resp. } S : (\mathscr{W})C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{\{\mathfrak{M}\}}(\mathbb{R}^d)) \to \mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(b\mathbb{Z}^d \times \mathbb{Z}^d))$$

$$(\varphi_\beta)_{\beta \in b\mathbb{Z}^d} \mapsto (a^{-d} \int_{[0,a]^d} \varphi_\beta(x) e^{-i\frac{2\pi m \cdot x}{a}} dx)_{m \in \mathbb{Z}^d, \beta \in b\mathbb{Z}^d}$$

is continuous.

Next, for any $\lambda, \mu \in \mathbb{R}_+$ take an arbitrary $(c_{m,\beta})_{m \in \mathbb{Z}^d, \beta \in b\mathbb{Z}^d} \in C_{w^\lambda \otimes e^{\omega_M \mu}}(\mathbb{Z}^d \otimes b\mathbb{Z}^d)$. Suppose $\eta \in \mathbb{R}_+$ is such that

$$\sum_{m \in \mathbb{Z}^d} \frac{e^{\omega_{M^\eta}(2\pi m/a)}}{e^{\omega_{M^\mu}(m)}} < \infty.$$

We now have for any $\alpha \in \mathbb{N}^d$ and $\beta \in b\mathbb{Z}^d$:

$$\frac{w^\lambda(\beta)}{M_\alpha^\eta} \left| \partial^\alpha \sum_{m \in \mathbb{Z}^d} c_{m,\beta} e^{i\frac{2\pi m \cdot x}{a}} \right| \leqslant \sum_{m \in \mathbb{Z}^d} |c_{m,\beta}| w^\lambda(\beta) e^{\omega_{M^\eta}(2\pi m/a)}$$

$$\leqslant C \left\| (c_{m,\beta}) \right\|_{C_{w^\lambda \otimes e^{\omega_M \mu}}}$$

for some $C > 0$ depending on μ and η. By Lemma 3.2.4 and Lemma 3.2.5, we have that the injective mapping

$$R : \mathcal{A}_{\mathscr{W}_\mathfrak{M} \otimes \mathscr{W}} C(\mathbb{Z}^d \times b\mathbb{Z}^d) \to \{\mathscr{W}\}C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{(\mathfrak{M})}(\mathbb{R}^d))$$

$$(R : \mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_\mathfrak{M}} C(b\mathbb{Z}^d \times \mathbb{Z}^d) \to (\mathscr{W})C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{\{\mathfrak{M}\}}(\mathbb{R}^d)))$$

$$(c_{m,\beta})_{m \in \mathbb{Z}^d, \beta \in b\mathbb{Z}^d} \mapsto \sum_{m \in \mathbb{Z}^d} c_{m,\beta} e^{i\frac{2\pi m \cdot x}{a}}$$

is continuous. Since clearly $R \circ S = \mathrm{id}$, our proof is complete. $\qquad \square$

The next proposition, based on an idea of Vogt [144, Theorem 5.1], allows us to embed $\langle\mathscr{W}\rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d))$ into $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$.

Proposition 5.4.10. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system such that Assumption 5.4.2 holds for some weight function system $\mathscr{V}$. Let $a, b > 0$ and let $\psi, \gamma \in \mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]}$. Then, the mappings*

$$\Psi = \Psi_\psi : \langle\mathscr{W}\rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d)) \to \mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]} : (\varphi_\beta)_{\beta\in b\mathbb{Z}^d} \mapsto \sum_{\beta\in b\mathbb{Z}^d} (T_\beta\psi)\varphi_\beta,$$

and

$$\Phi = \Phi_\gamma : \mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]} \to \langle\mathscr{W}\rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d)) : \varphi \mapsto \left(\sum_{\delta\in a\mathbb{Z}^d} T_\delta((T_\beta\gamma)\varphi)\right)_{\beta\in b\mathbb{Z}^d},$$

are well-defined and continuous.

Proof. We first consider Ψ. We have $\psi \in \mathcal{S}_{v^{\mu_3},\infty}^{M^{\lambda_3}}$ for any (resp. for some) $\lambda_3, \mu_3 \in \mathbb{R}_+$. It suffices to show

$$\forall\lambda_1 \in \mathbb{R}_+ \ \exists\lambda_2 \in \mathbb{R}_+ \ \forall\mu_2 \in \mathbb{R}_+ \ \exists\mu_1 \in \mathbb{R}_+$$
$$(\forall\mu_1 \in \mathbb{R}_+ \ \exists\mu_2 \in \mathbb{R}_+ \ \forall\lambda_2 \in \mathbb{R}_+ \ \exists\lambda_1 \in \mathbb{R}_+)$$
$$\Psi : Cw^{\mu_2}(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^{\lambda_2}}(\mathbb{R}^d)) \to \mathcal{Z}_{w^{\mu_1}}^{M^{\lambda_1}} \text{ is well-defined and continuous.}$$

Using [L], for any $\lambda_1 \in \mathbb{R}_+$ there exists a $\lambda_2, \lambda_3 \in \mathbb{R}_+$ (for any $\lambda_2, \lambda_3 \in \mathbb{R}_+$ there exists a $\lambda_1 \in \mathbb{R}_+$) such that $2^{|\alpha|}\max(M_\alpha^{\lambda_2}, M_\alpha^{\lambda_3}) \leqslant C_1 M_\alpha^{\lambda_1}$ for some $C_1 > 0$. For any $\mu_2 \in \mathbb{R}_+$ we have a $\mu_1, \mu_3' \in \mathbb{R}_+$ (for any $\mu_1, \mu_3' \in \mathbb{R}_+$ we have $\mu_2 \in \mathbb{R}_+$) such that $w^{\mu_1}(x+y) \leqslant C_2 w^{\mu_2}(x)v^{\mu_3'}(y)$ for some $C_2 > 0$ and any $x, y \in \mathbb{R}^d$. Also, by Lemma 3.2.3, there is a $\mu_3 \in \mathbb{R}_+$ (for a fixed $\mu_3 \in \mathbb{R}_+$ there exists a $\mu_3' \in \mathbb{R}_+$) such that $\sum_{\beta\in b\mathbb{Z}^d} v^{\mu_3'}(t-\beta)/v^{\mu_3}(t-\beta) \leqslant C_3$ for some $C_3 > 0$ and all $t \in \mathbb{R}^d$. Now, for an arbitrary $(\varphi_\beta)_{\beta\in b\mathbb{Z}} \in Cw^{\mu_2}(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^{\lambda_2}}(\mathbb{R}^d))$ we have that

for any $\alpha \in \mathbb{N}^d$ and $t \in \mathbb{R}^d$,

$$\left| \partial^\alpha \sum_{\beta \in b\mathbb{Z}^d} \psi(t - \beta) \varphi_\beta(t) \right| \frac{w^{\mu_1}(t)}{M_\alpha^{\lambda_1}}$$

$$\leq C_1 C_2 M_0^{\lambda_1} \sum_{\beta \in b\mathbb{Z}^d} 2^{-|\alpha|}$$

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \frac{|\psi^{(\alpha')}(t - \beta)| v^{\mu_3'}(t - \beta)}{M_{\alpha'}^{\lambda_3}} \cdot \frac{|\varphi_\beta^{(\alpha - \alpha')}(t)| w^{\mu_2}(\beta)}{M_{\alpha - \alpha'}^{\lambda_2}}$$

$$\leq \left(C_1 C_2 C_3 M_0^{\lambda_1} \|\psi\|_{\mathcal{S}_{v^{\mu_3},\infty}^{M^{\lambda_3}}} \right) \left\| (\varphi_\beta)_{\beta \in b\mathbb{Z}^d} \right\|_{w^{\mu_2}, \mathcal{E}_{p,a}^{M^{\lambda_2}}} .$$

Whence the continuity of Ψ follows.

Next, we consider Φ. Again we have $\gamma \in \mathcal{S}_{v^{\mu_3},\infty}^{M^{\lambda_3}}$ for any $\lambda_3, \mu_3 \in \mathbb{R}_+$ (for some $\lambda_3, \mu_3 \in \mathbb{R}_+$). It suffices to show

$$\forall \lambda_1 \in \mathbb{R}_+ \; \exists \lambda_2 \in \mathbb{R}_+ \; \forall \mu_2 \in \mathbb{R}_+ \; \exists \mu_1 \in \mathbb{R}_+$$
$$(\forall \mu_1 \in \mathbb{R}_+ \; \exists \mu_2 \in \mathbb{R}_+ \; \forall \lambda_2 \in \mathbb{R}_+ \; \exists \lambda_1 \in \mathbb{R}_+)$$

$$\Phi : \mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}} \to Cw^{\mu_1}(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{M^{\lambda_1}}(\mathbb{R}^d)) \text{ is well-defined and continuous.}$$

Using similar considerations on the parameters as above, we have for arbitrary $\varphi \in \mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}}$ that for any $\alpha \in \mathbb{N}^d$, $\beta \in b\mathbb{Z}^d$ and $t \in \mathbb{R}^d$,

$$\left| \partial^\alpha \sum_{\delta \in a\mathbb{Z}^d} \gamma(t - \beta - \delta) \varphi(t - \delta) \right| \frac{w^{\mu_1}(\beta)}{M_\alpha^{\lambda_1}}$$

$$\leq C_1 C_2 M_0^{\lambda_1} \sum_{\delta \in a\mathbb{Z}^d} 2^{-|\alpha|}$$

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \frac{|\gamma^{(\alpha')}(t - \beta - \delta)| v^{\mu_3'}(t - \beta - \delta)}{M_{\alpha'}^{\lambda_3}} \frac{|\varphi^{(\alpha - \alpha')}(t - \delta)| w^{\mu_2}(t - \delta)}{M_{\alpha - \alpha'}^{\lambda_2}}$$

$$\leq \left(C_1 C_2 C_3 M_0^{\lambda_1} \|\gamma\|_{\mathcal{S}_{v^{\mu_3},\infty}^{M^{\lambda_3}}} \right) \|\varphi\|_{\mathcal{Z}_{w^{\mu_2}}^{M^{\lambda_2}}} ,$$

where we used the fact that $\mathscr{V}$ is symmetric. As $\sum_{\delta \in a\mathbb{Z}^d} T_\delta((T_\beta \gamma)\varphi)$ is clearly $a\mathbb{Z}^d$-periodic, we see that Φ is well-defined and continuous. $\square$

Corollary 5.4.11. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system such that Assumption (5.4.2) holds for some weight function system $\mathscr{V}$. If there exists $\psi, \gamma \in \mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]}$ and $a, b > 0$ such that*

$$\sum_{k \in \mathbb{Z}^d} T_{ak+bj} \psi T_{ak+bl} \gamma = \delta_{j,l}, \qquad (j, l) \in \mathbb{Z}^{2d}, \qquad (5.3)$$

then $\langle \mathscr{W} \rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d))$ is isomorphic to a complemented subspace of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$.

Proof. Condition (5.3) implies $\Phi \circ \Psi = \mathrm{id}_{\langle \mathscr{W} \rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d))}$, so this is a direct consequence of Proposition 5.4.10. $\qquad\square$

In the following lemma, we establish the connection between (5.3) and Gabor frames.

Lemma 5.4.12. *Let $a, b > 0$. For $\psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$ the following statements are equivalent:*

(i) ψ *and* γ *satisfy* (5.3).

(ii) $\frac{1}{a^d} V_{\overline{\gamma}} \psi \left(bj, \frac{k}{a} \right) = \delta_{j,0} \delta_{k,0}$ *for all* $(k, j) \in \mathbb{Z}^{2d}$.

(iii) $\mathcal{G} \left(\psi, a, \frac{1}{b} \right)$ *is a Gabor frame and* $\frac{1}{b^d} \overline{\gamma}$ *is a dual window of* ψ.

Proof. By the Poisson summation formula, we have that for any $(l, j) \in \mathbb{Z}^{2d}$:

$$\sum_{k \in \mathbb{Z}^d} T_{ak+bj} \gamma(x) T_{ak+bl} \psi(x) = \frac{1}{a^d} \sum_{k \in \mathbb{Z}^d} V_{\overline{\gamma}} \psi \left(b(j - l), \frac{k}{a} \right) e^{2\pi i \frac{k}{a}(x-bl)}.$$

This shows $(i) \Leftrightarrow (ii)$. The equivalence $(ii) \Leftrightarrow (iii)$ is a consequence of the the Wexler-Raz biorthogonality relations, see Lemma 3.4.4. $\quad\square$

We now arrive at the proof of part B.

Proof of Theorem 5.4.3 Part B. If $\mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]}$ is Gabor accessible, then (5.3) holds for certain $\psi, \gamma \in \mathcal{S}_{[\mathscr{V}]}^{[\mathfrak{M}]}$ by Lemma 5.4.12. Consequently, the

space $\langle \mathscr{W} \rangle C(b\mathbb{Z}^d; \mathcal{E}_{p,a}^{[\mathfrak{M}]}(\mathbb{R}^d))$ is isomorphic to a complemented subspace of $\mathcal{Z}_{\mathscr{W}}^{[\mathfrak{M}]}$ by Corollary 5.4.11. As a complemented subspace of a barrelled space is again barrelled, it follows from Lemma 5.4.9 that $\mathcal{A}_{\mathscr{W}_{\mathfrak{M}} \otimes \mathscr{W}} C(b\mathbb{Z}^d \times \mathbb{Z}^d)$ (resp. $\mathcal{A}_{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}} C(\mathbb{Z}^d \times b\mathbb{Z}^d)$) is barrelled. Combining Theorem 5.3.3 with Lemma 5.3.5, we see that $\mathscr{W}_{|\mathbb{Z}^d}$ satisfies $\{DN\}$ (resp. $(\overline{\overline{\Omega}})$). The proof is now completed by Lemma 5.2.4. $\quad\square$

Chapter 6

The spaces $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$

6.1 Introduction

The space $\mathcal{B}'$ of bounded distributions and its subspace $\dot{\mathcal{B}}'$ of distributions vanishing at infinity, introduced by Schwartz [125], play an important role in the convolution theory for distributions [97, 99, 100] and the asymptotic analysis of generalized functions [114]. Their analogues in the setting of ultradistributions were first considered in [25, 108] and further studied in [7, 24, 46, 48]. In [46], the second structure theorem for these spaces (and their weighted variants) was shown for weight sequences by means of the parametrix method. This technique imposes heavy restrictions on the defining weight sequence, namely, the assumptions $(M.1)$, $(M.2)$, and $(M.3)$. The main goal of this chapter is to show the first structure theorem for the bounded ultradistributions $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and the space of ultradistributions vanishing at infinity $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ (both with respect to a weight function ω). Our main results are Theorem 6.4.1 and Theorem 6.4.12, both crucial for Part II.

In the case of $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$, it is important to point out that none of the methods available in the literature applies to deliver a proof for Theorem 6.4.1. We develop here a new approach to the problem whose core consists in combining a criterion for the surjectivity of a continuous linear mapping in terms of its transpose (Lemma 6.4.3) with the computation of the dual of $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$. The latter computation

is achieved by exploiting the mapping properties of the short-time Fourier transform. In fact, we shall show that the strong dual of $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ is given by $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$.

The chapter is organized as follows. We start in Section 6.2 by studying the mapping properties of the STFT on the tempered ultra-distributions $\mathcal{S}'^{[\mathfrak{M}]}_{[\mathscr{W}]}$. Our main result will be Proposition 6.2.7, where in particular we have that the so-called desingularisation formula (6.2) holds in $\mathcal{S}'^{[\mathfrak{M}]}_{[\mathscr{W}]}$. After this, in Section 6.3, we formally intro-duce the spaces we will be working with and consider their inherent topological properties. In particular, we characterize these spaces by evaluating the decay of the STFT. The main results here will be Theorem 6.3.10 and Theorem 6.3.11 where we determine the spaces $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ by the limit behavior of their translates. Finally, in Section 6.4 we deliver the aforementioned structural theorems as well as discuss the projective description of $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$.

6.2 The short-time Fourier transform of tempered ultradistributions

In this preliminary section we discuss the mapping properties of the STFT on the duals of the Gelfand-Shilov spaces $\mathcal{S}'^{(\mathfrak{M})}_{(\mathscr{W})}$. In particular, our aim is to obtain the so-called desingularisation formula (6.2). In order to do this, we must first consider the differentiability of the STFT on $\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$. To this purpose, we introduce the following condition on $\mathscr{W}$:

(sN) $\forall \lambda \in \mathbb{R}_+ \; \forall k \in \mathbb{N} \; \exists \mu \in \mathbb{R}_+ : \int_{\mathbb{R}^d}(1+|t|)^k w^\lambda(t)/w^\mu(t)dt < \infty$.

We will work under the following strengthening of Assumption 3.4.2 on a weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$.

Assumption 6.2.1. $\mathfrak{M}$ *is isotropically decomposable satisfying* (L) *and* $(\mathfrak{M}.2)'$, $\mathscr{W}$ *is symmetric and satisfies* (M) *and* (sN) *and* $\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \neq \{0\}$.

For a weight function system $\mathscr{W}$, we define $\mathcal{S}_{(\mathscr{W})}$ as the Fréchet space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{|\alpha| \leqslant k} \sup_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)|w^\lambda(x) < \infty, \qquad \forall k \in \mathbb{N}, \lambda \in \mathbb{R}_+.$$

We may then described he continuity of the STFT on $\mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$ as follows.

Proposition 6.2.2. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system such that Assumption 6.2.1 holds. For any $\psi \in \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$ the mappings*

$$V_\psi : \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})} \to \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}$$

and

$$V_\psi^* : \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})} \to \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$$

are well-defined and continuous.

Proof. In view of Proposition 3.4.1 it suffices to show $V_\psi : \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})} \to \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}$ is well-defined an continuous. We have that $\psi \in \mathcal{S}_{w^{\mu_3},\infty}^{M^{\lambda_3}}$ for every $\lambda_3, \mu_3 \in \mathbb{R}_+$. It suffices to show

$$\forall \lambda_1, \mu_1 \in \mathbb{R}_+ \ \forall k \in \mathbb{N} \ \exists \lambda_2, \mu_2 \in \mathbb{R}_+ :$$

$\mathcal{S}_{w^{\mu_2},\infty}^{M^{\lambda_2}} \to \mathcal{S}_{w^{\mu_1} \otimes \exp \omega_{M^{\lambda_1}}}^{k}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ is well-defined and continuous.

Fix $\lambda_1, \mu_1 \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Let $\mu_2', \mu_3 \in \mathbb{R}_+$ be such that $w^{\mu_1}(x+y) \leqslant C_1 w^{\mu_2'}(x) w^{\mu_3}(y)$ for some $C_1 > 0$ and any $x, y \in \mathbb{R}^d$. Using (sN), let $\mu_2 \in \mathbb{R}_+$ be such that $(1+|t|)^\beta w^{\mu_2'}(t)/w^{\mu_2}(t) \in L^1(\mathbb{R}^d)$ for all $\beta \in \mathbb{N}^d$ such that $|\beta| \leqslant k$. Using (L) and $(\mathfrak{M}.2)'$ there is some $\lambda_2 \in \mathbb{R}_+$ such that $M_{\alpha+\nu}^{\lambda_2} \leqslant C_2 M_\alpha^{\lambda_1}$ for some $C_2 > 0$ and any $\alpha, \nu \in \mathbb{N}^d$ with $|\nu| \leqslant k$. We set $\lambda_3 = \lambda_2$. Take any $\varphi \in \mathcal{S}_{w^{\mu_2},\infty}^{M^{\lambda_2}}$, then for all $\gamma \in \mathbb{N}^d$ we have

$$\max_{|\alpha|,|\beta| \leqslant k} w^{\mu_1}(x) \left| \frac{\xi^\gamma}{M_\gamma^{\lambda_1}} \partial_\xi^\beta \partial_x^\alpha V_\psi \varphi(x, \xi) \right|$$

$$\leqslant C_1 M_0^{\lambda_1} (2\pi)^{k-|\gamma|}$$

$$\max_{|\alpha|,|\beta| \leqslant k} \sum_{\kappa \leqslant \gamma} \binom{\gamma}{\kappa} \int_{\mathbb{R}^d} \frac{|(t^\beta \varphi(t))^{(\kappa)}| w^{\mu_2'}(t)}{M_\kappa^{\lambda_1}} \frac{|\psi^{(\alpha+\gamma-\kappa)}(x-t)| w^{\mu_3}(x-t)}{M_{\gamma-\kappa}^{\lambda_1}} dt$$

$$\leqslant C \|\varphi\|_{\mathcal{S}_{w^{\mu_2},\infty}^{M^{\lambda_2}}},$$

for some $C > 0$. We may conclude the continuity of $V_\psi : \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})} \to \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$. $\square$

For any $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$ and $f \in \mathcal{S}'^{[\mathfrak{M}]}_{[\mathscr{W}]}$ we define the STFT of f as

$$V_\psi f(x, \xi) = \left\langle f, \overline{M_\xi T_x \psi} \right\rangle = e^{-2\pi i \xi \cdot x} (f * M_\xi \check{\bar{\psi}})(x), \qquad (x, \xi) \in \mathbb{R}^{2d}.$$

Then, $V_\psi f$ is obviously a smooth function on $\mathbb{R}^{2d}$.

Lemma 6.2.3. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and $\mathscr{W}$ be a weight function system satisfying* [M]. *Let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W}),\infty}$ and $f \in \mathcal{S}'^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}$. Then for some $\lambda \in \mathbb{R}_+$ (for any $\lambda \in \mathbb{R}_+$) there is a $C = C_\lambda > 0$ such that*

$$|V_\psi f(x, \xi)| \leqslant C w^\lambda(x) e^{\omega_{M^\lambda}(\xi)}.$$

In particular, if $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ *and $\mathscr{W}$ satisfies* [N], *then, $V_\psi f$ defines an element of $\mathcal{S}'_{(\mathscr{W} \otimes \mathscr{W}_\mathfrak{M})}(\mathbb{R}^{2d}_{x,\xi})$ via*

$$\langle V_\psi f, \Phi \rangle := \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) \Phi(x, \xi) dx d\xi, \qquad \Phi \in \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_\mathfrak{M})}(\mathbb{R}^{2d}_{x,\xi}).$$

Proof. For some $\mu > 0$ (for any $\mu > 0$) there exists a $C = C_\mu > 0$ such that

$$|V_\psi f(x, \xi)| \leqslant C \left\| M_\xi T_x \psi \right\|_{\mathcal{S}^{M^\mu}_{w^\mu,\infty}}.$$

There exists a $\lambda > 0$ (for every $\lambda > 0$ there exists a $\mu > 0$) such that $w^\mu(x + y) \leqslant C_1 w^\lambda(x) w^\lambda(y)$ and $(4\pi)^{|\alpha|} M^\lambda_\alpha \leqslant C_2 M^\mu_\alpha$ for some $C_1, C_2 > 0$. Then

$$\left\| M_\xi T_x \psi \right\|_{\mathcal{S}^{M^\mu}_{w^\mu,\infty}}$$
$$\leqslant C_1 w^\lambda(x) \sup_{\alpha \in \mathbb{N}^d} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{|(2\pi\xi)^\beta| \sup_{t \in \mathbb{R}^d} |\psi^{(\alpha-\beta)}(t-x)| w^\lambda(t-x)}{M^\mu_\alpha}$$
$$\leqslant C_1 C_2 \left\| \psi \right\|_{\mathcal{S}^{M^\lambda}_{w^\lambda,\infty}} w^\lambda(x) e^{\omega_{M^\lambda}(\xi)}.$$

The second assertion now follows from Lemma 3.2.5. $\qquad \square$

Corollary 6.2.4. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ be a weight function system such that Assumption 6.2.1 holds. Let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$ and $f \in \mathcal{S}'^{(\mathfrak{M})}_{(\mathscr{W})}$, then*

$$\langle V_\psi f, \Phi \rangle = \left\langle f, \overline{V^*_\psi \bar{\Phi}} \right\rangle, \qquad \Phi \in \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_\mathfrak{M})}.$$

Proof. For any $\Phi \in \mathcal{S}_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}$ and $h > 0$ consider

$$\varphi_h(t) = \sum_{k,l \in \mathbb{Z}^d} \Phi(hk, hl) e^{-2\pi i h l \cdot t} \overline{\psi(t - hk)} h^{2d}.$$

By Proposition 3.4.5 we see that the sum in each φ_n converges absolutely in $\mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$. Moreover, by applying the mean value theorem, one sees that

$$\varphi_h \to \int\int_{\mathbb{R}^{2d}} \Phi(x, \xi) e^{-2\pi i \xi \cdot t} \overline{\psi(t - x)} dx d\xi \quad \text{in } \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}.$$

Consequently, we see that

$$\left\langle f(t), \int\int_{\mathbb{R}^{2d}} \Phi(x, \xi) e^{-2\pi i \xi \cdot t} \overline{\psi(t - x)} dx d\xi \right\rangle$$

$$= \int\int_{\mathbb{R}^{2d}} \Phi(x, \xi) \left\langle f, \overline{M_\xi T_x \psi} \right\rangle dx d\xi$$

which completes the proof. $\qquad\qquad\square$

Under Assumption 6.2.1, for $\psi \in \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$, we define the adjoint STFT of $F \in \mathcal{S}'_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}$ as

$$\left\langle V_\psi^* F, \varphi \right\rangle = \left\langle F, \overline{V_\psi \overline{\varphi}} \right\rangle, \qquad \varphi \in \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}.$$

Then, $V_\psi^* F \in \mathcal{S}_{(\mathscr{W})}'^{(\mathfrak{M})}$ by Proposition 6.2.2. We are now able to establish the mappings properties of the STFT on $\mathcal{S}_{(\mathscr{W})}'^{(\mathfrak{M})}$ and obtain the desingularisation formula.

Proposition 6.2.5. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ a weight function system such that Assumption 6.2.1 holds. For any $\psi \in \mathcal{S}_{(\mathscr{W})}^{(\mathfrak{M})}$ the mappings*

$$V_\psi : \mathcal{S}_{(\mathscr{W})}'^{(\mathfrak{M})} \to \mathcal{S}'_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}$$

and

$$V_\psi^* : \mathcal{S}'_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})} \to \mathcal{S}_{(\mathscr{W})}'^{(\mathfrak{M})}$$

are continuous. Moreover, if $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \setminus \{0\}$ and $\gamma \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$ is a synthesis window for ψ, then

$$\frac{1}{(\gamma, \psi)_{L^2}} V^*_\gamma \circ V_\psi = \mathrm{id}_{\mathcal{S}'^{(\mathfrak{M})}_{(\mathscr{W})}} \tag{6.1}$$

and the desingularisation formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\overline{\gamma}} \varphi(x, -\xi) dx d\xi \tag{6.2}$$

holds for any $f \in \mathcal{S}'^{(\mathfrak{M})}_{(\mathscr{W})}$ and $\varphi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$.

Proof. Proposition 6.2.2 directly yields the continuity of V^*_ψ, and the continuity of V_ψ also follows from here in view of Corollary 6.2.4. Next, let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \setminus \{0\}$ and $\gamma \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$ be a synthesis window for ψ, then by Corollary 6.2.4 and the reconstruction formula (3.1), we infer that for any $\varphi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$

$$\left\langle V^*_\gamma(V_\psi f), \varphi \right\rangle = \left\langle V_\psi f, \overline{V_\gamma \overline{\varphi}} \right\rangle = \left\langle f, \overline{V^*_\psi(V_\gamma \overline{\varphi})} \right\rangle = (\gamma, \psi)_{L^2} \left\langle f, \varphi \right\rangle$$

which shows (6.1) and (6.2). $\qquad\qquad\qquad\square$

The final goal of this section is to show (6.2) also holds for any $f \in \mathcal{S}'^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ and $\varphi \in \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$. This would follow easily from Lemma 6.2.3 if we can show that $\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})}$ is dense in $\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$.

Lemma 6.2.6. *Let $\mathfrak{M}$ be a weight sequence system satisfying* (L), *{L}, $(\mathfrak{M}.2)'$ and $\{\mathfrak{M}.2\}'$ and $\mathscr{W}$ a symmetric weight function system satisfying* (M), *{M}, (N) and {N}. If $\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \neq \{0\}$, then the following dense inclusions hold*

$$\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \hookrightarrow \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} \hookrightarrow \mathcal{S}'^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} \hookrightarrow \mathcal{S}'^{(\mathfrak{M})}_{(\mathscr{W})}.$$

Proof. We start by showing the first density. Of course we have the continuous inclusion $\mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \subseteq \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$. Take any $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathscr{W})} \setminus \{0\}$ such that $\|\psi\|_{L^2} = 1$. For any $\varphi \in \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ we have $V_\psi \varphi \in C_{\{\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}}\}}(\mathbb{R}^{2d}_{x,\xi})$ by Proposition 3.4.1. By our assumptions, $C_{(\mathscr{W} \otimes \mathscr{W}_{\mathfrak{M}})}(\mathbb{R}^{2d}_{x,\xi})$ is dense

in $C_{\{\mathcal{W} \otimes \mathcal{W}_\mathfrak{M}\}}(\mathbb{R}^{2d}_{x,\xi})$, so take any sequence $(\varphi_n)_n \subset C_{(\mathcal{W} \otimes \mathcal{W}_\mathfrak{M})}(\mathbb{R}^{2d}_{x,\xi})$ such that $\varphi_n \to V_\psi \varphi$ in $C_{\{\mathcal{W} \otimes \mathcal{W}_\mathfrak{M}\}}(\mathbb{R}^{2d}_{x,\xi})$. Then $V^*_{\tilde{\psi}} \varphi_n \to V^*_{\tilde{\psi}} \circ V_\psi \varphi = \varphi$ in $\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathcal{W}\}}$ by Proposition 3.4.1. Now, once again by Proposition 3.4.1, $V^*_{\tilde{\psi}} \varphi_n \in \mathcal{S}^{(\mathfrak{M})}_{(\mathcal{W})}$, so that the first density follows.

We now show $\mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$ is dense in $\mathcal{S}'^{[\mathfrak{M}]}_{[\mathcal{W}]}$, which would complete the proof. The embedding $\iota : \mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]} \to \mathcal{S}'^{[\mathfrak{M}]}_{[\mathcal{W}]}; \varphi \mapsto (\phi \mapsto \int_{\mathbb{R}^d} \phi\varphi)$ is continuous, and as $\mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$ is reflexive we have that $\iota^t = \iota$. Suppose that for some $\varphi \in \mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$ we have $\int_{\mathbb{R}^d} \phi\varphi = 0$ for any $\phi \in \mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$. Take $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathcal{W})}$ as before, then $V_\psi \varphi = 0$, so that it follows from Proposition 3.4.1 that $\varphi = 0$. Hence the mapping ι^t is injective, whence it follows that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$ is dense in $\mathcal{S}'^{[\mathfrak{M}]}_{[\mathcal{W}]}$. $\qquad\square$

Proposition 6.2.7. *Let $\mathfrak{M}$ be a weight sequence system satisfying* (L), {L}, $(\mathfrak{M}.2)'$ *and* $\{\mathfrak{M}.2\}'$ *and $\mathcal{W}$ a symmetric weight function system satisfying* (M), {M}, (sN) *and* {N}. *If $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(\mathcal{W})} \setminus \{0\}$ and $\gamma \in \mathcal{S}^{(\mathfrak{M})}_{(\mathcal{W})}$ is a synthesis window for ψ, then the desingularisation formula* (6.2) *holds for any $f \in \mathcal{S}'^{[\mathfrak{M}]}_{[\mathcal{W}]}$ and $\varphi \in \mathcal{S}^{[\mathfrak{M}]}_{[\mathcal{W}]}$.*

Proof. We need only verify the Roumieu case. Take any $f \in \mathcal{S}'^{\{\mathfrak{M}\}}_{\{\mathcal{W}\}}$. We define $\tilde{f}$ by

$$\left\langle \tilde{f}, \varphi \right\rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\overline{\gamma}} \varphi(x, -\xi) dx d\xi,$$

then by combining Proposition 3.4.1 with Lemma 6.2.3 we see that $\tilde{f} \in \mathcal{S}'^{[\mathfrak{M}]}_{[\mathcal{W}]}$. By (6.2) it follows that f and $\tilde{f}$ coincide on $\mathcal{S}^{(\mathfrak{M})}_{(\mathcal{W})}$, so that the proof is completed by Lemma 6.2.6. $\qquad\square$

6.3 The space $\mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]}$ and its dual

In this chapter, by a *weight function* we mean a measurable function $\omega : \mathbb{R}^d \to (0, \infty)$ such that ω and ω^{-1} are locally bounded. Given an isotropically decomposable weight sequence A, a weight function ω is said to be (A)-*admissible* ($\{A\}$-*admissible*) if

$$\exists q > 0 \ (\forall q > 0) \ \exists C > 0 \ \forall x, t \in \mathbb{R}^d \ : \ \omega(x + t) \leqslant C\omega(x)e^{\omega_A(qt)}.$$

Next, we introduce various function and ultradistribution spaces associated to a weight function ω (cf. [48]). We define $L_\omega^1(\mathbb{R}^d)$ as the Banach space consisting of all measurable functions φ on $\mathbb{R}^d$ such that

$$\|\varphi\|_{L_\omega^1} := \int_{\mathbb{R}^d} |\varphi(x)|\omega(x)dx < \infty.$$

Its dual is given by the space $L_\omega^\infty(\mathbb{R}^d)$ of all those measurable functions φ on $\mathbb{R}^d$ such that

$$\|\varphi\|_{L_\omega^\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \frac{|\varphi(x)|}{\omega(x)} < \infty.$$

We write $\mathcal{D}_{L_\omega^1}(\mathbb{R}^d)$ for the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\varphi^{(\alpha)} \in L_\omega^1(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$. Let M be a weight sequence. We denote by $\mathcal{D}_{L_\omega^1}^M$ the Banach space consisting of all $\varphi \in \mathcal{D}_{L_\omega^1}(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{D}_{L_\omega^1}^M} := \sup_{\alpha\in\mathbb{N}^d} \frac{\left\|\varphi^{(\alpha)}\right\|_{L_\omega^1}}{M_\alpha} < \infty.$$

For a weight sequence system $\mathfrak{M}$, we define

$$\mathcal{D}_{L_\omega^1}^{(\mathfrak{M})} := \varprojlim_{\lambda\to 0^+} \mathcal{D}_{L_\omega^1}^{M^\lambda}, \qquad \mathcal{D}_{L_\omega^1}^{\{\mathfrak{M}\}} := \varinjlim_{\lambda\to\infty} \mathcal{D}_{L_\omega^1}^{M^\lambda}.$$

We introduce the following set of assumptions on a weight sequence system $\mathfrak{M}$ and a weight function ω.

Assumption 6.3.1. $\mathfrak{M}$ *is an isotropically decomposable weight sequence system satisfying* (L), {L}, $(\mathfrak{M}.2)'$ *and* $\{\mathfrak{M}.2\}'$ *and there exists an isotropically decomposable weight sequence A satisfying* $(M.1)$ *and* $(M.2)'$ *such that ω is $[A]$-admissible and $\mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$ is non-trivial.*

Remarks 6.3.2. If a weight function ω is $(p!)$-admissible, then Assumption 6.3.1 is fulfilled for M and ω, whenever M is an isotropic weight sequence that satisfies $(M.1)$, $(M.2)'$, and $(\log p)^p \prec M_p$, as follows from [41, Proposition 2.7 and Theorem 5.9]. We point out that [44, Remark 5.3] ω is $(p!)$-admissible if and only if L_ω^1 is translation-invariant if and only if

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \frac{\omega(\,\cdot\,+x)}{\omega(x)} \in L_{\text{loc}}^\infty. \tag{6.3}$$

In the rest of this section, we fix a weight sequence system $\mathfrak{M}$ and a weight function ω such that Assumption 6.3.1 holds for some weight sequence A. Note that in particular the assumptions posed by Proposition 6.2.7 hold for $\mathfrak{M}$ and $\mathscr{W}_A$.

We first discuss the topological properties of the space $\mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]}$. In the Roumieu case we will employ the following projective version of $\mathcal{D}_{L^1_\omega}^{\{\mathfrak{M}\}}$:

$$\widetilde{\mathcal{D}}_{L^1_\omega}^{\{\mathfrak{M}\}} = \varprojlim_{M \in \overline{V}(\mathfrak{M})} \mathcal{D}_{L^1_\omega}^M.$$

In view of Lemma 4.2.12(i), the spaces $\mathcal{D}_{L^1_\omega}^{\{\mathfrak{M}\}}$ and $\widetilde{\mathcal{D}}_{L^1_\omega}^{\{\mathfrak{M}\}}$ coincide as sets. It is clear that the topology of $\mathcal{D}_{L^1_\omega}^{\{\mathfrak{M}\}}$ is finer than that of $\widetilde{\mathcal{D}}_{L^1_\omega}^{\{\mathfrak{M}\}}$. Later, in Section 6.4, we will discuss when $\mathcal{D}_{L^1_\omega}^{\{\mathfrak{M}\}} = \widetilde{\mathcal{D}}_{L^1_\omega}^{\{\mathfrak{M}\}}$ as locally convex spaces.

