Quantum gravity in two dimensions

Andreas Blommaert

UNIVERSITEIT
GENT

This thesis bundles the doctoral research by the author on JT gravity. Work on related topics is included as supplementary material. JT gravity is a model of two dimensional AdS quantum gravity. It captures the low energy dynamics of a large class of higher dimensional black holes. An exact quantization of different versions of JT gravity is presented based on a rewriting as a topological gauge theory. In particular we study three different versions. One is topologically trivial and corresponds to a continuous quantum system. The second includes a sum over baby universes and corresponds to an ensemble average of discrete quantum chaotic systems. The third version includes baby universes and the possibility of baby universes to be emitted and absorbed by spacetime D branes. Only the latter version is an accurate proxy for AdS quantum gravity. We address a version of the information paradox due to Maldacena within JT gravity. We furthermore argue that the cluster decomposition principle must break down in quantum gravity and we discuss dramatic effects of quantum gravity on physics near black holes.

Quantum gravity in two dimensions

Andreas Blommaert

Preface

Many have contributed to this work, some more knowingly than others. Before getting started, let me give credit where credit is due.

Let me start by thanking those who have contributed research wise. First and foremost, I am deeply indebted to Thomas Mertens for several reasons. He has been my guide in the world of academia from day one and is a great colleague to have. In terms of research he has had a huge impact. In that respect I want to thank him for introducing me to JT gravity and for various explanations on the basics of high energy physics. Thomas has thought me a lot and I look up to him in many ways. On a personal level I want to thank him for advise regarding post doc applications and for the fun collaboration. Without him I would have had to essentially find my way all alone in this once new world of theoretical physics. Consequently my research would most likely not have been as cool as I find it today.

I owe a great deal to Henri Verschelde. Of course he has helped me plenty research wise. More importantly Henri gave me the opportunity to pursue my life goals four years ago. In the same context let me thank Frank Verstraete who provided me with funds during my first year as a graduate student. Henri was also the one who introduced me to the beauty of theoretical high energy physics as an undergraduate. It was a time when I was just about to lose faith in the beauty of the universe. I was starting to wonder if there were any physics courses that were not about diodes or Hartree-Fock. Without Henri I would honestly not be a physicist today. He can not be thanked enough.

Furthermore I should thank a handful of people for useful communication or discussion over the years. In pseudo random order thanks to Gertian Roose, Valya Zakharov, Phil Saad, Steve Shenker, Douglas Stanford, Jordan Cotler, Joaquin Turiaci, Juan Maldacena, Herman Verlinde and William Donnelly. Finally let me thank Steve Shenker, Stanford University and the BAEF for providing me with an exciting opportunity to continue my journey in physics. Let me also take the time to thank the government of China for making physically getting to Stanford University perhaps as challenging as getting a job there.

In a separate category let me thank Manu for reading through an earlier version of this work. She helped to get rid of most typos. It was a horror show. Certainly there will still be some typos left for which I take full responsibility.

Then there are of course the people in my personal life. They may not have contributed directly to the work I am presenting here. However the have certainly contributed indirectly. Let me not go into too much detail here. Certain things should not go on official record. First of all in this category I want to thank my parents. Words are not enough. Let me just thank them for a wonderful life. I could have asked for nothing more. Second, moving up one knot in the old family tree, let me thank my bothers for enduring life with me. I am sorry for all the collateral damage such as the fact that you now know who Simon Ammann, Martin Fourcade and Christian McCaffrey are. Let me also thank my grandparents for always rooting for me.

Furthermore let me thank my friends. Thanks to the boys from the KBK for making life fun, and for sticking with me at all time. Thanks also for the fun chats and for some great games over time. Nothing beats the feeling of winning. My life expectancy has certainly been shortened by several years due to our numerous escapades, worth it. In this same category let me also thank Marie. She has impacted some professional life choices, but more importantly she makes my everyday life a happier place.

Finally let me thank two guys who have unknowingly succeeded in motivating me time and time again and who most certainly will never find out about this thank you. Motivation is key in life and a necessity in research. It is valuable to have examples who never seem to give up. Those who keep a positive attitude even in the darkest of days. It is therefore a pleasure to thank Sam Mills for reminding me to keep pounding and Johannes Klæbo for reminding me to keep smiling.

English summary

This section contains an abstract, a motivation in layman's terms and a short update on the current status of this line of research. A reading guide is presented in the introductory chapter 1.

Abstract

This thesis bundles the doctoral research by the author on JT gravity [1, 2, 3, 4]. Work on related topics is included as supplementary material [5, 6]. JT gravity is a model of AdS_2 quantum gravity. It captures the universal low energy dynamics of a large class of higher dimensional black holes. An exact quantization of different versions of JT gravity is presented based on a rewriting as a topological gauge theory. In particular we study three different versions. One is topologically trivial and corresponds to a continuous quantum system. The second includes a sum over baby universes and corresponds to an ensemble average of discrete quantum chaotic systems. The third version includes baby universes and the possibility of baby universes to be emitted and absorbed by so called eigenbrane boundaries. Only the latter is an accurate proxy for more generic models of quantum gravity. We address Maldacena's version of the information paradox within JT gravity. Finally we discuss matter probes and the dramatic quantum effects of gravity on physics close to the semiclassical black hole horizon.