Lemma 6.3.3. $\mathcal{D}_{L^1_\omega}^{(\mathfrak{M})}$ *is a quasinormable and thus distinguished Fréchet space, and $\mathcal{D}_{L^1_\omega}^{\{\mathfrak{M}\}}$ is a complete and thus regular (LB)-space.*

Proof. To verify that $\mathcal{D}_{L^1_\omega}^{(\mathfrak{M})}$ is quasinormable, it suffices to show that [92, Lemma 26.14, p. 315]

$$\forall \lambda > 0 \; \exists \mu > 0 \; \forall \eta > 0 \; \forall \varepsilon \in (0,1] \; \exists R > 0$$

$$\forall \varphi \in \mathcal{D}_{L^1_\omega}^{(\mathfrak{M})} \text{ with } \|\varphi\|_{\mathcal{D}_{L^1_\omega}^{M^\mu}} \leqslant 1$$

$$\exists \psi \in \mathcal{D}_{L^1_\omega}^{(\mathfrak{M})} \text{ with } \|\psi\|_{\mathcal{D}_{L^1_\omega}^{M^\eta}} \leqslant R \text{ such that } \|\varphi - \psi\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}} \leqslant \varepsilon.$$

Let $\lambda > 0$ be arbitrary and $\mu > 0$ be such that $M_{\alpha+e_j}^\mu \leqslant C_0 M_\alpha^\lambda$ for any $\alpha \in \mathbb{N}^d$ and $j \in \{1,\ldots,d\}$. Take any $\varphi \in \mathcal{D}_{L^1_\omega}^{(\mathfrak{M})}$ such that $\|\varphi\|_{\mathcal{D}_{L^1_\omega}^{M^\mu}} \leqslant 1$. Choose some $\chi \in \mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \chi(x)dx = 1$ and put $\chi_\varepsilon = \varepsilon^{-d}\chi(\cdot/\varepsilon)$ for $\varepsilon \in (0,1]$. Take any $\eta > 0$. For any $\varepsilon \in (0,1]$ let $\kappa > 0$ be such that $M_\alpha^\kappa \leqslant C_0' \varepsilon^{|\alpha|} M_\alpha^\eta$ for any $\alpha \in \mathbb{N}^d$. Then

$$\|\varphi * \chi_\varepsilon\|_{\mathcal{D}_{L^1_\omega}^{M^\eta}} = \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M_\alpha^\eta} \int_{\mathbb{R}^d} |\varphi * \chi_\varepsilon^{(\alpha)}(x)|\omega(x)dx$$

$$\leqslant C \|\varphi\|_{L^1_\omega} \sup_{\alpha \in \mathbb{N}^d} \frac{1}{\varepsilon^{|\alpha|} M_\alpha^\eta} \int_{\mathbb{R}^d} |\chi^{(\alpha)}(x)|e^{\omega_A(q\varepsilon x)}dx$$

$$\leqslant CC_0' M_0^\mu \|\chi\|_{\mathcal{S}_{\exp \omega_A(q\cdot),1}^{M^\kappa}}.$$

On the other hand, applying the mean-value theorem, we obtain that

$$\|\varphi - \varphi * \chi_\varepsilon\|_{\mathcal{D}^{M^\lambda}_{L^1_\omega}}$$

$$= \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \|\varphi^{(\alpha)} - \varphi^{(\alpha)} * \chi_\varepsilon\|_{L^1_\omega}$$

$$\leqslant \varepsilon \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \int_{\mathbb{R}^d} \omega(x) \left(\int_{\mathbb{R}^d} |\chi(t)||t| \sum_{j=1}^d \int_0^1 |\varphi^{(\alpha+e_j)}(x - \gamma\varepsilon t)| d\gamma dt \right) dx$$

$$\leqslant \left(CC_0 d \int_{\mathbb{R}^d} \chi(t)|t| e^{\omega_A(qt)} dt \right) \varepsilon,$$

from which the result for $\mathcal{D}^{(\mathfrak{M})}_{L^1_\omega}$ easily follows.

It suffices to show that $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ is sequentially retractive in order to conclude that it is complete by (2.1). Let $(\varphi_n)_{n \in \mathbb{N}}$ be a null sequence in $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$. As $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ may be continuously included into $\widetilde{\mathcal{D}}^{\{\mathfrak{M}\}}_{L^1_\omega}$, $(\varphi_n)_n$ is also a null sequence in $\widetilde{\mathcal{D}}^{\{\mathfrak{M}\}}_{L^1_\omega}$. Consequently, for any $M \in \overline{V}(\mathfrak{M})$ and $\varepsilon > 0$ there exists a $n_{M,\varepsilon} \in \mathbb{N}$ such that $\sup_{\alpha \in \mathbb{N}^d} \|\varphi_n^{(\alpha)}\|_{L^1_\omega}/M_\alpha \leqslant \varepsilon$, for all $n \geqslant n_{M,\varepsilon}$. By Lemma 4.2.12$(i)$ we may already conclude there exists a $\lambda > 0$ and $C > 0$ such that $\sup_{\alpha \in \mathbb{N}^d} \|\varphi_n\|_{L^1_\omega}/M^\lambda_\alpha \leqslant C$ for all $n \in \mathbb{N}$. Choose $\mu \in \mathbb{R}_+$ so that $2^{|\alpha|} M^\lambda_\alpha \leqslant C_0 M^\mu_\alpha$ for any $\alpha \in \mathbb{N}^d$ and take $p_0 \in \mathbb{N}$ for which $2^{-p} \leqslant \varepsilon/(CC_0)$ for any $p \geqslant p_0$. One may easily construct a $M \in \overline{V}(\mathfrak{M})$ such that $M_\alpha = M^\mu_\alpha$ for any $|\alpha| < p_0$. For any $n \geqslant n_{M,\varepsilon}$ we have for $|\alpha| < p_0$,

$$\frac{\|\varphi_n^{(\alpha)}\|_{L^1_\omega}}{M^\mu_\alpha} = \frac{\|\varphi_n^{(\alpha)}\|_{L^1_\omega}}{M_\alpha} \leqslant \varepsilon,$$

while for $|\alpha| \geqslant p_0$,

$$\frac{\|\varphi_n^{(\alpha)}\|_{L^1_\omega}}{M^\mu_\alpha} \leqslant \frac{C_0 \|\varphi_n^{(\alpha)}\|_{L^1_\omega}}{2^{|\alpha|} M^\lambda_\alpha} \leqslant \varepsilon.$$

We see that $(\varphi_n)_n$ is a null sequence in $\mathcal{D}^{M^\mu}_{L^1_\omega}$. Whence $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ is sequentially retractive. $\qquad \square$

We will need the ensuing basic density property.

Proposition 6.3.4. *We have the following dense continuous inclusions,*
$$\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \hookrightarrow \mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]} \hookrightarrow \mathcal{S}_{[A]}'^{[\mathfrak{M}]}(\mathbb{R}^d).$$

Proof. We adapt the idea from [46, Proof of Proposition 5.2]. It is clear that $\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \subset \mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]} \subset \mathcal{S}_{[A]}'^{[\mathfrak{M}]}(\mathbb{R}^d)$ with continuous inclusions. By Lemma 6.2.6 $\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)$ is dense in $\mathcal{S}_{[A]}'^{[\mathfrak{M}]}(\mathbb{R}^d)$, whence it suffices to show that $\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)$ is dense in $\mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]}$. Choose $\chi \in \mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \chi(x)dx = 1$ and $\psi(0) = 1$. Next, set $\chi_n = n^d\chi(n\cdot)$ and $\psi_n = \psi(\cdot/n)$ for $n \geqslant 1$. For arbitrary $\lambda > 0$ and $\varphi \in \mathcal{D}_{L^1_\omega}^{M^\lambda}$, in view of the inequality $e^{\omega_A(x+y)} \leqslant e^{\omega_A(2\sqrt{d}x) + \omega_A(2\sqrt{d}y)}$, it is clear that $\varphi_{n,j} = \chi_n * (\psi_j\varphi) \in \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)$. Take any $\varphi \in \mathcal{D}_{L^1_\omega}^{[\mathfrak{M}]}$, we shall show that for every $\lambda > 0$ (for some $\lambda > 0$) and any $\varepsilon > 0$ there are $n, j \in \mathbb{N}$ such that $\|\varphi - \varphi_{n,j}\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}} \leqslant \varepsilon$, which will complete the proof. Obviously,

$$\|\varphi - \varphi_{n,j}\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}} \leqslant \|\varphi - \chi_n * \varphi\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}} + \|\chi_n * (\varphi - \psi_j\varphi)\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}}. \quad (6.4)$$

Analogously to the first part of the proof of Lemma 6.3.3 one shows that for every $\lambda > 0$ (for some $\lambda > 0$),

$$\|\varphi - \chi_n * \varphi\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}} \leqslant \frac{\varepsilon}{2},$$

for sufficiently large n. For such a fixed n, we now proceed to estimate the second term in the right-hand side of (6.4). There is some $\mu \in \mathbb{R}_+$ such that (we may potentially enlarge λ such that for $\mu = 1$) $n^{|\alpha|} M_\alpha^\mu \leqslant C_0 M_\alpha^\lambda$ for any $\alpha \in \mathbb{N}^d$. We have that

$$\|\chi_n * (\varphi - \psi_j\varphi)\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}}$$
$$\leqslant C \|\varphi - \psi_j\varphi\|_{L^1_\omega} \sup_{\alpha \in \mathbb{N}^d} \frac{n^{|\alpha|}}{M_\alpha^\lambda} \int_{\mathbb{R}^d} |\chi^{(\alpha)}(x)| e^{\omega_A(qx)} dx$$
$$\leqslant C \|\varphi - \psi_j\varphi\|_{L^1_\omega} \|\chi\|_{\mathcal{S}_{\exp\omega_A(q\cdot),1}^{M^\mu}} \leqslant \frac{\varepsilon}{2},$$

for large enough j. $\qquad\square$

The strong dual of $\mathcal{D}_{L_\omega^1}^{[\mathfrak{M}]}$ is denoted by $\mathcal{B}_\omega'^{[\mathfrak{M}]}$. By the previous proposition, we may view $\mathcal{B}_\omega'^{[\mathfrak{M}]}$ as a subspace of $\mathcal{S}_{[A]}'^{[\mathfrak{M}]}(\mathbb{R}^d)$. We define $\dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}$ as the closure in $\mathcal{B}_\omega'^{[\mathfrak{M}]}$ of the space of compactly supported continuous functions on $\mathbb{R}^d$. Notice that $\dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}$ coincides with the closure in $\mathcal{B}_\omega'^{[\mathfrak{M}]}$ of $\mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$.

6.3.1 Characterization via the STFT

The goal of this subsection is to characterize $\mathcal{D}_{L_\omega^1}^{[\mathfrak{M}]}$, $\mathcal{B}_\omega'^{[\mathfrak{M}]}$ and $\dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}$ in terms of the STFT. We first consider $\mathcal{D}_{L_\omega^1}^{[\mathfrak{M}]}$. The following two lemmas are needed in our analysis.

Lemma 6.3.5. *Let* $\psi \in \mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$*. Then, for any* $\lambda > 0$ *(for some* $\lambda > 0$*) there is a* $C' = C'_\lambda > 0$ *such that*

$$\|V_\psi \varphi(\,\cdot\,, \xi)\|_{L_\omega^1} \leqslant C' \, \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M^\lambda}} \, e^{-\omega_{M^\lambda}(\xi)}, \qquad \xi \in \mathbb{R}^d,$$

for all $\varphi \in \mathcal{D}_{L_\omega^1}^{M^\lambda}$.

Proof. Let $\varphi \in \mathcal{D}_{L_\omega^1}^{M^\lambda}$ be arbitrary. For any $\alpha \in \mathbb{N}^d$ we have that

$$\int_{\mathbb{R}^d} |\xi^\alpha V_\psi \varphi(x, \xi)| \omega(x) dx$$

$$\leqslant (2\pi)^{-|\alpha|} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \omega(x) \left(\int_{\mathbb{R}^d} |\varphi^{(\beta)}(t)| |\psi^{(\alpha-\beta)}(x-t)| dt \right) dx$$

$$\leqslant C(2\pi)^{-|\alpha|}$$

$$\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} |\varphi^{(\beta)}(t)| \omega(t) \left(\int_{\mathbb{R}^d} |\psi^{(\alpha-\beta)}(x-t)| e^{\omega_A(q(x-t))} dx \right) dt$$

$$\leqslant C'' \, \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M^\lambda}} \, M_\alpha^\lambda,$$

whence

$$\int_{\mathbb{R}^d} |V_\psi \varphi(x, \xi)| \omega(x) dx \leqslant M_0^\lambda C'' \, \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M^\lambda}} \inf_{\alpha \in \mathbb{N}^d} \frac{M_\alpha^\lambda}{|\xi^\alpha| M_0^\lambda}$$

$$= C' \, \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M^\lambda}} \, e^{-\omega_{M^\lambda}(\xi)}.$$

$\square$

Lemma 6.3.6. *Let $\psi \in \mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d)$. For every $\lambda > 0$ there is a $\mu > 0$ (for every $\mu > 0$ there is a $\lambda > 0$) such that if F is a measurable function on $\mathbb{R}^{2d}$ for which*

$$\sup_{\xi \in \mathbb{R}^d} e^{\omega_{M^\mu}(\xi)} \int_{\mathbb{R}^d} |F(x,\xi)|\omega(x)dx < \infty,$$

then, the function

$$t \mapsto \int\int_{\mathbb{R}^{2d}} F(x,\xi) M_\xi T_x \psi(t) dx d\xi$$

belongs to $\mathcal{D}_{L^1_\omega}^{M^\lambda}$.

Proof. For any $\lambda > 0$ there is a $\eta > 0$ (for every $\eta > 0$ there is a $\lambda > 0$) such that $(4\pi)^{|\alpha|}M_\alpha^\eta \leqslant C_0 M_\alpha^\lambda$. Then for any $\alpha \in \mathbb{N}^d$ we have that

$$|\partial_t^\alpha [M_\xi T_x \psi(t)]| \leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (2\pi)^{|\beta|} |\xi^\beta| |\psi^{(\alpha-\beta)}(t-x)|$$

$$\leqslant C_0 M_0^\eta \|\psi\|_{\mathcal{S}_{\exp \omega_A(q'\cdot),\infty}^{M^\eta}} M_\alpha^\lambda e^{\omega_{M^\eta}(\xi) - \omega_A(q'(t-x))},$$

where $q' > 0$ is such that $e^{\omega_A(q\cdot)}/e^{\omega_A(q'\cdot)} \in L^1(\mathbb{R}^d)$. Now there exists a $\mu > 0$ (for every $\mu > 0$ there exists a $\eta > 0$) such that $e^{\omega_{M^\eta}(\cdot)}/e^{\omega_{M^\mu}(\cdot)} \in L^1(\mathbb{R}^d)$. Whence,

$$\left\| \int\int_{\mathbb{R}^{2d}} F(x,\xi) M_\xi T_x \psi dx d\xi \right\|_{\mathcal{D}_{L^1_\omega}^{M^\lambda}}$$

$$\leqslant C_0 M_0^\eta \|\psi\|_{\mathcal{S}_{\exp \omega_A(q'\cdot)}^{M^\eta}}$$

$$\int_{\mathbb{R}^d} \omega(t) \left(\int\int_{\mathbb{R}^{2d}} |F(x,\xi)| \exp[\omega_{M^\eta}(\xi) - \omega_A(q'(t-x))] dx d\xi \right) dt$$

$$\leqslant CC_0 M_0^\eta \|\psi\|_{\mathcal{S}_{\exp \omega_A(q'\cdot)}^{M^\eta}}$$

$$\int_{\mathbb{R}^d} e^{\omega_{M^\eta}(\xi)} \left(\int_{\mathbb{R}^d} |F(x,\xi)|\omega(x) \left(\int_{\mathbb{R}^d} e^{\omega_A(q(t-x)) - \omega_A(q'(t-x))} dt \right) dx \right) d\xi$$

$$\leqslant C'' \int_{\mathbb{R}^d} e^{\omega_{M^\eta}(\xi) - \omega_{M^\mu}(\xi)} d\xi < \infty.$$

$\square$

We are now able to characterize $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ via the STFT.

Proposition 6.3.7. *Let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d) \setminus \{0\}$ and let $f \in \mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$. Then, $f \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ if and only if*

$$\forall \lambda > 0 \ (\exists \lambda > 0) \ : \ \sup_{\xi \in \mathbb{R}^d} e^{\omega_{M^\lambda}(\xi)} \|V_\psi f(\,\cdot\,, \xi)\|_{L^1_\omega} < \infty. \qquad (6.5)$$

If $B \subset \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ is a bounded set, then (6.5) holds uniformly over B.

Proof. The direct implication and the fact that (6.5) holds uniformly over bounded sets follows immediately from Lemma 6.3.5 (and, in the Roumieu case, Lemma 6.3.3). Conversely, suppose that (6.5) holds and choose $\gamma \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d)$ such that $(\gamma, \psi)_{L^2} = 1$. By (6.2), we have that, for all $\varphi \in \mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle f, \varphi \rangle &= \int\!\!\int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\overline{\gamma}} \varphi(x, -\xi) dx d\xi \\
&= \int\!\!\int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) \left(\int_{\mathbb{R}^d} \varphi(t) M_\xi T_x \gamma(t) dt \right) dx d\xi \\
&= \int_{\mathbb{R}^d} \left(\int\!\!\int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) M_\xi T_x \gamma(t) dx d\xi \right) \varphi(t) dt,
\end{aligned}$$

where the switching of the integrals in the last step is permitted because of (6.5). Hence,

$$f = \int\!\!\int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) M_\xi T_x \gamma \, dx d\xi$$

and we may conclude that $f \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ by applying Lemma 6.3.6 to $F = V_\psi f$. $\qquad \square$

Next, we treat $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$. We again need some preparation. We consider the space $C_\omega(\mathbb{R}^d)$ of all $\varphi \in C(\mathbb{R}^d)$ such that $\|\varphi/\omega\|_{L^\infty} < \infty$ and its closed subspace $C_{0,\omega}(\mathbb{R}^d)$ of all elements f such that $\lim_{|x| \to \infty} f(x)/\omega(x) = 0$. We endow $C_{0,\omega}(\mathbb{R}^d)$ with the norm $\|\cdot\|_{L^\infty_\omega}$. The dual of $C_{0,\omega}(\mathbb{R}^d)$ is denoted by $\mathcal{M}^1_\omega$. For every $\mu \in \mathcal{M}^1_\omega$

there is a unique regular complex Borel measure $\nu \in \mathcal{M}^1 = (C_0(\mathbb{R}^d))'$ such that

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \frac{\varphi(x)}{\omega(x)} d\nu(x), \qquad \varphi \in C_{0,\omega}(\mathbb{R}^d).$$

Moreover, $\|\mu\|_{\mathcal{M}^1_\omega} = \|\nu\|_{\mathcal{M}^1} = |\nu|(\mathbb{R}^d)$. By [122, Theorem 6.13], the natural inclusion $L^1_\omega \subset \mathcal{M}^1_\omega$ holds topologically, that is,

$$\|\varphi\|_{L^1_\omega} = \sup_{f \in B_{C_{0,\omega}}} \left| \int_{\mathbb{R}^d} \varphi(x) f(x) dx \right|, \qquad \varphi \in L^1_\omega, \qquad (6.6)$$

where $B_{C_{0,\omega}}$ denotes the unit ball in $C_{0,\omega}(\mathbb{R}^d)$. We define

$$C_{(\mathfrak{M})}(\mathbb{R}^d) := \varprojlim_{\lambda\to\infty} C_{e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^d), \quad C_{\{\mathfrak{M}\}}(\mathbb{R}^d) := \varinjlim_{\lambda\to 0^+} C_{e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^d).$$

The following canonical isomorphisms of lcHs hold

$$C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{(\mathfrak{M})}(\mathbb{R}^d_\xi) \cong \varprojlim_{\lambda\to 0^+} C_{\omega\otimes e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^{2d}_{x,\xi})$$

and

$$C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{(\mathfrak{M})}(\mathbb{R}^d_\xi) \cong \varprojlim_{\lambda\to 0^+} C_{0,\omega\otimes e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^{2d}_{x,\xi}).$$

Similarly, in view of Lemma 3.2.5, [9, Theorem 3.1(d)] and [9, Theorem 3.7] yield the following canonical isomorphisms of lcHs

$$C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\{\mathfrak{M}\}}(\mathbb{R}^d_\xi) \cong \varinjlim_{\lambda\to\infty} C_{\omega\otimes e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^{2d}_{x,\xi})$$

and

$$C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\{\mathfrak{M}\}}(\mathbb{R}^d_\xi) \cong \varinjlim_{\lambda\to\infty} C_{0,\omega\otimes e^{\omega_{M^\lambda}(\cdot)}}(\mathbb{R}^{2d}_{x,\xi}).$$

We are ready to establish the mapping properties of the STFT on $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$. Recall that $\langle \cdot \rangle$ mean the opposite of $[\cdot]$.

Proposition 6.3.8. *Let* $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d)$. *The following mappings*

$$V_\psi : \mathcal{B}'^{[\mathfrak{M}]}_\omega \to C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi),$$
$$V_\psi : \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega \to C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi),$$

and

$$V^*_\psi : C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi) \to \mathcal{B}'^{[\mathfrak{M}]}_\omega,$$

$$V^*_\psi : C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi) \to \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega,$$

are well-defined and continuous. In particular, if $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d) \setminus \{0\}$ and $\gamma \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d)$ is a synthesis window for ψ, then the desingularisation formula (6.2) holds for any $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\varphi \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$.

Proof. We first consider V_ψ. It suffices to show that the mapping $V_\psi : \mathcal{B}'^{[\mathfrak{M}]}_\omega \to C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$ is continuous. In fact, as the space $C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$ is a closed topological subspace of the tensor product $C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$, the result would then follow from Proposition 6.2.2 and $\mathcal{S}_{(A)}(\mathbb{R}^d_x)\widehat{\otimes}\mathcal{S}_{(\mathfrak{M})}(\mathbb{R}^d_\xi) \subset C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$. Since $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ is bornological (see Lemma 6.3.3 in the Beurling case), it suffices to show that $V_\psi(B)$ is bounded in $C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$ for all bounded sets $B \subset \mathcal{B}'^{[\mathfrak{M}]}_\omega$. For some $\lambda > 0$ (for all $\lambda > 0$) it holds that $\sup_{f\in B}\sup_{\varphi\in A} |\langle f, \varphi\rangle| < \infty$ for all $A \subset \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ bounded with respect to the norm $\|\cdot\|_{\mathcal{D}^{M^\lambda}_{L^1_\omega}}$. As

$$\{e^{-\omega_{M^\lambda}(4\pi\xi)}\omega^{-1}(x)\overline{M_\xi T_x\psi} : (x,\xi) \in \mathbb{R}^{2d}\} \subset \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$$

is bounded with respect to $\|\cdot\|_{\mathcal{D}^{M^\lambda}_{L^1_\omega}}$, it follows that

$$\sup_{f\in B}\sup_{(x,\xi)\in\mathbb{R}^{2d}} e^{-\omega_{M^\lambda}(4\pi\xi)}\omega^{-1}(x)|V_\psi f(x,\xi)| < \infty.$$

The continuity of V_ψ thus follows from Lemma 3.2.4(i).

Next, we treat V^*_ψ. Lemma 6.3.5 implies that the mapping $V^*_\psi : C_\omega(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi) \to \mathcal{B}'^{[\mathfrak{M}]}_\omega$ is continuous. As $\mathcal{S}_{(A)}(\mathbb{R}^d_x)\widehat{\otimes}\mathcal{S}_{(\mathfrak{M})}(\mathbb{R}^d_\xi)$ is dense in $C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$, $V^*_\psi : C_{0,\omega}(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi) \to \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ is continuous by Proposition 6.2.2.

For the desingularisation formula, for any $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$ we define the mapping $\tilde{f}$ as

$$\tilde{f}(\varphi) = \frac{1}{(\gamma, \psi)_{L^2}} \int\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) V_{\overline{\gamma}}\varphi(x, -\xi)dxd\xi, \qquad \varphi \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega},$$

which is well-defined and continuous by Proposition 6.3.7. By Proposition 6.2.5 f and $\tilde{f}$ coincide on $\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)$ so that by Proposition 6.3.4 f and $\tilde{f}$ define the same element in $\mathcal{B}_\omega^{\prime[\mathfrak{M}]}$. □

Corollary 6.3.9. $\dot{\mathcal{B}}_\omega^{\prime(\mathfrak{M})}$ *is a complete (LB)-space, and* $\dot{\mathcal{B}}_\omega^{\prime\{\mathfrak{M}\}}$ *is a quasinormable Fréchet space.*

Proof. Proposition 6.3.8 and the reconstruction formula (6.1) imply that $\dot{\mathcal{B}}_\omega^{\prime[\mathfrak{M}]}$ is isomorphic to a complemented subspace of the space $C_{0,\omega}(\mathbb{R}^d) \hat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d)$. Hence, as $C_{0,\omega}(\mathbb{R}^d) \hat{\otimes}_\varepsilon C_{\{\mathfrak{M}\}}(\mathbb{R}^d)$ is an (LB)-space that is complete and $C_{0,\omega}(\mathbb{R}^d) \hat{\otimes}_\varepsilon C_{(\mathfrak{M})}(\mathbb{R}^d)$ is a quasinormable Fréchet space by [5, Proposition 2], the proof is complete. □

Proposition 6.3.8 allows for the following characterizations of $\mathcal{B}_\omega^{\prime[\mathfrak{M}]}$ and $\dot{\mathcal{B}}_\omega^{\prime[\mathfrak{M}]}$ via the STFT.

Theorem 6.3.10. *Let* $\psi \in \mathcal{S}_{(A)}^{(\mathfrak{M})}(\mathbb{R}^d) \setminus \{0\}$ *and let* $f \in \mathcal{S}_{[A]}^{\prime[\mathfrak{M}]}(\mathbb{R}^d)$. *The following statements are equivalent:*

(i) $f \in \mathcal{B}_\omega^{\prime[\mathfrak{M}]}$.

(ii) $\{T_{-h}f/\omega(h) \mid h \in \mathbb{R}^d\}$ *is bounded in* $\mathcal{S}_{[A]}^{\prime[\mathfrak{M}]}(\mathbb{R}^d)$.

(iii) *For some* $\lambda > 0$ *(for all* $\lambda > 0$*) it holds that*

$$\sup_{(x,\xi)\in\mathbb{R}^d} e^{-\omega_{M^\lambda}(\xi)} \frac{|V_\psi f(x,\xi)|}{\omega(x)} < \infty. \tag{6.7}$$

Proof. (i) $\Rightarrow$ (ii) Since $\mathcal{S}_{[A]}^{[\mathfrak{M}]}$ is barrelled, it suffices to show that $\{T_{-h}f/\omega(h) \mid h \in \mathbb{R}^d\}$ is weakly bounded. This follows however immediately by observing that $\{\varphi(x-h)/\omega(h) \mid h \in \mathbb{R}^d\}$ is a bounded set in $\mathcal{D}_{L_\omega^1}^{[\mathfrak{M}]}$ for any $\varphi \in \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)$.
(ii) $\Rightarrow$ (iii) As the mapping

$$*_f : \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \to C_{\langle A\rangle}(\mathbb{R}^d), \qquad \varphi \mapsto f * \varphi$$

is continuous and our assumption yields that $*_f(\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)) \subseteq C_\omega(\mathbb{R}^d)$, we may infer from the closed graph theorem that $*_f : \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \to$

$C_\omega(\mathbb{R}^d)$ is continuous. Hence for some $M = M^\lambda$ and $v = e^{\omega_A(\cdot/\lambda)}$ (for some $M \in \overline{V}(\mathfrak{M})$ and $v \in \overline{V}(\mathscr{W}_A)$ in view of Theorem 4.2.14) we have that $*_f$ can be uniquely extended to a continuous linear mapping $*_f : \overline{\mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)}^{\mathcal{S}^M_v} \to C_\omega(\mathbb{R}^d)$. As $\{e^{-\omega_M(4\pi\xi)}M_\xi \check{\overline{\psi}} : \xi \in \mathbb{R}^d\}$ is bounded in $\overline{\mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)}^{\mathcal{S}^M_v}$, we obtain that

$$\{e^{-\omega_M(4\pi\xi)}|V_\psi f(x,\xi)| : \xi \in \mathbb{R}^d\} = \{*_f(e^{-\omega_M(4\pi\xi)}M_\xi \check{\overline{\psi}}) : \xi \in \mathbb{R}^d\}$$

is bounded in $C_w(\mathbb{R}^d)$. This implies that

$$\sup_{(x,\xi)\in\mathbb{R}^d} e^{-\omega_M(4\pi\xi)} \frac{|V_\psi f(x,\xi)|}{\omega(x)} < \infty.$$

The result now follows from Lemma 3.2.4(i).

$(iii) \Rightarrow (i)$ As (6.7) implies that $V_\psi f \in C_\omega(\mathbb{R}^d_x) \hat{\otimes}_\varepsilon C_{\langle\mathfrak{M}\rangle}(\mathbb{R}^d_\xi)$ it follows from Proposition 6.3.8 and (6.1) that $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$. $\qquad\square$

Theorem 6.3.11. *Let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d) \setminus \{0\}$ and let $f \in \mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$. The following statements are equivalent:*

(i) $f \in \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$.

(ii) $\lim_{|h|\to\infty} T_{-h} f / \omega(h) = 0$ *in* $\mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$.

(iii) For some $\lambda > 0$ (for all $\lambda > 0$) it holds that

$$\lim_{|(x,\xi)|\to\infty} e^{-\omega_{M^\lambda}(\xi)} \frac{|V_\psi f(x,\xi)|}{\omega(x)} = 0. \qquad (6.8)$$

Proof. $(1) \Rightarrow (2)$: Since $\mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ is Montel, it suffices to show that $\lim_{|h|\to\infty} T_{-h} f / \omega(h) = 0$ weakly in $\mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$. Take any $\varphi \in \mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ and let $\varepsilon > 0$ be arbitrary. The set $\{T_h \varphi / \omega(h) : h \in \mathbb{R}^d\}$ is bounded in $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$. Hence, there is $\chi \in \mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ such that $|\langle T_{-h}(f - \chi), \varphi \rangle| \leqslant \varepsilon \omega(h)$ for all $h \in \mathbb{R}^d$. We obtain that

$$\limsup_{|h|\to\infty} \frac{|\langle T_{-h} f, \varphi \rangle|}{\omega(h)} \leqslant \varepsilon + \lim_{|h|\to\infty} \frac{1}{\omega(h)} \left| \int_{\mathbb{R}^d} \varphi(t-h)\chi(t)dt \right| = \varepsilon.$$

(2) $\Rightarrow$ (3): Since the mapping

$$*_f : \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \to C_{\langle A \rangle}(\mathbb{R}^d), \qquad \varphi \mapsto f * \varphi$$

is continuous and our assumption yields that $*_f(\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)) \subset C_{0,\omega}(\mathbb{R}^d)$, we may infer from the closed graph theorem that $*_f : \mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d) \to C_{0,\omega}(\mathbb{R}^d)$ is continuous. Hence for some $M = M^\lambda$ and $v = e^{\omega_A(\cdot/\lambda)}$ (for some $M \in \overline{V}(\mathfrak{M})$ and $v \in \overline{V}(\mathscr{W}_A)$ in view of Theorem 4.2.14) we have that $*_f$ can be uniquely extended to a continuous linear mapping $*_f : \overline{\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)}^{\mathcal{S}_v^M(\mathbb{R}^d)} \to C_{0,\omega}(\mathbb{R}^d)$. Fix $q' > 4\pi$. As $\{e^{-\omega_M(q'\xi)}M_\xi \check{\overline{\psi}} : \xi \in \mathbb{R}^d\}$ is relatively compact in $\overline{\mathcal{S}_{[A]}^{[\mathfrak{M}]}(\mathbb{R}^d)}^{\mathcal{S}_v^M(\mathbb{R}^d)}$, we obtain that

$$\{e^{-\omega_M(q'\xi)}|V_\psi f(x,\xi)| : \xi \in \mathbb{R}^d\} = \{*_f(e^{-\omega_M(q'\xi)}M_\xi \check{\overline{\psi}}) : \xi \in \mathbb{R}^d\}$$

is relatively compact in $C_{0,\omega}(\mathbb{R}^d)$. This implies that

$$\lim_{|x| \to \infty} \sup_{\xi \in \mathbb{R}^d} e^{-\omega_M(q'\xi)} \frac{|V_\psi f(x,\xi)|}{\omega(x)} = 0,$$

whence the result follows from Lemma 3.2.4.

(3) $\Rightarrow$ (1): (3) means that $V_\psi f \in C_{0,\omega}(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon C_{\langle \mathfrak{M} \rangle}(\mathbb{R}_\xi^d)$. The result therefore follows from Proposition 6.3.8 and the reconstruction formula (6.1). $\qquad \square$

6.4 Structural theorems

We provide here first structural theorems for the spaces $\mathcal{B}'^{[\mathfrak{M}]}_\omega$ and $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$. These results will form the cornerstones of the theory we construct in Part II, where we consider the structure for a variety of types of asymptotic behavior of ultradistributions. The main results here are Theorem 6.4.1 and Theorem 6.4.12.

6.4.1 The structure of $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$

The goal of this section is to obtain structural theorems for $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$. In particular, we shall prove the following result.

Theorem 6.4.1. *Let $\mathfrak{M}$ be a weight sequence system and ω be a weight function such that Assumption 6.3.1 holds. Then $f \in \dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}$ if and only if there exist continuous functions $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ on $\mathbb{R}^d$ such that*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)}, \tag{6.9}$$

the limits

$$\lim_{|x| \to \infty} \frac{f_\alpha(x)}{\omega(x)} = 0, \qquad \forall \alpha \in \mathbb{N}^d, \tag{6.10}$$

hold, and for some $\lambda > 0$ (for any $\lambda > 0$) we have that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{M_\alpha^\lambda |f_\alpha(x)|}{\omega(x)} < \infty. \tag{6.11}$$

We will work with the following spaces of vector-valued multi-sequences. Let E be a Banach space. For a weight sequence M we define $\Lambda_M(E)$ as the Banach space consisting of all (multi-indexed) sequences $(e_\alpha)_{\alpha \in \mathbb{N}^d} \in E^{\mathbb{N}^d}$ such that

$$\|(e_\alpha)_{\alpha \in \mathbb{N}^d}\|_{\Lambda_M(E)} := \sup_{\alpha \in \mathbb{N}^d} M_\alpha \|e_\alpha\|_E < \infty.$$

We set

$$\Lambda_{(\mathfrak{M})}(E) := \varinjlim_{\lambda \to 0^+} \Lambda_{M^\lambda}(E), \qquad \Lambda_{\{\mathfrak{M}\}}(E) := \varprojlim_{\lambda \to \infty} \Lambda_{M^\lambda}(E).$$

$\Lambda_{(\mathfrak{M})}(E)$ is a complete (LB)-space by [9, Theorem 2.6], and $\Lambda_{\{\mathfrak{M}\}}(E)$ is a Fréchet space. Given a Banach space F, we set $\Lambda'_{(\mathfrak{M})}(F) := \Lambda_{\{\mathfrak{M}^\circ\}}(F)$ and $\Lambda'_{\{\mathfrak{M}\}}(F) := \Lambda_{(\mathfrak{M}^\circ)}(F)$. We then have the following canonical isomorphisms of lcHs

$$(\Lambda_{(\mathfrak{M})}(E))' \cong \Lambda'_{(\mathfrak{M})}(E'), \qquad (\Lambda_{\{\mathfrak{M}\}}(E))' \cong \Lambda'_{\{\mathfrak{M}\}}(E').$$

Theorem 6.4.1 may now be reformulated as follows.

Theorem 6.4.2. *The mapping*

$$S : \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d)) \to \dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}, \quad (f_\alpha)_{\alpha \in \mathbb{N}^d} \mapsto \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)}$$

is surjective.

We will prove Theorem 6.4.2 by employing an abstract surjectivity criterion. A continuous linear mapping between Fréchet spaces is surjective if and only if its transpose is injective and has weakly closed range [131, Theorem 37.2, p. 382]. We will make use of the following generalization of this result.

Lemma 6.4.3. *Let E and F be lcHs and let $S : E \to F$ be a continuous linear mapping. Suppose that E is Mackey, $E/\ker S$ is complete, and $\operatorname{Im} S$ is Mackey for the topology induced by F. Then, S is surjective if the following two conditions are satisfied:*

(1) $S^t : F' \to E'$ *is injective;*

(2) $\operatorname{Im} S^t$ *is weakly closed in E'.*

Proof. If S^t is injective, then $\operatorname{Im} S$ is dense in F. Hence, it suffices to show that $\operatorname{Im} S$ is closed in F. As $\operatorname{Im} S^t$ is weakly closed, S is a weak homomorphism [131, Lemma 37.4]. Since $\sigma(E/\ker S, (E/\ker S)')$ coincides with the quotient of $\sigma(E, E')$ modulo $\ker S$ [131, p. 385] and $\sigma(\operatorname{Im} S, (\operatorname{Im} S)')$ coincides with the topology induced by $\sigma(F, F')$, we obtain that $\widetilde{S} : E/\ker S \to \operatorname{Im} S$ is a weak isomorphism. Consequently, $\widetilde{S}$ is also an isomorphism if we equip $E/\ker S$ and $\operatorname{Im} S$ with their Mackey topology [123, p. 158]. From this we may infer that S is a homomorphism because $E/\ker S$ is Mackey as E is so [123, p. 136] and $\operatorname{Im} S$ is Mackey by assumption. Finally, since $E/\ker S$ is complete, we have that $\operatorname{Im} S \cong E/\ker S$ is complete and, thus, closed in F. $\square$

We need several preliminary results.

Lemma 6.4.4. *S is a well-defined continuous linear mapping.*

Proof. One easily verifies that $S : \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d)) \to \mathcal{B}'^{[\mathfrak{M}]}_{\omega}$ is a continuous linear mapping and that $\lim_{|h|\to\infty} T_{-h}S((f_\alpha)_{\alpha\in\mathbb{N}^d})/\omega(h) = 0$ in $\mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ for all $(f_\alpha)_{\alpha\in\mathbb{N}^d} \in \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))$. Hence, the result follows from Theorem 6.3.11. $\square$

Our next goal is to determine the transpose of S. To this end, we first show that, similarly as in the distributional case [125], the dual of $\mathcal{B}'^{[\mathfrak{M}]}_{\omega}$ is given by $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$.

Proposition 6.4.5. *The canonical inclusion mapping*

$$\iota : \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega} \to (\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega)'_b, \quad \varphi \mapsto (f \mapsto \langle f, \varphi \rangle)$$

is a topological isomorphism.

Proof. Clearly, ι is continuous and injective. Since $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ is webbed and $(\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega)'_b$ is ultrabornological (Corollary 6.3.9), it suffices, by De Wilde's open mapping theorem [45], to show that ι is surjective. Let $\Phi \in (\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega)'$ be arbitrary. Denote by $\rho : \mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d) \to \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ the canonical inclusion and set $f = \Phi \circ \rho \in \mathcal{S}'^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$. As $\Phi(\rho(\chi)) = \langle f, \chi \rangle$ for every $\chi \in \mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ and $\mathcal{S}^{[\mathfrak{M}]}_{[A]}(\mathbb{R}^d)$ is dense in $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$, it is enough to show that $f \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}(\mathbb{R}^d)$. Let $\psi \in \mathcal{S}^{(\mathfrak{M})}_{(A)}(\mathbb{R}^d)$ be a fixed non-zero window function. Since Φ is continuous, there is a bounded set $B \subset \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ such that

$$|V_\psi f(x, \xi)| = |\Phi(\rho(\overline{M_\xi T_x \psi}))| \leqslant \sup_{\varphi \in B} |\langle \overline{M_\xi T_x \psi}, \varphi \rangle| = \sup_{\varphi \in B} |V_\psi \varphi(x, \xi)|.$$

Proposition 6.3.7 implies that for every $\lambda > 0$ (for some $\lambda > 0$)

$$\sup_{\xi \in \mathbb{R}^d} e^{\omega_{M^\lambda}(\xi)} \|V_\psi f(\,\cdot\,, \xi)\|_{L^1_\omega} \leqslant \sup_{\varphi \in B} \sup_{\xi \in \mathbb{R}^d} e^{\omega_{M^\lambda}(\xi)} \|V_\psi \varphi(\,\cdot\,, \xi)\|_{L^1_\omega} < \infty,$$

so that another application of Proposition 6.3.7 shows that $f \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$. $\square$

Corollary 6.4.6. *The transposed mapping S^t may be identified with the continuous linear mapping*

$$\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega} \to \Lambda'_{[\mathfrak{M}]}(\mathcal{M}^1_\omega) : \quad \varphi \mapsto ((-1)^{|\alpha|} \varphi^{(\alpha)})_{\alpha \in \mathbb{N}^d}.$$

Proof of Theorem 6.4.2. We shall show that S is surjective via Lemma 6.4.3. The space $\Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))$ is clearly Mackey, while the quotient $\Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))/\ker S$ is complete as $\Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))$ is complete. Next, we show that $\operatorname{Im} S$ is Mackey. In the Romieu case this is trivial because $\dot{\mathcal{B}}'^{\{\mathfrak{M}\}}_\omega$ is a Fréchet space. We now consider the Beurling case. We shall prove that $X = \operatorname{Im} S$ is infrabarreled and thus Mackey. We

need to show that every strongly bounded set B in X' is equicontinuous. Since X is dense in $\dot{\mathcal{B}}'^{(\mathfrak{M})}_{\omega}$ (as S^t is injective), Proposition 6.4.5 implies that $X' = \mathcal{D}^{(\mathfrak{M})}_{L^1_{\omega}}$. For arbitrary $\lambda > 0$ we consider the set

$$V_\lambda = \left\{ \frac{f^{(\alpha)}}{M^\lambda_\alpha} : \alpha \in \mathbb{N}^d, f \in B_{C_{0,\omega}} \right\} \subseteq X.$$

The set V_λ is bounded in X because S is continuous, so that we have $\sup_{\varphi \in B} \sup_{g \in V_\lambda} |\langle \varphi, g \rangle| < \infty$. The relation (6.6) yields that

$$
\sup_{\varphi \in B} \sup_{g \in V_\lambda} |\langle \varphi, g \rangle| = \sup_{\varphi \in B} \sup_{\alpha \in \mathbb{N}^d} \sup_{f \in B_{C_{0,\omega}}} \left| \left\langle \varphi, \frac{f^{(\alpha)}}{M^\lambda_\alpha} \right\rangle \right|
$$
$$
= \sup_{\varphi \in B} \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M^\lambda_\alpha} \sup_{f \in B_{C_{0,\omega}}} \left| \int_{\mathbb{R}^d} \varphi^{(\alpha)}(x) f(x) dx \right|
$$
$$
= \sup_{\varphi \in B} \sup_{\alpha \in \mathbb{N}^d} \frac{\left\| \varphi^{(\alpha)} \right\|_{L^1_\omega}}{M^\lambda_\alpha}.
$$

Hence,
$$
\sup_{\varphi \in B} \|\varphi\|_{\mathcal{D}^{M^\lambda}_{L^1_\omega}} < \infty, \qquad \forall \lambda > 0,
$$

which means that B is bounded in $\mathcal{D}^{(\mathfrak{M})}_{L^1_\omega}$. Then, B is equicontinuous because of Proposition 6.4.5 and the fact that $\dot{\mathcal{B}}'^{(\mathfrak{M})}_{\omega}$ is barreled (Corollary 6.3.9). We already noticed that S^t is injective. Finally, we show that $\operatorname{Im} S^t$ is weakly closed in $\Lambda'_{[\mathfrak{M}]}(\mathcal{M}^1_\omega)$. Let $(\varphi_j)_j$ be a net in $\mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$ and $(\mu_\alpha)_{\alpha \in \mathbb{N}^d} \in \Lambda'_{[\mathfrak{M}]}(\mathcal{M}^1_\omega)$ such that $((-1)^{|\alpha|} \varphi_j^{(\alpha)})_{\alpha \in \mathbb{N}^d} \to (\mu_\alpha)_{\alpha \in \mathbb{N}^d}$ weakly in $\Lambda'_{[\mathfrak{M}]}(\mathcal{M}^1_\omega)$. In particular, $\varphi_j^{(\alpha)} \to (-1)^{|\alpha|} \mu_\alpha$ weakly in $\mathcal{M}^1_\omega$ for all $\alpha \in \mathbb{N}^d$. Consequently, we have that $\mu_0^{(\alpha)} = (-1)^{|\alpha|} \mu_\alpha \in \mathcal{M}^1_\omega$ for all $\alpha \in \mathbb{N}^d$ (the derivatives should be interpreted in the sense of distributions). The equality (6.6) implies that $\mu_0 \in \mathcal{D}_{L^1_\omega}$ and that

$$
(\|\mu_0^{(\alpha)}\|_{L^1_\omega})_{\alpha \in \mathbb{N}^d} = (\|\mu_0^{(\alpha)}\|_{\mathcal{M}^1_\omega})_{\alpha \in \mathbb{N}^d} = (\|\mu_\alpha\|_{\mathcal{M}^1_\omega})_{\alpha \in \mathbb{N}^d} \in \Lambda'_{[\mathfrak{M}]}(\mathbb{C}),
$$

which means that $\mu_0 \in \mathcal{D}^{[\mathfrak{M}]}_{L^1_\omega}$. Hence, $(\mu_\alpha)_{\alpha \in \mathbb{N}^d} = ((-1)^{|\alpha|} \mu_0^{(\alpha)})_{\alpha \in \mathbb{N}^d} \in \operatorname{Im} S^t$. $\qquad \square$

6.4.2 The structure of $\mathcal{B}'^{[\mathfrak{M}]}_\omega$

We now consider the first structural theorem for the space $\mathcal{B}'^{[\mathfrak{M}]}_\omega$. In the Beurling case, our proof will be straightforward along the lines of Komatsu's proof for [81, Theorem 8.1, p. 76] using the Hahn-Banach theorem. In the Roumieu case, we employ a similar tactic, however here we must rely on the projective description of $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$. For this reason we will say $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ *allows a projective description* if

$$\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega} = \varprojlim_{M \in \overline{V}(\mathfrak{M})} \mathcal{D}^M_{L^1_\omega}$$

as locally convex spaces. Note that by Lemma 4.2.12(*i*), these spaces always coincide as sets. In the case of isotropic weight sequences, the description is immediate.

Theorem 6.4.7. *Let M and A be isotropic weight sequences satisfying $(M.1)$ and $(M.2)'$ and let ω be an $\{A\}$-admissible weight function. Then, $\mathcal{D}^{\{M\}}_{L^1_\omega}$ allows a projective description.*

Proof. Set, as lcHs,

$$\widetilde{\mathcal{D}}^{\{M\}}_{L^1_\omega} := \varprojlim_{(r_p) \in [\mathfrak{R}]} \mathcal{D}^{M_{r_p}}_{L^1_\omega}.$$

Trivially, $\mathcal{D}^{\{M\}}_{L^1_\omega}$ is continuously contained in $\widetilde{\mathcal{D}}^{\{M\}}_{L^1_\omega}$. Take any $p \in \mathrm{csn}(\mathcal{D}^{\{M\}}_{L^1_\omega})$. Let $B \subset \mathcal{B}'^{\{M\}}_\omega$ be the polar of the closed unit ball of p, so that, by the bipolar theorem,

$$p(\varphi) = \sup_{f \in B} |\langle f, \varphi \rangle|, \qquad \varphi \in \mathcal{D}^{\{M\}}_{L^1_\omega}.$$

The set B is strongly bounded in $\mathcal{B}'^{\{M\}}_\omega$, so that by employing Proposition 6.3.8, we get that for each $\psi \in \mathcal{S}^{(M)}_{(A)}$ and $q > 0$ there is some $C'_B = C'_{B,\psi,q} > 0$ such that

$$\sup_{f \in B} |V_\psi f(x, \xi)| \leqslant C'_B \omega(x) e^{\omega_M(q\xi)}.$$

If we now apply (2.8), there is some $(k_p) \in [\mathfrak{R}]$ for which

$$\sup_{f \in B} |V_\psi f(x, \xi)| \leqslant C_B \omega(x) e^{\omega_{M_{k_p}}(\xi)}.$$

Let $\psi \in \mathcal{S}_{(A)}^{(M)} \backslash \{0\}$ and $\gamma \in \mathcal{S}_{(A)}^{(M)}$ be a synthesis window for ψ. If we select $h_p = k_p H^{d+1}$, then, for any $\varphi \in \widetilde{\mathcal{D}}_{L_\omega^1}^{\{M\}}$, it follows that from Lemma 6.3.5 and Proposition 6.3.8

$$\sup_{f \in B} |\langle f, \varphi \rangle| \leqslant \frac{C_B}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} \omega(x) |V_{\overline{\gamma}} \varphi(x, -\xi)| e^{\omega_{M_{k_p}}(\xi)} dx d\xi$$

$$\leqslant \frac{C_B C_\gamma}{(\gamma, \psi)_{L^2}} \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M_{h_p}}} \int_{\mathbb{R}^d} e^{\omega_{M_{k_p}}(\xi) - \omega_{M_{k_p}}(H^{d+1}\xi)} d\xi$$

$$\leqslant C_{h_p} \|\varphi\|_{\mathcal{D}_{L_\omega^1}^{M_{h_p}}},$$

for some $C_{h_p} > 0$, where we have made use of [81, Proposition 3.4, p. 50]. Consequently, p is also a continuous seminorm on $\widetilde{\mathcal{D}}_{L_\omega^1}^{\{M\}}$, so that the spaces also coincide topologically as claimed. $\square$

Whether Theorem 6.4.7 also holds in the general case of a weight function system $\mathfrak{M}$ remains an open question. In fact, there is an interesting connection between this problem and the lifting properties of the map $S : \Lambda_{\{\mathfrak{M}\}}(C_{0,\omega}(\mathbb{R}^d)) \to \dot{\mathcal{B}}_\omega'^{\{\mathfrak{M}\}}$. One could ask in general whether S lifts bounded sets, i.e. for every bounded subset $B \subset \dot{\mathcal{B}}_\omega'^{[\mathfrak{M}]}$, does there exist a bounded set $A \subset \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))$ such that $S(A) = B$. Due to our non-constructive approach, a direct solution to this is not apparent. In the Roumieu case, this is true if $\mathcal{D}_{L_\omega^1}^{\{\mathfrak{M}\}}$ lifts bounded sets, and moreover it is equivalent to it.

Proposition 6.4.8. *The following statements are equivalent:*

   (i) S lifts bounded sets;

   (ii) $\mathcal{D}_{L_\omega^1}^{\{\mathfrak{M}\}}$ allows a projective description.

Proof. Put $\widetilde{\mathcal{D}}_{L_\omega^1}^{\{\mathfrak{M}\}} = \varprojlim_{M \in \overline{V}(\mathfrak{M})} \mathcal{D}_{L_\omega^1}^M$. By using the projective description of $\Lambda_{[\mathfrak{M}]}'(\mathcal{M}_\omega^1)$ [9, Theorem 2.3], we easily see that $\widetilde{\mathcal{D}}_{L_\omega^1}^{\{\mathfrak{M}\}} \cong \operatorname{Im} S^t$.

Whence $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ allows a projective description if and only if S^t is a topological homomorphism. By [92, Lemma 26.7, p. 310], the latter is equivalent to S lifting bounded sets. $\qquad\square$

For the general case, we may now state the following important open problem.

Open problem 6.4.9. Determine whether the map

$$S : \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d)) \to \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega.$$

lifts bounded sets. In the Roumieu case, by Proposition 6.4.8, this is equivalent to determining when $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ allows a projective description, which is always true for isotropic weight sequences by Theorem 6.4.7.

Another alluring question, which would automatically imply the former, is whether $\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$ possesses a continuous structural representation.

Open problem 6.4.10. Does S have a continuous right inverse, i.e. a continuous map $R : \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega \to \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d))$ such that $S \circ R = \mathrm{id}_{\dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega}$? In other words we would like to know whether the short exact sequence

$$0 \longrightarrow \ker S \longrightarrow \Lambda_{[\mathfrak{M}]}(C_{0,\omega}(\mathbb{R}^d)) \longrightarrow \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega \longrightarrow 0$$

splits. For the Roumieu case, when dealing with Fréchet spaces, abstract conditions for this to hold true have been found [92, 151], which might be a good starting point for tackling this problem. We also remark that such a solution for other spaces of ultradistributions would be interesting as well.

We now move on to the first structural theorem of $\mathcal{B}'^{[\mathfrak{M}]}_\omega$. We will need the following simple auxiliary lemma that allows us to preserve certain growth properties when regularizing functions.

Lemma 6.4.11. *Given $R > 0$ there are absolute constants $c_{0,R}$ and $c_{1,R}$ such that each function $g \in L^\infty_{\mathrm{loc}}(W_R)$ satisfying the bound $\sup_{x \in W, |h| < R} |g(x + h)|/\omega(x) < \infty$, where $W \subset \mathbb{R}^d$ and ω is a positive*

function defined on W, can be written as $g = \Delta g_1 + g_0$ in W_R for some functions $g_j \in C(\mathbb{R}^d)$ that satisfy

$$\sup_{x \in W} \frac{|g_j(x)|}{\omega(x)} \leqslant c_{j,R} \sup_{x \in W, \, |h| < R} \frac{|g(x+h)|}{\omega(x)}, \qquad j = 0, 1.$$

Proof. To show this, we make use of the fact that the fundamental solutions of the Laplacian belong to $L^1_{loc}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \backslash \{0\})$. By cutting-off a fundamental solution in the ball $B(0, R)$, this implies we can select functions $\chi_1 \in L^1(\mathbb{R}^d)$ and $\chi_0 \in \mathcal{D}(\mathbb{R}^d)$ both supported on $B(0, R)$, such that $\delta = \Delta\chi_1 + \chi_0$. Extend g off W_R as 0 and keep calling this extension by g. We obtain the claim if we set $g_j = g * \chi_j$ so that the desired inequalities hold with $c_{j,R} = \int_{|x| \leqslant R} |\chi_j(-x)| dx$. $\qquad \square$

We are now ready to describe the structure of $\mathcal{B}'^{[\mathfrak{M}]}_\omega$.

Theorem 6.4.12. *Let $\mathfrak{M}$ be a weight sequence system and ω be a weight function such that Assumption 6.3.1 holds. In the Roumieu case we additionally assume that $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ allows a projective description. Then, $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$ if and only if there exist continuous functions $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ on $\mathbb{R}^d$ such that*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)}, \tag{6.12}$$

and for some $\lambda > 0$ (for any $\lambda > 0$) we have that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{M_\alpha^\lambda |f_\alpha(x)|}{\omega(x)} < \infty. \tag{6.13}$$

Proof. Clearly if (6.12) and (6.13) hold then $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$. We now show that it is also necessary. In view of Lemma 6.4.11 and $[\mathfrak{M}.2]'$ it suffices to show (6.12) and (6.13) hold for measurable functions f_α. Given a $M = M^\lambda$ (a $M \in \overline{V}(\mathfrak{M})$), define X_M as the Banach space of all smooth functions φ such that

$$\|\varphi\|_{X_M} = \sum_{\alpha \in \mathbb{N}^d} \int_{\mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|}{M_\alpha} \omega(x) dx < \infty.$$

As $\mathfrak{M}$ satisfies [L] (where in the Roumieu we use Corollary 4.2.13(i) and the assumption that $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega}$ allows a projective description) we have that $\mathcal{D}^{(\mathfrak{M})}_{L^1_\omega} = \varprojlim_{\lambda\to 0^+} X_{M^\lambda}$ (resp. $\mathcal{D}^{\{\mathfrak{M}\}}_{L^1_\omega} = \varprojlim_{M\in\overline{V}(\mathfrak{M})} X_M$). Let then $f \in X'_M$. Consider the weight $w_M(\alpha, x) = w(x)/M_\alpha$ on $\mathbb{N}^d \times \mathbb{R}^d$ and the weighted space $L^1_{\omega_M}$. The mapping $j : X_M \to L^1_{\omega_M}$ given by $j(\varphi)(\alpha, x) = (-1)^{|\alpha|}\varphi^{(\alpha)}(x)$ is an isometry so that $\langle f, j(\varphi)\rangle = \langle f, \varphi\rangle$ defines a continuous linear functional on $j(X_M)$. The representation (6.12) with functions as in (6.13) then follows by applying the Hanh-Banach theorem (and Lemma 4.2.12(ii) in the Roumieu case). $\qquad\square$

We conclude this chapter with the following interesting problem.

Open problem 6.4.13. We have that $f \in \mathcal{B}'^{[\mathfrak{M}]}_\omega$, resp. $f \in \dot{\mathcal{B}}'^{[\mathfrak{M}]}_\omega$, if and only if $f * \varphi \in C_\omega(\mathbb{R}^d)$, resp. $f * \varphi \in C_{0,\omega}(\mathbb{R}^d)$, for any $\varphi \in \mathcal{S}^{[\mathfrak{M}]}_{[A]}$. If now exchange $C_\omega(\mathbb{R}^d)$ or $C_{0,\omega}(\mathbb{R}^d)$ with some other translation-invariant Banach space E of measurable functions, could we then provide first structural theorems for all those tempered ultradistributions f such that $f * \varphi \in E$? Second structure theorems for such spaces were obtained in [46], however first structural theorems will require novel methods, perhaps similar to those explored here. A particularly interesting example would be the almost periodic ultradistributions, see [25, 86].

Part II

Asymptotic behavior of generalized functions

Chapter 7

Introduction

Asymptotic analysis encompasses a wide branch of pure and applied mathematics, with a long history and a strong promise of continued importance. Due to the general development of various areas of mathematical analysis, especially the theory of differential equations, it has obtained several new impulses resulting in novel approaches and methods. One such particular field is the asymptotic behavior of generalized functions, which has had an important role in quantum physics [6, 13, 140, 141], where rigorous proofs for foundational results were provided by the use of generalized asymptotic behavior. This prompted mathematicians to further develop the theory, see e.g. the monographs [113, 114]. In this part we will be concerned with the asymptotic behavior of ultradistributions, where in particular we will provide structural theorems for several types of asymptotics.

The asymptotic behavior of a generalized function is usually analyzed via its parametric behavior, mostly with respect to translation or dilation. Moreover, there exist three prominent approaches: that of Vladimirov, Drozhinov and Zav'yalov [140], of Pilipović and Stanković [114], and of Kanwal and Estrada [57]. This first and second approach follow the direction of S-asymptotic and quasiasymptotic behavior. The third approach is related to the moment asymptotic expansion and the Cesàro behavior. All three paths will be traversed in this text in the framework of ultradistributions.

The idea of looking at the translates of a distribution goes back to Schwartz [125, Chapter VII], who used it to measure the order of growth of tempered distributions at infinity. Pilipović and Stanković later introduced a generalization, the so called *S-asymptotic behavior*, and thoroughly investigated its properties for distributions, ultradistributions, and Fourier hyperfunctions. There are deep connections between S-asymptotics and Wiener Tauberian theorems for generalized functions [112]. In [114] the second structure theorem for *S*-asymptotics was shown using Komatsu's parametrix. However, this poses often unnecessarily strong restrictions on the weight sequences. In Chapter 9 this is remedied by providing the first structure theorem for *S*-asymptotics using the results obtained in Chapter 6. Furthermore, in Chapter 10, we consider the concept of *S*-asymptotic boundedness and obtain both the first and second structural theorem for it.

There are two very well-established approaches to asymptotics of generalized functions related to dilation. The first one is the *quasiasymptotic behavior*, which employs regularly varying functions [10] as gauges in the asymptotic comparisons. The concept of quasiasymptotic behavior for Schwartz distributions was introduced by Zav'yalov in [155] and further developed by him, Drozhzhinov, and Vladimirov in connection with their powerful multidimensional Tauberian theory for Laplace transforms [140]. A significant milestone for the theory were the complete structural theorems Vindas and Pilipović provided [135, 136, 137] for the quasiasymptotic behavior of distributions on the real line. An important consequence of these characterizations were the extension results to the tempered distributions: a distribution which has quasiasymptotic behavior at infinity is automatically a tempered distribution an its asymptotic behavior holds there; a similar yet local result also holds for quasiasymptotics at the origin. In Chapter 9 we will give an ultradistributional analog of these structural results, both at infinity and the origin. Furthermore, we also provide extension results, however not to the canonical tempered ultradistributions. Their specific quasiasymptotic behavior is studied in Chapter 11, where we extend the so-called general Tauberian theorem for the dilation group [140, Chapter 2] from tempered distributions to tempered ultradistributions.

The second important approach to asymptotic behavior related to dilation is the so-called *moment asymptotic expansion (MAE)*. As explored in the monograph [58], the MAE supplies a unified approach to several aspects of asymptotic analysis and its applications. In the distributional case, this behavior has been extensively studied by Estrada and Kanwal [57, 58]. In particular, in the one dimensional case, in [56] Estrada showed that a distribution satisfies the MAE if and only if it lies in the dual of the space of so-called GLS symbols [68]. Some recent developments may be found in [124, 153]. The subject of Chapter 10 will be the study of the MAE in the ultradistributional case. One of our main results there provides a counterpart of Estrada's full characterization in the one-dimensional case. In addition to that, we also consider a uniform analog of the MAE and give a partial characterization on the real line.

As opposed to Part I, we will only work under the condition of non-quasianalytitcity. This is motivated by the fact that asymptotic behavior is in large part a local behavior. Also, for the sake of simplicity, we define our spaces via isotropic weight sequences instead of weight sequence systems.

Chapter 8

Preliminaries

In this Chapter, we build upon the preparations we have made in Chapter 2 and specify further towards the framework of asymptotic behavior of generalized functions.

8.1 Cones

A set $\Gamma \subseteq \mathbb{R}^d$ is called a *cone* if $u \in \Gamma$ implies $\lambda u \in \Gamma$ for any $\lambda > 0$. The cone Γ is called *solid* if $\operatorname{int} \Gamma \neq \varnothing$, while Γ is said to be *acute* if there exists some $y \in \mathbb{R}^d$ such that

$$y \cdot u > 0, \qquad \forall u \in \Gamma \setminus \{0\}.$$

The conjugate cone Γ^* is the set

$$\Gamma^* := \{y \in \mathbb{R}^d : y \cdot u \geqslant 0, \forall u \in \Gamma\}.$$

Then Γ^* is a closed convex cone with vertex at the origin. We set $C = \operatorname{int} \Gamma^*$, then Γ is acute if and only if $C \neq \varnothing$, i.e. if and only if Γ^* is solid (cfr. [140, Lemma 1, p. 27]). If Γ is closed and convex, then $(\Gamma^*)^* = \Gamma$.

Suppose Γ is a closed convex acute cone, then we denote the distance of a point to the boundary of C by $\Delta_C(\,\cdot\,)$, i.e.

$$\Delta_C(x) := d(x, \partial C), \qquad \forall x \in \mathbb{R}^d.$$

We will often make use of the following estimate ([139, p. 61])

$$y \cdot u \geqslant \Delta_C(y)|u|, \qquad \forall u \in \Gamma, y \in C. \tag{8.1}$$

The *tube domain* T^C with base C is the set

$$T^C := \mathbb{R}^d + iC \subseteq \mathbb{C}^d.$$

8.2 Ultradistributions

We consider once more weight sequences and the spaces of ultradifferentiable functions associated to them in the specific context of this part. In particular, we only work in the isotropic case.

8.2.1 Weight sequences

A (isotropic) *weight sequence* $M = (M_p)_{p \in \mathbb{N}}$ is a sequence of positive numbers. To it we associated the sequence M^* with $M_p^* = M_p/p!$ for any $p \in \mathbb{N}$. Furthermore, for $p \in \mathbb{Z}_+$, we set $m_p = M_p/M_{p-1}$. We will make use of the following conditions on weight sequences:

$(M.1)$ $M_p^2 \leqslant M_{p-1}M_{p+1}$, $p \geqslant 1$;

$(M.1)^*$ $(M_p^*)^2 \leqslant M_{p-1}^* M_{p+1}^*, p \geqslant 1$;

$(M.2)'$ $M_{p+1} \leqslant AH^p M_p$, $p \in \mathbb{N}$, for constants $A, H \geqslant 1$;

$(M.2)$ $M_{p+q} \leqslant AH^{p+q} M_p M_q$, $p, q \in \mathbb{N}$, for constants $A, H \geqslant 1$;

$(M.3)'$ $\sum_{p=1}^{\infty} 1/m_p < \infty$;

$(M.3)$ $\sum_{p=q}^{\infty} 1/m_p \leqslant c_0 q/m_q$, $q \geqslant 1$, for a constant c_0.

Whenever we consider weight sequences, we assume they satisfy at least $(M.1)$. For multi-indices $\alpha \in \mathbb{N}^d$, we will simply denote $M_{|\alpha|}$ by M_α. As usual the relation $M \prec N$ between two such sequences means that for any $h > 0$ there is an $L = L_h > 0$ for which $M_p \leqslant Lh^p N_p$, $p \in \mathbb{N}$. We may then introduce the condition

(NA) $p! \prec M$.

Note that if M satisfies $(M.1)$ and $(M.3)'$, it automatically satisfies (NA) [81, Lemma 4.1, p. 55].

The associated function of the sequence M is given by

$$\omega_M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p M_0}{M_p}, \qquad t > 0,$$

and $\omega_M(0) = 0$. It increases faster than $\log t$ as $t \to \infty$ (cf. [81, p. 48]). We define ω_M on $\mathbb{R}^d$ as the radial function $\omega_M(x) := \omega_M(|x|)$, $x \in \mathbb{R}^d$ (note that this differs from the definition used in Part I). Throughout this text we shall often exploit the following bounds:

- If M satisfies $(M.2)'$, then for any $k > 0$

$$\omega_M(t) - \omega_M(kt) \leqslant -\frac{\log(t/A) \log k}{\log H}, \qquad t > 0. \qquad (8.2)$$

- M_p satisfies $(M.2)$ if and only if

$$2\omega_M(t) \leqslant \omega_M(Ht) + \log(AM_0). \qquad (8.3)$$

- If M_p satisfies $(M.1)^*$, we have, for some $A' > 0$,

$$\omega_{M^*}\left(\frac{t}{4(m_1 + 1)\omega_M(t)}\right) \leqslant \omega_M(t) + A', \qquad t \geqslant m_1 + 1. \quad (8.4)$$

Indeed, the first and second statement are [81, Proposition 3.4, p. 50] and [81, Proposition 3.6, p. 51] (see also Lemma 3.2.4), while the third one is shown in [24, Lemma 5.2.5, p. 96].

Throughout this part, we will regularly employ the set $[\mathfrak{R}]$. See Section 2.3.3 for its definition and basic properties.

An *ultrapolynomial of type* $[M]$ is an entire function

$$P(z) = \sum_{m=0}^{\infty} a_m z^m, \qquad a_m \in \mathbb{C},$$

where the coefficients satisfy $|a_m| \leqslant L/\ell^p M_m$ for some $\ell > 0$ (for any $\ell > 0$) and some $L > 0$. Note that by Lemma $4.2.12(ii)$ this is equivalent to $|a_m| \leqslant L/L_m M_m$ for some $(\ell_p) \in [\mathfrak{R}]$ and $L > 0$. If M satisfies $(M.2)$, the multiplication of two ultrapolynomials is again an ultrapolynomial (cf. [81, Proposition 4.5, p. 58]).

8.2.2   Spaces of ultradifferentiable functions and ultradistributions

Let M be a weight sequence and $K \Subset \mathbb{R}^d$ a regular compact subset. For $\ell > 0$ we define $\mathcal{E}^{M,\ell}(K)$ as the Banach space of all $\varphi \in C^\infty(K)$ such that

$$\|\varphi\|_{\mathcal{E}^{M,\ell}(K)} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|\varphi^{(\alpha)}(x)|}{\ell^{|\alpha|} M_\alpha} < \infty.$$

We set

$$\mathcal{E}^{(M)}(K) = \varprojlim_{\ell \to 0^+} \mathcal{E}^{M,\ell}(K), \qquad \mathcal{E}^{\{M\}}(K) = \varinjlim_{\ell \to \infty} \mathcal{E}^{M,\ell}(K).$$

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $(K_N)_{N \in \mathbb{N}}$ be an exhaustion by regular compact sets of Ω. We define

$$\mathcal{E}^{[M]}(\Omega) = \varprojlim_{N \in \mathbb{N}} \mathcal{E}^{[M]}(K_N).$$

These definitions are independent of the chosen exhaustion by regular compact sets of Ω. The elements of $\mathcal{E}^{(M)}(\Omega)$ are called *ultradifferentiable functions of class (M) (of Beurling type) in Ω* while the elements of $\mathcal{E}^{\{M\}}(\Omega)$ are called *ultradifferentiable functions of class $\{M\}$ (of Roumieu type) in Ω*.

For any $K \Subset \Omega$ and $\ell > 0$ we write $\mathcal{D}_K^{M,\ell}$ for the closed subspace of elements in $\mathcal{E}^{M,\ell}(K)$ with support contained in K. Then we set

$$\mathcal{D}_K^{(M)} = \varprojlim_{\ell \to 0^+} \mathcal{D}_K^{M,\ell}, \qquad \mathcal{D}_K^{\{M\}} = \varinjlim_{\ell \to \infty} \mathcal{D}_K^{M,\ell},$$

and

$$\mathcal{D}^{[M]}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{[M]}.$$

If M satisfies $(M.1)$, then $\mathcal{D}^{[M]}(\Omega)$ is non-trivial if and only if M satisfies $(M.3)'$ [81, Theorem 4.2, p. 56].

When M satisfies $(M.1)$ and $(M.3)'$, the dual $\mathcal{D}'^{[M]}(\mathbb{R}^d)$ of $\mathcal{D}^{[M]}(\mathbb{R}^d)$ is called the *space of ultradistributions of Beurling (resp. Roumieu) type* . Then $\mathcal{E}'^{[M]}(\mathbb{R}^d)$ is exactly the subspace of $\mathcal{D}'^{[M]}(\mathbb{R}^d)$ of all compactly supported ultradistributions [81, Theorem 5.9, p. 64]. Moreover, its structure may be described as follows.

Theorem 8.2.1. *Let M be a weight sequence satisfying $(M.1)$, $(M.2)'$ and $(M.3)'$. Then, $f \in \mathcal{E}'^{[M]}(\mathbb{R}^d)$ with $\operatorname{supp} f = K$ if and only if for every open neighborhood U of K there are continuous $f_\alpha \in C_c(U)$, $\alpha \in \mathbb{N}^d$, such that*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \qquad in \ \mathcal{E}'^{[M]}(\mathbb{R}^d).$$

Proof. This is a direct consequence of Komatsu's first structural theorem[1] for ultradistributions [81, Theorem 8.1 and Theorem 8.7]. $\square$

For any two weight sequences M and N and $\ell, q > 0$ we define $\mathcal{S}_{N,q}^{M,\ell}(\mathbb{R}^d)$ as the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{S}_{N,q}^{M,\ell}} = \sup_{\alpha,\beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|x^\beta \varphi^{(\alpha)}(x)|}{\ell^{|\alpha|} M_\alpha q^{|\beta|} N_\beta} < \infty.$$

Then, we define the test function spaces

$$\mathcal{S}_{(N)}^{(M)}(\mathbb{R}^d) = \varprojlim_{\ell \to 0^+} \mathcal{S}_{N,\ell}^{M,\ell}(\mathbb{R}^d), \qquad \mathcal{S}_{\{N\}}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{\ell \to \infty} \mathcal{S}_{N,\ell}^{M,\ell}(\mathbb{R}^d)$$

and they are called the *Gelfand-Shilov spaces*. Note that by using the weight function system $\mathscr{W}_N$, this definition coincides with the one used in Chapter 3. The elements of its dual $\mathcal{S}_{[N]}'^{[M]}(\mathbb{R}^d)$ are called *tempered ultradistributions* .

We conclude this section with two projective descriptions.

- For a weight sequence M satisfying $(M.1)$, $(M.2)'$ and $(M.3)'$ and an open subset $\Omega \subseteq \mathbb{R}^d$ we have as locally convex spaces [83, Proposition 3.5]:

$$\mathcal{E}^{[M]}(\Omega) = \varprojlim_{K \Subset \Omega} \varprojlim_{(\ell_p) \in [\mathfrak{R}]} \mathcal{E}^{M_{\ell_p},1}(K).$$

- For two weight sequences M and N satisfying $(M.1)$ and $(M.2)'$ we have as locally convex spaces [39, Theorem 3] (see also Theorem 4.2.14):

$$\mathcal{S}_{[N]}^{[M]}(\mathbb{R}^d) = \varprojlim_{(a_p),(b_p) \in [\mathfrak{R}]} \mathcal{S}_{N_{b_p},1}^{M_{a_p},1}(\mathbb{R}^d).$$

---

[1]The first structural theorem even holds true under the conditions $(M.1)$ and $(M.2)'$ if one were to use [37, Theorem 1].

8.3 Asymptotic behavior of generalized functions

We discuss here two types of asymptotic behavior for generalized functions, for a more thorough overview we refer the reader to the monograph [114].