Motivation in layman's terms

In order to motivate research on JT quantum gravity let me first motivate research on quantum gravity in general. One immediate reason why you should care about quantum gravity is that the so called theory of everything describing everything we see around us is a theory of quantum gravity. Another reason is that we will need to understand the intricacies of working with quantum gravity if we want to develop a satisfactory understanding of black holes and cosmology. Let me point out why our current understanding of black holes and cosmology is not quite satisfactory. One fact in this regard is that at the moment there is not a soul alive who knows for a fact what would happen if you were to jump into a black hole. Certainly we do not know what you would find inside a black hole. If you would make it that far. A potentially more important fact is that to date there is no satisfactory description of the origin of the universe itself. We do not even know if it has an origin. Of course we have a model for the "creation" of the universe in the big bang theory. That model though is only reliable as a description of what went on if we wait a sufficiently long time after the actual "moment of creation". We do not have a clue what went on at very early times. The question of the origin of the universe is actually surprisingly similar to some of our black hole issues. A model of the interior of black holes does exist but just like the big bang it is not a particularly good one. The model in this case is Einstein gravity and it predicts the existence of a singularity inside black holes. This singularity in a black hole is remarkably similar to the big bang model. Both are certainly at best an approximation. One way to appreciate this is as follows. We have good reasons to believe that sensible models of quantum gravity are discrete. In more basic terms this technical statement means that everything we see around us can be described by a sum of a finite number of terms. The result therefore can never be infinite. The big bang and the singularity are in a very precise sense infinite. Therefore neither description can be precise. We would like a more appropriate description. We would like to understand what really happens inside black holes. More importantly we would like to know what went on at the "beginning" of the universe. To do so we will need to familiarize ourselves with quantum gravity.

So what is the catch? Unfortunately realistic descriptions of quantum gravity are very hard to come by. There is of course string theory. String theory might just have a realistic variant that describes our universe. One major problem with string theory though is that it is very hard to do even the simplest of calculations. By consequence we would argue that it has proven difficult to answer deep black hole questions directly in string theory. Due to this complexity of string theory we believe it might not be advisable to tackle problems such as that of the black hole interior via direct head on string computations. More precisely it seems unlikely that doing string calculations is the most efficient way to make any progress on our understanding of black holes and the big bang. We will do essentially the complete opposite. Let us back off from this unwieldy model of quantum gravity. Let us look instead for a ridiculously simple model of quantum gravity and see if we can find at least some traction.

This is where JT gravity comes in. JT gravity is the theory of quantum gravity in two dimensions. This might seem like a giant leap in the wrong direction. The universe we live in has four dimensions so why in the world would we be interested in quantum gravity in two dimensions? By the way. String theory is a theory of quantum gravity in *eleven* dimensions. Just a sketch of context. Enough with the negatives though. There are many good reasons to be interested in two dimensional quantum gravity. One reason to be interest in JT gravity is that the theory is rather universal. Many models of quantum gravity reduce to precisely JT gravity in a low energy limit. Examples are quantum gravity in three dimensions and large spinning black holes in four dimensions. When you thing about it is is not very surprising that the low energy sector of a theory has a "simpler" description as compared to the full theory. In this context "simpler" just means less dimensions. The following point is extremely important to appreciate. Low energies is not necessarily a serious constraint. There are certain important questions in quantum gravity which do not probe high energy physics. By consequence as long as we are asking questions about black holes that do not probe for very high energy physics

we can get along just fine with asking them in JT gravity.
Forget about quantum gravity for a while. Let us just focus on the word quantum. Think about a hydrogen atom with its energy levels. For such a quantum system we expect classical behavior at high energies and genuinely quantum effects at low energies. Similarly a decay process between two very nearby energy levels will result in a very low energy byproduct. By consequence our intuition about quantum mechanics is that interesting stuff happens at low energies or involves small energy differences. For example many materials have the most amazing features when we cool them to almost zero temperature. This is because at low energies the quantum effects do not get washed out and we are actually sensitive to the individual energy levels of the system. Of course in quantum gravity there are a lot of interesting questions about the behavior at high energies. JT gravity is clearly not well suited to tackle such questions. The point which we are trying to make here is that there are also a lot of interesting questions about the behavior of quantum gravity at incredibly low energies or involving the tiniest of energy differences. To get interesting results out of JT quantum gravity the trick is clearly to probe for such genuine quantum effects. For example there is the fact that the spectrum of a quantum theory is discrete. Hence originally the word quantum. Can we understand the origin of this discreteness in quantum gravity? What does this tell us about quantum black holes? Answering this question within JT gravity is actually our main goal in this work. A second question which we would like to answer is what is going on close to the horizon of black holes. General intuition suggests quantum effects become important. The intuition is that when something falls into a black hole it seems to freeze. Something that moves slowly has a low energy. By consequence we expect genuine quantum effects to kick in.
To get to these question we will first need to solve the theory. This is a bit technical but good fun nonetheless. Once we are able to play around with the calculations we might just be able to answer these physically relevant questions. The answers which we will find are rather cool if you like science fiction. It turns out for example that demanding discreteness in JT gravity implies there must be very exotic dynamical processes in quantum gravity. For example it must be possible for our universe to spit out or absorb baby universes at will. When two baby universes are born far apart in time or space they can act like what Morgan Freeman imagines when he uses the word wormhole. They can create a shortcut to travel to far away regions or times. Furthermore as it turns out the emergence of these baby universes tends to proliferate close to the horizon. They give rise to the most violent quantum effects just outside the horizon.