Definition 8.3.1. Let $\mathcal{X}$ be a lcHs of smooth functions on $\mathbb{R}^d$ provided with continuous action of the translation operator. Let $\Gamma \subseteq \mathbb{R}^d$ be a cone with vertex at the origin and $\omega : \Gamma \to \mathbb{R}_+$. Then $f \in \mathcal{X}'$ has *S-asymptotic behavior with respect to ω on Γ with limit $g \in \mathcal{X}'$* if

$$\lim_{h \in \Gamma, |h| \to \infty} \frac{\langle f(x+h), \varphi(x) \rangle}{\omega(h)} = \langle g(x), \varphi(x) \rangle, \qquad \forall \varphi \in \mathcal{X}.$$

In such a case we write $f(x+h) \sim \omega(h)g(x)$.

When considering ultradistributions, the S-asymptotic behavior imposes a certain structure on the limit and gauge function.

Proposition 8.3.2 ([114, Proposition 1.2, p. 12])**.** *Let M be a weight sequence satisfying $(M.1)$ and $(M.3)'$ and $\Gamma \subseteq \mathbb{R}^d$ be a solid convex cone with vertex at the origin. If $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ has S-asymptotics $f(x+h) \sim \omega(h)g(x)$ on Γ for some function $\omega : \Gamma \to \mathbb{R}_+$ and non-zero limit $g \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$, then for some $y \in \mathbb{R}^d$:*

(i) *For every $h_0 \in \mathbb{R}^d$ we have*

$$\lim_{\substack{h \in (h_0 + \Gamma) \cap \Gamma \\ |h| \to \infty}} \frac{\omega(h + h_0)}{\omega(h)} = e^{h_0 \cdot y}.$$

(ii) *There exists $C \in \mathbb{R}$ such that $g(x) = C \exp(y \cdot x)$.*

The second type we consider is the so-called quasiasymptotic behavior. In order to introduce this concept, we first recall the notion of regularly varying functions.

A function $\rho : (a, \infty) \to \mathbb{R}$, $a > 0$, is called *regularly varying at infinity* [10, 128] if it is positive, measurable, and if there exists a real number $\alpha \in \mathbb{R}$ such that for each $x > 0$

$$\lim_{\lambda \to \infty} \frac{\rho(\lambda x)}{\rho(\lambda)} = x^\alpha. \tag{8.5}$$

The number α is called the *degree* of regular variation. If $\alpha = 0$, then ρ is called *slowly varying at infinity* and will be denoted by L. A function ρ is called *regularly (resp. slowly) varying at the origin* if $\widetilde{\rho}(x) := \rho(1/x)$ is regularly (resp. slowly) varying at infinity. Any regularly varying function may then be written as $\rho(x) = x^{\alpha}L(x)$, $x > a$. The convergence of (8.5) is uniform on every fixed compact interval $[b, c]$, $a < b < c < \infty$, and ρ is bounded (hence integrable) on it [128, Theorem 1.1 and Lemma 1.2]. As we will only be interested in the terminal behavior of ρ, one may assume [10] without any loss of generality that the regularly varying function at infinity (resp. at the origin) ρ is continuous on $[0, \infty)$ (resp. on $(0, \infty]$).

For a slowly varying function L we will often make use of Potter's estimate [10, Theorem 1.5.4]: for any $\varepsilon > 0$ there exists a $C_{\varepsilon} > 0$ such that

$$\frac{L(\lambda x)}{L(\lambda)} \leqslant C_{\varepsilon} \max\{x^{-\varepsilon}, x^{\varepsilon}\}, \qquad \forall x, \lambda > 0. \tag{8.6}$$

We may now define the quasiasymptotic behavior of a generalized function.

Definition 8.3.3. Let $\mathcal{X}$ be a lcHs of smooth functions on $\mathbb{R}^d$ provided with continuous action of the dilation operator. Let L be a slowly varying function at infinity (resp. at the origin). Then $f \in \mathcal{X}'$ has *quasiasymptotic behavior at infinity (resp. at the origin) in $\mathcal{X}'$ with respect to L of degree α, $\alpha \in \mathbb{R}$, and limit $g \in \mathcal{X}'$* if for all $\varphi \in \mathcal{X}$

$$\lim_{\lambda \to \infty} \frac{\langle f(\lambda x), \varphi(x) \rangle}{\lambda^{\alpha} L(\lambda)} = \langle g(x), \varphi(x) \rangle \qquad \left(\text{resp. } \lim_{\lambda \to 0^+}\right). \tag{8.7}$$

If (8.7) holds, we also say that f has *quasiasymptotics of degree α at infinity (at the origin) with respect to L* and write in short: $f(\lambda x) \sim \lambda^{\alpha}L(\lambda)g(x)$ in $\mathcal{X}'$ as $\lambda \to \infty$ (resp. $\lambda \to 0^+$).

Remark 8.3.4. It is not necessary to define the quasiasymptotic behavior via a gauge function that is regularly varying. However, in case the limit is non-zero, this is automatically so [114, Proposition 2.1, p. 83]. As our arguments to come will be based on the degree of the quasiasymptotic, this motivates our choice in the definition.

The quasiasymptotic behavior imposes a structure on the limit.

Proposition 8.3.5 ([114, Proposition 2.1, p. 83]). *If $f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ in $\mathcal{X}'$, then g is a homogeneous element of $\mathcal{X}'$ of degree α, i.e. $g(\lambda x) = \lambda^\alpha g(x)$.*

Chapter 9

Quasiasymptotic behavior

9.1 Introduction

The quasiasympotic behavior of generalized functions, which employs regularly varying functions [10] as gauges in the asymptotic comparisons, provides a framework for the study of local properties of generalized functions. The behavior was first introduced by Zav'yalov for tempered distributions in [155] and further developed by him, Drozhzhinov, and Vladimirov in connection with their powerful multidimensional Tauberian theory for Laplace transforms [140]. Notably, this behavior is commonly employed to express Tauberian theorems for generalized functions. Starting from the 1970s until the present, Tauberian theorems for integral transforms of generalized functions has been an extensively studied subject, see e.g. [55, 111, 114, 140], with applications to research areas such as probability theory, number theory, and mathematical physics.

A key aspect in the understanding of this concept is its description via so-called structural theorems and complete results in that direction were achieved in [135, 136, 137] (cf. [90, 114]). The purpose of this chapter is to present a detailed structural study of the so-called quasiasymptotics of ultradistributions. In [110] Pilipović and Stanković naturally extended the definition of quasiasymptotic behavior to the context of one-dimensional ultradistributions and studied its basic properties. We shall obtain here complete structural theorems for quasiasymptotics of non-quasianalytic ultradistributions

that generalize their distributional counterparts. Our main goal is thus to characterize those ultradistributions having quasiasymptotic behavior as infinite sums of derivatives of functions satisfying classical pointwise asymptotic relations.

This chapter is organized as follows. We first establish in Section 9.2 structural theorems for the S-asymptotic behavior of ultradistributions. These will be a direct consequence of the results found in Chapter 6, in particular Theorem 6.4.1. Section 9.3 studies the quasiasymptotic behavior at infinity. A key idea we apply here will be to connect the quasiasymptotic behavior with the S-asymptotic behavior via an exponential change of variables. The nature of the problem under consideration requires to split our treatment in two cases, depending on whether the degree of the quasiasymptotic behavior is a negative integer or not. We obtain in Section 9.4 structural theorems for the quasiasymptotic behavior at the origin. Our technique there is based on a reduction to the results from Section 9.3 by means of a change of variables and then regularization. Our method also yields asymptotic properties of regularizations at the origin of ultradistributions having prescribed asymptotic properties, generalizing results for distributions from [134]. It is also worth mentioning that our approach here differs from the one employed in the literature to deal with Schwartz distributions, and in fact can be used to produce new proofs for the classical structural theorems for the quasiasymptotic behavior of distributions. We conclude this chapter by studying extensions of quasiasymptotics to new ultradistributions spaces of Gelfand-Shilov type that we shall introduce in Section 9.5.

9.2   The structure of S-asymptotics

We start by obtaining first structural theorems for the S-asymptotic behavior of ultradistributions. Second structural theorems were shown in [114, Theorem 1.10, p. 46], however our results will hold under considerably less restrictions on the weight sequence. We fix for the remaining sections in this chapter a weight sequence M that satisfies $(M.1)$, $(M.2)'$ and $(M.3)'$.

Let ω be a weight function. We consider a convex cone Γ (with

vertex at the origin). We will work with the following assumption on ω: the limits

$$\lim_{\substack{|h|\to\infty \\ h\in\Gamma}} \frac{\omega(x+h)}{\omega(h)} \qquad \text{exist for all } x\in\mathbb{R}^d. \tag{9.1}$$

Suppose that for some $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ the S-asymptotic $f(x+h) \sim \omega(h)g(x)$ holds on Γ for some limit $g \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$. If $g \neq 0$, then clearly (9.1) must hold uniformly for x in compact sets. However, we will always require that (6.3) holds, so that the uniformity of (9.1) is always true.

The idea is to apply Theorem 6.4.1 to find the structure of the S-asymptotic behavior of ultradistributions. We start by noting that in the non-quasianalytic case, the space $\dot{\mathcal{B}}'^{[M]}_\omega$ may be characertized via the limit behavior of the convolution with elements in $\mathcal{D}^{[M]}(\mathbb{R}^d)$.

Theorem 9.2.1. *Let $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$. Then, $f \in \dot{\mathcal{B}}'^{[M]}_\omega$ if and only if the limit $\lim_{|h|\to\infty} T_{-h}f/\omega(h) = 0$ holds in $\mathcal{D}'^{[M]}(\mathbb{R}^d)$.*

Proof. Necessity follows immediately from Theorem 6.3.11. To show sufficiency, we notice that ω is $(p!)$-admissible (see Remark 6.3.2). As $(M.1)$ and $(M.3)'$ imply that $p! \prec M$, we have that M and ω satisfy Assumption 6.3.1. Whence $f \in \mathcal{B}'^{[M]}_\omega \subset \mathcal{S}'^{[M]}_{[p!]}(\mathbb{R}^d)$. Next, one may obtain (6.8) for some $M_p^\lambda = \lambda^p M_p > 0$ (for all $M_p^\lambda = \lambda^p M_p > 0$) by taking a window function $\psi \in \mathcal{D}^{(M)}(\mathbb{R}^d) \setminus \{0\}$ and making minor adjustments in the proof of $(2) \Rightarrow (3)$ in Theorem 6.3.11. Hence, the result follows from Theorem 6.3.11. $\qquad\qquad\square$

We now find the following structural theorems.

Theorem 9.2.2. *Let $\Gamma \subseteq \mathbb{R}^d$ be a solid convex cone and let ω be a weight function satisfying (6.3) and (9.1). Then, $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ has S-asymptotic behavior with respect to ω on Γ if and only if for each $R > 0$ there exist $f_\alpha \in C(\mathbb{R}^d)$, $\alpha \in \mathbb{N}^d$, such that*

$$f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha^{(\alpha)} \quad \text{on } \Gamma_R,$$

the limits

$$\lim_{\substack{|x|\to\infty \\ x\in\Gamma_R}} \frac{f_\alpha(x)}{\omega(x)}, \qquad \alpha\in\mathbb{N}^d,$$

exist, and for some $\ell > 0$ (for all $\ell > 0$) it holds that

$$\sup_{\alpha\in\mathbb{N}^d,\ x\in\Gamma_R} \frac{\ell^{|\alpha|}M_\alpha|f_\alpha(x)|}{\omega(x)} < \infty.$$

Proof. The conditions are clearly sufficient. To show necessity, we first make a few reductions. Suppose $f(x+h) \sim \omega(h)g(x)$ on Γ. By Proposition 8.3.2, there is $y\in\mathbb{R}^d$ such that the limits (9.1) equal $e^{y\cdot x}$ for each $x\in\mathbb{R}^d$ and $g(x) = Ce^{y\cdot x}$ for some constant C. Hence, if we put $f_0 = f - C\omega$, we see that $f_0(x+h) \sim \omega(h)\cdot 0$ on Γ. Establishing the structure for f_0 will provide us with the structure of f, so we may assume without loss of generality that $f(x+h) \sim \omega(h)\cdot 0$ on Γ. Next, note that we may reduce the general case to that of $\Gamma = \mathbb{R}^d$. Indeed, for any $R > 0$, take some $\psi\in\mathcal{D}^{(M)}(\mathbb{R}^d)$ such that supp $\psi\subset B(0,R/2)$ and $\int_{\mathbb{R}^d}\psi = 1$. Let $\chi_{\Gamma_{3R/2}}$ be the characteristic function of $\Gamma_{3R/2}$. Then $\chi_R = \psi * \chi_{\Gamma_{3R/2}}$ is a smooth function such that $\chi_R \equiv 1$ on Γ_R, χ_R vanishes off Γ_{2R} and

$$\sup_{\alpha\in\mathbb{N}^d,\xi\in\mathbb{R}^d} \frac{|\chi_R^{(\alpha)}(\xi)|}{\ell^{|\alpha|}M_\alpha} < \infty.$$

If we set $\widetilde{f} := \chi_R \cdot f$, then $\widetilde{f}$ and f coincide on Γ_R. We now verify that $\widetilde{f}(x+h) \sim \omega(h)\cdot 0$ on $\mathbb{R}^d$. Take any $\varphi\in\mathcal{D}^{[M]}(\mathbb{R}^d)$ and let $r > 0$ be such that supp $\varphi\subset B(0,r)$. Take any $h\in\mathbb{R}^d$. If $h\notin\Gamma_{r+2R}$ then $\left\langle \widetilde{f}(x+h)/\omega(h), \varphi(x)\right\rangle = 0$. Suppose now $h\in\Gamma_{r+2R}$, then $h = h_1 + h_2$ with $h_1\in\Gamma$ and $h_2\in B(0,r+2R)$. Then, employing (6.3), (9.1) and the Banach-Steinhaus theorem

$$\lim_{|h|\to\infty,h\in\Gamma_{r+2R}} \frac{\left\langle \widetilde{f}(x+h),\varphi(x)\right\rangle}{\omega(h)}$$

$$= \lim_{|h_1|\to\infty,h_1\in\Gamma} \sup_{h_2\in B(0,r+2R)} \frac{\omega(h_1)}{\omega(h_1+h_2)} \frac{\left\langle \widetilde{f}(x+h_1),\varphi(x-h_2)\right\rangle}{\omega(h_1)} = 0$$

because $\{T_{h_2}\varphi : h_2 \in B(0, r + 2R)\}$ is a bounded family in $\mathcal{D}^{[M]}(\mathbb{R}^d)$. Consequently, $\widetilde{f}(x + h) \sim \omega(h) \cdot 0$ on $\mathbb{R}^d$.

If $f(x + h) \sim \omega(h) \cdot 0$ on $\mathbb{R}^d$, then $f \in \dot{\mathcal{B}}_\omega'^{[M]}$ by Theorem 9.2.1. Hence, the desired structure of f follows from Theorem 6.4.1. $\qquad\square$

9.3 The structure of quasiasymptotics at infinity

This section is devoted to the study of the quasiasymptotic behavior at infinity. Our main results are Theorem 9.3.6 and Theorem 9.3.7, where we provide a full description of the structure of quasiasymptotics at infinity. Some auxiliary lemmas used in their proofs are shown in Section 9.3.1. Throughout this section and the ones following it, we will work in dimension 1 and M will denote a weight sequence satisfying $(M.1)$, $(M.2)'$ and $(M.3)'$. We will also work with the notation $\mathcal{D}^{[M]}$ for $\mathcal{D}^{[M]}(\mathbb{R})$ and similar for other spaces. In this specific section L stands for a slowly varying function at infinity.

9.3.1 Some lemmas

We start with the ensuing useful estimates for the weight sequence M, which we shall often exploit throughout the chapter. Hereafter $S(n, k)$ stand for the *Stirling numbers of the second kind* (see e.g. [78]).

Lemma 9.3.1. *For any $\ell > 0$ there is $C_\ell > 0$ (independent of p) such that*

$$\sum_{k=p}^{\infty} \frac{k!\ell^k}{M_k} \leqslant C_\ell \frac{p!}{M_p}\ell^p \tag{9.2}$$

and

$$\sum_{k=p}^{\infty} S(k + 1, p + 1)\frac{\ell^k}{M_k} \leqslant C_\ell \frac{(2\ell)^p}{M_p}. \tag{9.3}$$

Proof. Clearly, it is enough to show (9.2) just for sufficiently large p. Using [81, Lemma 4.1, p. 55], there is p_0 such that for any $p \geqslant p_0$ we

have $p/m_p = pM_{p-1}/M_p \leqslant (2\ell)^{-1}$. Hence, it follows that for p in this range

$$\sum_{k=p}^{\infty} \frac{k!\ell^k}{M_k} = \frac{p!}{M_p}\left(\ell^p + \sum_{k=p+1}^{\infty} \frac{(p+1)\cdot\ldots\cdot k\cdot\ell^k}{m_{p+1}\cdot\ldots\cdot m_k}\right) \leqslant 2\frac{p!}{M_p}\ell^p.$$

For (9.3), in view of [120, Theorem 3], we have $S(k+1, p+1) \leqslant 2^{k+1}(p+1)^{k-p} \leqslant 2^{k+1}k!/p!$ for $k \geqslant p$. The rest follows by application of (9.2). $\qquad\square$

If $f(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)$ in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$ ($\lambda \to 0^+$) then g is a homogeneous ultradistribution by Proposition 8.3.5. We first show that all the homogeneous ultradistributions are exactly the homogeneous distributions. We will employ the notation $H(x) = x_+^0$ for the Heaviside function.

Proposition 9.3.2. *Let $g \in \mathcal{D}'^{[M]}$ be a homogeneous ultradistribution of degree α. If $\alpha \neq -1, -2, -3, \ldots$, there exist constants c_+ and c_- such that*

$$g(x) = c_+ x_+^\alpha + c_- x_-^\alpha.$$

If $\alpha = -n$, with $n \in \mathbb{Z}_+$, then there are constants c_1 and c_2 such that

$$g(x) = c_1 x^{-n} + c_2\delta^{(n-1)}(x).$$

Proof. Suppose that $g(\lambda x) = \lambda^\alpha g(x)$ for all $\lambda > 0$, then one verifies that

$$xg'(x) = \alpha g(x).$$

This differential equation can be solved locally on $\mathbb{R} \setminus \{0\}$, so that g takes the form

$$g(x) = c_+ x_+^\alpha + c_- x_-^\alpha + f(x)$$

if $\alpha \notin \mathbb{Z}_-$, or

$$g(x) = c_1 x^\alpha + f(x)$$

if $\alpha \in \mathbb{Z}_-$, where $f \in \mathcal{D}'^{[M]}$ is homogeneous of degree α with support in $\{0\}$. Then, the Fourier-Laplace transform $\widehat{f}$ is an entire function of exponential type 0, homogeneous of degree $-\alpha - 1$. Since homogeneous entire functions are polynomials, it follows that $\widehat{f} = 0$ if $\alpha \notin \mathbb{Z}_-$ or $\widehat{f}(\xi) = (-i)^{-\alpha-1}c_2\xi^{-\alpha-1}$ for some constant c_2 if $\alpha \in \mathbb{Z}_-$. As $\{e^{ix\xi} : \xi \in \mathbb{R}\}$ is a dense subspace of $\mathcal{E}^{[M]}$ (cf. [81, Theorem 7.3, p. 75]), it follows that $f = 0$ if $\alpha \notin \mathbb{Z}_-$ or $f = c_2\delta^{(-\alpha-1)}$ if $\alpha \in \mathbb{Z}_-$. $\qquad\square$

In [135], the structure of distributional quasiasymptotics at infinity was found by noting that certain primitives preserve the asymptotic behavior, being of a higher degree, and using the fact that eventually the primitives are continuous functions. As the latter part does not hold in general for ultradistributions, a more careful analysis is needed, although we may carry over some of the distributional results. In fact, one may retread the proofs from [135, Section 2] (see also [114, Section 2.10])

Lemma 9.3.3. *Let $f \in \mathcal{D}'^{[M]}$. Suppose f has quasiasymptotics with respect to $\lambda^\alpha L(\lambda)$.*

(i) *If $\alpha \notin \mathbb{Z}_-$: for any $n \in \mathbb{N}$ and any n-primitive F_n of f there exists a polynomial P of degree at most $n - 1$ such that $F_n + P$ has quasiasymptotics with respect to $\lambda^{\alpha+n} L(\lambda)$ in $\mathcal{D}'^{[M]}$.*

(ii) *If $\alpha = -k$, $k \in \mathbb{Z}_+$: there is some $(k-1)$-primitive F of f such that F has quasiasymptotics with respect to $\lambda^{-1} L(\lambda)$ in $\mathcal{D}'^{[M]}$.*

The previous lemma roughly speaking shows that in order to find the structure of quasiasymptotics for arbitrary degree, it suffices to discover the structure for degrees $\geqslant -1$, where extra care is needed for the case -1. It should also be noticed that the converse statements for (i) and (ii) from Lemma 9.3.3 trivially hold true.

The next lemma, a direct consequence of the well-known moment asymptotic expansion [58, 124] (see also Chapter 10), states that the quasiasymptotic behavior of degree > -1 is a local property at infinity, which in some arguments enables us to remove the origin from the support of the ultradistribution in our analysis.

Lemma 9.3.4. *Suppose that $f_1, f_2 \in \mathcal{D}'^{[M]}$ and that for some $a > 0$, f_1 and f_2 coincide on $\mathbb{R} \setminus [-a, a]$. Suppose that $f_1(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$, where $\alpha > -1$. Then, also $f_2(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$.*

Proof. By our assumptions we have that $h := f_2 - f_1 \in \mathcal{E}'^{[M]}$, so that by [124, Theorem 4.4] (see also Theorem 10.4.3) $h(\lambda x) = o(\lambda^{-\alpha} L(\lambda))$ as $\lambda \to \infty$. Whence

$$f_2(\lambda x) = f_1(\lambda x) + h(\lambda x) = \lambda^\alpha L(\lambda) g(\lambda x) + o(\lambda^\alpha L(\lambda))$$

in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$. □

9.3.2 Structural theorem for $\alpha \notin \mathbb{Z}_-$

We study in this subsection quasiasymptotics of degree $\alpha \notin \mathbb{Z}_-$. Part of our analysis reduces the general case to that when $\alpha > -1$, i.e., the case when the quasiasymptotic behavior is local. Consequently, we may restrict our discussion to those ultradistributions whose support lie in the complement of some zero neighborhood. As both the negative and positive half-line can be treated symmetrically, it is natural to start the analysis with ultradistributions that are supported on the positive half-line. In the next crucial lemma we further normalize the situation by assuming that our ultradistribution is supported in (e, ∞).

Lemma 9.3.5. *Let $\alpha \in \mathbb{R}$ and let $f \in \mathcal{D}'^{[M]}$ be such that $\operatorname{supp} f \subset (e, \infty)$ and f has quasiasymptotic behavior at infinity with respect to $\lambda^\alpha L(\lambda)$ in $\mathcal{D}'^{[M]}(0, \infty)$. Then, there are continuous functions f_m such that $\operatorname{supp} f_m \subset (e, \infty)$,*

$$f = \sum_{m=0}^{\infty} f_m^{(m)},$$

the limits

$$\lim_{x \to \infty} \frac{f_m(x)}{x^{\alpha+m} L(x)}$$

exist, and furthermore, for some $\ell > 0$ (any $\ell > 0$) there is a $C = C_\ell > 0$ such that,

$$|f_m(x)| \leqslant C \frac{\ell^m}{M_m} x^{\alpha+m} L(x), \qquad m \in \mathbb{N}, \ x > 0.$$

Proof. Suppose $f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ in $\mathcal{D}'^{[M]}(0, \infty)$ as $\lambda \to \infty$. Since composition with a real analytic function induces continuous mappings between spaces of ultradifferentiable functions (see e.g. [72, Prop. 8.4.1, p. 281]), we obtain that the composition $f(e^x)$ is an element of $\mathcal{D}'^{[M]}$. Also, $\psi \in \mathcal{D}^{[M]}$ if and only if $\psi(x) = \varphi(e^x)$ with $\varphi \in \mathcal{D}^{[M]}(0, \infty)$.

These key observations allow us to make a change of variables in order to apply the structural theorem for S-asymptotics. In fact, we set $u(x) := f(e^x)$, $w(x) := g(e^x)$ and $c(h) := e^{\alpha h} L(e^h)$ (notice that

w has actually the form $w(x) = Be^{\alpha x}$ for some $B > 0$). A quick computation shows that

$$u(x + h) \sim c(h)w(x) \qquad \text{in } \mathcal{D}'^{[M]} \text{ as } h \to \infty.$$

Theorem 9.2.2 yields the existence of continuous functions u_n on $\mathbb{R}$, $n \in \mathbb{N}$, with supports on $(1, \infty)$ such that $u = \sum_{n=0}^{\infty} u_n^{(n)}$ on $(0, \infty)$ and $\lim_{h \to \infty} u_n(x+h)/c(h)$ exist uniformly for x on compacts of $(0, \infty)$ for any $n \in \mathbb{N}$ and for some $\ell > 0$ (for any $\ell > 0$) there exists a $C_0 = C_{0,\ell} > 0$ such that $\sup_{x>0} |u_n(x)|/c(x) \leqslant C_0 \ell^n / M_n$.

Take any $\varphi \in \mathcal{D}^{[M]}(0, \infty)$ and put $\psi(x) = e^x \varphi(e^x)$, then the substitution $y = e^x$ yields

$$\langle f(y), \varphi(y) \rangle = \left\langle f(y), \frac{\varphi(y)}{y} y \right\rangle = \langle u(x), \psi(x) \rangle = \sum_{n=0}^{\infty} \left\langle u_n^{(n)}, \psi(x) \right\rangle.$$

Let us now consider each term of the sum individually. We will need to explicitly calculate the derivatives of ψ. Using the Faà di Bruno formula [78, Eq. (2.2)],

$$\psi^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} (\varphi(e^x))$$

$$= e^x \sum_{m=0}^{n} \varphi^{(m)}(e^x) e^{mx} \sum_{k=m}^{n} \binom{n}{k} S(k, m)$$

$$= e^x \sum_{m=0}^{n} S(n+1, m+1) e^{mx} \varphi^{(m)}(e^x),$$

where we have applied [32, Theorem 5.3.B]. Then, for any $n \in \mathbb{N}$,

$$\int_1^{\infty} u_n(x) \psi^{(n)}(x) dx = \sum_{m=0}^{n} S(n+1, m+1) \int_e^{\infty} u_n(\log y) \varphi^{(m)}(y) y^m dy.$$

We define the functions $f_m(y) := y^m \sum_{n=m}^{\infty} (-1)^n S(n+1, m+1) u_n(\log y)$. In virtue of (9.3), we obtain the bounds

$$|f_m(y)| \leqslant C_0 y^m \sum_{n=m}^{\infty} S(n+1, m+1) \frac{\ell^n}{M_n} y^\alpha L(y) \leqslant C_0 C_\ell \frac{(2\ell)^m}{M_m} y^{\alpha+m} L(y)$$

for $y > 0$ and $m \in \mathbb{N}$. By the Lebesgue dominated convergence theorem we clearly have that $\lim_{y \to \infty} y^{-\alpha - m} f_m(y)/L(y)$ exists for every $m \in \mathbb{N}$. As, by our construction,

$$\langle f(y), \varphi(y) \rangle = \sum_{m=0}^{\infty} \left\langle f_m^{(m)}(y), \varphi(y) \right\rangle,$$

our proof is complete. □

We are ready to discuss the general case.

Theorem 9.3.6. *Suppose $\alpha \notin \mathbb{Z}_-$ and let $k \in \mathbb{N}$ be the smallest non-negative integer such that $-(k+1) < \alpha$. Then, an ultradistribution $f \in \mathcal{D}'^{[M]}$ has quasiasymptotic behavior*

$$f(\lambda x) \sim \lambda^\alpha L(\lambda)(c_- x_-^\alpha + c_+ x_+^\alpha) \quad in \ \mathcal{D}'^{[M]} \ as \ \lambda \to \infty \qquad (9.4)$$

if and only if there exist continuous functions f_m on $\mathbb{R}$, $m \geqslant k$, such that

$$f = \sum_{m=k}^{\infty} f_m^{(m)}, \qquad (9.5)$$

the limits

$$\lim_{x \to \pm\infty} \frac{f_m(x)}{x^m |x|^\alpha L(|x|)} = c_m^\pm, \qquad m \geqslant k, \qquad (9.6)$$

exist, and for some $\ell > 0$ (any $\ell > 0$) there is a $C = C_\ell > 0$ such that

$$|f_m(x)| \leqslant C \frac{\ell^m}{M_m} (1 + |x|)^{\alpha + m} L(|x|), \qquad x \in \mathbb{R}, \qquad (9.7)$$

for all $m \geqslant k$. Furthermore, in this case we have

$$c_\pm = \sum_{m=k}^{\infty} c_m^\pm \frac{\Gamma(\alpha + m + 1)}{\Gamma(\alpha + 1)}. \qquad (9.8)$$

Proof. In view of Lemma 9.3.3(i), we may assume that $\alpha > -1$ so that $k = 0$.

Suppose then first that f has quasiasymptotic behavior (9.4). We write $f = f_- + f_c + f_+$, where $f_c \in \mathcal{E}'^{[M]}$ coincides with f on an open interval containing $[-e, e]$ and supp $f_- \subset (-\infty, -e)$ and supp $f_+ \subset$

(e, ∞). Then, by Lemma 9.3.4 each $f_\pm$ has quasiasymptotic behavior with respect to $\lambda^\alpha L(\lambda)$ in $\mathcal{D}'^{[M]}(-\infty, 0)$ and $\mathcal{D}'^{[M]}(0, \infty)$, respectively. Using Lemma 9.3.5, we find continuous functions $f_{1,m}^\pm$, $m \in \mathbb{N}$, with supports in $(-\infty, -e)$ and (e, ∞), respectively, such that the identities

$$f_\pm = \sum_{m=0}^\infty (f_{1,m}^\pm)^{(m)}$$

hold, the limits

$$c_m^\pm = (-1)^m \lim_{x \to \infty} \frac{f_{1,m}^\pm(\pm x)}{x^{\alpha+m} L(x)}$$

exist, and the bounds $|f_{1,m}^\pm(x)| \leq C'\ell^m |x|^{\alpha+m} L(|x|)/M_m$ are satisfied for some $\ell > 0$ (any $\ell > 0$) and some $C' = C'_\ell > 0$. Applying Theorem 8.2.1 one can also find continuous functions g_m, whose supports lie in some (arbitrarily chosen) neighborhood of supp f_c, such that $f_c = \sum_{m=0}^\infty g_m^{(m)}$ in $\mathcal{D}'^{[M]}$ and $\sup_{x \in \mathbb{R}} |g_m(x)| \leq C''\ell^m/M_m$ for some $\ell > 0$ (for every $\ell > 0$) and $C'' = C''_\ell > 0$. The functions

$$f_m = g_m + f_{1,m}^- + f_{1,m}^+$$

satisfy all sought requirements. We verify the relation (9.8) below.

Conversely, assume that f satisfies all of the conditions above. Take any $\phi \in \mathcal{D}^{[M]}$ and suppose that for some $R > 1$ we have supp $\phi \subseteq [-R, R]$. Pick $\gamma > 0$ such that $\alpha - \gamma > -1$. Using Potter's estimate (8.6), we may assume that

$$\frac{L(\lambda x)}{L(\lambda)} \leq C_\gamma \max\{x^{-\gamma}, x^\gamma\} \tag{9.9}$$

holds for all $x, \lambda > 0$. Since $\phi \in \mathcal{D}^{[M]}$, for any $h > 0$ (for some $h > 0$) there exists a $C_{\phi,h}$ such that for all $m \in \mathbb{N}$ we have $\sup_{x \in \mathbb{R}} |\phi^{(m)}(x)| \leq$

$C_{\phi,h}h^m M_m$. Due to (9.7), we now have for any $m \in \mathbb{N}$ and $\lambda > 1$

$$\left| \frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{f_m(x)}{\lambda^\alpha L(\lambda)} \frac{\phi^{(m)}(x/\lambda)}{\lambda^m} dx \right|$$

$$\leqslant C_\ell \frac{(2\ell)^m}{M_m} \left(\int_{-\lambda^{-1}}^{\lambda^{-1}} \frac{|\phi^{(m)}(x)|}{\lambda^{m+\alpha}L(\lambda)} dx \right.$$

$$\left. + \int_{|x| \geqslant 1/\lambda} \frac{L(\lambda|x|)}{L(\lambda)} |x|^{\alpha+m} |\phi^{(m)}(x)| dx \right)$$

$$\leqslant 2C_\ell C_{\phi,h}(2h\ell)^m \left(\frac{1}{\lambda^{m+1+\alpha}L(\lambda)} + C_\gamma R^{m+\alpha+\gamma+1} + \frac{C_\gamma}{\alpha-\gamma+m+1} \right)$$

$$\leqslant C(2h\ell R)^m,$$

and, as $2h\ell$ may be chosen freely, this is absolutely summable over $m \in \mathbb{N}$. It follows by applying the Lebesgue dominated convergence theorem twice that

$$\lim_{\lambda \to \infty} \left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)}, \phi(x) \right\rangle = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{m=0}^{\infty} (-1)^m \int_{-\infty}^{\infty} \frac{f_m(x)}{\lambda^\alpha L(\lambda)} \frac{\phi^{(m)}(x/\lambda)}{\lambda^m} dx$$

$$= \sum_{m=0}^{\infty} (-1)^m \int_{-\infty}^{\infty} (c_m^- x_-^\alpha + c_m^+ x_+^\alpha) x^m \phi^{(m)}(x) dx$$

$$= c_- \int_{-\infty}^{0} |x|^\alpha \phi(x) dx + c_+ \int_{0}^{\infty} x^\alpha \phi(x) dx,$$

with c_- and c_+ given by (9.8). $\qquad \square$

9.3.3 Structural Theorem for negative integral degrees

We now address the case of quasiasymptotics of degree $\alpha \in \mathbb{Z}_-$. The next structural theorem is the second main result of this section.

Theorem 9.3.7. *Let $k \in \mathbb{Z}_+$ and $f \in \mathcal{D}'^{[M]}$. Then, f has the quasi-asymptotic behavior*

$$f(\lambda x) \sim \frac{L(\lambda)}{\lambda^k}(\gamma \delta^{(k-1)}(x) + \beta x^{-k}) \quad in \; \mathcal{D}'^{[M]} \; as \; \lambda \to \infty \quad (9.10)$$

if and only if there exist continuous functions f_m on $\mathbb{R}$, $m \geqslant k - 1$, such that

$$f = \sum_{m=k-1}^{\infty} f_m^{(m)}, \tag{9.11}$$

the limits

$$\lim_{x \to \pm\infty} \frac{f_m(x)}{x^{m-k} L(|x|)} = c_m^{\pm}, \qquad m \geqslant k - 1, \tag{9.12}$$

and

$$\lim_{x \to \infty} \frac{1}{L(x)} \int_{-x}^{x} f_{k-1}(t) dt = c_{k-1}^* \tag{9.13}$$

exist, and for some $\ell > 0$ (any $\ell > 0$) there is $C = C_\ell > 0$ such that

$$|f_m(x)| \leqslant C \frac{\ell^m}{M_m} (1 + |x|)^{m-k} L(|x|), \qquad x \in \mathbb{R}, \tag{9.14}$$

for all $m \geqslant k$. Furthermore, we must have

$$\gamma = c_{k-1}^* + \sum_{m=k}^{\infty} (c_m^+ - c_m^-) \tag{9.15}$$

$$\beta = (-1)^{k-1}(k-1)! c_{k-1}^+ = (-1)^{k-1}(k-1)! c_{k-1}^-. \tag{9.16}$$

Proof. In view of Lemma 9.3.3(*ii*) we may assume that $k = 1$.

Necessity. We start by showing the necessity of the conditions if f has the quasiasymptotic behavior (9.10). Our strategy consists of modifying the quasiasymptotics to one of order 0 by multiplying f by x, applying Lemma 9.3.5, and then studying the structure it imposes on f. Take a compactly supported ultradistribution f_c that coincides with f on $[-e, e]$ and consider $\tilde{f} = f - f_c$, so that $\text{supp}(f - f_c) \cap [-e, e] = \varnothing$. We set $g(x) = x(f(x) - f_c(x))$, which, in view of Lemma 9.3.4, has quasiasymptotic behavior

$$g(\lambda x) \sim \beta L(\lambda) \quad \text{in } \mathcal{D}'^{[M]} \text{ as } \lambda \to \infty.$$

Splitting g as the sum of two distributions supported on $(-\infty, -e)$ and (e, ∞) respectively, we can apply Lemma 9.3.5 to obtain its structure as

$$g = \sum_{m=0}^{\infty} g_m^{(m)},$$

where each of the functions has support in $(-\infty, -e) \cup (e, \infty)$, satisfies the corresponding bounds implied by the lemma, and is such that for any $m \in \mathbb{N}$ the limit $\lim_{x \to \pm\infty} x^{-m} g_m(x)/L(|x|)$ exists. Define, for any $j \in \mathbb{N}$, the following continuous functions

$$
\tilde{f}_j(x) = \begin{cases} 0, & x = 0, \\ \dfrac{x^{j-1}}{j!} \displaystyle\sum_{m=j}^{\infty} m! g_m(x) x^{-m}, & x \neq 0. \end{cases}
$$

Let us verify they satisfy the requirements that the f_j should satisfy. First of all, for some $\ell > 0$ (any $\ell > 0$) and $C = C_\ell > 0$,

$$
|\tilde{f}_j(x)| \leq C \frac{|x|^{j-1}}{j!} L(|x|) \sum_{m=j}^{\infty} \frac{m! \ell^m}{M_m} \leq C' |x|^{j-1} \frac{\ell^j}{M_j} L(|x|),
$$

by (9.2). This not only shows that each $\tilde{f}_j$ is well-defined and continuous on $\mathbb{R}$, but also provides the bounds (9.14) for them. From dominated convergence we infer the existence of

$$
\lim_{x \to \pm\infty} \frac{\tilde{f}_j(x)}{x^{j-1} L(|x|)} = \lim_{x \to \pm\infty} \frac{1}{j!} \sum_{m=j}^{\infty} \frac{m! g_m(x)}{x^m L(|x|)} = \frac{1}{j!} \sum_{m=j}^{\infty} \lim_{x \to \pm\infty} \frac{m! g_m(x)}{x^m L(|x|)}.
$$

Take an arbitrary $\phi \in \mathcal{D}^{[M]}$ and let $\varphi \in \mathcal{D}^{[M]}$ be another corresponding test function that coincides with ϕ on $\mathbb{R} \backslash (-e, e)$, while its support does not contain the origin. We then have

$$
\left\langle \tilde{f}(x), \phi(x) \right\rangle = \left\langle g(x), \frac{\varphi(x)}{x} \right\rangle
$$
$$
= \sum_{m=0}^{\infty} \sum_{j=0}^{m} (-1)^m \binom{m}{j} \left\langle g_m(x), (-1)^{m-j} (m-j)! \frac{\varphi^{(j)}(x)}{x^{m-j+1}} \right\rangle
$$
$$
= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{m=j}^{\infty} \left\langle m! x^{j-1} \frac{g_m(x)}{x^m}, \varphi^{(j)}(x) \right\rangle
$$
$$
= \sum_{j=0}^{\infty} \left\langle \tilde{f}_j^{(j)}, \phi(x) \right\rangle.
$$

Applying Theorem 8.2.1 to f_c as in the proof of Theorem 9.3.6, we obtain compactly supported continuous functions g_m such that $f_m = \tilde{f}_m + g_m$ satisfy (9.11), (9.12), and (9.14). The necessity of (9.13) follows from (9.17) below. That (9.15) and (9.16) must necessarily hold will also be shown below in the proof of the converse.

Sufficiency. Conversely, assume that (9.11) holds with f_m fulfilling (9.12), (9.13) and (9.14) (recall we work with the reduction $k = 1$). We assume without loss of generality that $L(x)$ is everywhere continuous and vanishes for $x \leqslant 1$. We consider

$$g = \sum_{m=1}^{\infty} f_m^{(m-1)}.$$

It follows from Theorem 9.3.6 that g has quasiasymptotic behavior of degree 0 with respect to $L(\lambda)$, and differentiation then yields

$$f(\lambda x) - f_0(\lambda x) = g'(\lambda x) \sim (\gamma - c_0^*)\frac{L(\lambda)}{\lambda}\delta(x) \quad \text{in } \mathcal{D}'^{[M]} \text{ as } \lambda \to \infty,$$

with γ precisely given as in (9.15). It thus remains to determine the quasiasymptotic properties of f_0. Write $F(x) = \int_0^x f_0(t)dt$. Since $f_0(\pm x) \sim \pm c_0^{\pm} L(x)/x$, $x \to \infty$, one readily shows that

$$F(\lambda x)H(\pm x) = F(\pm\lambda)H(\pm x) + c_0^{\pm} \int_\lambda^{\pm\lambda x} \frac{L(t)}{t}dt + o\left(L(\lambda)\right)$$

$$= F(\pm\lambda)H(\pm x) + c_0^{\pm}L(\lambda)H(\pm x)\log|x| + o\left(L(\lambda)\right),$$

as $\lambda \to \infty$ uniformly for x on compact intervals, and in particular the relation holds in $\mathcal{D}'^{[M]}$. Differentiating

$$F(\lambda x) = F(-\lambda)H(-x) + F(\lambda)H(x)$$
$$+ L(\lambda)\left(c_0^- H(-x) + c_0^+ H(x)\right)\log|x| + o\left(L(\lambda)\right),$$

we conclude that

$$f_0(\lambda x) = \frac{F(\lambda) - F(-\lambda)}{\lambda}\delta(x)$$
$$+ \frac{L(\lambda)}{\lambda}\left(c_0^- \operatorname{Pf}\left(\frac{H(-x)}{x}\right) + c_0^+ \operatorname{Pf}\left(\frac{H(x)}{x}\right)\right)$$
$$+ o\left(\frac{L(\lambda)}{\lambda}\right), \tag{9.17}$$

whence the result follows. $\qquad\square$

9.3.4   Extension from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$

The methods employed in the previous two subsections also allow us to study the following question. Suppose that the restriction of $f \in \mathcal{D}'^{[M]}$ to $\mathbb{R} \setminus \{0\}$ is known to have quasiasymptotic behavior in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$, what can we say about the quasiasymptotic properties of f? In view of symmetry considerations, it is clear that it suffices to restrict our attention to ultradistributions supported on $[0, \infty)$.

Theorem 9.3.8. *Suppose that $f \in \mathcal{D}'^{[M]}$ is supported in $[0, \infty)$ and has quasiasymptotic behavior*

$$f(\lambda x) \sim c\lambda^\alpha L(\lambda) x^\alpha \qquad in \; \mathcal{D}'^{[M]}(0, \infty) \; as \; \lambda \to \infty.$$

(i) *If $\alpha > -1$, then $f(\lambda x) \sim c\lambda^\alpha L(\lambda) x_+^\alpha$ in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$.*

(ii) *If $\alpha < -1$ and $N \in \mathbb{N}$ is such that $-(N+1) < \alpha < -N$, then there exist constants $a_0, \ldots, a_{N-1}$ such that*

$$f(\lambda x) - \sum_{n=0}^{N-1} a_n \frac{\delta^{(n)}(x)}{\lambda^{n+1}} \sim c\lambda^\alpha L(\lambda) x_+^\alpha \quad in \; \mathcal{D}'^{[M]} \; as \; \lambda \to \infty.$$

(iii) *If $\alpha = -k \in \mathbb{Z}_-$, then there is a function b satisfying[1] for each $a > 0$*

$$b(ax) = b(x) + c\frac{(-1)^{k-1}}{(k-1)!} L(x) \log a + o\left(L(x)\right), \qquad (9.18)$$

$x \to \infty$, and constants $a_0, \ldots, a_{k-1}$ such that

$$f(\lambda x) = c\frac{L(\lambda)}{\lambda^k} \operatorname{Pf}\left(\frac{H(x)}{x^k}\right)$$

$$+ \frac{b(\lambda)}{\lambda^k} \delta^{(k-1)}(x) + \sum_{j=0}^{k-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}}$$

$$+ o\left(\frac{L(\lambda)}{\lambda^k}\right), \qquad (9.19)$$

in $\mathcal{D}'^{[M]}$ as $\lambda \to \infty$.

---

[1]Such functions are called associate homogeneous of degree 0 with respect to L in [114, 135]. They coincide with functions of the so-called De Haan class [10].

Proof. The moment asymptotic expansion [124, Theorem 4.4] (see also Theorem 10.4.3) says that we may assume that, say, supp $f \subset (e, \infty)$ by removing a neighborhood of the origin. So, we can apply exactly the same argument as in the proof of Theorem 9.3.6 (via Lemma 9.3.5 and Lemma 9.3.3(i)) to show parts (i) and (ii). For (iii), we assume without loss of generality that $k = 1$ (Lemma 9.3.3(ii)) and apply the same argument as in the proof of Theorem 9.3.7 to conclude that

$$f(\lambda x) = f_0(\lambda x) + \gamma L(\lambda)\delta(\lambda x) + o\left(L(\lambda)/\lambda\right) \qquad \text{in } \mathcal{D}'^{[M]}$$

where the continuous function f_0 has also support in (e, ∞) and satisfies $f_0(x) \sim cL(x)/x$, $x \to \infty$, in the ordinary sense. At this point the result can be derived from [135, Theorem 4.3] (see also [114, Theorem 2.38, p. 155]), but we might argue directly as follows. In fact, we proceed in the same way we arrived at (9.17). Set $b(x) = \int_1^x f_0(t)dt$, then, uniformly for x in compact subsets of $(0, \infty)$,

$$b(\lambda x) = b(\lambda)H(x) + c\int_\lambda^{\lambda x} \frac{L(t)}{t}dt + o(L(\lambda))$$
$$= b(\lambda)H(x) + cL(\lambda)H(x)\log x + o(L(\lambda)),$$

so that differentiation finally shows

$$f_0(\lambda x) = \frac{b(\lambda)}{\lambda}\delta(x) + c\frac{L(\lambda)}{\lambda}\operatorname{Pf}\left(\frac{H(x)}{x}\right) + o\left(\frac{L(\lambda)}{\lambda}\right) \qquad \text{in } \mathcal{D}'.$$

$\square$

9.4 The structure of quasiasymptotics at the origin

We now focus our attention on quasiasymptotic behavior at the origin. The reader should notice that Lemma 9.3.3 holds for quasiasymptotics at the origin as well. Furthermore, it is a simple consequence of the definition that quasiasymptotics at the origin is a local property, in the sense that two ultradistributions that coincide

in a neighborhood of the origin must have precisely the same quasi-asymptotic properties. Throughout this section L stands for a slowly varying function at the origin and we set $\tilde{L}(x) = L(1/x)$. From now on, by convention the parameters $\varepsilon \to 0^+$ and $\lambda \to \infty$.

We will reduce the analysis of the structure of quasiasymptotics at the origin to that of the quasiasymptotics at infinity via a substitution. Our starting key observation is the following lemma:

Lemma 9.4.1. *If $f \in \mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$ has quasiasymptotic behavior with respect to $\varepsilon^\alpha L(\varepsilon)$, $\alpha \in \mathbb{R}$, then $\tilde{f}(x) := f(1/x)$ has quasiasymptotics in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$ with respect to $\lambda^{-\alpha} \tilde{L}(\lambda)$.*

Proof. Take any $\phi \in \mathcal{D}^{[M]}(\mathbb{R} \setminus \{0\})$ and set $\tilde{\phi}(x) := \phi(1/x)$. Suppose that $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x)$ in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$. If we set $\lambda := \varepsilon^{-1}$, then we get

$$
\begin{aligned}
\lim_{\lambda \to \infty} \left\langle \frac{\tilde{f}(\lambda x)}{\lambda^{-\alpha} \tilde{L}(\lambda)}, \phi(x) \right\rangle &= \lim_{\lambda \to \infty} \left\langle \frac{f(x)}{\lambda^{-\alpha+1} \tilde{L}(\lambda)}, \tilde{\phi}(\lambda x) x^{-2} \right\rangle \\
&= \lim_{\varepsilon \to 0+} \left\langle \frac{f(x)}{\varepsilon^{\alpha+1} L(\varepsilon)}, \tilde{\phi}\left(\frac{x}{\varepsilon}\right) \left(\frac{x}{\varepsilon}\right)^{-2} \right\rangle \\
&= \left\langle g(x), \tilde{\phi}(x) x^{-2} \right\rangle \\
&= \left\langle g(1/x), \phi(x) \right\rangle.
\end{aligned}
$$

$\square$

We would now like to proceed applying the structure theorem to $\tilde{f}$ and transform back via the change of variables $x \leftrightarrow 1/x$. We therefore need to see how this substitution acts on derivatives, which can be done via Faà di Bruno's formula.

Lemma 9.4.2. *Let $\phi \in C^\infty(\mathbb{R} \setminus \{0\})$ and set $\psi(x) := x^{-2} \phi(1/x)$. Then for any $m \in \mathbb{N}$, there exist constants $c_{m,0}, \ldots, c_{m,m}$ such that*

$$
\frac{d^m}{dx^m}(\psi(x)) = \sum_{j=0}^{m} c_{m,j} \frac{\phi^{(j)}(1/x)}{x^{m+j+2}}, \tag{9.20}
$$

where we have the bounds

$$
|c_{m,j}| \leqslant \frac{m!}{j!} 4^m, \qquad 0 \leqslant j \leqslant m. \tag{9.21}
$$

Proof. Applying the Faà di Bruno formula [78, Eq. (2.2)],

$$\frac{d^k}{dx^k}\left(\phi(1/x)\right) = \sum_{j=1}^{k}(-1)^k x^{-(k+j)}\phi^{(j)}(1/x)B_{k,j}(1!, 2!, \ldots, (k-j+1)!),$$

where $B_{k,j}$ are the Bell polynomials; from their generating function identity [32, (3a'), p. 133] we infer that

$$B_{k,j}(1!, \ldots, (k-j+1)!) = \left.\frac{d^k}{dt^k}\left(\frac{1}{j!}\left(\frac{t}{1-t}\right)^j\right)\right|_{t=0} = \frac{k!(k-1)!}{j!(j-1)!(k-j)!}.$$

Therefore, we obtain that (9.20) holds with

$$c_{m,0} = (-1)^m(m+1)! \quad \text{and} \quad c_{m,j} = (-1)^m\frac{m!}{j!}\sum_{k=j}^{m}(m-k+1)\binom{k-1}{j-1}$$

when $0 < j \leqslant m$. In the latter case,

$$|c_{m,j}| \leqslant \frac{m!}{j!}\frac{(m-j+1)(m-j+2)}{2}\binom{m-1}{j-1} \leqslant \frac{m!}{j!}m(m+1)2^{m-2},$$

which shows (9.21). $\qquad\square$

Theorem 9.4.3. *Let $\alpha \notin \mathbb{Z}_-$ and let $k \in \mathbb{N}$ be the smallest integer such that $-(k+1) < \alpha$. Then, $f \in \mathcal{D}'^{[M]}$ has quasiasymptotic behavior*

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon)(c_- x_-^\alpha + c_+ x_+^\alpha) \quad \text{in } \mathcal{D}'^{[M]} \text{ as } \varepsilon \to 0^+ \qquad (9.22)$$

if and only if there exist functions $f_m \in L^1(-1,1)$, $m \geqslant k$, that are continuous on $[-1,1]\setminus\{0\}$ such that

$$f = \sum_{m=k}^{\infty} f_m^{(m)}, \qquad \text{on } (-1,1), \qquad (9.23)$$

the limits

$$c_m^\pm = \lim_{x\to 0^\pm} \frac{f_m(x)}{x^m|x|^\alpha L(|x|)}, \qquad m \geqslant k, \qquad (9.24)$$

exist, and furthermore, for some $\ell > 0$ (for any $\ell > 0$) there is a $C = C_\ell > 0$ such that

$$|f_m(x)| \leqslant C\frac{\ell^m}{M_m}|x|^{\alpha+m}L(|x|), \qquad 0 < |x| \leqslant 1, \qquad (9.25)$$

for all $m \geqslant k$. Moreover, the relation (9.8) must hold.

Proof. The proof of sufficiency can be done analogously as in Theorem 9.3.6. Hence we are only left with necessity. If we can show the theorem for degree larger than -1, then the full structure theorem will follow from Lemma 9.3.3(i), hence we assume that $\alpha > -1$ (and thus $k = 0$). If f has quasiasymptotic behavior with respect to $\varepsilon^\alpha L(\varepsilon)$, then $\widetilde{f}(x) := f(1/x)$ has quasiasymptotic behavior in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$ with respect to $\lambda^{-\alpha}\widetilde{L}(\lambda)$, where $\widetilde{L}(x) := L(1/x)$. Then by Theorem 9.3.6 or Theorem 9.3.7 if $\alpha \in \mathbb{Z}_+$ and keeping in mind our observations from Section 9.3.4, there exist continuous $\widetilde{f}_m$ in $\mathbb{R} \setminus \{0\}$, $m \geqslant 0$, that satisfy (9.5), (9.6) and (9.7). Consider now for any $m \geqslant 0$,

$$f_m(x) := \sum_{k=m}^{\infty} (-1)^{k+m} c_{k,m} \widetilde{f}_k(1/x) x^{m+k},$$

where the $c_{k,m}$ are as in Lemma 9.4.2. By (9.7) and (9.21) it follows that for some $\ell > 0$ (for any $\ell > 0$) and any $0 < |x| \leqslant 1$,

$$
\begin{aligned}
|f_m(x)| &= \left| \sum_{k=m}^{\infty} (-1)^{k+m} c_{k,m} \widetilde{f}_k(1/x) x^{m+k} \right| \\
&\leqslant \sum_{k=m}^{\infty} \frac{k!}{m!} 4^k \cdot C \frac{\ell^k}{M_k} |x|^{\alpha-k} L(|x|) |x|^{m+k} \\
&= C|x|^{\alpha+m} L(|x|) \frac{1}{m!} \sum_{k=m}^{\infty} k! \frac{(4\ell)^k}{M_k} \leqslant C C_{4\ell} \frac{(4\ell)^m}{M_m} |x|^{\alpha+m} L(|x|),
\end{aligned}
$$

by (9.2). This not only shows existence and continuity in $[-1, 1] \setminus \{0\}$, but also shows that the f_m satisfy (9.25). By (9.6) and dominated convergence, it also follows that for these functions the limits (9.24) exist. Now take any $\phi \in \mathcal{D}^{[M]}(\mathbb{R} \setminus \{0\})$ with supp $\phi \subseteq (-1, 1)$ and set $\psi(x) := \phi(1/x) x^{-2}$. Then,

$$\langle f(x), \phi(x) \rangle = \left\langle \widetilde{f}(x), \psi(x) \right\rangle = \sum_{k=0}^{\infty} \left\langle \widetilde{f}_k(x), (-1)^k \psi^{(k)}(x) \right\rangle.$$

Since for any $k \in \mathbb{N}$, by Lemma 9.4.2,

$$
\int_{-\infty}^{\infty} \widetilde{f}_k(x) \psi^{(k)}(x) dx = \sum_{m=0}^{k} c_{k,m} \int_{-\infty}^{\infty} \widetilde{f}_k(x) \frac{\phi^{(m)}(1/x)}{x^{m+k+2}} dx
$$

$$
= \sum_{m=0}^{k} c_{k,m} \int_{-\infty}^{\infty} \widetilde{f}_k(1/x) \phi^{(m)}(x) x^{m+k} dx,
$$

it follows by switching the order of summation that

$$
f = \sum_{m=0}^{\infty} f_m^{(m)},
$$

in $\mathcal{D}'^{[M]}((-1,1)\setminus\{0\})$. Now as $\alpha > -1$, the latter sum is an element of $\mathcal{D}'^{[M]}$, so that there is some $g \in \mathcal{E}'^{[M]}$ with $\operatorname{supp} g \subseteq \{0\}$ for which

$$
f = \sum_{m=0}^{\infty} f_m^{(m)} + g,
$$

in $\mathcal{D}'^{[M]}_{[-1,1]}$. Since we have already shown sufficiency, the sum has quasi-asymptotics with respect to $\varepsilon^{\alpha} L(\varepsilon)$, implying that the same holds for g. As $\operatorname{supp} g \subseteq \{0\}$, its Fourier-Laplace transform $\widehat{g}$ is an entire function of exponential type 0. By the quasiasymptotic behavior of g, it follows that $\widehat{g}(r) = o(r^2)$ as $|r| \to \infty$. By [12, Theorem 10.2.11, p. 183]. we see that $\widehat{g}(z) = -4\pi^2 a_2 z^2 + (2\pi i a_1) z + a_0$ for certain $a_i \in \mathbb{C}$, $i = 0, 1, 2$. Whence $g = a_2 \delta^{(2)} + a_1 \delta^{(1)} + a_0 \delta$. However the latter function can only have quasiasymptotic behavior at the origin of degree α if and only if $a_0 = a_1 = a_2 = 0$, so we may conclude g must be identically 0 and this completes the proof of the theorem. $\qquad\square$

The structure for negative integral degree can be described as follows.

Theorem 9.4.4. *Let $f \in \mathcal{D}'^{[M]}$ and $k \in \mathbb{Z}_+$. Then, f has quasi-asymptotic behavior*

$$
f(\varepsilon x) \sim \frac{L(\varepsilon)}{\varepsilon^k}(\gamma \delta^{(k-1)}(x) + \beta x^{-k}) \quad in \ \mathcal{D}'^{[M]} \ as \ \varepsilon \to 0^+ \qquad (9.26)
$$

if and only if there are continuous functions F and f_m on $[-1, 1] \setminus \{0\}$, $m \geqslant k$, such that

$$f = F^{(k)} + \sum_{m=k}^{\infty} f_m^{(m)} \qquad on \ (-1, 1), \qquad (9.27)$$

the limits

$$c_m^{\pm} = \lim_{x \to 0^{\pm}} \frac{f_m(x)}{x^{m-k} L(|x|)}, \qquad m \geqslant k, \qquad (9.28)$$

exist, for some $\ell > 0$ (for any $\ell > 0$) there exists $C = C_\ell > 0$ such that

$$|f_m(x)| \leqslant C \frac{\ell^m}{M_m} |x|^{m-k} L(|x|), \qquad 0 < |x| \leqslant 1, \qquad (9.29)$$

for all $m \geqslant k$, and for any $a > 0$ the limit

$$\lim_{x \to 0^+} \frac{F(ax) - F(-x)}{L(x)} = c_1^* + c_2^* \log a \qquad (9.30)$$

exists. In this case,

$$\gamma = c_1^* + \sum_{m=k}^{\infty} (c_m^+ - c_m^-) \quad and \quad \beta = (-1)^{k-1}(k-1)! c_2^*. \qquad (9.31)$$

Proof. For the sufficiency, by applying Theorem 9.4.3 to the series $\sum_{m=k}^{\infty} f_m^{(m-1)}$, one deduces

$$f(\varepsilon x) - F^{(k)}(\varepsilon x) \sim (\gamma - c_1^*) \delta^{(k-1)}(\varepsilon x), \qquad \text{in } \mathcal{D}'^{[M]} \text{ as } \varepsilon \to 0^+.$$

In view of [137, Theorem 5.3] (see also [114, Theorem 2.33, p. 149]), we have

$$F^{(k)}(\varepsilon x) \sim L(\varepsilon)(c_1^* \delta^{(k-1)}(\varepsilon x) + \beta(\varepsilon x)^{-k}), \qquad \text{in } \mathcal{D}' \text{ as } \varepsilon \to 0^+,$$

which yields the result.

For the necessity, we may assume that $k = 1$ by Lemma 9.3.3(*ii*). We now apply Theorem 9.4.3 to $xf(x)$. Using the same reasoning as in the proof of Theorem 9.3.7, one can write $f(x) = f_0 + \sum_{m=1}^{\infty} f_m^{(m)}$ on $(-1, 1) \setminus \{0\}$, with continuous functions $f_0, f_1, \ldots$ on $[-1, 1] \setminus \{0\}$

such that the limits (9.28) exist including the case $m = 0$. Applying again Theorem 9.4.3 to the series $\sum_{m=1}^{\infty} f_m^{(m-1)}$, we deduce that f_0 has an extension g_0 to $\mathbb{R}$ with quasiasymptotic behavior of order -1 with respect to $L(\varepsilon)$. Let F be a first order primitive of g_0. Due to the fact that $F' = f_0$ off the origin and the quasiasymptotic behavior of F', it is clear that F is integrable at the origin and that it must have the form

$$F(x) = -H(x)\left(\int_x^1 f_0(t)dt + C_+\right) + H(-x)\left(\int_{-1}^x f_0(t)dt + C_-\right).$$

Similarly as in the proof of Theorem 9.3.7, we conclude that

$$c_1^* = \lim_{x \to 0^+} \frac{F(x) - F(-x)}{L(x)}$$

must exist by comparing with the quasiasymptotics of g_0. Hence, for each $a > 0$

$$\lim_{x \to 0^+} \frac{F(ax) - F(-x)}{L(x)} = c_1^* + \lim_{x \to 0^+} \frac{1}{L(x)} \int_x^{ax} f_0(t)dt = c_1^* + c_0^+ \log a.$$

$\square$

Our method also yields:

Theorem 9.4.5. *Suppose that $f_0 \in \mathcal{D}'^{[M]}(0, \infty)$ has quasiasymptotic behavior*

$$f_0(\varepsilon x) \sim c\varepsilon^\alpha L(\varepsilon) x^\alpha \quad \text{in } \mathcal{D}'^{[M]}(0, \infty) \text{ as } \varepsilon \to 0^+.$$

Then f_0 admits extensions to $\mathbb{R}$. Let $f \in \mathcal{D}'^{[M]}$ be any of such extensions with support in $[0, \infty)$. Then:

(i) *If $\alpha \notin \mathbb{Z}_-$, then there is $g \in \mathcal{D}'^{[M]}$ with $\operatorname{supp} g \subseteq \{0\}$ such that*

$$f(\varepsilon x) - g(\varepsilon x) \sim c\varepsilon^\alpha L(\varepsilon) x_+^\alpha \quad \text{in } \mathcal{D}'^{[M]} \text{ as } \varepsilon \to 0^+.$$

(ii) *If $\alpha = -k \in \mathbb{Z}_-$, then there are a function b satisfying (9.18) as $x \to 0^+$ for each $a > 0$ and an ultradistribution $g \in \mathcal{D}'^{[M]}$ with $\operatorname{supp} g \subseteq \{0\}$ such that*

$$f(\varepsilon x) = c\frac{L(\varepsilon)}{\varepsilon^k} \operatorname{Pf}\left(\frac{H(x)}{x^k}\right) + \frac{b(\varepsilon)}{\varepsilon^k}\delta^{(k-1)}(x) + g(\varepsilon x) + o\left(\frac{L(\varepsilon)}{\varepsilon^k}\right)$$

in $\mathcal{D}'^{[M]}$ as $\varepsilon \to 0^+$.

We conclude this section with an open problem.

Open problem 9.4.6. Provide structural theorems for the quasi-asymptotic behavior of (ultra-)distributions, both at infinity and the origin, for dimension 2 or higher. Note that our previous techniques, which mainly consist out of a suitable change of variables, do not work in general if $d \geqslant 2$. One possibility for solving this problem could be via an application of spherical representations, where the work of Drozhzhinov and Zav'yalov [53] is of relevance (we also refer to [147] for the ultradistributional case). Another possibility would be to characterize the dual of the space of all ultradistributions with quasiasymptotic behavior of degree α and employ similar techniques as in Chapter 6.

9.5 Extension of quasiasymptotic behavior

As an application of our structural theorems, we now discuss some other extension results for quasiasymptotics of ultradistributions. For distributions, the connection between tempered distributions and the quasiasymptotic behavior has been extensively studied [114, 115, 135, 137, 154]. The following properties are well known:

1. If $f \in \mathcal{D}'$ has quasiasymptotic behavior at infinity, then $f \in \mathcal{S}'$ and it has the same quasiasymptotic behavior in $\mathcal{S}'$.

2. If $f \in \mathcal{S}'$ has quasiasymptotic behavior at the origin in $\mathcal{D}'$, then it has the same quasiasymptotic behavior in $\mathcal{S}'$.

Our goal here is to obtain ultradistributional analogs of these results. For this, we introduce new ultradistribution spaces $\mathcal{Z}'^{[M]}$, who generalize $\mathcal{S}'$ however differ from the canonical tempered ultradistributions. They resemble (in the Roumieu case) the spaces we considered in Chapter 5, however exhibit interference in their time-frequency decay (see Open Problem 5.4.7). The spaces are defined as follows. For any $n \in \mathbb{N}$ and $\ell > 0$, $\mathcal{Z}_n^{M,\ell}$ denotes the Banach space of all $\varphi \in C^\infty$

for which the norm

$$\|\varphi\|_{\mathcal{Z}_n^{M,\ell}} := \sup_{x\in\mathbb{R},m\in\mathbb{N}} \frac{(1+|x|)^{n+m}|\varphi^{(m)}(x)|}{\ell^m M_m}$$

is finite. Then we consider the following locally convex spaces

$$\mathcal{Z}_n^{(M)} = \varprojlim_{\ell\to 0^+} \mathcal{Z}_n^{M,\ell}, \qquad \mathcal{Z}_n^{\{M\}} = \varinjlim_{\ell\to\infty} \mathcal{Z}_n^{M,\ell},$$

and finally we define

$$\mathcal{Z}^{[M]} = \varprojlim_{n\in\mathbb{N}} \mathcal{Z}_n^{[M]}.$$

The aim of this section is to show that quasiasymptotic behavior in $\mathcal{D}'^{[M]}$ naturally extends to quasiasymptotic behavior in $\mathcal{Z}'^{[M]}$. Let us first consider the case at infinity.

Theorem 9.5.1. *If $f \in \mathcal{D}'^{[M]}$ has quasiasymptotic behavior with respect to $\lambda^\alpha L(\lambda)$, with L slowly varying at infinity and $\alpha \in \mathbb{R}$, then $f \in \mathcal{Z}'^{[M]}$ and it has the same quasiasymptotic behavior in $\mathcal{Z}'^{[M]}$.*

Proof. Let $k \in \mathbb{N}$ be the smallest natural number such that $-(k+1) \leqslant \alpha$. Then by either Theorem 9.3.6 or Theorem 9.3.7 we find for some $\ell > 0$ (for any $\ell > 0$) a $C = C_\ell > 0$ such that (9.5) and (9.7) hold. Wet set $n = \lceil \alpha + 1 \rceil$. Employing Potter's estimate (8.6) (with $\varepsilon = \lambda = 1$), we find that for any $\varphi \in \mathcal{Z}^{[M]}$ and any $m \geqslant k$ we have

$$\left|\int_{-\infty}^{\infty} f_m(x)\varphi^{(m)}(x)dx\right| \leqslant C\frac{\ell^m}{M_m} \int_{\mathbb{R}} (1+|x|)^{m+n}|\varphi^{(m)}(x)|dx$$
$$\leqslant C' \|\varphi\|_{\mathcal{Z}_{n+2}^{M,h}} (h\ell)^m,$$

and as $h\ell$ may be chosen freely, it follows that this is absolutely summable over $m \geqslant k$. Consequently, $f = \sum_{m=k}^{\infty} f_m^{(m)} \in \mathcal{Z}'^{[M]}$.

For the quasiasymptotic behavior of f, the case where α is not a negative integer can be shown in a similar fashion as the sufficiency proof of Theorem 9.3.6. For $\alpha = -k \in \mathbb{Z}_-$, it is clear that we only need to treat the case $k = 1$, as the general case then automatically follows by differentiating. By Theorem 9.3.7, there exist continuous functions f_m, $m \in \mathbb{N}$, satisfying (9.12), (9.13), and (9.14) such that

$$f = f_0 + \sum_{m=1}^{\infty} f_m^{(m)}.$$

The infinite sum in the previous identity clearly has a primitive with quasiasymptotic behavior with respect to $L(\lambda)$, so that its quasi-asymptotic behavior may be extended to the whole of $\mathcal{Z}'^{[M]}$, and in turn its derivative $\sum_{m=1}^{\infty} f_m^{(m)}$ has quasiasymptotic behavior with respect to $\lambda^{-1} L(\lambda)$ in $\mathcal{Z}'^{[M]}$. By (9.12) and (9.13), f_0 has quasiasymptotic behavior with respect $\lambda^{-1} L(\lambda)$ in $\mathcal{D}'$, hence, by [135, Remark 3.1] (see also [114, Theorem 2.41, p. 158]), it has the same quasiasymptotic behavior in $\mathcal{S}'$, hence certainly also in $\mathcal{Z}'^{[M]}$. Therefore, the same also holds for f. $\square$

Let us now turn our attention to the case at the origin. The next lemma proves that the quasiasymptotic at the origin in $\mathcal{Z}'^{[M]}$ is a local property.

Lemma 9.5.2. *Let L be a slowly varying function at the origin and $\alpha \in \mathbb{R}$. Suppose $f_1, f_2 \in \mathcal{Z}'^{[M]}$ are such that for some $a > 0$, f_1 and f_2 coincide on $(-a, a)$. Suppose that $f_1(\varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) g(x)$ in $\mathcal{Z}'^{[M]}$ as $\varepsilon \to 0^+$, then, also $f_2(\varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) g(x)$ in $\mathcal{Z}'^{[M]}$.*

Proof. We only show the Beurling case; the Roumieu case can be shown analogously by employing a projective description for $\mathcal{Z}^{\{M\}}$ obtained similarly as in [24]. It suffices to show that if $f \in \mathcal{Z}'^{(M)}$ vanishes near the origin, then $f(\varepsilon x) \sim \varepsilon^N \cdot 0$ for all $N \in \mathbb{N}$. Let f be as described, then there exist $0 < R < 1$, $n \in \mathbb{N}$, $\ell, C > 0$ such that

$$|\langle f(x), \phi(x) \rangle| \leqslant C \sup_{|x| \geqslant R, m \in \mathbb{N}} \frac{|x|^{n+m} |\phi^{(m)}(x)|}{\ell^m M_m}, \qquad \phi \in \mathcal{Z}^{(M)}.$$

Taking $\phi(x) = \varepsilon^{-1} \varphi(x/\varepsilon)$ with $\varphi \in \mathcal{Z}^{(M)}$ and arbitrary $0 < \varepsilon < 1$ we have for $N \geqslant n$

$$\varepsilon^{-N} |\langle f(\varepsilon x), \varphi \rangle| \leqslant C \sup_{|x| \geqslant R, m \in \mathbb{N}} \frac{|x|^{n+m} |\varphi^{(m)}(x/\varepsilon)|}{\varepsilon^{N+m+1} \ell^m M_m}$$

$$\leqslant C R^{-N+n-1} \sup_{|x| \geqslant R/\varepsilon, m \in \mathbb{N}} \frac{|x|^{N+m+1} |\varphi^{(m)}(x)|}{\ell^m M_m} \to 0,$$

as $\varepsilon \to 0^+$. $\square$

Theorem 9.5.3. *Suppose $f \in \mathcal{Z}'^{[M]}$ has quasiasymptotic behavior in $\mathcal{D}'^{[M]}$ with respect to $\varepsilon^\alpha L(\varepsilon)$, with L slowly varying at the origin and $\alpha \in \mathbb{R}$, then f has the same quasiasymptotic behavior in $\mathcal{Z}'^{[M]}$.*

Proof. By Lemma 9.5.2 we may assume that $\operatorname{supp} f \subset [-1, 1]$. Suppose first that $\alpha \notin \mathbb{Z}_-$ and let $k \in \mathbb{N}$ be the smallest integer such that $-(k+1) < \alpha$. From Theorem 9.4.3 we find continuous functions f_m on $[-1, 1] \setminus \{0\}$, satisfying (9.23), (9.24) and (9.25). Take any $\psi \in \mathcal{Z}^{[M]}$ and decompose it as $\psi = \psi_- + \psi_c + \psi_+$ where $\operatorname{supp} \psi_- \subseteq (-\infty, -1]$, ψ_c has compact support and $\operatorname{supp} \psi_+ \subseteq [1, \infty)$. Then by the hypothesis

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \psi_c(x) \right\rangle = c_+ \left\langle x_+^\alpha, \psi_c(x) \right\rangle + c_- \left\langle x_-^\alpha, \psi_c(x) \right\rangle.$$

It suffices to show that the same limit holds for $\psi_\pm$ placed instead of ψ_c. As the two cases are symmetrical, we only look at ψ_+. It follows from (9.9), (9.25) and the Lebesgue dominated convergence theorem that for any $m \geqslant k$,

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f_m^{(m)}(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \psi_+(x) \right\rangle$$

$$= \lim_{\varepsilon \to 0^+} \int_1^{1/\varepsilon} \frac{L(\varepsilon x)}{L(\varepsilon)} \left(\frac{f_m(\varepsilon x)}{(\varepsilon x)^{\alpha+m} L(\varepsilon x)} \right) x^{\alpha+m} \psi_+^{(m)}(x) dx$$

$$= c_m^+ \int_0^\infty x^\alpha \psi_+(x) dx.$$

Then another application of dominated convergence shows that

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \psi_+(x) \right\rangle = c_+ \left\langle x_+^\alpha, \psi_+(x) \right\rangle.$$

This shows the case for $\alpha \notin \mathbb{Z}_-$. The case of negative integral degree can then be done as in the proof of [137, Theorem 6.1]. $\square$

We have thus shown that, similar to the distributional case, there are extension principles for the quasiasymptotic behavior of ultradistributions. However, the space of extension differs to that of the tempered ultradistributions. As quasiasymptotic behavior over the

Gelfand-Shilov spaces is of great significance for applications, its theoretical study becomes of interest to us. In Chapter 11, we will determine when a tempered ultradistributions has quasiasymptotic behavior via its Laplace transform. Structural theorems are at the moment of writing still lacking. Whence the following interesting open problem.