What has been achieved

We have certainly not reached the point where we can make claims about the fate of the interior singularity and the beginning of time and all that. However, as a community in general we have arguably made some significant progress in understanding how quantum gravity works in general. For example as pointed out above it is now clear that wormholes and baby universes play a pivotal role in understanding all kinds of genuine "quantum" aspects of black holes. Examples of such quantum aspects include the quantum chaotic nature of black holes [7, 8, 9, 4] and their unitary evaporation [10, 11]. These are definitely interesting times for those of us interested in quantum black holes. For example it does not seem totally out of reach for someone to find some sort of precise claim about the fate of the singularity in the foreseeable future. That would be so cool.

Contents

1 Introduction

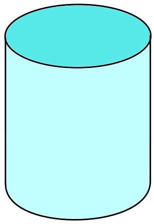
We introduce the questions which we will be investigating in this work and the model in which we will be tackling them.

The questions

How did the universe begin? What is inside of a black hole? What comes of objects that are thrown into a black hole? These are some of the most important open questions in modern high energy physics. Notably they are gravitational questions, asking for an inherently gravitational answer. To address them we will need to figure out how to deal with gravity as a quantum theory. What are the rules of the game for quantum gravity? There is at least one major guiding principle for the behavior of quantum gravity to which we can resort. There are old and quite general arguments [12, 13] suggesting that any gravitational theory in a given number of dimensions has a holographically dual description as a theory living on a spacetime with one less dimension. This fact is known as holography. The behavior of the dual theory is essentially a rule book for quantum gravity, but it is written in a foreign language. One of the major points of focus in modern high energy physics is to establish a dictionary by which we can translate the rules for the dual theory into a set of rules for quantum gravity. The stakes are high. If successful this translation will allow us to answer at least two out of our three initial questions (the ones associated to black holes), and shed light on the third.

Holography has been explicitly realized realized in a variety of different setups about two decades ago. For example asymptotically AdS quantum gravity has a dual description as a unitary CFT living on the timelike boundary of AdS. For universes with a positive cosmological constant we have that asymptotically dS quantum quantum gravity is dual to a non unitary CFT on the spacelike future boundary of global dS. Finally for flat universes the holographic screen are the future lightlike boundaries of the Penrose diagram.
This holographic duality is especially compelling for AdS quantum gravity. Most of us high energy physicists have decided for ourselves that the universe ought to be a unitary quantum theory, where information is not lost. However it is not understood how this is realized in terms of gravitational dynamics. What happens inside of a black hole such that information is not lost in the singularity? The dual unitary CFT on the other

hand is in some sense an explicit realization of unitary quantum gravity. So holography proves that AdS quantum gravity is indeed a unitary quantum theory. This should be compared for example to the situation in dS quantum gravity where unitarity is not a manifest consequence of holography. Unitarity of dS bulk quantum gravity should hence universally emerge in another manner, which we don't yet understand. For this reason we will proceed in this work with AdS quantum gravity. We define quantum gravity to be the bulk dual to a unitary CFT, with some additional constraints on the CFT which are not important here.
A second important observation is that the CFT dual to AdS quantum gravity lives on a compact spatial manifold Σ cross time. For example in 3d Σ is naturally a circle



(1.1)

Because $\mathrm{Vol}(\Sigma)$ is finite the spectrum of the Hamiltonian $H(\Sigma)$ on these Cauchy slices is discrete. Think for example about the Laplacian on a sphere versus the Laplacian on an infinite plane. The holographic duality then implies that the spectrum of AdS quantum gravity $H(\mathcal{M})$ on some Cauchy slice $\mathcal{M}$ with $\partial\mathcal{M} = \Sigma$ is also discrete.

The rules of the dual CFT imply that unitarity and discreteness are universal properties of AdS quantum gravity. But how do these rules translate into features of bulk AdS quantum gravity? What is the bulk gravitational explanation of this unitarity and of this discreteness? How are these hallmark properties of the dual CFT encoded in the bulk gravitational path integral? The goal of this work is to partially address these question.
One way to sharpen these questions is to consider an argument by Maldacena [14] about late time correlators. Consider two dimensional AdS quantum gravity. The boundary two point function in a thermal state is that of a discrete quantum mechanical system

$$\langle \mathcal{O}(0)\mathcal{O}(t)\rangle_\beta = \sum_{ij=1}^{L} \lvert \mathcal{O}_{ij}\rvert^2 \,\cos t(E_i - E_j)\, e^{-\beta(E_i+E_j)}. \tag{1.2}$$

Given plausible assumptions about the density of the energies and the matrix elements [15, 16] one proves that at early times this decays exponentially

$$\langle \mathcal{O}(0)\mathcal{O}(t)\rangle_\beta \sim \exp\left(-\frac{4\pi\ell}{\beta}t\right). \tag{1.3}$$

However at late times the discreteness shines through and the correlator oscillates erratically around an in general nonzero averaged value. Semiclassical physics (defined throughout this work as quantum field theory on the saddle point of the gravitational action, which is a classical black hole) would claim that the exponential decay continues on all time scales. This can't be true in quantum gravity, due to its fundamental discreteness. Therefore there must exist some universal bulk gravitational explanation for this late time behavior of correlators near the boundary. In particular quantum gravity should explain the erratic oscillations around a generically nonzero average. What is this explanation?