Open problem 9.5.4. Provide structural theorems for the quasiasymptotic behavior, both at infinity and the origin, of tempered ultradistributions.

Chapter 10

The moment asymptotic expansion

10.1 Introduction

Another important approach to asymptotic behavior related to dilation is the so-called *moment asymptotic expansion (MAE)*, whose properties have been extensively investigated by Estrada and Kanwal [57, 58]. Some recent contributions can be found in [124, 153]. A generalized function f is said to satisfy the MAE if there is a certain multi-sequence $\{\mu_\alpha\}_{\alpha\in\mathbb{N}^d}$, called the moments of f, such that the following asymptotic expansion holds

$$f(\lambda x) \sim \sum_{\alpha\in\mathbb{N}^d} \frac{(-1)^{|\alpha|}\mu_\alpha\delta^{(\alpha)}(x)}{\alpha!\lambda^{|\alpha|+d}}, \qquad \lambda \to \infty. \qquad (10.1)$$

As is shown in the monograph [58], the MAE supplies a unified approach to several aspects of asymptotic analysis and its applications. Interestingly, Estrada characterized [56] the largest space of distributions where the MAE holds as the dual of the space of so-called GLS symbols [68]. We will consider in this chapter the MAE for ultradistributions.

The chapter is organized as follows. We start in Section 10.2 with a discussion on the structure of asymptotic boundedness for ultradistributions. Using the same techniques as in Chapter 9, we

obtain characterizations for both S-asymptotic and quasiasymptotic boundedness. In Section 10.3 we provide a counterpart of Estrada's full characterization in the one-dimensional case. We shall also study a uniform version of (10.1) in Section 10.4, which we call the UMAE. Our considerations naturally lead us to introduce the ultradistribution spaces $\mathcal{K}'^{[M]}(\mathbb{R}^d)$ and $\mathcal{K}'^{[M]}_{[N]}(\mathbb{R}^d)$, which are intimately connected with the MAE and UMAE, respectively. We note that in even dimension our space $\mathcal{K}'^{[M]}_{[N]}(\mathbb{R}^{2d})$ arises as the dual of one of the spaces of symbols of 'infinite order' pseudo-differential operators from [118].

10.2 Asymptotic boundedness

We provide in this section structural theorems for the bounded variants of the asymptotic behavior considered in Chapter 9.

10.2.1 S-asymptotic boundedness

We aim to study the structure of those ultradistributions that satisfy

$$f(x + h) = O\left(\omega(h)\right), \quad h \in W, \quad \text{in } \mathcal{D}'^{[M]}(\mathbb{R}^d), \tag{10.2}$$

where $W \subseteq \mathbb{R}^d$ is simply an unbounded set and ω is a positive function. Explicitly this means that for each test function $\varphi \in \mathcal{D}^{[M]}(\mathbb{R}^d)$,

$$\sup_{h \in W} \frac{\langle f(x + h), \varphi(x) \rangle}{\omega(h)} = \sup_{h \in W} \frac{(f * \check{\varphi})(h)}{\omega(h)} < \infty. \tag{10.3}$$

We will impose the following mild regularity condition on the gauge function ω,

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(\cdot + x)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d). \tag{10.4}$$

To find structural theorems for the behavior (10.2) one may follow the proof of Theorem 9.2.2 and apply the structural theorems found for the space $\mathcal{B}'^{[M]}_\omega$, i.e. Theorem 6.4.12. However, we present here an alternative proof, based on a technique by Gómez-Collado that she applied to obtain various characterizations of the space of bounded ultradistributions in [63].

Theorem 10.2.1. *Let $W \subset \mathbb{R}^d$ be an unbounded set and let ω be a positive measurable function on $\mathbb{R}^d$ that satisfies (10.4). Suppose $(M.1)$, $(M.2)'$, and $(M.3)'$ hold. Then, an ultradistribution $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ satisfies (10.2) if and only if for each $R > 0$ there are continuous functions $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ defined on W_R such that for some $\ell > 0$ (for each $\ell > 0$) there exists $C_\ell > 0$ for which the bounds*

$$|f_\alpha(x)| \leqslant C_\ell \frac{\ell^{|\alpha|}}{M_\alpha} \omega(x), \qquad x \in W_R, \ \alpha \in \mathbb{N}^d, \tag{10.5}$$

hold and

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \qquad in \ W_R. \tag{10.6}$$

Proof. The sufficiency of the conditions is easily verified. To show the necessity, similar as in the proof of Theorem 9.2.2, we may assume $W = \mathbb{R}^d$. Also, by Lemma 6.4.11, it suffices to show (10.5) and (10.6) hold for measurable functions f_α. Hence, suppose $\{f(x+h)/\omega(h) : h \in \mathbb{R}^d\}$ is a bounded subset of $\mathcal{D}'^{[M]}(\mathbb{R}^d)$. Let $\psi \in \mathcal{D}^{[M]}_{[-1,1]^d}$ be such that $\sum_{n \in \mathbb{Z}^d} \psi(x - n) = 1$ for each $x \in \mathbb{R}^d$. We have that $\{\psi f(\cdot + n)/\omega(n) : n \in \mathbb{Z}^d\}$ is now a bounded set in the space $\mathcal{E}'^{[M]}(\mathbb{R}^d)$. Using $(M.2)'$, we obtain the existence of some $(\ell_p) \in [\mathfrak{R}]$ such that for some $C > 0$ and all $n \in \mathbb{Z}^d$ and $\phi \in \mathcal{E}^{[M]}(\mathbb{R}^d)$

$$|\langle f, T_n \psi \phi \rangle| \leqslant C \sum_{\alpha \in \mathbb{N}^d} \frac{\omega(n)}{L_\alpha M_\alpha} \int_{[-1,1]^d} |\phi^{(\alpha)}(x)| dx, \tag{10.7}$$

where in the Roumieu case we have used the projective description of $\mathcal{E}^{\{M\}}(\mathbb{R}^d)$. We consider the Banach space X of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_X = \sum_{\alpha \in \mathbb{N}^d} \int_{\mathbb{R}^d} |\varphi^{(\alpha)}(x)| \frac{\omega(x)}{L_\alpha M_\alpha} dx < \infty.$$

Let $\varphi \in \mathcal{D}^{[M]}(\mathbb{R}^d)$ be arbitrary. Applying (10.7) to each $\phi(x) = \varphi(x + n)$ and using the hypothesis (10.4), we obtain, with $C' = C \sup_{x \in \mathbb{R}^d, \ y \in [-1,1]^d} \omega(x + y)/\omega(x)$,

$$|\langle f, \varphi \rangle| \leqslant \sum_{n \in \mathbb{Z}^d} |\langle f(x), \psi(x - n)\varphi(x) \rangle| \leqslant 3^d C' \|\varphi\|_X.$$

By using the Hahn-Banach theorem, we may then extend f to an element of X'. Embedding X into $L^1(\mathbb{N}^d \times \mathbb{R}^d, d\mu)$ via the isometry $j(\varphi)(\alpha, x) = (-1)^{|\alpha|}\varphi^{(\alpha)}(x)$, where the measure is given by $d\mu = \omega(x)/(L_\alpha M_\alpha)d\alpha dx$ with $d\alpha$ the natural counting measure on $\mathbb{N}^d$, we can apply the Hahn-Banach theorem to get the representation (10.6) with measurable functions f_α on $\mathbb{R}^d$ that satisfy bounds $|f_\alpha(x)| \leqslant C''\omega(x)/(L_\alpha M_\alpha)$. This yields already the result in the Beurling case. In the Roumieu case we finally employ Lemma 4.2.12(ii) to obtain the bounds (10.5) for each $\ell > 0$ and some $C_\ell > 0$. $\qquad\square$

In applications it is very useful to combine Theorem 10.2.1 with the ensuing proposition, which provides conditions under which one might essentially apply Theorem 10.2.1 with a function ω that is just defined on the set W.

Proposition 10.2.2. *Let $W \subset \mathbb{R}^d$ be a closed convex set. Any positive function ω on W satisfying*

$$(\forall R > 0) \quad \sup_{\substack{x,\, x+h \in W \\ |h| \leqslant R}} \frac{\omega(x+h)}{\omega(x)} < \infty \qquad (10.8)$$

can be extended to a positive function on $\mathbb{R}^d$ satisfying (10.4). *In addition, if ω is measurable (or continuous), the extension can be chosen measurable (or continuous) as well.*

Proof. For any $x \in \mathbb{R}^d$ we denote by $\widetilde{x} \in W$ the (unique in view of convexity) closest point to x in W. Then, we set $\widetilde{\omega}(x) := \omega(\widetilde{x})$. Since $x \mapsto \widetilde{x}$ is continuous, $\widetilde{\omega}$ inherits measurability or continuity if ω has the property. We now verify (10.4) for $\widetilde{\omega}$. Let $R > 0$ and let C_R be an upper bound for $\omega(t+y)/\omega(y)$, where $y, t+y \in W$ and $t \in \overline{B}(0, R)$. Let $x \in \mathbb{R}^d$ and $h \in \overline{B}(0, R)$ be arbitrary. Consider the points $x, x+h, \widetilde{x}$ and $\widetilde{x+h}$. By the obtuse angle criterion, the angles defined by the line segments $[x, \widetilde{x}, \widetilde{x+h}]$ and $[x+h, \widetilde{x+h}, \widetilde{x}]$ are at least $\pi/2$, whence $|\widetilde{x} - \widetilde{x+h}| \leqslant |x - (x+h)| \leqslant R$. It then follows that $\widetilde{\omega}(x+h) \leqslant C_R\widetilde{\omega}(x)$, as required. $\qquad\square$

If the weight sequence satisfies stronger assumption, one can drop any regularity assumption on ω, as stated in the next result.

Theorem 10.2.3. *Let $W \subset \mathbb{R}^d$ be an unbounded set and let ω be a positive function on W. Suppose that $(M.1)$, $(M.2)$, and $(M.3)$ hold. An ultradistribution $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ satisfies (10.2) if and only if for each $R > 0$ there are continuous functions $\{f_\alpha\}_{\alpha \in \mathbb{N}^d}$ defined on W_R such that for some $\ell > 0$ (for each $\ell > 0$) there exists $C_\ell > 0$ such that*

$$|f_\alpha(x+h)| \leqslant C_\ell \frac{\ell^{|\alpha|}}{M_\alpha} \omega(x), \qquad x \in W, \ |h| < R, \ \alpha \in \mathbb{N}^d, \qquad (10.9)$$

and the representation (10.6) holds.

Proof. The proof is similar to that of [114, Theorem 1.10, p. 46], but we provide some simplifications. The converse is easy to show, so we concentrate on showing the necessity of the conditions for the S-asymptotic boundedness relation (10.2). Let $R > 0$. We consider the linear mapping $A : \mathcal{D}^{[M]}(\mathbb{R}^d) \to X$, with values in the Banach space $X = \{g : W \to \mathbb{C} : \sup_{x \in W} |g(x)|/\omega(x) < \infty\}$, given by $A\varphi = f * \varphi$. It follows from the closed graph theorem that A is continuous. Consequently, we obtain from the Banach-Steinhaus theorem the existence of $(\ell_p) \in [\mathfrak{R}]$ such that $A \in L(\mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,2R)}, X)$ and $f * \varphi \in X$ for each $\varphi \in \mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,2R)}$. Since for each $\varphi \in \mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,R)}$ the set $\{T_x\varphi : |x| \leqslant R\}$ is compact in $\mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,2R)}$, we conclude that for any such a φ the function $f * \varphi$ is continuous on W_R and

$$\sup_{h \in W, \, |x| < R} \frac{(f * \varphi)(x+h)}{\omega(h)} < \infty.$$

We now employ the parametrix method. As shown in [79, p. 199], there is an ultradifferential operator $P(D)$ of class $[M]$ that admits a $\mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,R)}$-parametrix, namely, for which there are $\chi \in \mathcal{D}^{[M]}_{\bar{B}(0,R)}$ and $\varphi \in \mathcal{D}^{M_{\ell_p},1}_{\bar{B}(0,R)}$ such that $\delta = P(D)\varphi + \chi$. Setting $f_0 = f * \chi$ and $g = f * \varphi$, we obtain the decomposition $f = P(D)g + f_0$, which in particular establishes the representation (10.6) with functions f_α satisfying the bounds (10.9). $\qquad \square$

10.2.2   Quasiasymptotic boundedness

Similar as in Chapter 9, our results on the S-asymptotic boundedness of ultradistributions may be used to obtain structural theorems for ultradistributions being quasiasymptotically bounded in dimension 1. Let ρ be a positive function defined on an interval of the form $[\lambda_0, \infty)$. We are interested in the relation

$$f(\lambda x) = O(\rho(\lambda)), \qquad \lambda \to \infty$$

in ultradistribution spaces. The analog of the condition (10.8) for a function ρ in this multiplicative setting is being O-regularly varying (at infinity) [10, p. 65]. The latter means (cf. [10, Theorem 2.0.4, p. 64]) that ρ is measurable and for each $R > 1$

$$\limsup_{x \to \infty} \sup_{\lambda \in [R^{-1}, R]} \frac{\rho(\lambda x)}{\rho(\lambda)} < \infty.$$

The next proposition can be established with the aid of Theorem 10.2.1 and Theorem 10.2.3 via an exponential change of variables as in the proof of Lemma 9.3.5; we leave its verification to the reader.

Proposition 10.2.4. *Let $f \in \mathcal{D}'^{[M]}(\mathbb{R})$ and ρ be a positive function. Suppose that*

$$f(\lambda x) = O(\rho(\lambda)), \qquad as \ \lambda \to \infty$$

in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$.

(i) *If $(M.1)$, $(M.2)'$, and $(M.3)'$ hold and ρ is O-regularly varying at infinity, then there are continuous functions f_m and $x_0 > 0$ such that*

$$f = \sum_{m=0}^{\infty} f_m^{(m)} \qquad on \ \mathbb{R} \setminus [-x_0, x_0] \qquad (10.10)$$

and for some $\ell > 0$ (for any $\ell > 0$) there is $C_\ell > 0$ such that

$$|f_m(x)| \leqslant C_\ell \frac{\ell^m}{M_m} |x|^m \rho(|x|), \qquad |x| > x_0, \ m \in \mathbb{N}. \qquad (10.11)$$

(*ii*) *If* (*M*.1), (*M*.2), *and* (*M*.3) *hold, for each* $R > 1$ *one can find* x_0 *and continuous functions such that* f *has the representation* (10.10), *where the* f_m *satisfy the bounds*

$$|f_m(ax)| \leqslant C_\ell \frac{\ell^m}{M_m} |x|^m \rho(|x|), \quad |x| > x_0, a \in [R^{-1}, R] \quad (10.12)$$

for all $m \in \mathbb{N}$ *and some* $\ell > 0$ *(for any* $\ell > 0$*).*

Remark 10.2.5. Clearly, (10.11) implies (10.12) for an *O*-regularly varying function ρ. Assume (*M*.1), (*M*.2)′, and (*M*.3)′ hold. Notice the representations (10.10) with bounds (10.12) are also sufficient to yield $f(\lambda x) = O(\rho(\lambda))$ as $\lambda \to \infty$ in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$, so that the converses of both parts (*i*) and (*ii*) of Proposition 10.2.4 are valid.

For the remainder of this section we are interested in describing quasiasymptotic boundedness in the full space $\mathcal{D}'^{[M]}(\mathbb{R})$. For it, we need to impose stronger variation assumptions on the gauge function ρ. We call a positive measurable function *O-slowly varying at infintiy* if for each $\varepsilon > 0$ there are $C_\varepsilon, c_\varepsilon, R_\varepsilon > 0$ such that

$$\frac{c_\varepsilon}{\lambda^\varepsilon} \leqslant \frac{L(\lambda x)}{L(x)} \leqslant C_\varepsilon \lambda^\varepsilon, \qquad \lambda \geqslant 1, \ x > R_\varepsilon. \qquad (10.13)$$

In the terminology from [10] this means that the upper and lower Matuszewska indices of L are both equal to 0. Thus, a function of the form $\rho(\lambda) = \lambda^q L(\lambda)$ is an *O-regularly varying function* with both upper and lower Matuszewska indices equal to $q \in \mathbb{R}$.

Employing the same technique[1] as in Chapter 9, where we simply need to exchange Lemma 9.3.5 with Proposition 10.2.4, leads to two ensuing structural theorems for quasiasymptotic boundedness.

Theorem 10.2.6. *Assume* (*M*.1), (*M*.2)′, *and* (*M*.3)′ *hold. Let* $f \in \mathcal{D}'^{[M]}(\mathbb{R})$, $\alpha \in \mathbb{R}$, *and let* $L \in L^\infty_{loc}[0, \infty)$ *be O-slowly varying*

---

[1]One still needs an *O*-version of Lemma 9.3.3; however, careful inspection in the arguments given in [114, Subsection 2.10.2 and Proposition 2.17] shows that having the inequalities (10.13) is all one needs to establish the validity of such an *O*-version.

at infinity. Let k be the smallest positive integer such that $-k \leqslant \alpha$. Then,

$$f(\lambda x) = O\left(\lambda^\alpha L(\lambda)\right) \qquad as \ \lambda \to \infty \ in \ \mathcal{D}'^{[M]}(\mathbb{R}) \qquad (10.14)$$

holds if and only if there are continuous functions f_m on $\mathbb{R}$ such that

$$f = \sum_{m=k-1}^{\infty} f_m^{(m)},$$

for some $\ell > 0$ (for any $\ell > 0$) there exists $C_\ell > 0$ such that

$$|f_m(x)| \leqslant C_\ell \frac{\ell^m}{M_m}(1 + |x|)^{\alpha+m} L(|x|), \qquad m \geqslant k - 1, \qquad (10.15)$$

and additionally (only) when $\alpha = -k$

$$\int_{-x}^{x} f_{k-1}(x)dx = O\left(L(x)\right), \qquad x \to \infty. \qquad (10.16)$$

A function L is O-regularly varying at the origin if $L(1/x)$ is O-regularly varying at infinity.

Theorem 10.2.7. *Assume $(M.1)$, $(M.2)'$, and $(M.3)'$. Let $f \in \mathcal{D}'^{[M]}(\mathbb{R})$, $\alpha \in \mathbb{R}$, and let L be O-slowly varying at the origin. Let k be the smallest positive integer such that $-k \leqslant \alpha$. Then, we have that*

$$f(\varepsilon x) = O\left(\varepsilon^\alpha L(\varepsilon)\right) \qquad as \ \varepsilon \to 0^+ \ in \ \mathcal{D}'^{[M]}(\mathbb{R}) \qquad (10.17)$$

holds if and only if there exist $x_0 > 0$ and continuous functions F and f_m on $[-x_0, x_0] \setminus \{0\}$, $m \geqslant k$, such that

$$f(x) = F^{(k)} + \sum_{m=k}^{\infty} f_m^{(m)}, \qquad on \ (-x_0, x_0),$$

for some $\ell > 0$ (for any $\ell > 0$) there exists $C_\ell > 0$ such that

$$|f_m(x)| \leqslant C_\ell \frac{\ell^m}{M_m}|x|^{\alpha+m} L(|x|), \qquad 0 < |x| \leqslant x_0,$$

for all $m \geqslant k$, and $F = 0$ when $\alpha > -k$ while if $\alpha = -k$ the function F satisfies, for each $a > 0$, the bounds

$$F(ax) - F(-x) = O_a(L(x)), \qquad x \to 0^+.$$

We end this section with a brief discussion on the extension properties of quasiasymptotic boundedness. Let $\mathcal{Z}'^{[M]}(\mathbb{R})$ be the space of ultradifferentiable functions introduced in Section 9.3.4. Without much alteration of the proofs shown in Section 9.3.4 one finds.

Proposition 10.2.8. *Assume* $(M.1)$, $(M.2)'$, $(M.3)'$.

(i) *If* (10.14) *holds with an O-regularly varying function at infinity L, then $f \in \mathcal{Z}'^{[M]}(\mathbb{R})$ and the quasiasymptotic boundedness relation* (10.14) *actually holds true in* $\mathcal{Z}'^{[M]}(\mathbb{R})$.

(ii) *If $f \in \mathcal{Z}'^{[M]}(\mathbb{R})$ and* (10.17) *holds with an O-regularly varying function at the origin L, then* (10.17) *is actually valid in* $\mathcal{Z}'^{[M]}(\mathbb{R})$.

We now obtain the following characterization of $\mathcal{Z}'^{[M]}(\mathbb{R})$.

Theorem 10.2.9. *An ultradistribution $f \in \mathcal{D}'^{[M]}(\mathbb{R})$ belongs to the space $\mathcal{Z}'^{[M]}(\mathbb{R})$ if and only if there is some $\alpha \in \mathbb{R}$ such that $f(\lambda x) = O(\lambda^\alpha)$ as $\lambda \to \infty$ in $\mathcal{D}'^{[M]}(\mathbb{R})$.*

Proof. As any constant function is O-slowly varying, sufficiency follows immediately from Proposition 10.2.8. Suppose now that $f \in \mathcal{Z}'^{[M]}(\mathbb{R})$. Then there is some $q \in \mathbb{N}$ such that $f \in (\mathcal{Z}_{q+1}^{[M]})'$. In particular, there is an $\ell > 0$ such that (for any $\ell > 0$ we have that) for some $C = C_\ell > 0$, any $R > 1$ and all $\varphi \in \mathcal{D}^{[M]}(B(0, R))$:

$$
\begin{aligned}
|\langle f(\lambda x), \varphi(x) \rangle| &\leqslant \frac{C}{\lambda} \|\varphi(x/\lambda)\|_{\mathcal{Z}_{q+1}^{M,\ell}} \\
&= \frac{1}{\lambda} \sup_{x \in \mathbb{R}, m \in \mathbb{N}} \frac{(1+|x|)^{q+1+m}|\varphi^{(m)}(x/\lambda)|}{(\lambda\ell)^m M_m} \\
&\leqslant C(2R)^{q+1} \|\varphi\|_{\mathcal{D}^{M,\ell/2R}} \lambda^q
\end{aligned}
$$

where in the Roumieu case ℓ is fixed by φ. Whence we may conclude that $f(\lambda x) = O(\lambda^q)$ as $\lambda \to \infty$ in $\mathcal{D}'^{[M]}(\mathbb{R})$. $\qquad\square$

10.3 The moment asymptotic expansion

This section is devoted to the study of the moment asymptotic expansion (10.1), which in general we interpret in the sense of the following definition.

Definition 10.3.1. Let $\mathcal{X}$ be a lcHs of smooth functions provided with continuous actions of the dilation operators and the Dirac delta and all its partial derivatives. An element $f \in \mathcal{X}'$ is said to satisfy the *moment asymptotic expansion (MAE)* in $\mathcal{X}'$ if there are $\mu_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^d$, called its *moments*, such that for any $\varphi \in \mathcal{X}$ and $k \in \mathbb{N}$ we have

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{|\alpha| < k} \frac{\mu_\alpha \varphi^{(\alpha)}(0)}{\alpha! \lambda^{|\alpha|+d}} + O\left(\frac{1}{\lambda^{k+d}}\right), \qquad \lambda \to \infty. \quad (10.18)$$

Similarly as in the case of compactly supported distributions [57, 58] or analytic functionals [124], one can show that any compactly supported ultradistribution satisfies the MAE in $\mathcal{E}'^{[M]}(\mathbb{R}^d)$ (we will actually state a stronger result in Proposition 10.4.3 below). Naturally, as in the distributional case, we expect the MAE to be also valid in larger ultradistribution spaces. In dimension 1, Estrada gave in [56, Theorem 7.1] (cf. [58]) a full characterization of the largest distribution space where the moment asymptotic expansion holds; in fact, he showed that $f \in \mathcal{D}'(\mathbb{R})$ satisfies the MAE (in $\mathcal{D}'(\mathbb{R})$) if and only if $f \in \mathcal{K}'(\mathbb{R})$ (and the MAE holds in this space), where $\mathcal{K}'(\mathbb{R})$ is the dual of the so-called space of GLS symbols of pseudodifferential operators [68]. One of our goals here is to give an ultradistributional counterpart of Estrada's result.

We start by introducing an ultradistributional version of $\mathcal{K}(\mathbb{R}^d)$. For each $q \in \mathbb{N}$ and $\ell > 0$ we denote by $\mathcal{K}_q^{M,\ell}(\mathbb{R}^d)$ the Banach space of all smooth functions φ for which the norm

$$\|\varphi\|_{\mathcal{K}_q^{M,\ell}} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{(1 + |x|)^{|\alpha|-q} |\varphi^{(\alpha)}(x)|}{\ell^{|\alpha|} M_\alpha}$$

is finite. From this we construct the spaces

$$\mathcal{K}_q^{(M)}(\mathbb{R}^d) = \varprojlim_{\ell \to 0^+} \mathcal{K}_q^{M,\ell}(\mathbb{R}^d), \qquad \mathcal{K}_q^{\{M\}}(\mathbb{R}^d) = \varinjlim_{\ell \to \infty} \mathcal{K}_q^{M,\ell}(\mathbb{R}^d),$$

and finally the test function space

$$\mathcal{K}^{[M]}(\mathbb{R}^d) = \varinjlim_{q \in \mathbb{N}} \mathcal{K}_q^{[M]}(\mathbb{R}^d).$$

It should be noticed that this is space is never trivial; in fact, $\mathcal{K}^{[M]}(\mathbb{R}^d)$ contains the space of polynomials.

Our first important result in this subsection asserts that the elements of $\mathcal{K}'^{[M]}(\mathbb{R}^d)$ automatically satisfy the MAE. Interestingly, no restriction on the weight sequence M is needed to achieve this.

Theorem 10.3.2. *Any element $f \in \mathcal{K}'^{[M]}(\mathbb{R}^d)$ satisfies the MAE in $\mathcal{K}'^{[M]}(\mathbb{R}^d)$ and its moments are exactly $\mu_\alpha = \langle f(x), x^\alpha \rangle$, $\alpha \in \mathbb{N}^d$.*

Proof. Let $f \in \mathcal{K}'^{[M]}(\mathbb{R}^d)$. We keep $\lambda \geqslant 1$ and fix $k \in \mathbb{N}$. Take any arbitrary $\varphi \in \mathcal{K}_q^{[M]}(\mathbb{R}^d)$, where we may assume $q \geqslant k$. Consider the $(k-1)$th order Taylor polynomial of φ at the origin, that is, $\varphi_k(x) := \sum_{|\alpha| < k} \varphi^{(\alpha)}(0) x^\alpha / \alpha!$. Since $\varphi_k \in \mathcal{K}^{[M]}(\mathbb{R}^d)$,

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{|\alpha| < k} \frac{\mu_\alpha \varphi^{(\alpha)}(0)}{\alpha! \lambda^{|\alpha|+d}} + \langle f(\lambda x), \varphi(x) - \varphi_k(x) \rangle .$$

Thus, we need to show $\langle f(\lambda x), \varphi(x) - \varphi_k(x) \rangle = O(1/\lambda^{k+d})$. This bound does not require any uniformity in k; therefore, we may just assume that $\varphi^{(\alpha)}(0) = 0$ for any $|\alpha| < k$ so that our problem reduces to estimate $|\langle f(\lambda x), \varphi(x) \rangle|$. There exists some $\ell = \ell_f > 0$ (some $\ell = \ell_\phi > 0$) such that $\varphi \in \mathcal{K}_q^{M,\ell}(\mathbb{R}^d)$ and

$$|\langle f(\lambda x), \varphi(x) \rangle| \leqslant \frac{\|f\|_{(\mathcal{K}_q^{M,\ell}(\mathbb{R}^d))'}}{\lambda^d} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{(1+|x|)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|}\ell^{|\alpha|} M_\alpha} .$$

If $|\alpha| \geqslant q$, we have

$$\sup_{x \in \mathbb{R}^d} \frac{(1+|x|)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|}\ell^{|\alpha|} M_\alpha}$$

$$= \frac{1}{\lambda^q} \sup_{x \in \mathbb{R}^d} \left(\frac{1+|x|}{\lambda+|x|} \right)^{|\alpha|-q} \frac{(1+|x|/\lambda)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\ell^{|\alpha|} M_\alpha}$$

$$\leqslant \frac{\|\varphi\|_{\mathcal{K}_q^{M,\ell}}}{\lambda^k} .$$

We further consider $|\alpha| < q$. When $|x| \geqslant \lambda$, obviously

$$\frac{1}{2} \leqslant \frac{1+|x|}{\lambda+|x|}$$

and we obtain

$$\sup_{|x|\geqslant\lambda} \frac{(1+|x|)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|}\ell^{|\alpha|}M_\alpha} \leqslant 2^q \frac{\|\varphi\|_{\mathcal{K}_q^{M,\ell}}}{\lambda^k}$$

We are left with the case $|x| \leqslant \lambda$ and $|\alpha| < q$. If $k \leqslant |\alpha| < q$ we get

$$\sup_{|x|\leqslant\lambda} \frac{(1+|x|)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|}\ell^{|\alpha|}M_\alpha} \leqslant \frac{1}{\lambda^k} \sup_{|x|\leqslant\lambda} \frac{|\varphi^{(\alpha)}(x/\lambda)|}{\ell^{|\alpha|}M_\alpha} \leqslant 2^{q-k} \frac{\|\varphi\|_{\mathcal{K}_q^{M,\ell}}}{\lambda^k}.$$

Finally, for $|\alpha| < k$, the Taylor formula yields

$$\sup_{|x|\leqslant\lambda} \frac{(1+|x|)^{|\alpha|-q}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|}\ell^{|\alpha|}M_\alpha}$$

$$\leqslant \sup_{|x|\leqslant\lambda} \frac{(1+|x|)^{|\alpha|-q}}{\lambda^{|\alpha|}\ell^{|\alpha|}M_\alpha} \sum_{\substack{\alpha\leqslant\beta \\ |\beta|=k}} \frac{|\varphi^{(\beta)}(\xi_{x/\lambda})|}{(\beta-\alpha)!} \frac{|x|^{|\beta-\alpha|}}{\lambda^{|\beta-\alpha|}}$$

$$\leqslant 2^q \frac{C_{\ell,k}}{\lambda^k} \|\varphi\|_{\mathcal{K}_p^{M,\ell}}.$$

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\square$

Next, we describe the structure of the elements of $\mathcal{K}'^{[M]}(\mathbb{R}^d)$. We first need the ensuing lemma.

Lemma 10.3.3. *Let $\omega : \mathbb{R}^d \to \mathbb{R}_+$ be such that $\sup_{x\in\mathbb{R}^d} \omega(x+\cdot)/\omega(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. There exists a $k \in \mathbb{Z}_+$, independent of ω, such that for any $p \in [1, \infty]$ there is a $C = C_p > 0$ so that for each $f \in L^p_{1/\omega}$ there exist $f_j \in C(\mathbb{R}^d)$, $j \in \{0, 1\}$, such that*

$$f = \Delta^k f_1 + f_0$$

and

$$|f_j(x)| \leqslant C \|f/\omega\|_{L^p} \omega(x), \qquad x \in \mathbb{R}^d, \ j \in \{0, 1\}.$$

Proof. Using Schwartz's parametrix method [125] we find a $k \in \mathbb{Z}_+$ such that $\delta = \Delta^k \chi_1 + \chi_0$ where $\chi_0 \in \mathcal{D}(\mathbb{R}^d)$ and χ_1 is a compactly supported continuous function. Take any symmetric compact subset $K \Subset \mathbb{R}^d$ containing $\operatorname{supp}\chi_0$ and $\operatorname{supp}\chi_1$ and let $C_K > 0$ be such

that $\sup_{y \in K} \omega(x + y) \leqslant C_K \omega(x)$. Suppose f is a function for which $f/\omega \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty]$ and let $q = p/(p - 1)$. By Hölder's inequality we have

$$\frac{|f * \chi_j(t)|}{\omega(t)} \leqslant C \int_K |\chi_j(x)| \frac{|f(t - x)|}{\omega(t - x)} dx \leqslant C_K \|\chi_j\|_{L^q} \|f/\omega\|_{L^p}$$

for $j \in \{0, 1\}$. The claim now follows by setting $f_j = f * \chi_j$ and $C_p = C_K \max(\|\chi_0\|_{L^q}, \|\chi_1\|_{L^q})$. $\square$

Going from here we obtain the following structural theorem. We point out that the converse of Proposition 10.3.4 holds unconditionally, that is, without having to impose any assumption on M.

Proposition 10.3.4. *Let M satisfy $(M.1)$ and $(M.2)'$. Let $f \in \mathcal{K}'^{[M]}(\mathbb{R}^d)$. Then, given any $q \in \mathbb{N}$ one can find a multi-sequence of continuous functions $f_\alpha = f_{q,\alpha} \in C(\mathbb{R}^d)$ such that*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \tag{10.19}$$

and for some $\ell > 0$ (for any $\ell > 0$) there is $C = C_{q,\ell} > 0$ such that

$$|f_\alpha(x)| \leqslant C \frac{\ell^{|\alpha|}}{M_\alpha} (1 + |x|)^{|\alpha| - q}, \qquad x \in \mathbb{R}^d, \ \alpha \in \mathbb{N}^d. \tag{10.20}$$

Proof. We only show the Roumieu case, the proof of the Beurling case is standard. For any $r \in [1, \infty)$ and some $\ell, q > 0$ we denote by $\mathcal{K}_{L^r,q}^{M,\ell}(\mathbb{R}^d)$ the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{K}_{L^r,q}^{M,\ell}} = \left(\sum_{\alpha \in \mathbb{N}^d} \left(\frac{\|(1 + |\cdot|)^{|\alpha| - q} \varphi^{(\alpha)}\|_{L^r}}{\ell^{|\alpha|} M_\alpha} \right)^r \right)^{1/r} < \infty.$$

We also write $\mathcal{K}_{L^\infty,q}^{M,\ell}(\mathbb{R}^d) = \mathcal{K}_q^{M,\ell}(\mathbb{R}^d)$. For any $r \in [1, \infty]$ and $j \in \mathbb{Z}_+$ we put $X_{r,j} = \mathcal{K}_{L^r,j}^{M,j}$. Then, using Jensen's inequality and Sobolev's theorem, one may easily verify that for any $r \in [1, \infty]$

$$\mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{j \to \infty} X_{r,j}$$

as locally convex spaces. Next, for any $r \in (1, \infty)$ and $j \in \mathbb{Z}_+$ we consider the Banach space $Y_{r,j}$ of all sequences of functions $(\varphi_\alpha)_{\alpha \in \mathbb{N}^d}$ such that

$$\|(\varphi_\alpha)\|_{Y_{r,j}} = \left(\sum_{\alpha \in \mathbb{N}^d} \left(\frac{\|(1 + |\cdot|)^{|\alpha|-j} \varphi_\alpha\|_{L^r}}{\ell^{|\alpha|} M_\alpha} \right)^r \right)^{1/r} < \infty.$$

Note that both $X_{r,j}$ and $Y_{r,j}$ are reflexive. The mapping $\rho_{r,j} : X_{r,j} \to Y_{r,j}, \varphi \mapsto ((-1)^{|\alpha|} \varphi^{(\alpha)})_\alpha$ is a topological embedding. We set $Z_{r,j} = Y_{r,j}/\rho_{r,j}(X_{r,j})$, then $Z_{r,j}$ is a reflexive Banach space. We denote by $\pi_j : Y_{r,j} \to Z_{r,j}$ the quotient mapping. The natural linking mappings $Z_{r,j} \to Z_{r,j+1}$ are injective since $\rho_{r,j+1}(X_{r,j+1}) \cap Y_{r,j} = \rho_{r,j}(X_{r,j})$. Consider the following injective inductive sequence of short topologically exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_{r,1} & \xrightarrow{\rho_{r,1}} & Y_{r,1} & \xrightarrow{\pi_{r,1}} & Z_{r,1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{r,2} & \xrightarrow{\rho_{r,2}} & Y_{r,2} & \xrightarrow{\pi_{r,2}} & Z_{r,2} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

The linking mappings of the inductive spectra $(X_{r,j})_{j \in \mathbb{Z}_+}$, $(Y_{r,j})_{j \in \mathbb{Z}_+}$ and $(Z_{r,j})_{j \in \mathbb{Z}_+}$ are weakly compact as continuous linear mappings between reflexive Banach spaces. In particular, these inductive spectra are regular [80, Lemma 3]. One may easily verify the embedding $X_{\infty,j} \to X_{\infty,j+1}$ is compact for any $j \in \mathbb{Z}_+$, whence $X = \varinjlim_{j \in \mathbb{Z}_+} X_{\infty,j}$ is a (DFS)-space. Consequently, X is Montel. Applying the dual Mittag Leffler theorem 2.2.2, we have that $\rho_r = \varinjlim_{j \in \mathbb{Z}_+} \rho_{r,j} : X \to Y_r = \varinjlim_{j \in \mathbb{Z}_+} Y_{r,j}$ is a topological embedding. From here the structure of $\mathcal{K}'^{\{M\}}(\mathbb{R}^d)$ follows easily from the Hahn-Banach theorem and Lemma 10.3.3. $\qquad \square$

Notice that when $(M.1)$ and $(M.3)'$ hold, then one has the continuous and dense inclusions $\mathcal{D}^{[M]}(\mathbb{R}^d) \hookrightarrow \mathcal{K}^{[M]}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{[M]}(\mathbb{R}^d)$, so

that in particular $\mathcal{K}'^{[M]}(\mathbb{R}^d) \subseteq \mathcal{D}'^{[M]}(\mathbb{R}^d)$. Upon combining Proposition 10.2.4(*i*) with Theorem 10.3.2, one obtains the following complete characterization of those one-dimensional ultradistributions $f \in \mathcal{D}'^{[M]}(\mathbb{R}^d)$ satisfying the MAE:

Theorem 10.3.5. *Suppose* M *satisfies* $(M.1)$, $(M.2)'$, *and* $(M.3)'$. *An ultradistribution* $f \in \mathcal{D}'^{[M]}(\mathbb{R})$ *satisfies the MAE in* $\mathcal{D}'^{[M]}(\mathbb{R})$ *if and only if* $f \in \mathcal{K}'^{[M]}(\mathbb{R})$.