A further salient feature of generic theories of quantum gravity is that their local eigenvalue statistics are described by random matrix theory [7]. This is so because black holes are quantum chaotic systems. The local level statistics of quantum chaotic systems are universally described by random matrix theory [17]. Random matrix theory gives predictions about the averaged behavior of late time correlators, but does not fix the details of the erratic oscillations. As a first major step towards explaining the late time behavior of correlators one might ask the following question. What is the explanation of random matrix statistics from the bulk gravitational path integral? This question was largely addressed in recent years [7, 9].

The answer is that random matrix universality is explained in bulk quantum gravity by the fact that in quantum gravity we cannot neglect the possibility of baby universes detaching from and reattaching to our parent universe. These baby universes act like wormholes, creating shortcuts between two distant points in time and or space. Evidence for this was recently gathered in [8, 9, 4, 18]. So not only do wormholes exist, they actually play a key role in the working of gravity when it's at it most violent, for example close to or inside a black hole. The erratic oscillations and associated discreteness on the other hand are explained by the presence of branes in the gravitational theory on which these baby universes can be born and can come to die. Evidence for this type of physics was recently gathered in [4, 19]. Each of these recent papers was written in the context of a particular theory of two dimensional AdS gravity to which we will turn from hereon.

The model

In this work we will be studying quantum aspects of a specific theory of dilaton gravity in two spacetime dimensions with action:

$$S[g,\Phi] = -S_0\chi - \frac{1}{2}\int dx\sqrt{g}\,\Phi(R+2) - \int_\partial d\tau\sqrt{h}\,\Phi(K-1). \tag{1.4}$$

The first term is the Einstein Hilbert action for two manifolds. We see that we only get topological dynamics in two dimensional quantum gravity by just considering Einstein-Hilbert gravity. The above theory can be considered the simplest model of two dimensional quantum gravity with more than just topological dynamics. This theory was first considered by Jackiw and Teitelboim in the mid eighties [24, 25]. Henceforth they will be known by their initials J and T. This theory of two dimensional gravity is therefore known as JT gravity. We will be considering the theory on two manifolds with a boundary ∂ on which we impose amongst others the boundary condition that Φ_∂ is some constant. The dilaton Φ should not quite be considered as a physically interesting field in this model but rather as just a Lagrange multiplier. Doing the path integral over Φ localizes on hyperbolic Riemann surfaces with fixed local negative curvature $R+2=0$. The precise value of the cosmological constant here is but a matter of choosing units. Anyway. We see that JT gravity is a theory of hyperbolic Riemann surfaces with action:

$$S[g] = -S_0\chi - \int_\partial d\tau\sqrt{h}(K-1), \qquad R+2=0. \tag{1.5}$$

As compared to two dimensional Einstein-Hilbert gravity we have essentially just included a slightly more interesting boundary term on top of the usual Gibbons Hawking term to provide the model with some non-topological boundary dynamics. In this sense we would argue that JT gravity is really what one would most naturally point to as two dimensional quantum gravity. Before arguing further in favor of this model let us note one thing. Rather surprisingly it turns out [19] that actually the purely topological Einstein-Hilbert gravity in two dimensions in not so dull after all. Rather miraculously one can find for example traces in ordinary two dimensional Einstein-Hilbert gravity of the discreteness we will spends most of chapter 4 talking about [19]. For those who have followed the developments in this field though it should be obvious that those results would have not been obtained had we not first had several seminal breakthroughs in JT gravity [20, 21, 22, 23, 8, 9].

A brief history of JT gravity

Moving on. We have explained in the summary section above why on general grounds one should be interested in studying aspects of quantum gravity in two dimensions. Let us not repeat that discussion as a whole here. Rather let us be more specific about the claimed universality of JT gravity. One might argue that JT gravity only truly came to the spotlight due to the work of Kitaev, Maldacena and Stanford [26, 27]. The reason being that JT gravity emerges as an effective low energy description of the SYK model. For more on the SYK model see among others [26, 28, 29, 30, 27, 31, 7, 32, 33, 34, 35, 36]. The SYK model is a model of N Majorana fermions with all to all random interactions:

$$H = \sum_{i,j,k,l=1}^{N} \mathcal{J}_{ijkl}\psi_i\psi_j\psi_k\psi_l. \tag{1.6}$$

For every fixed choice of couplings $\mathcal{J}_{ijkl}$ this is just your ordinary quantum chaotic system with some finite dimensional Hilbert space of dimensions 2^N [7]. Typically though what is referred to as the SYK model is obtained by ensemble averaging over different such discrete quantum mechanical systems. To do so one averages over each of the coupling $\mathcal{J}_{ijkl}$ with a Gaussian weight. So to calculate a certain observable in the SYK model we just choose some fixed $\mathcal{J}_{ijkl}$ and solve the system. One now does this for a large number of different $\mathcal{J}_{ijkl}$ and averages with a Gaussian weight on the $\mathcal{J}_{ijkl}$. This gives an answer for the correlator in question. Anyway. As it turns out the dynamics of the SYK model at low energies reduces to the dynamics of JT gravity [27]. In fact it reduces to Schwarzian quantum mechanics which we will discuss in chapter 2. It did not take long for people to realize though that Schwarzian quantum mechanics is holographically dual to JT gravity [20, 21, 22, 23, 37, 38, 39]. This is extremely exciting in the sense that in the SYK model we have a finite dimensional quantum mechanical system that reduces to pure quantum gravity at low energies. Therefore by definition the SYK model is a UV complete theory of quantum gravity. At least its bulk dual would be. Unfortunately we do not know precisely what that dual is. Besides the SYK model it turns out that JT gravity appears rather universally as the low energy limit of physics around near extremal black holes in higher dimensions [40, 41, 42]. Furthermore we can understand that full fledged JT gravity is in some very precise sense the classical limit of AdS_3 quantum gravity [43, 44, 1, 45, 46]. In this sense, by learning more about JT gravity we are learning universal lessons. Roughly speaking whatever lesson we can get from JT gravity should translate into a lesson on black holes in higher dimensions as well.

Anyway all these developments have sparked a high degree of interest in JT gravity itself over the last five years or so. For a list of references see [24, 25, 20, 21, 22, 23, 47, 27, 48, 49, 50, 51, 43, 44, 52, 53, 1, 54, 55, 8, 56, 57, 2, 58, 3, 9, 59, 60, 61, 62, 18, 63].[1] Originally the main motivation was the relation of JT gravity to the SYK model. However over the years this has shifted somewhat as people started to realize that JT gravity in itself is actually quite an interesting model of quantum gravity. The main reason for that is twofold. On the one hand it turns out that JT gravity is both incredibly rich and also incredibly simple at the same time. It is rich because it has black holes. It is simple because it turns out that we can do basically any calculation in JT gravity exact including all backreaction. This means we have a completely solvable model of quantum gravity in which we can study dynamical processes that involve black holes. This will be discussed in detail in this work. A second important reason is that JT gravity is in itself much better a proxy for quantum gravity than we would expect a priori. It is much more than just the low energy limit of a good model of quantum gravity. This shines through in the work of [8, 9]. The low energy limit of the SYK model is dual to the topologically trivial version of JT gravity discussed in chapter 2. However as we will discuss in chapter 3 and further on it is possible to define a more complete version of JT gravity by summing over topologies. The resulting theory is structurally not half bad as compared to the SYK model. It turns out that this version of JT gravity is dual to an ensemble average of discrete quantum chaotic system, just like the SYK model. The only structural difference with the SYK model is that the quantum mechanical systems in question for pure JT gravity have an infinite dimensional Hilbert space. Because of this the theory is not UV complete. By consequence JT gravity is a bad proxy for quantum gravity when we are asking UV questions. However it is a great representative for quantum gravity when we are asking low energy questions. Alternatively is does very well when we are asking questions that probe locally on the energy axis. As explained in the summary above, such questions are probing genuine quantum questions in gravity. In the second part of this work we will try to address question of these types as much as possible. As a closing remark let us note that taking JT gravity seriously as a model of quantum gravity has recently sparked some amazing developments such as [10, 11].

[1]This list is far from exhaustive though.

By taking JT gravity seriously and using it as our testing case for ideas about quantum gravity it seems that we *can* really get to deep lessons about the behavior of quantum black holes in the real world.

Structure of this work

This work contains four main chapters and two supplementary chapters. They are organized as follows.
In **chapter 2** we solve a version of JT gravity where we constrain the metrics g in the path integral to be of the simplest possible topology consistent with the boundary conditions. This version of JT gravity is dual to Schwarzian quantum mechanics and features as the low energy limit of SYK. In particular we point out that JT gravity has a quantum mechanically completely equivalent formulation as a particular two dimensional topological gauge theory. Such topological field theories are well known to be exactly solvable. We apply this to JT gravity and compute all correlation functions. This chapter is based on [1, 2]. In order to present a complete story we do mix in some features of the model that appeared elsewhere in for example [22, 64].
In **chapter 3** we solve JT gravity for metrics on any fixed but possibly complicated topology consistent with the boundary conditions. We then sum over all such possible topologies. We do so by studying in detail how we can cut and glue the path integral of two dimensional topological field theories. This is then applied to JT gravity which comes with an additional difficulty associated with invariance under large diffeomorphisms. As before we are able to get a precise answer for generic correlation functions. We furthermore comment on the question of factorization in the topologically trivial model. This chapter is based on parts of [2, 3, 4]. We do borrow in this particular chapter a significant number of results from work by others [9, 65, 18]. In a quickly evolving field our work on this topic in [2, 3, 4] can not be considered as logically independent of [9, 65, 18]. A self contained story forces us to mix elements from both parties. A particularly important point to understand the ensuing chapters is that when we sum over topologies the theory becomes structurally equivalent to a random matrix theory [9]. This is an ensemble average of discrete quantum chaotic systems. We add a significant amount of additional discussion in this chapter as compared to the story presented in the papers [2, 3, 4]. For example we exploit maximally our understanding of cutting and gluing in gauge theories obtained in chapter B.
In **chapter 4** we solve a third version of JT gravity. Here we allow all topological types of spacetimes or metrics g that end on a fixed number of asymptotic boundaries. Furthermore we also allow them to end on so called "eigenbranes". They act like D-branes in string theory. The difference is that in string theory it's worldsheets that are ending on the branes. Here it's the spacetimes or universes themselves which can end on these branes. The eigenbranes can emit and absorb universes at will. We prove that this version of JT gravity is structurally equivalent to a single discrete quantum chaotic system. This provides with a bulk gravitational interpretation to go with Maldacena's information paradox [14]. That paradox is basically a call for a bulk gravitational interpretation of the late time erratic oscillations in boundary correlation functions. We provide with