Proof. If f satisfies the MAE, then in particular $f(\lambda x) = O(\lambda^{-q})$ in $\mathcal{D}'^{[M]}(\mathbb{R} \setminus \{0\})$ for each $q \in \mathbb{N}$. Hence, for a fixed but arbitrary $q \in \mathbb{N}$, using Proposition 10.2.4(*i*) and Theorem 8.2.1, we can write $f = \sum_{m=1}^{\infty} f_m^{(m)}$ in $\mathcal{D}'^{[M]}(\mathbb{R})$ with $f_m = f_{q,m} \in C(\mathbb{R})$ such that for some (for each) $\ell > 0$ they fulfil bounds $f_m(x) = O_{q,\ell}(\ell^m (|x| + 1)^{m-q-2}/M_m)$. Clearly, this representation yields $f \in \mathcal{K}'^{[M]}_q(\mathbb{R})$. Since q was arbitrary, we conclude that $f \in \mathcal{K}'^{[M]}(\mathbb{R})$. For the converse, Theorem 10.3.2 shows that a stronger conclusion actually holds. $\qquad\square$

Remark 10.3.6. In dimension $d = 1$, this argument gives an alternative way for proving Proposition 10.3.4 in the non-quasianalytic case without having to resort in the dual Mittag-Leffler theorem.

Evidently, for higher dimensions we now get the following interesting problem.

Open problem 10.3.7. Show whether or not Theorem 10.3.5 holds for dimension $d \geqslant 2$. A possible avenue to solve this would be to find structural theorems for multidimensional quasiasymptotic boundedness, see also Open Problem 9.4.6.

10.4 The uniform moment asymptotic expansion

The bound in (10.18) is not uniform in general, but in the ultradistributional case it is natural to expect that some sort of uniformity could be present. For instance, we see below in Proposition 10.4.3 that this is the case for compactly supported ultradistributions. Let us introduce the following uniform variant of the MAE.

Definition 10.4.1. Let A be a weight sequence and let $\mathcal{X}$ be a lcHs of smooth functions provided with continuous actions of the dilation operators and the Dirac delta and all its partial derivatives. An element $f \in \mathcal{X}'$ satisfies the *uniform moment asymptotic expansion (UMAE) in $\mathcal{X}'$ with respect to $[A]$* if there are $\mu_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^d$, such that for any $\varphi \in \mathcal{X}$ and each $\ell > 0$ (for some $\ell = \ell_\varphi > 0$) the asymptotic formula

$$\langle f(\lambda x), \varphi(x) \rangle = \sum_{|\alpha| < k} \frac{\mu_\alpha \varphi^{(\alpha)}(0)}{\alpha! \lambda^{|\alpha|+d}} + O\left(\frac{\ell^k A_k}{\lambda^{k+d}} \right), \qquad \lambda \to \infty, \quad (10.21)$$

holds uniformly for $k \in \mathbb{N}$.

Fix three weight sequences M, N and A for the remainder of this section. We now introduce ultradistribution spaces that are closely related to the UMAE. Given $q, \ell > 0$ we denote by $\mathcal{K}_{N,q}^{M,\ell}(\mathbb{R}^d)$ the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ for which

$$\|\varphi\|_{\mathcal{K}_{N,q}^{M,\ell}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{e^{-\omega_N(q|x|)}(1 + |x|)^{|\alpha|} \left| \varphi^{(\alpha)}(x) \right|}{\ell^{|\alpha|} M_\alpha} \qquad (10.22)$$

is finite. We then define

$$\mathcal{K}_{(N)}^{(M)}(\mathbb{R}^d) = \varinjlim_{q \to \infty} \varprojlim_{\ell \to 0^+} \mathcal{K}_{N,q}^{M,\ell}(\mathbb{R}^d), \qquad \mathcal{K}_{\{N\}}^{\{M\}}(\mathbb{R}^d) = \varprojlim_{\ell \to \infty} \varinjlim_{q \to 0^+} \mathcal{K}_{N,q}^{M,\ell}(\mathbb{R}^d),$$

and consider the dual $\mathcal{K}_{[N]}^{\prime[M]}(\mathbb{R}^d)$, whose elements satisfy the UMAE as stated in the next theorem.

Theorem 10.4.2. *Suppose M and N satisfy $(M.1)$ and $(M.2)$. Set $A_p = N_p \max_{j \leqslant p}(M_j/j!)$. Then, any element $f \in \mathcal{K}_{[N]}^{\prime[M]}(\mathbb{R}^d)$ satisfies the UMAE in $\mathcal{K}_{[N]}^{\prime[M]}(\mathbb{R}^d)$ w.r.t. $[A]$.*

Proof. By replacing it by an equivalent sequence, we may assume that $N_p > 1$ for each $p \in \mathbb{N}$. Fix an arbitrary $0 < \varepsilon \leqslant 1$ in the Beurling case, while we put $\varepsilon = 1$ in the Roumieu case. We will always assume $\lambda \geqslant H \geqslant 1$, where H is the parameter in $(M.2)$ (for both sequences M and N). Take any $f \in \mathcal{K}_{[N]}^{\prime[M]}(\mathbb{R}^d)$ and $\varphi \in \mathcal{K}_{[N]}^{[M]}(\mathbb{R}^d)$. Arguing as in the proof of Theorem 10.3.2, we need to find a uniform bound

for $|\langle f(\lambda x), \varphi(x) - \varphi_k(x) \rangle|$, where φ_k is the $(k-1)$th order Taylor polynomial of φ at the origin. There exist $q = q_\varphi > 0$ and $\ell = \ell_f > 0$ ($\ell = \ell_\varphi > 0$ and $q = q_f > 0$) such that $\varphi \in \mathcal{K}_{N,q}^{M,\ell}(\mathbb{R}^d)$ and for some $C > 0$

$$
|\langle f(\lambda x), \varphi(x) - \varphi_k(x) \rangle|
$$
$$
\leqslant \frac{C}{\lambda^d} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{e^{-\omega_N(q|x|)}(1 + |x|)^{|\alpha|} \left| \partial^\alpha [\varphi\left(\frac{x}{\lambda}\right) - \varphi_k\left(\frac{x}{\lambda}\right)] \right|}{\ell^{|\alpha|} M_\alpha}.
$$

We split according to the size of $\alpha \in \mathbb{N}^d$.

First suppose that $|\alpha| < k$. Set $\ell_0 := \max(1, \ell)$. From the Taylor expansion and (8.3) applied to the sequence N,

$$
\frac{(1 + |x|)^{|\alpha|} |\varphi^{(\alpha)}(x/\lambda) - \varphi_k^{(\alpha)}(x/\lambda)|}{e^{\omega_N(q|x|)} \lambda^{|\alpha|} \ell^{|\alpha|} M_\alpha}
$$
$$
\leqslant \frac{e^{-\omega_N(q|x|)}(1 + |x|)^{|\alpha|}}{(\lambda \ell)^{|\alpha|} M_\alpha} \sum_{\substack{\alpha \leqslant \beta \\ |\beta| = k}} \frac{|\varphi^{(\beta)}(\xi_{\frac{x}{\lambda}})|}{(\beta - \alpha)!} \left(\frac{|x|}{\lambda}\right)^{|\beta - \alpha|}
$$
$$
\leqslant \lambda^{-k} \|\varphi\|_{\mathcal{K}_{N,q}^{M,\varepsilon\ell}} \varepsilon^k (1 + |x|)^k e^{\omega_N(q|x|/\lambda) - \omega_N(q|x|)} \sum_{\substack{\alpha \leqslant \beta \\ |\beta| = k}} \frac{\ell^{|\beta - \alpha|} M_\beta}{M_\alpha (\beta - \alpha)!}
$$

$$
\leqslant \lambda^{-k} A N_0 \|\varphi\|_{\mathcal{K}_{N,q}^{M,\varepsilon\ell}} (dH\ell_0\varepsilon)^k (1 + |x|)^k e^{-\omega_N(q|x|/H)} \sum_{\substack{\alpha \leqslant \beta \\ |\beta| = k}} \frac{M_{\beta - \alpha}}{|\beta - \alpha|!}
$$

$$
\leqslant \lambda^{-k} 2^{d-1} A^2 \|\varphi\|_{\mathcal{K}_{N,q}^{M,\varepsilon\ell}} \left(4 d q^{-1} H^2 \ell_0 \varepsilon\right)^k N_k \max_{0 \leqslant j \leqslant k} \frac{M_j}{j!}.
$$

Now let $|\alpha| \geqslant k$. For $|x| \geqslant \lambda$, one has

$$
\lambda^k \exp[\omega_N(q|x|/\lambda)] = \lambda^k \sup_{p \in \mathbb{N}} \frac{(q|x|/\lambda)^p N_0}{N_p}
$$
$$
\leqslant \max \left\{ \sup_{p \geqslant k} \frac{q^p |x|^p N_0}{N_p}, \sup_{0 \leqslant p < k} \frac{q^p |x|^k N_0}{N_p} \right\}
$$
$$
\leqslant q_0^{-k} N_k \exp[\omega_N(q|x|)],
$$

where $q_0 = \min(1, q)$. Then, since $(1 + |x|)^{|\alpha|}/(1 + |x|/\lambda)^{|\alpha|} \leqslant \lambda^{|\alpha|}$ for any $\alpha \in \mathbb{N}^d$, we have

$$\sup_{|\alpha| \geqslant k} \sup_{|x| \geqslant \lambda} \frac{e^{-\omega_N(q|x|)}(1 + |x|)^{|\alpha|} \left|\varphi^{(\alpha)}(x/\lambda)\right|}{(\lambda \ell)^{|\alpha|} M_\alpha} \leqslant \lambda^{-k} \|\varphi\|_{\mathcal{K}_{N,q}^{M,\varepsilon\ell}} (\varepsilon/q_0)^k N_k.$$

In the case $|x| \leqslant \lambda$, we have for $|\alpha| \geqslant k$,

$$\frac{e^{-\omega_N(q|x|)}(1 + |x|)^{|\alpha|} \left|\varphi^{(\alpha)}(x/\lambda)\right|}{(\lambda \ell)^{|\alpha|} M_\alpha}$$

$$\leqslant \frac{e^{-\omega_N(q|x|)}(1 + |x|)^k}{\lambda^k} \frac{(1 + |x|)^{|\alpha|-k}|\varphi^{(\alpha)}(x/\lambda)|}{\lambda^{|\alpha|-k} \ell^{|\alpha|} M_\alpha}$$

$$\leqslant \lambda^{-k} N_0^{-1} e^{\omega_N(q)} \|\varphi\|_{\mathcal{K}_{N,q}^{M,\varepsilon\ell}} (2\varepsilon/q)^k N_k,$$

which concludes the proof. $\qquad\square$

The next result describes the UMAE for compactly supported ultradistributions. The proof goes alone the same lines as that of Theorem 10.4.2 and we therefore leave details to the reader.

Proposition 10.4.3. *Any element $f \in \mathcal{E}'^{[M]}(\mathbb{R}^d)$ satisfies the UMAE in $\mathcal{E}'^{[M]}(\mathbb{R}^d)$ w.r.t. $[A]$, where $A_p = \max_{j \leqslant p}(M_j/j!)$.*

Via an analogous argument as in the proof of Proposition 10.3.4, one shows the ensuing structural description for $\mathcal{K}_{[N]}'^{[M]}(\mathbb{R}^d)$.

Proposition 10.4.4. *Let M and N satisfy (M.1) and (M.2)′. Let $f \in \mathcal{K}_{[N]}'^{[M]}(\mathbb{R}^d)$. Then, for each $q > 0$, there is some $\ell = \ell_q$ (for each ℓ there some $q_\ell > 0$) such that one can find a multi-sequence of continuous functions $f_\alpha = f_{q,\ell,\alpha} \in C(\mathbb{R}^d)$ for which*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha^{(\alpha)} \tag{10.23}$$

and there is a $C = C_{q,\ell} > 0$ such that

$$|f_\alpha(x)| \leqslant C \frac{\ell^{|\alpha|}}{M_\alpha}(1 + |x|)^{|\alpha|} e^{-\omega_N(q|x|)}, \qquad x \in \mathbb{R}^d, \ \alpha \in \mathbb{N}^d. \tag{10.24}$$

Let us now consider the one-dimensional case. The ensuing theorem is a counterpart of Theorem 10.3.5 for the UMAE; notice however that a full characterization is lacking in this case. We mention that if $(M.1)$ and $(M.3)'$ hold, one verifies that $\mathcal{D}^{[M]}(\mathbb{R}^d) \hookrightarrow \mathcal{K}_{[N]}^{[M]}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{[M]}(\mathbb{R}^d)$.

Theorem 10.4.5. *Suppose that N satisfies $(M.1)$ and that $(M.1)$, $(M.2)$, and $(M.3)$ hold for the weight sequence M. Set $A_p = M_p N_p / p!$. If $f \in \mathcal{D}'^{[M]}(\mathbb{R})$ satisfies the UMAE in $\mathcal{D}'^{[M]}(\mathbb{R})$ with respect to $[N]$, then $f \in \mathcal{K}_{[N]}'^{[M]}(\mathbb{R})$ and if in addition N satisfies $(M.2)$, the UMAE holds for f in $\mathcal{K}_{[N]}'^{[M]}(\mathbb{R})$ w.r.t. $[A]$.*

Proof. It suffices to show that $f \in \mathcal{K}_{[N]}'^{[M]}(\mathbb{R})$. In the Beurling case we take an arbitrary constant sequence $r_p = 1/q > 0$ and in the Roumieu case an arbitrary $(r_p) \in \{\mathfrak{R}\}$. We have that, whenever $\varphi \in \mathcal{D}^{[M]}(\mathbb{R} \setminus \{0\})$,

$$|\langle f(\lambda x), \varphi(x) \rangle| \leqslant O\left(\frac{R_{k-1} N_{k-1}}{\lambda^{k+1} N_0} \right),$$

which implies, taking infimum over k,

$$|\langle f(\lambda x), \varphi(x) \rangle| = O\left(\lambda^{-2} \exp\left(-\omega_{N_{r_p}}(\lambda) \right) \right).$$

Applying Proposition 10.2.4(ii), we can write $f = \sum_{m \in \mathbb{N}} f_m^{(m)}$ with continuous functions f_m satisfying the bounds

$$|f_m(x)| \leqslant C_\ell \frac{\ell^m}{M_m} (1 + |x|)^{m-2} e^{-\omega_{N_{r_q}}(|x|)}, \qquad x \in \mathbb{R}, \ m \in \mathbb{N},$$

for some $\ell > 0$ (for each $\ell > 0$). This yields $f \in \mathcal{K}_{[N]}'^{[M]}(\mathbb{R})$ in both cases, as required (in the Roumieu case we apply (2.8)). It has been proved by Petzsche [105, Proposition 1.1] that $(M.3)$ implies the so-called Rudin condition, namely, there is $C > 0$ such that

$$\max_{j \leqslant p} \left(\frac{M_j}{j!} \right)^{1/j} \leqslant C \left(\frac{M_p}{p!} \right)^{1/p}, \qquad p \in \mathbb{N};$$

therefore, the rest follows from Theorem 10.4.2. $\qquad\square$

Chapter 11

A multidimensional Tauberian theorem for the Laplace transform

11.1   Introduction

In 1976, Vladimirov obtained an important multidimensional generalization of the Hardy-Littlewood-Karamata Tauberian theorem [138]. Multidimensional Tauberian theorems were then systematically investigated by him, Drozhzhinov, and Zav'yalov, and their approach resulted in a powerful Tauberian machinery for multidimensional Laplace transforms of Schwartz distributions. Such results have been very useful in probability theory [152] and mathematical physics [6, 52, 141]. Tauberian theorems for other integral transforms of generalized functions have been extensively studied by several authors as well, see e.g. [51, 54, 111, 115, 116]. We refer to the monographs [114, 139, 140] for accounts on the subject and its applications; see also the recent survey article [50].

The aim of this chapter is to extend the so-called general Tauberian theorem for the dilation group [140, Chapter 2] from distributions to ultradistributions. Our considerations apply to Laplace transforms of elements in $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$, the space of Gelfand-Shilov ultradistributions with supports in a closed convex acute cone Γ of $\mathbb{R}^d$. We start in Section 11.2 with a formal definition of the Laplace trans-

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form and some preliminary discussions. Then, in Section 11.3, we provide characterizations of bounded sets and convergent sequences in $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ in terms of Laplace transform growth estimates; interestingly, our approach to the desired Laplace transform characterization is based on a useful convolution average description of bounded sets of $\mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d)$, originally established in [46] (cf. [109]) but improved here by relaxing hypotheses on the weight sequences. Those results are employed in Section 11.4 to derive a Tauberian theorem in which the quasiasymptotic behavior of an ultradistribution is deduced from asymptotic properties of its Laplace transform. Finally, as a natural refinement of the main result of Section 11.4 when the weight sequences and the cone satisfy stronger regularity conditions, we prove in Section 11.5 that the Laplace transform is an isomorphism of locally convex spaces between $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ and a certain space of holomorphic functions on the tube domain $\mathbb{R}^d + i\operatorname{int}\Gamma^*$, with Γ^* the conjugate cone of Γ.

11.2 The Laplace transform of tempered ultradistributions

Throughout this chapter $\Gamma \subseteq \mathbb{R}^d$ stands for a (non-empty) closed, convex and acute cone with vertex at the origin. We denote by T^C the tube domain with base $C = \operatorname{int}\Gamma^*$, see Section 8.1. Additionally, M and N will always denote two weight sequences, where M satisfies $(M.1)$ and $(M.3)'$. We define

$$\mathcal{S}'^{[M]}_{[N]}[\Gamma] := \{f \in \mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d) \mid \operatorname{supp} f \subseteq \Gamma\},$$

which is a closed subspace of $\mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d)$. We formally define the Laplace transform on $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ as follows.

Definition 11.2.1. Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a function such that $\eta(\xi) = 1$ for ξ in an open neighbourhood of Γ and for which $\eta(\xi)e^{iz\cdot\xi} \in \mathcal{S}^{[M]}_{[N]}(\mathbb{R}^d)$ for any $z \in T^C$. The Laplace transform of $f \in \mathcal{S}'^{[M]}_{[N]}[\Gamma]$ is then the holomorphic function

$$\mathcal{L}\{f; z\} := \left\langle f(\xi), \eta(\xi)e^{iz\cdot\xi} \right\rangle, \qquad z \in T^C.$$

As supp $f \subseteq \Gamma$, this definition is independent of the function η as long as such a function exists.

We first verify whether a function η as in the definition always exists. For this, we introduce the following concept. We recall that the set $[\mathfrak{R}]$ was introduced in Section 2.3.3.

Definition 11.2.2. A family $\{\eta_\varepsilon\}_{\varepsilon>0}$ of non-negative smooth functions $\eta_\varepsilon : \mathbb{R}^d \to [0, \infty)$ is called a $[M]$-Γ-*mollifier* if for every $\varepsilon > 0$ the ensuing conditions hold

(a) $\eta_\varepsilon(\xi) = 1$ for $\xi \in \Gamma_\varepsilon$ while $\eta_\varepsilon(\xi) = 0$ for $\xi \notin \Gamma_{2\varepsilon}$;

(b) for every $(\ell_p) \in [\mathfrak{R}]$ there is a constant $H_{\ell_p,\varepsilon} > 0$ such that

$$\left| \eta_\varepsilon^{(\alpha)}(\xi) \right| \leqslant H_{\ell_p,\varepsilon} L_\alpha M_\alpha, \qquad \forall \xi \in \mathbb{R}^d, \forall \alpha \in \mathbb{N}^d. \tag{11.1}$$

Lemma 11.2.3. *If M satisfies $(M.1)$ and $(M.3)'$, then there are $[M]$-Γ-mollifiers.*

Proof. The existence of such functions is guaranteed by the non-quasianalyticity. Take any non-negative $\varphi \in \mathcal{D}^{[M]}(\mathbb{R}^d)$ such that $\operatorname{supp} \varphi \subset B(0, 1/2)$ and $\int_{\mathbb{R}^d} \varphi(\xi)d\xi = 1$. Set $\varphi_\varepsilon(\xi) := \varepsilon^{-d}\varphi(\xi/\varepsilon)$ and let $\chi_{\Gamma_{3\varepsilon/2}}$ be the characteristic function of $\Gamma_{3\varepsilon/2}$. Taking $\eta_\varepsilon = \varphi_\varepsilon * \chi_{\Gamma_{3\varepsilon/2}}$, one easily verifies that $\{\eta_\varepsilon\}_{\varepsilon>0}$ is a $[M]$-Γ-mollifier. $\square$

The next result shows that in particular we may define the Laplace transform via $[M]$-Γ-mollifiers.

Lemma 11.2.4. *Assume $(M.1)$ and $(M.3)'$ on M. Let $(a_p), (b_p) \in [\mathfrak{R}]$ and $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a $[M]$-Γ-mollifier. Then there is $(\ell_p) \in [\mathfrak{R}]$ such that, for any $\varepsilon > 0$, we have*

$$\left\| \eta_\varepsilon(\xi) e^{iz \cdot \xi} \right\|_{\mathcal{S}_{N_{b_p},1}^{M_{a_p},1}}$$

$$\leqslant H_{\ell_p,\varepsilon} \exp\left(4\varepsilon |\operatorname{Im} z| + \omega_{M_{\ell_p}}(|z|) + \omega_{N_{\ell_p}^*}\left(\frac{1}{\Delta_C(\operatorname{Im} z)} \right) \right).$$

for any $z \in T^C$. In particular, we have $\eta_\varepsilon(\xi) e^{iz \cdot \xi} \in \mathcal{S}_{N_{b_p},1}^{M_{a_p},1}(\mathbb{R}^d)$ for all $z \in T^C$.

Proof. Set $\ell'_p := \min\{a_p, b_p\}$. Due to the support assumption on η_ε, we may assume below that $\xi \in \Gamma_{2\varepsilon}$. Then for any $z \in T^C$, $\alpha, \beta \in \mathbb{N}^d$, we have

$$
\frac{\left| \xi^\beta \frac{\partial^\alpha}{\partial \xi^\alpha} \left(\eta_\varepsilon(\xi) e^{iz\cdot\xi} \right) \right|}{A_\alpha M_\alpha B_\beta N_\beta}
$$

$$
\leqslant \frac{|\xi|^\beta e^{-y\cdot\xi}}{L'_\beta N_\beta} 2^{-|\alpha|} \sum_{0 \leqslant \alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} \frac{(2|z|)^{|\alpha'|}}{L'_{\alpha'} M_{\alpha'}} \left(\frac{2^{|\alpha-\alpha'|}}{L'_{\alpha-\alpha'} M_{\alpha-\alpha'}} \left| \eta_\varepsilon^{(\alpha-\alpha')}(\xi) \right| \right)
$$

$$
\leqslant H_{\ell_p,\varepsilon} e^{\omega_{M_{\ell_p}}(|z|)} \frac{|\xi|^\beta e^{-y\cdot\xi}}{L_\beta N_\beta},
$$

where we have set $\ell_p := \ell'_p/2$. Now $\xi = u + v$ for certain $u \in \Gamma$ and $v \in B(0, 2\varepsilon)$, so that by the Cauchy-Schwarz inequality

$$
\frac{|\xi|^\beta e^{-y\cdot\xi}}{L_\beta N_\beta} \leqslant \frac{(|u| + 2\varepsilon)^\beta e^{-y\cdot u} e^{-y\cdot v}}{L_\beta N_\beta}
$$

$$
\leqslant \frac{(|u| + 2\varepsilon)^{|\beta|} e^{-\Delta_C(y)|u|} e^{2\varepsilon|y|}}{L_\beta N_\beta}
$$

$$
\leqslant \frac{\left(\frac{1}{\Delta_C(y)} \right)^\beta \left(\frac{|\beta|}{e} \right)^\beta}{L_\beta N_\beta} e^{2\varepsilon\Delta_C(y) + 2\varepsilon|y|}
$$

$$
\leqslant \exp\left(\omega_{N^*_{\ell_p}} \left(\frac{1}{\Delta_C(y)} \right) + 4\varepsilon|y| \right),
$$

where we have used (8.1) and the elementary inequality $m^m \leqslant e^m m!$.
□

The ε-term that appears in the bound of Lemma 11.2.4 is a direct consequence of our construction via $[M]$-Γ-mollifiers. These terms will prevent us from finding an isomorphism between $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ and and a certain space of holomorphic functions on the tube domain $\mathbb{R}^d + i\,\text{int}\,\Gamma^*$ unless we impose heavy restrictions on the weight sequences, see Section 11.5. This now raises the question whether we may define the Laplace transform on $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ in an alternate way, avoiding the ε-terms altogether. For distributions, such an alternate definition was given using a Whitney type extension theorem for the

Schwartz space of rapidly decreasing smooth functions defined on an unbounded closed set [140] (see also [129]). Continuing from there, we could now ask ourselves if the same can be done in the ultradifferentiable context. Work towards this has already been done, we refer to [19, 22, 91], however in the case of Gelfand-Shilov spaces no satisfying answer is at hand. Hence the following open problem.

Open problem 11.2.5. Determine for an unbounded closed set $V \subset \mathbb{R}^d$ under which conditions on the weight sequences a Whitney type extension theorem on V holds in the context of Gelfand-Shilov spaces. From here, provide an alternate definition for the Laplace transform of elements in $\mathcal{S}_{[N]}^{\prime[M]}[\Gamma]$.

11.3 Laplace transform characterization of bounded sets in $\mathcal{S}_{[A]}^{\prime[M]}(\mathbb{R}^d)$

In this section we shall characterize those subsets of $\mathcal{S}_{[N]}^{\prime[M]}[\Gamma]$ that are bounded (with respect to the relative topology inherited from $\mathcal{S}_{[N]}^{\prime[M]}(\mathbb{R}^d)$) via bounds on the Laplace transforms of their elements. Hereafter, we assume M satisfies $(M.1)$, $(M.2)'$ and $(M.3)'$ while our assumptions on N are $(M.1)^*$ and $(M.2)$. Furthermore, whenever considering the Beurling case we assume in addition that N fulfils (NA). Note that these assumptions ensure that $\omega_{N_{\ell_p}}(t) = o(t)$ [81, Lemma 3.8 and Lemma 3.10, p. 52–53], $\omega_{N_{\ell_p}^*}(t) < \infty$ for all $t \geqslant 0$, and $\omega_{N_{\ell_p}^*}(t) \to \infty$ as $t \to \infty$ for any sequence $(\ell_p) \in [\mathfrak{R}]$. If stronger assumptions on the weight sequences are needed, this will be explicitly stated in the corresponding statement. The following theorem is our main result in this section.

Theorem 11.3.1. *Let $B \subseteq \mathcal{S}_{[N]}^{\prime[M]}[\Gamma]$.*

(i) *If B is a bounded set, then, there is $(\ell_p) \in [\mathfrak{R}]$ for which, given any $\varepsilon > 0$, there is $L = L_\varepsilon > 0$ such that for all $f \in B$*

$$|\mathcal{L}\{f; z\}|$$
$$\leqslant L \exp\left(\varepsilon|\operatorname{Im} z| + \omega_{M_{\ell_p}}(|z|) + \omega_{N_{\ell_p}^*}\left(\frac{1}{\Delta_C(\operatorname{Im} z)}\right)\right), \quad (11.2)$$

for all $z \in T^c$.

(ii) *Conversely, suppose there are $\theta \in C$, $\sigma_0 > 0$, $L = L_B > 0$, and $(\ell_p) \in [\mathfrak{R}]$ such that*

$$|\mathcal{L}\{f; x + i\sigma\theta\}| \leqslant L \exp\left(\omega_{M_{\ell_p}}(|x|) + \omega_{N^*_{\ell_p}}\left(\frac{1}{\sigma}\right)\right), \quad (11.3)$$

for all $f \in B$, $x \in \mathbb{R}^d$, and $\sigma \in (0, \sigma_0]$, then B is a bounded subset of $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$.

Before proving Theorem 11.3.1, let us discuss an important consequence. Namely, we shall derive from it a characterization of convergent sequences of $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$. Notice first that if a sequence $f_k \to g$ in $\mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d)$ and supp $f_k \subseteq \Gamma$ for each k, one easily shows that

$$\lim_{k \to \infty} \mathcal{L}\{f_k; z\} = \mathcal{L}\{g; z\},$$

and this limit holds uniformly for z in compact subsets of T^C; furthermore, by Theorem 11.3.1, the Laplace transforms of the f_k satisfy bounds of the form (11.2) uniformly in k. The converse also holds. In fact, the next result might be interpreted as a sort of Tauberian theorem.

Corollary 11.3.2. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$. Suppose that there is a non-empty open subset $\Omega \subseteq C$ such that for each $y \in \Omega$ the limit*

$$\lim_{k \to \infty} \mathcal{L}\{f_k; iy\} \quad (11.4)$$

exists. If there are $\theta \in C$, $\sigma_0 > 0$, and $(\ell_p) \in [\mathfrak{R}]$ such that

$$\sup_{k \in \mathbb{N}, \, x \in \mathbb{R}^d, \, \sigma \in (0, \sigma_0]} \frac{|\mathcal{L}\{f_k; x + i\sigma\theta\}|}{\exp\left(\omega_{M_{\ell_p}}(|x|) + \omega_{N^*_{\ell_p}}\left(\frac{1}{\sigma}\right)\right)} < \infty \quad (11.5)$$

then

$$\lim_{k \to \infty} f_k = g \quad in \; \mathcal{S}'^{[M]}_{[N]}[\Gamma], \quad (11.6)$$

for some $g \in \mathcal{S}'^{[M]}_{[N]}[\Gamma]$. In particular, the limit (11.4) is given by $\mathcal{L}\{g; iy\}$.

Proof. Notice first that if two subsequences converge, respectively, to ultradistributions g and h, the limits (11.4) tell us $\mathcal{L}\{g; iy\} = \mathcal{L}\{h; iy\}$ for all $y \in \Omega$. By uniqueness of holomorphic functions and the injectivity of the Laplace transform (which follows from that of the Fourier transform), we conclude $g = h$. It therefore suffices to show that every arbitrary subsequence of the f_k possesses a convergent subsequence in $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$, but this follows from the fact that $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ is Montel because, in view of Theorem 11.3.1, the estimate (11.5) is equivalent to $\{f_k : k \in \mathbb{N}\}$ being bounded in $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ (and hence relatively compact). $\square$

Let us now prepare ourselves to prove Theorem 11.3.1. Part (i) will be an easy consequence of our application of $[M]$-Γ-mollifiers, in particular Lemma 11.2.4. In preparation for the proof of part (ii), we first need to extend [46, Proposition 3.1] (cf. [109, Lemma 2.7]) by relaxing assumptions on the weight sequences. This provides a useful convolution characterization of bounded sets in $\mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d)$. Our approach to this convolution characterization employs the short-time Fourier transform, see Section 3.4.1.

Lemma 11.3.3. *A subset $B \subset \mathcal{S}'^{[M]}_{[N]}(\mathbb{R}^d)$ is bounded if and only if there exists $(\ell_p) \in [\mathfrak{R}]$ such that*

$$\sup_{f \in B,\, x \in \mathbb{R}^d} e^{-\omega_{N_{\ell_p}}(|x|)} |(f * \psi)(x)| < \infty, \qquad \forall \psi \in \mathcal{D}^{[M]}(\mathbb{R}^d). \quad (11.7)$$

Proof. We only make use here of the assumptions $(M.1)$ and $(M.2)'$ on N. The necessity is easily obtained. Hence suppose that (11.7) holds for some $(\ell_p) \in [\mathfrak{R}]$. We may assume the sequence N_{ℓ_p} satisfies $(M.2)'$. We consider the weighted Banach space $X = \{g \in C(\mathbb{R}^d) : g(\xi) = O(\exp(\omega_{N_{\ell_p}}(|\xi|)))\}$ and fix a compact set $K \Subset \mathbb{R}^d$ with non-empty interior.

The assumption (11.7) implies that for each $f \in B$ the mapping $L_f : \varphi \mapsto f * \varphi$ is continuous from $\mathcal{D}^{[M]}(\mathbb{R}^d)$ into X, so that in particular, in view of the Banach-Steinhaus theorem, $\tilde{B} = \{(L_f)_{|\mathcal{D}^{[M]}_K} : f \in B\}$ is an equicontinuous subset of $L_b(\mathcal{D}^{[M]}_K, X)$. This implies that there is $(h_p) \in [\mathfrak{R}]$ such that $\tilde{B} \subset L_b(\mathcal{D}^{M_{h_p}, 1}_K, X)$ and it is equicontinuous there. Fix $\psi \in \mathcal{D}^{(M)}_K$ with $\|\psi\|_{L^2} = 1$. Since $\{e^{-\omega_{M_{h_p}}(4\pi|\xi|)} e^{2\pi i \xi \cdot} \overline{\breve{\psi}} :$

$\xi \in \mathbb{R}^d\}$ is a bounded family in $\mathcal{D}_K^{M_{h_p},1}$, we conclude that, for some $C_B > 0$, independent of $f \in B$,

$$\begin{aligned}
|V_\psi f(x,\xi)| &= \left| e^{-2\pi i \xi \cdot x} \left(f * (e^{2\pi i \xi \cdot} \overset{\smile}{\overline{\psi}}) \right)(x) \right| \\
&\leqslant C_B \exp \left(\omega_{N_{\ell_p}}(|x|) + \omega_{M_{h_p}}(4\pi|\xi|) \right).
\end{aligned}$$

On the other hand, let now $\varphi \in \mathcal{S}_{[N]}^{[M]}(\mathbb{R}^d)$. For any $(\ell_p') \in [\mathfrak{R}]$ it follows from Proposition 3.4.1 that there is some $C_\varphi > 0$ such that

$$\left| V_{\overline{\psi}}\varphi(x,-\xi) \right| \leqslant C_\varphi \exp \left(-\omega_{N_{\ell_p'}}(|x|) - \omega_{M_{\ell_p'}}(|\xi|) \right).$$

Moreover, according to the desingularization formula (3.2) for the STFT,

$$\langle f, \varphi \rangle = \int\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) V_{\overline{\psi}}\varphi(x,-\xi) dx d\xi.$$

Let $h > 0$ be such that $\log h / \log H \geqslant d + 1$ (with H the corresponding constant occurring in $(M.2)'$ for N_{ℓ_p} and M_{h_p}) and set $\ell_p' := h^{-1} \min(\ell_p, (4\pi)^{-1}h_p)$, then applying (8.2) one gets

$$\begin{aligned}
&\sup_{f \in B} |\langle f, \varphi \rangle| \\
&\leqslant C_B C_\varphi \int_{\mathbb{R}^d} e^{\omega_{M_{h_p}}(4\pi|\xi|) - \omega_{M_{\ell_p'}}(|\xi|)} d\xi \int_{\mathbb{R}^d} e^{\omega_{N_{\ell_p}}(|x|) - \omega_{N_{\ell_p'}}(|x|)} dx < \infty,
\end{aligned}$$

which concludes the proof of the sufficiency. $\qquad\square$

We are now ready to present a proof of Theorem 11.3.1.