such an explanation in JT gravity in terms of eigenbranes. This chapter is based on [4]. In **chapter 5** we move to understand a bulk version of Maldacena's information paradox. We prove via explicit computations that the late time behavior of boundary correlators maps to the large distance behavior of correlators of matter probes in the bulk. The late time erratic oscillations are found to map to a breakdown of cluster decomposition in bulk quantum gravity. One way to obtain large distances in the bulk is to probe very close to the semiclassical black hole horizon. We find clearly that quantum effects proliferate close to the horizon. This counters the intuition that quantum effects would only be important when the gravitational curvature is large such as close to a singularity. The more appropriate statement is that UV modifications of gravity will only be important when the gravitational curvature is large such as close to a singularity. Quantum effects though are rather naturally associated with very small energies, or very small energy differences. The latter will be important at late times and large distances. The UV corrections on the other hand will be important at early times and short distances. One of the conclusions is that quantum fields in Rindler is a very poor approximation to physics very close to the semiclassical black hole horizon. Quantum effects cannot or rather should not be neglected close to a black hole. This chapter is based on [3]. It furthermore contains certain comments and some calculations which are yet to appear [66, 67].

In the **supplementary chapter A** we discuss electromagnetic edge states. These are degrees of freedom which arise when we cut a gauge theory on some cutting surface. Understanding how this works in gauge theories in general was instrumental in understanding how cutting and gluing works in JT gravity. This chapter is based on [5].

In the **supplementary chapter B** we discuss a path integral perspective on edge states in gauge theories. This more fundamental way of looking at edge degrees of freedom can in particular be applied rather easily to topological field theories such as 3d Chern-Simons theory or 2d BF theory. JT gravity is a special case of the latter. The cutting and gluing in JT gravity, which is instrumental in understanding JT gravity on complicated topologies in chapter 3, is logically an application of the discussion in this supplementary chapter. It is also phrased in the same language. This chapter is based on [6].

Comments on the purpose of this work

The goal of this doctoral thesis it to provide a more or less self contained and streamlined story of recent developments in JT gravity. Of course this is written from a biased perspective. The purpose here is not to present all possible computations in the context of JT gravity. Rather we try to follow one story line all the way through. For this reason we have chosen to leave out a significant number of the calculations that have appeared in publications by the author [5, 6, 1, 2, 3, 4]. Furthermore style wise we have tried to opt as much as possible for intuitive explanations in favor of rigorous derivations. The hope is that doing so lowers the level of technicality and makes this work accessible to a wider audience. By consequence the chapters are written in a somewhat more fluent style than we would usually opt for in a paper directed at a specialist audience. Another consequence is that we have often neglected overall constants including minus sign. They are easily restored but in our opinion do not add significantly to the presentation. On the other hand we have tried to make this work interesting also for the more specialist reader who might be already familiar with the contents of [5, 6, 1, 2, 3, 4]. To this end we have often pursued a different presentation and different emphasis versus how certain topics were presented in the papers. Furthermore we have included many up to date comments and remarks as well as several new calculations.

2 Euclidean disks and Schwarzian quantum mechanics

This chapter combines two publications [1, 2] by the author in collaboration with Thomas Mertens and Henri Verschelde concerning JT gravity on a disk. We identify JT gravity as a particular two dimensional topological field theory. We exploit this correspondence to completely solve the theory by calculating essentially all correlation functions of known observables. In this chapter we constrain the topology of the spacetimes that contribute in the model to be that of a disk. This constraint is lifted in chapter 3.

2.1 Introduction

Ever since the early work on JT gravity is has been known that the action of JT gravity is identical to that of a 2d $\mathrm{SL}(2,\mathbb{R})$ BF theory [68, 69, 70, 71]. This is a topological quantum field theory much like 3d Chern-Simons. This identification was originally only considered at the level of the action and locally and does not guarantee quantum equivalence of both theories. The latter requires an identification of Euclidean path integrals. Schematically:

$$\int [\mathcal{D}A] e^{-S[A]} = \int [\mathcal{D}B] e^{-S[B]}. \tag{2.1}$$

Such an equivalence depends obviously on the contour choice for each of the integrals. For example the action of an $\mathrm{SL}(2,\mathbb{R})$ BF theory depends only on the local properties of $\mathrm{SL}(2,\mathbb{R})$. In other words it is characterized by the algebra. Path integrals of $\mathrm{SL}(2,\mathbb{R})$ BF on the other hand depend on the allowed range of the group elements g which is the exponentiation of the algebra. Different sensible exponentiations exist, only one of which is actually the group $\mathrm{SL}(2,\mathbb{R})$. So there are many sensible inequivalent "$\mathrm{SL}(2,\mathbb{R})$" BF theories all of which have the same action.
A similar ambiguity in the choice of integration contour exists in JT gravity, or quantum gravity in general. First there is the question when we path integrate over metrics g what we consider to be allowed metrics? What are the boundary conditions? How exotic a metric do we allow? Do we allow singular metrics in the path integral?[1] Do we allow

[1] No.

zero metrics $g = 0$?[2] Do we allow topology changes in the metric?[3] When considering perturbative quantum gravity none of these questions are relevant, modulo the one about the boundary conditions, because we are perturbing around some smooth metric of fixed topology. In a genuine path integral description of quantum gravity though, these questions are at the very heart of the discussion.
Gravity has a first order formulation where instead of the metric field g we path integrate over vielbeins e and spin connection ω. This transformation has some desirable effects. For one it rewrites quantum gravity as a gauge theory such as 2d BF theory. In such theories it is easier to control the integration space. Moreover we have natural path integral measures. The downside is that it also somewhat obscures the natural integration space in gravity. Certain inequivalent configurations of e and ω correspond to equivalent metrics g. Should we integrate over inequivalent e and ω or only over inequivalent metrics g?[4] Furthermore perfectly sensible connections e and ω from the gauge theory point of view may correspond to singular or zero metrics.

We would like to address as much of these confusions as possible within JT gravity throughout this work. As it turns out, there is a somewhat natural manner not to have singular metrics in the gauge theoretic path integral. Furthermore we will take the point of view that we want to integrate only over inequivalent metrics, which constrains the integration space in terms of e and ω somewhat. Finally, the question of whether or not we should allow topology changes is a guiding line throughout this work. Our point of view is that there are certain general expectations about the structure we expect from a generic theory of quantum gravity. For example, on account of the holographic dictionary, a proper theory of quantum gravity should end up being unitary and having a discrete spectrum. We read these properties as implementing a constraint on sensible choices of the integration contour over bulk metrics. This will ultimately select a very specific family of contours which do allow topology changes amongst other things.
In this chapter we will take a more modest first step and consider JT gravity with a single asymptotic boundary and with the path integral over metrics constrained such that there are no topology changes. The idea is to more of less define the theory to be completely quantum mechanically equivalent to a particular "$\text{SL}(2,\mathbb{R})$" BF theory and to check that this theory is precisely dual to a 1d Schwarzian quantum mechanics. We will first advocate for this equivalence via path integral manipulations. We then prove the equivalence by matching all correlators in 1d Schwarzian quantum mechanics to all correlators in JT gravity with particular boundary conditions. It is important to note that it is possible to prove rigorously the quantum mechanical equivalence of JT gravity to this "$\text{SL}(2,\mathbb{R})$" BF theory. The piece of the proof that we leave out here is proving that the path integral measure over metrics $[\mathcal{D}g]\,\delta(R+2)$ implies the natural path integral measure in the first order formulation. This is basically the Haar measure. The reason we leave this part of the proof out is that, although important, it's a fairly technical computation and would distract from the story presented here. We kindly

[2] Yes.
[3] Yes.
[4] The latter is the case in JT gravity. For AdS_3 gravity the situation is more complicated [45].

refer the interested reader to sections 3.1, 3.2 and 3.3 in [9] where this is explained in a rather didactic manner, however for the key part of the calculation they refer to [72, 73], which are rather technical papers.

To summarize, our goal here is solve for all correlators of JT gravity on a Euclidean disk by resorting to its first order "$\text{SL}(2,\mathbb{R})$" BF formulation. The specific exponentiation can be fixed by comparing with exact amplitudes in the dual 1d Schwarzian quantum mechanics which have been calculated using different techniques that are unrelated to bulk quantum gravity in [43]. The specifics of this exponentiation "$\text{SL}(2,\mathbb{R})$" automatically capture several nice properties, for example the $\text{SL}(2,\mathbb{R})$ connections associated with singular metrics with conical deficits are not included in the integration contour.

The remainder of this chapter is organized as follows.
In **section 2.1** we introduce 2d BF theories and establish a holographi correspondence to quantum mechanics on a group manifold, including a mapping of all significant operators. We then briefly introduce JT gravity on the disk in the second order formulation and its known equivalence to 1d Schwarzian quantum mechanics on the level of the partition function [21, 22, 23]. We spell out the first order formulation at the level of the action which points to an equivalence with "$\text{SL}(2,\mathbb{R})$" BF theory. We write out the boundary conditions in the first order formulation that result in an equivalence of this "$\text{SL}(2,\mathbb{R})$" BF theory to 1d Schwarzian mechanics. We then explain from the path integral point of view how natural operators in the second order formulation map to operators in the gauge theoretic formulation and how these further map to operators in the Schwarzian formulation.
In **section 2.2** we discuss the exact answers for the relevant correlators in BF theory for compact groups. This is extended to cosets and noncompact groups.
In **section 2.3** we propose to consider a particular "$\text{SL}(2,\mathbb{R})$" BF theory as quantum mechanical description of JT gravity and calculate its path integrals using the techniques of the previous section. The resulting answers for the correlators match exactly with the dual correlators of Schwarzian quantum mechanics calculated first in [43]. This proves the equivalence.
In **section 2.4** we conclude by providing further motivation for considering this particular "$\text{SL}(2,\mathbb{R})$" BF theory as first order description of JT gravity and comment on the relation to quantum gravity in AdS_3 and Chern-Simons theory whose path integrals have a very similar structure.