Proof of Theorem 11.3.1. Suppose that $B \subseteq \mathcal{S}_{[N]}'^{[M]}[\Gamma]$ is bounded in $\mathcal{S}_{[N]}'^{[M]}(\mathbb{R}^d)$. By equicontinuity, there are certain $(a_p), (b_p) \in [\mathfrak{R}]$ such that $B \subseteq \left(\mathcal{S}_{N_{b_p},1}^{M_{a_p},1}(\mathbb{R}^d) \right)'$ and it is bounded there. Then, (11.2) follows directly from Lemma 11.2.4 (in particular, one does not employ $(M.2)$ for N in this implication).

We now show that (11.3) is sufficient to guarantee boundedness. We are going to do this employing Lemma 11.3.3. We may assume that $(\ell_p) \in [\mathfrak{R}]$ is such that M_{ℓ_p} satisfies $(M.2)'$ and $N_{\ell_p}^*$ fulfills $(M.2)$

(the constants occurring in these conditions are denoted by A and H below). We may also suppose that $|\theta| = 1$. Fix $\varphi \in \mathcal{D}^{[M]}(\mathbb{R}^d)$. Find $R > 0$ such that $\operatorname{supp} \varphi \subset B(0, R)$. We keep $f \in B$. Take a bounded function $\gamma : \mathbb{R}^d \to (0, \sigma_0]$, which will be specified later. Inverting the Laplace transform of $f * \varphi$,

$$(f * \varphi)(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d + i\gamma(t)\theta} e^{-iz\cdot t} \mathcal{L}\{f; z\} \mathcal{L}\{\varphi; z\} dz.$$

By [81, Lemma 3.3, p. 49] and (2.9), we have that for any $(h_p) \in [\mathfrak{R}]$

$$|\mathcal{L}\{\varphi; x + i\gamma(t)\theta\}| \leqslant L_\varphi \exp\left(-\omega_{M_{h_p}}(|x|) + R\gamma(t)\right), \qquad x \in \mathbb{R}^d.$$

Choose $h > 0$ such that $\log h \geqslant (d+1)\log H$. Taking $h_p = \ell_p/h$, the condition $(M.2)'$ in the form of estimate (8.2) yields

$$\omega_{M_{\ell_p}}(|x|) - \omega_{M_{h_p}}(|x|) = \omega_{M_{\ell_p}}(|x|) - \omega_{M_{\ell_p}}(h|x|) \leqslant -(d+1)\log(|x|/A),$$

whence we infer the exponential function of this expression is integrable on $\mathbb{R}^d$. Let $\delta = \Delta_C(\theta)$. Employing (11.3) we then obtain

$$\left| \int_{\mathbb{R}^d + i\gamma(t)\theta} e^{-iz\cdot t} \mathcal{L}\{f; z\} \mathcal{L}\{\varphi; z\} dz \right|$$

$$\leqslant L_B L_\varphi \exp\left(\gamma(t)(\theta \cdot t) + \omega_{N^*_{\ell_p}}\left(\frac{1}{\delta\gamma(t)}\right) + R\gamma(t)\right)$$

$$\cdot \int_{\mathbb{R}^d} e^{\omega_{M_{\ell_p}}(|x|) - \omega_{M_{h_p}}(|x|)} dx$$

$$\leqslant L \exp\left(\omega_{N^*_{\ell_p}}\left(\frac{1}{\delta\gamma(t)}\right) + |t|\gamma(t)\right),$$

for some $L > 0$. Note that N_{ℓ_p} satisfies $(M.1)^*$, so that (8.4) holds for it. Also, since $\omega_{N_{\ell_p}}(t) = o(t)$, there is a sufficiently large r_0 such that

$$\frac{4(n_1\ell_1 + 1)\omega_{N_{\ell_p}}(|t|)}{\delta|t|} \leqslant \sigma_0 \qquad \text{for } |t| > r_0.$$

Set $r = \max\{r_0, n_1\ell_1 + 1\}$, we then define

$$\gamma(t) = \begin{cases} \sigma_0, & |t| < r, \\ \dfrac{4(n_1\ell_1 + 1)\omega_{N_{\ell_p}}(|t|)}{\delta|t|}, & |t| \geqslant r. \end{cases}$$

For $|t| < r$ obviously

$$\exp\left(\omega_{N^*_{\ell_p}}\left(\frac{1}{\delta\gamma(t)}\right) + |t|\gamma(t)\right) \leqslant \exp\left(r\sigma_0 + \omega_{N^*_{\ell_p}}\left(\frac{1}{\sigma_0\delta}\right)\right).$$

If $|t| \geqslant r$, the inequality (8.4) yields

$$\exp\left(\omega_{N^*_{\ell_p}}\left(\frac{1}{\delta\gamma(t)}\right) + |t|\gamma(t)\right)$$

$$\leqslant \exp\left(\omega_{N^*_{\ell_p}}\left(\frac{|t|}{4(n_1\ell_1 + 1)\omega_{N_{\ell_p}}(|t|)}\right) + \frac{4(n_1\ell_1 + 1)}{\delta}\omega_{N_{\ell_p}}(|t|)\right)$$

$$\leqslant \exp\left(2^k\omega_{N_{\ell_p}}(|t|) + A'\right)$$

for some $A' > 0$ and $k = \lceil \log_2(1 + 4(n_1\ell_1 + 1)/\delta) \rceil$. By repeated application of (8.3) for $\omega_{N_{\ell_p}}$, one obtains

$$\exp\left(2^k\omega_{N_{\ell_p}}(|t|)\right) \leqslant \exp(\omega_{N_{\ell_p}}\left(H^k|t|\right) + A''),$$

for some $A''' > 0$. Let $a_p = \ell_p H^{-pk}$. Summing up, we have shown that

$$\sup_{f\in B,\, t\in\mathbb{R}^d} e^{-\omega_{N_{a_p}}(|t|)}|(f * \varphi)(t)| < \infty.$$

Since φ was arbitrary, Lemma 11.3.3 applies to conclude that B is bounded. $\hfill\square$

11.4   The Tauberian theorem

We shall now use our results from the previous section to generalize the Drozhzhinov-Vladimirov-Zav'yalov multidimensional Tauberian theorem for Laplace transforms [139, 140] from distributions to ultradistributions. Our goal is to devise a Laplace transform criterion for the quasiasymptotic behavior of tempered ultradistributions, see also Chapter 9. Our Tauberian theorem is the inverse to the ensuing Abelian statement that readily follows from the definition:

If an ultradistribution $f \in \mathcal{S}_{[N]}^{\prime[M]}[\Gamma]$ has quasiasymptotic behavior $f(\lambda x) \sim \rho(\lambda)g(x)$ in $\mathcal{S}_{[N]}^{\prime[M]}(\mathbb{R}^d)$, then

$$\lim_{r \to 0^+} \frac{r^d}{\rho(1/r)} \mathcal{L}\{f; rz\} = \mathcal{L}\{g; z\} \tag{11.8}$$

uniformly for z in compact subsets of T^C.

Theorem 11.4.1. *Assume that M and N both satisfy (M.1) and (M.2), while M also satisfies (M.3)′ and N satisfies (M.1)*. Set $A_p = M_p N_p$. Let $f \in \mathcal{S}_{[A]}^{\prime[M]}[\Gamma]$ and let ρ be regularly varying of degree α. Suppose that there is a non-empty solid subcone Γ' with $C' = \operatorname{int}\Gamma' \subset C$ such that for each $y \in C'$ the limit*

$$\lim_{r \to 0^+} \frac{r^d}{\rho(1/r)} \mathcal{L}\{f; riy\} \tag{11.9}$$

exists. If there are $\theta_0 \in C$ and $(\ell_p) \in [\mathfrak{R}]$ such that

$$\limsup_{r \to 0^+} \sup_{\substack{|x|^2 + \sin^2\theta = 1 \\ \theta \in (0, \pi/2]}} \frac{e^{-\omega_{A_{\ell_p}^*}\left(\frac{1}{\sin\theta}\right)}}{r^{-d}\rho(1/r)} |\mathcal{L}\{f; r(x + i\sin\theta_0\theta)\}| < \infty, \tag{11.10}$$

then f has quasiasymptotic behavior with respect to ρ in $\mathcal{S}_{[N]}^{\prime[M]}(\mathbb{R}^d)$.

Proof. In view of Corollary 11.3.2, it suffices to show that the Laplace transform of f satisfies a bound of the form

$$\frac{r^d}{\rho(1/r)} |\mathcal{L}\{f; r(x + i\sigma\theta_0)\}|$$

$$\leqslant L \exp\left(\omega_{M_{\ell_p'}}(|x|) + \omega_{N_{\ell_p'}}\left(\frac{1}{\sigma}\right)\right) \tag{11.11}$$

for some $(\ell_p') \in [\mathfrak{R}]$, $L, \sigma_0 > 0$ and all $x \in \mathbb{R}^d$ and $0 < \sigma < \sigma_0$. We may assume (M.2) holds for both M_{ℓ_p} and $N_{\ell_p}^*$ (with constants A and H). We can also assume that $L_p \geqslant 1$ for all $p \in \mathbb{N}$. Using (11.10), there are $0 < r_0 < 1$ and L_1 such that for any $0 < r < r_0$

$$\frac{r^d}{\rho(1/r)} |\mathcal{L}\{f; r(x + i\sin\theta_0\theta)\}| \leqslant L_1 \exp\left(\omega_{A_{\ell_p}^*}\left(\frac{1}{\sin\theta}\right)\right), \tag{11.12}$$

whenever $|x|^2 + \sin^2\theta = 1$, where we always keep $0 < \theta < \pi/2$. On the other hand, applying Theorem 11.3.1 to the singleton $B = \{f\}$ and possibly enlarging (ℓ_p),

$$|\mathcal{L}\{f; r(x + i\theta_0\sigma)\}| \leqslant L_2 \exp\left(\omega_{M_{\ell_p}}(|x|) + \omega_{A^*_{\ell_p}}\left(\frac{1}{r\sigma}\right)\right), \quad (11.13)$$

for any $0 < r < 1$, $x \in \mathbb{R}^d$ and $\sigma < r_0 < 1$. We may assume that $\rho(\lambda) = 1$ for $\lambda < r_0$. Furthermore, Potter's estimate (8.6) yields

$$\frac{\rho(\lambda t)}{\rho(\lambda)} \leqslant L_3 t^\alpha \max\{t^{-1}, t\}, \qquad t, \lambda > 0. \quad (11.14)$$

We keep arbitrary $r < 1$, $x \in \mathbb{R}^d$, $0 < \sigma < r_0$, and write $r' = \sqrt{|x|^2 + \sigma^2}$, $x' = x/r'$, and $\sin\theta = \sigma/r'$. If $rr' < r_0$, we obtain from (11.14), (11.12), and the fact that $\omega_{M_{\ell_p}}(t)$ increases faster than $\log t$,

$$\frac{r^d}{\rho(1/r)}\,|\mathcal{L}\{f; r(x + i\sigma\theta_0)\}|$$

$$\leqslant L_1 L_3 \left(\frac{1}{r'}\right)^{\alpha+d} \max\left\{r', \frac{1}{r'}\right\} \exp\left(\omega_{A^*_{\ell_p}}\left(\frac{r'}{\sigma}\right)\right)$$

$$= O\left(\exp\left(\omega_{M_{\ell_p}}(2|x|) + \omega_{A^*_{\ell_p}}\left(\frac{2|x|}{\sigma}\right)\right)\right).$$

Similarly, if $rr' \geqslant r_0$, we employ (11.13), (11.14), $\rho(1/(rr')) = 1$, and $(M.2)'$ for both M_{ℓ_p} and A_{ℓ_p} to conclude that for some $h' > 0$

$$\frac{r^d}{\rho(1/r)}\,|\mathcal{L}\{f; r(x + i\sigma\theta_0)\}| = O\left(\exp\left(\omega_{M_{\ell_p}}(h'|x|) + \omega_{A^*_{\ell_p}}\left(\frac{h'|x|}{\sigma}\right)\right)\right).$$

We have found in all cases

$$\frac{r^d}{\rho(1/r)}\,|\mathcal{L}\{f; r(x + i\sigma\theta_0)\}| \leqslant L_4 \exp\left(\omega_{M_{\ell_p}}(h|x|) + \omega_{A^*_{\ell_p}}\left(\frac{h|x|}{\sigma}\right)\right)$$

for some L_4 and $h = \max\{h', 2\}$. It remains to observe that

$$\omega_{A^*_{\ell_p}}(h|x|/\sigma) \leqslant \omega_{M_{\ell_p}}(h|x|) + \omega_{N^*_{\ell_p}}(h/\sigma),$$

so that (11.11) holds with $\ell'_p = \ell_p/(Hh)$, by $(M.2)$. $\qquad\square$

11.5   Sharpening the bound (11.2)

If the sequence M and the cone Γ satisfy stronger conditions, it turns out that the bound (11.2) can be considerably improved. In fact, we shall show here how to remove the ε term from (11.2).

   We start with three lemmas, from which our improvement of Theorem 11.3.1 will follow.

Lemma 11.5.1. *Let $\{F_j\}_{j\in I}$ be a family of holomorphic functions on T^C. Suppose that for some $(\ell_p) \in [\mathfrak{R}]$ and each $\varepsilon > 0$ there is $L = L_\varepsilon > 0$ such that for all $j \in I$ and any $z \in T^C$:*

$$\left|(1 + |\operatorname{Re} z|)^{d+2} F_j(z)\right|$$

$$\leqslant L \exp\left(\varepsilon|\operatorname{Im} z| + \omega_{N^*_{\ell_p}}\left(\frac{1}{\Delta_C(\operatorname{Im} z)}\right)\right). \quad (11.15)$$

Then there are $(h_p) \in [\mathfrak{R}]$ and $f_j \in C^1(\mathbb{R}^d)$ with $\operatorname{supp} f_j \subseteq \Gamma$, $\forall j \in I$, such that $\{e^{-\omega_{N_{h_p}}(|\cdot|)} f_j\}_{j\in I}$ is a bounded set in $L^\infty(\mathbb{R}^d)$ and $F_j(z) = \mathcal{L}\{f_j; z\}$, $j \in I$.

Proof. We closely follow the proof of the lemma in [139, Section 10.5, p. 148]. We may assume that N_{ℓ_p} satisfies $(M.1)$ and $(M.2)$. From (11.15) it follows in particular that

$$(1 + |\cdot|) F_j(\cdot + iy) \in L^1(\mathbb{R}^d), \qquad \forall y \in C, j \in I.$$

From the Cauchy formula we obtain for each compact subset $K \Subset C$ and each $j \in I$

$$\sup_{y \in K}\left|\frac{\partial}{\partial y_k} F_j(x + iy)\right| = O\left(\frac{1}{(1 + |x|)^{d+2}}\right), \qquad k \in \{1, \ldots, d\}.$$

Therefore,

$$g_j(\xi, y) = e^{2\pi\xi\cdot y}\mathcal{F}\{F_j(\cdot + iy); \xi\} \in C^1(\mathbb{R}^d \times C), \qquad j \in I.$$

Furthermore, for each $k \in \{1, \ldots, d\}$,

$$\frac{\partial}{\partial y_k} g_j(\xi, y) = e^{\xi\cdot y}\left[2\pi i\xi_k \mathcal{F}\{F_j(\cdot + iy); \xi\} + i\mathcal{F}\left\{\frac{\partial}{\partial x_k} F_j(\cdot + iy); \xi\right\}\right]$$

$$= 0,$$

so that the C^1-functions $f_j(\xi) := (2\pi)^{-d}g_j(\xi/2\pi, y/2\pi)$ do not depend on $y \in C$. By (11.15), there is $L' = L'_\varepsilon > 0$ such that

$$|f_j(\xi)| \leqslant L' \exp\left(\xi \cdot y + \varepsilon|y| + \omega_{N^*_{\ell_p}}\left(\frac{1}{\Delta_C(y)}\right)\right), \qquad (11.16)$$

for all $\xi \in \mathbb{R}^d$, $y \in C$, and $j \in I$. Take any $\xi_0 \notin \Gamma$. As $(\Gamma^*)^* = \Gamma$, there is some $y_0 \in C$ such that $\xi_0 \cdot y_0 = -1$. Since $\Delta_C(\lambda y_0) = \lambda\Delta_C(y_0)$ for $\lambda > 0$, we conclude from (11.16) for $\varepsilon = (2|y_0|)^{-1}$ and $y = \lambda y_0$ that

$$|f_j(\xi_0)| \leqslant L' \exp\left(-\frac{\lambda}{2} + \omega_{N^*_{\ell_p}}\left(\frac{1}{\lambda\Delta_C(y_0)}\right)\right), \qquad \lambda > 0.$$

By letting $\lambda \to \infty$, it follows that this is only possible if $f(\xi_0) = 0$. We conclude that supp $f_j \subseteq \Gamma$ for each $j \in I$.

Now take an arbitrary $y_0 \in C$ such that $|y_0| = 1$, then (11.16) gives us for $\varepsilon = 1/2$ and $y = \lambda y_0$, $\lambda > 0$,

$$|f_j(\xi)| \exp\left(-(1 + |\xi|)\lambda - \omega_{N^*_{\ell_p}}\left(\frac{1}{\lambda\Delta_C(y_0)}\right)\right) \leqslant L'e^{-\frac{\lambda}{2}}.$$

We now integrate this inequality with respect to λ on $(0, \infty)$ in order to gain an estimate on the f_j. The 1-dimensional case of [24, Lemma 5.2.6, p. 97] applied to the open cone $(0, \infty)$, yields the existence of constants $L'', c > 0$ such that

$$\int_0^\infty \exp\left(-(1 + |\xi|)\lambda - \omega_{N^*_{\ell_p}}\left(\frac{1}{\lambda\Delta_C(y_0)}\right)\right) d\lambda$$
$$\geqslant L'' \exp\left(-\omega_{N_{\ell_p}}(c(1 + |\xi|))\right).$$

Hence, using (2.5), it follows for any $\xi \in \mathbb{R}^d$ and $j \in I$ that

$$|f_j(\xi)| \leqslant \frac{2L'}{L''} \exp\left(\omega_{N_{\ell_p}}(c(1 + |\xi|))\right)$$
$$\leqslant \frac{2L'}{L''} \exp\left(\omega_{N_{\ell_p}}(2c) + \omega_{N_{\ell_p}}(2c|\xi|)\right).$$

The proof is complete noticing that by the Fourier inversion $F_j(z) = \mathcal{L}\{f_j; z\}$. $\qquad\square$

Lemma 11.5.2. *Let Γ be a solid cone and let $(\ell_p) \in [\mathfrak{R}]$. Suppose that M_{ℓ_p} satisfies (M.1), (M.2), and (M.3). Then, there are an ultrapolynomial P of type $[M]$ and constants $L, L' \geqslant 1$ such that*

$$e^{\omega_{M_{\ell_p}}(|z|)} \leqslant |P(z)| \leqslant L' e^{\omega_{M_{\ell_p}}(L|z|)}, \qquad \forall z \in T^C. \qquad (11.17)$$

Proof. Set

$$\widetilde{P}(z) := \prod_{p=1}^{\infty} \left(1 + \frac{z}{\ell_p m_p}\right), \qquad z \in \mathbb{C},$$

which is an ultrapolynomial of type $[M]$ satisfying the bound $\widetilde{P}(z) = O(\exp[\omega_{M_{\ell_p}}(L''|z|)])$ [81, Proposition 4.5 and Proposition 4.6, pp. 58–59]. Now for $\operatorname{Re} z \geqslant 0$ as in [81, p. 89]

$$\left|\widetilde{P}(z)\right| \geqslant \sup_{p \in \mathbb{N}} \prod_{q=1}^{p} \frac{|z|}{\ell_q m_q} = \sup_{p \in \mathbb{N}} \frac{M_0 |z|^p}{L_p M_p} = e^{\omega_{M_{\ell_p}}(|z|)}.$$

Since we assumed $\operatorname{int} \Gamma \neq \varnothing$, there is a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^d$ such that $e_j \in \operatorname{int} \Gamma$ for $1 \leqslant j \leqslant n$. Find also $\lambda > 0$ such that $\lambda \min_j |e_j \cdot z| \geqslant |z|$ for all $z \in C$. Now define

$$P(z) := \prod_{j=1}^{d} \widetilde{P}(-\lambda d^{1/2} i e_j \cdot z),$$

which is an ultrapolynomial of type $[M]$ as well and the upper bound in (11.17) holds because of (8.3) applied to $\omega_{M_{\ell_p}}$. Since for any $z \in T^C$ we have $\operatorname{Re}(-d i e_j \cdot z) > 0$, $1 \leqslant j \leqslant d$, one then obtains for any z in the tube domain

$$|P(z)| \geqslant \exp\left(\sum_{j=1}^{d} \omega_{M_{\ell_p}}(d^{1/2} \lambda |e_j \cdot z|)\right) \geqslant \exp\left(\omega_{M_{\ell_p}}(|z|)\right).$$

$\square$

Lemma 11.5.3. *Let $(\ell_p) \in [\mathfrak{R}]$. It holds that for any $y \in C$*

$$\sup_{\xi \in \Gamma} \exp\left(\omega_{N_{\ell_p}}(|\xi|) - y \cdot \xi\right) \leqslant \exp\left(\omega_{N_{\ell_p}^*}\left(\frac{1}{\Delta_C(y)}\right)\right). \qquad (11.18)$$

Proof. We only make use of $(M.1)^*$. Using the estimate (8.1), we obtain for any $y \in C$

$$\sup_{\xi \in \Gamma} e^{\omega_{N_{\ell_p}}(|\xi|) - y \cdot \xi} \leqslant \sup_{t \geqslant 0} e^{\omega_{N_{\ell_p}}(t) - \Delta_C(y)t},$$

so that (11.18) follows from $[106,$ Lemma $5.6]$

$$\sup_{t > 0} \left\{ \omega_{N_{\ell_p}}(t) - st \right\} \leqslant \omega_{N_{\ell_p}^*} \left(\frac{1}{s} \right), \qquad s > 0.$$

$\square$

Theorem 11.5.4. *Suppose that the cone Γ is solid, M and N both satisfy $(M.1)$ and $(M.2)$, and M also satisfies $(M.3)$. Then, a set $B \subset \mathcal{S}'^{[M]}_{[N]}[\Gamma]$ is bounded if and only if there are $L > 0$ and $(\ell_p) \in [\mathfrak{R}]$ such that for all $f \in B$*

$$|\mathcal{L}\{f; z\}| \leqslant L \exp \left(\omega_{M_{\ell_p}}(|z|) + \omega_{N_{\ell_p}^*} \left(\frac{1}{\Delta_C(\operatorname{Im} z)} \right) \right), \qquad (11.19)$$

for all $z \in T^C$.

Proof. We only need to show that if $B = \{f_j\}_{j \in I}$ is bounded then (11.19) holds. By Theorem 11.3.1, there is $(\ell_p) \in [\mathfrak{R}]$ such that for any $\varepsilon > 0$ there is $L = L_\varepsilon > 0$ such that for all $j \in I$

$$|\mathcal{L}\{f_j; z\}| \leqslant L \exp \left(\varepsilon |y| + \omega_{M_{\ell_p}}(|z|) + \omega_{N_{\ell_p}^*} \left(\frac{1}{\Delta_C(y)} \right) \right), \qquad \forall z \in T^C.$$

We may assume M_{ℓ_p} satisfies $(M.1)$, $(M.2)$ and $(M.3)$. Let P be the ultrapolynomial constructed as in Lemma 11.5.2. Fix $k \geqslant H^{d+2}$, where H is the constant occurring in $(M.2)'$ for M_{ℓ_p}. We consider the ultrapolynomial $Q(z) = P(kz)$, so that it satisfies the bounds

$$e^{\omega_{M_{\ell_p}}(k|z|)} \leqslant |Q(z)| \leqslant L' e^{\omega_{M_{\ell_p}}(\nu|z|)}$$

for all $z \in T^C$ and some $\nu > 0$. Set now $F_j(z) = \mathcal{L}\{f_j; z\}$, which are holomorphic functions on T^C. In view of (8.2), the family $\{F_j/Q\}_{j \in I}$ satisfies the conditions of Lemma 11.5.1, so that there are $g_j \in C^1(\mathbb{R}^d)$

with supp $g_j \subseteq \Gamma$ for which there is some $(\ell'_p) \in [\mathfrak{R}]$ such that $\{\exp(-\omega_{N_{\ell'_p}}(|\cdot|))g_j\}_{j \in I}$ is a bounded subset of $L^\infty(\mathbb{R}^d)$ and $F_j(z) = Q(z)\mathcal{L}\{g_j; z\}$ for each $j \in I$. Now, taking into account (8.2) (we may assume H is the same constant for both M_{ℓ_p} and N_{ℓ_p}) and Lemma 11.5.3, there are some $L'', L''' > 0$ such that for all $j \in I$

$$
|F_j(z)| \leqslant L'' e^{\omega_{M_{\ell_p}}(\nu|z|)} \int_\Gamma e^{\omega_{N_{\ell'_p}}(|\xi|)-y\cdot\xi} d\xi
$$

$$
\leqslant L''' e^{\omega_{M_{\ell_p}}(\nu|z|)} \sup_{\xi\in\Gamma} e^{\omega_{N_{\ell'_p}}(k|\xi|)-y\cdot\xi} \int_\Gamma \frac{d\xi}{(1+|\xi|)^{d+2}}
$$

$$
\leqslant AL' \left(\int_\Gamma \frac{d\xi}{(1+|\xi|)^{d+2}}\right) \exp\left(\omega_{M_{\ell_p}}(\nu|z|) + \omega_{N^*_{\ell'_p}}\left(\frac{k}{\Delta_C(y)}\right)\right).
$$

Hence, we obtain a bound of type (11.19) for the sequence $k_p = \min\{\ell_p/\nu, \ell'_p/k\}$. $\qquad\square$

Theorem 11.5.4 can be used to draw further topological information. In fact, it leads to an isomorphism between $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ and analogs of the Vladimirov algebra [139, Chapter 12] $H(T^C)$ of holomorphic functions on T^C. Given $\ell > 0$, we define the Banach space $\mathcal{O}_\ell(T^C)$ of all holomorphic functions F on the tube domain T^C that satisfy the bounds

$$
\|F\|_\ell = \sup_{z\in T^C} |F(z)| e^{-\omega_M(\ell|z|)-\omega_{N^*}\left(\frac{\ell}{\Delta_C(\mathrm{Im}\, z)}\right)} < \infty.
$$

We then introduce the (DFS)- and (FS)-spaces

$$
\mathcal{O}^{(M)}_{(N)}(T^C) = \varinjlim_\ell \mathcal{O}_\ell(T^C) \qquad \text{and} \qquad \mathcal{O}^{\{M\}}_{\{N\}}(T^C) = \varprojlim_\ell \mathcal{O}_\ell(T^C).
$$

The arguments we have given above actually show that the Laplace transform maps $\mathcal{S}'^{[M]}_{[N]}[\Gamma]$ bijectively into $\mathcal{O}^{[M]}_{[N]}(T^C)$ and that this mapping and its inverse maps bounded sets into bounded sets (cf. the property (2.8) in the Roumieu case). Since the spaces under consideration are all bornological, we might summarize the results from this section as follows,

Theorem 11.5.5. *Let Γ be a solid convex acute cone and suppose that M and N both satisfy $(M.1)$ and $(M.2)$, while M_p also satisfies $(M.3)$. Then, the Laplace transform*

$$\mathcal{L} : \mathcal{S}'^{[M]}_{[N]}[\Gamma] \to \mathcal{O}^{[M]}_{[N]}(T^C)$$

is an isomorphism of locally convex spaces.

English summary

This dissertation contains several new results situated in the theory of ultradifferentiable functions and ultradistributions. Specifically, we consider their topological invariants and asymptotic behavior.

The main goal of Part I is to characterize topological properties of ultradifferentiable function spaces with respect to their defining weight sequences and weight functions. In particular we consider the topological invariants of the Gelfand-Shilov spaces or spaces that contain them as a dense subspace. This results in several theorems that completely characterize certain locally convex properties, such as nuclearity and ultrabornologicity, of the spaces in question.

In Chapter 3 we introduce the Gelfand-Shilov spaces via weight sequence and weight function systems. This is done with respect to a parameter $q \in [1, \infty]$, where the ultradifferentiability is measured through the L^q-norm. In the main result of this chapter, we determine exactly when the definition of the Gelfand-Shilov space is independent of the parameter q that is used, which later we show to be equivalent to the nuclearity of this space. The chapter is then closed by a time-frequency analysis of the Gelfand-Shilov spaces, where we discuss the continuity of the short-time Fourier transform (STFT) and Gabor frames.

Chapter 4 considers nuclearity. More specific, we characterize exactly for two versions of the Gelfand-Shilov spaces when they are nuclear. The determination of the nuclearity of the first kind, the Gelfand-Shilov spaces considered in Chapter 3, is the main result of this chapter. Particular corollaries of this result are the kernel theorems for the Gelfand-Shilov spaces and their duals. The second type we consider are the so-called Beurling-Björck spaces. In a concise

manner we determine their nuclearity, which significantly extends the known results from the literature.

Next, in Chapter 5, we study the topological properties of certain variants of the Gelfand-Shilov spaces, whose topological structure take the form of (PLB)-spaces. A particular example are the multiplier spaces of the Gelfand-Shilov spaces. In this chapter we characterize exactly when such spaces are ultrabornological and barrelled. This is done via similar conditions as those of Vogt and Wagner for the splitting of short exact sequences of Fréchet spaces. To show that the conditions are sufficient we apply the STFT. Our technique for demonstrating that the conditions are necessary depends surprisingly enough on the existence of Gabor frames whose windows have specific rapid decay in time and frequency.

In Chapter 6, the final chapter of Part I, we look at the spaces of bounded ultradistributions and ultradistributions vanishing at infinity. The main results here are the so-called first structural theorems we obtain for both spaces. These results will serve as the cornerstones for the structural theorems we obtain in Part II.

Part II considers the asymptotic behavior of ultradistributions. In particular we obtain structural theorems for three types of asymptotic behavior related to translation and dilation. Moreover we also prove a general Tauberian theorem for the Laplace transform.

Chapter 9 looks at the quasiasymptotic behavior of ultradistributions. Specifically, we provide structural theorems for the quasiasymptotic behavior on the real line, both at infinity and at the origin. The crux of our proof is to convert the quasiasymptotics into the so-called S-asymptotic behavior, whose structure we may describe using the results obtained in Chapter 6. Once the structural theorems have been obtained, we analyse extension results. For instance we show that every ultradistribution having quasiasymptotics at infinity is in fact an element of an extension of the dual of the Schwartz space $\mathcal{S}'$ and her quasiasymptotic behavior holds there; an analogous yet local result is also obtained for quasiasymptotics at the origin.

The moment asymptotic expansion (MAE) is another major form of asymptotic behavior related to dilation, and in Chapter 10 we study

it in the context of ultradistributions. We introduce an ultradifferentiable version of the space of so-called GLS symbols and show that every element in its dual has the MAE. Moreover, we show that in the one-dimensional case every ultradistribution that has the MAE is contained in this dual. We also introduce a uniform analog of the MAE and demonstrate a partial characterization on the real line.

Finally, in Chapter 11, we extend a general Tauberian theorem for the dilation group from the context of distributions into that of ultradistributions.

Nederlandstalige samenvatting

Deze dissertatie omvat verschillende nieuwe resultaten gesitueerd in de theorie van ultradifferentieerbare functies and ultradistributies. Meer bepaald behandelen we hun topologische invarianten en hun asymptotisch gedrag.

Het hoofddoel van Deel I is het karakteriseren van topologische eigenschappen van ultradifferentieerbare functieruimten ten opzichte van hun bepalende gewichtsrijen en gewichtsfuncties. Specifiek bekijken we de topologische invariant van de Gelfand-Shilov ruimtes of ruimtes die deze bevatten als een dichte deelruimte. Dit resulteert in meerdere stellingen die bepaalde lokaal convexe eigenschappen, zoals nucleariteit of ultrabornologiciteit, van de ruimtes in kwestie compleet bepalen.

In Hoodstuk 3 introduceren we de Gelfand-Shilov ruimtes aan de hand van gewichtsrij- en gewichtsfunctiesystemen. Dit doen we via een parameter $q \in [1, \infty]$, waarbij de ultradifferentieerbaarheid gemeten wordt door middel van de L^q-norm. In het voornaamste resultaat van dit hoofdstuk bepalen we exact wanneer de definitie van de Gelfand-Shilov ruimte onafhankelijk is van de gebruikte q, wat later equivalent blijkt te zijn met de nucleariteit van deze ruimte. We sluiten het hoofdstuk af met een tijd-frequentieanalyse van de Gelfand-Shilov ruimtes, waar we de continuiteit van de short-time Fourier transform (STFT) en Gabor frames bespreken.

Hoofdstuk 4 staat in het teken van nucleariteit. Meer bepaald karakteriseren we voor twee varianten van de Gelfand-Shilov ruimtes exact wanneer deze nuclear zijn. De karakterisatie van nucleariteit bij de

eerste variant, de Gelfand-Shilov ruimtes beschouwd in Hoofdstuk 3, is het hoofdresultaat van dit hoofdstuk. In het bijzonder leiden deze resultaten tot kernstellingen voor de Gelfand-Shilov ruimtes en hun dualen. Het tweede type die we in beschouwing nemen zijn de zogehete Beurling-Björck ruimtes. Kort maar krachtig bepalen we hun nucleariteit, waarmee we de reeds gekende resultaten uit de literatuur significant uitbreiden.

Daarna, in Hoofdstuk 5, bestuderen we de topologische eigenschappen van varianten op de Gelfand-Shilov ruimtes, waar de topologische structuur de vorm aanneemt van (PLB)-ruimtes. Onder meer de vermenigvuldigersruimte van de Gelfand-Shilov ruimte is een voorbeeld van dit type ruimte. In dit hoofdstuk karakteriseren we exact wanneer deze ruimtes ultrabornologisch en barrelled zijn. Dit doen we aan de hand van condities gelijkaardig als deze van Vogt en Wagner voor de splitsing van korte exacte rijen van Fréchet ruimtes. Om aan te tonen dat de condities voldoende zijn maken we gebruik van de STFT. Onze methode voor de noodzaak van de condities aan te tonen hangt verassend genoeg af van het bestaan van Gabor frames waarvan de vensters een specifiek snel verval hebben in tijd en frequentie.

In Hoofdstuk 6, het laatste hoofdstuk van Deel I, bekijken we de ruimtes van begrensde ultradistributies en ultradistributies die verdwijnen op oneindig. De hoofdresultaten hier zijn de zogenaamde eerste structuurstellingen die we behalen voor beide ruimtes. Deze resultaten zullen de dienen als de bouwstenen voor de structuurstellingen uit Deel II.

Deel II behandelt het asymptotisch gedrag van ultradistributies. Meer bepaald bewijzen we structuurstellingen voor drie vormen van asymptotisch gedrag gerelateerd aan translatie en dilatatie. Tevens bewijzen we alsook een algemene Tauberse stelling voor de Laplacetransformatie.

Hoofdstuk 9 bekijkt het quasiasymptotisch gedrag van ultradistributies. In het bijzonder geven we structuurstellingen voor het quasiasymptotisch gedrag op de reële rechte, zowel op oneindig als in de oorsprong. De kern van onze techniek is het herleiden van het quasiasymptotisch gedrag naar zogenaamd S-asymptotisch gedrag, wiens

structuur we kunnen beschrijven aan de hand van de resultaten uit Hoofdstuk 6. Eens we de structuur verkregen hebben, analyseren we extensie resultaten. Zo tonen we aan dat elke ultradistributie die quasasymptotisch gedrag heeft op oneindig in feite een element is van een veralgemening van de duale Schwartz ruimte $\mathcal{S}'$ en haar quasiasymptotisch gedrag daar ook geldt; een analoog maar lokaal resultaat wordt verkregen voor quasiasymptotisch gedrag in de oorsprong.

De moment asymptotic expansion (MAE) is een tweede voorname vorm van asymptotisch gedrag gerelateerd aan dilatatie, en wordt in Hoofdstuk 10 bestudeerd in de context van ultradistributies. We introduceren een ultradifferentieerbare versie van de ruimte van zogenoemde GLS symbolen en bewijzen dat elk element in diens duale de MAE heeft. Bovendien tonen we aan dat in het ééndimensionaal geval elke ultradistributie die de MAE heeft bevat zit in deze duale. Verder introduceren we een uniforme variant van de MAE en demonstreren we op de reële rechte een partiële karakterisering.

Tot slot breiden we in Hoofdstuk 11 de algemene Tauberse stelling voor de dilatatiegroep uit van distributies naar ultradistributies.

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