2.1.1 The holographic dual to BF theory

We define 2d BF theory on some generic Riemann surface $\mathcal{M}$ with boundary ∂ as:

$$S[\chi, A] = \int_{\mathcal{M}} \text{Tr}(\chi F) - \frac{1}{2}\int_{\partial} \text{Tr}(\chi A). \tag{2.2}$$

We could choose to add a type of string coupling to this model that counts the genus by adding to the action a term proportional to $\chi(\mathcal{M})$. Here we will be interested in fixed

topology so such a term would add no interesting dynamics to this story. In particular we are interested in the theory on Riemann surface that is topologically a disk.
In order to completely define the theory, we need to choose boundary conditions. Variation of the action results in a boundary term proportional to:

$$\int d\tau \, \mathrm{Tr}(\chi \delta A_\tau - A_\tau \delta \chi). \tag{2.3}$$

One possible consistent set of boundary conditions is:

$$A_\tau^a|_\partial = \chi^a|_\partial. \tag{2.4}$$

The labels a denote that both objects are algebra valued. Other sensible boundary conditions exist, see for example (2.155). They result in a fundamentally different theory. Anyway, we can path integrate over the bulk field χ which enforces $F = 0$. This localizes on flat connections $A = g^{-1}dg$ with g a periodic $g(t+\beta) = g(t)$ trajectory on the group manifold. Such trajectories make up the loop group LG. There is clearly a redundancy in description $g \sim Vg$ for constant group elements V when going from A to g. Therefore the integration space is over the loop group modulo constant functions LG/G. The path integral with no operator insertions reduces to:

$$S[g] = -\frac{1}{2} \int d\tau \, \mathrm{Tr}\big(g^{-1}\partial_\tau g\big)^2. \tag{2.5}$$

The measure in the BF path integral implies the Haar measure $[\mathcal{D}g]$ for these boundary degrees of freedom.[5] The resulting dynamics is nothing but the quantum mechanics of a particle travelling on the group manifold studied for example in [75, 76, 77]. Wavefunctions in this quantum mechanical model are obviously by definition the square integrable functions on the group manifold. A basis for such functions are the representation matrices of the group. Indeed, Schur's orthogonality relation is:

$$\int dg R_{i,ab}(g) R_{j,cd}(g^{-1}) = \frac{\delta_{ij}}{\dim R_j} \delta_{ad}\delta_{bc}. \tag{2.6}$$

The Peter-Weyl theorem now states that a square integrable function on a group manifold decomposes into a basis of representation matrices. From (2.6) we find the appropriate completeness relation:

$$\sum_{R,m,n} \dim \mathrm{R} \, R_{mn}(g_1) R_{nm}(g_2^{-1}) = \delta(g_1 - g_2). \tag{2.7}$$

Here $\delta(g)$ is defined with respect to the Haar measure. An equivalent statement is to identify the normalized wavefunctions on the group as:[6]

$$\psi^R_{mn}(g) = \langle g|R,m,n\rangle = \sqrt{\dim \mathrm{R}}\, R_{mn}(g), \qquad \langle R,m,n|g\rangle = \sqrt{\dim \mathrm{R}}\, R_{nm}(g^{-1}). \tag{2.8}$$

[5] In particular this measure can be derived from the symplectic form on the space of gauge fields associated with BF theory. This is discussed in [74, 9].

[6] Notice that only unitary representations are included, because the property $\langle a|b\rangle = \langle b|a\rangle^*$ implies in this context $R(g^{-1}) = R(g)^*$.

Let us prove this in a bit more detail. At least the representation matrices are indeed eigenfunctions of the Casimir operator $\text{Tr}(\mathcal{J}\mathcal{J})$ with $\mathcal{J} = A_\tau|_\partial = g^{-1}\partial_\tau g$. This is the Hamiltonian of our quantum mechanical system and essentially is like the generalization of a Laplace operator to group manifolds. In the quantum theory the currents $\mathcal{J}^a$ become operators that satisfy the algebra of the group. They act on wavefunctions as differential operators in the same sense that $\pi_\phi = i\partial_\phi$ for a particle on the real line ϕ. For example in the case of $\text{SL}(2,\mathbb{R})$ we have from (2.58) $\mathcal{J}^+ = \partial_\phi/2 + \partial_+\gamma_+$ etcetera. The point is that given some fixed group G solving this theory is only as difficult as solving a set of differential equations. Say now that the group elements are labeled by a collection of fields $\phi_1 \ldots \phi_n$. Obviously these all commute, so one possible maximally commuting set of operators in the quantum theory are just the group element operators, therefore one basis of the Hilbert space would be $|g\rangle$. Alternatively we can observe that the system (2.5) enjoys a twofold G symmetry with the algebraic components of $\mathcal{J}$ and of $\mathcal{T} = \partial_\tau g g^{-1}$ as generators. If one writes down the canonical quantization then one finds that $\mathcal{J}^a$ and $\mathcal{T}^b$ commute for any a and b and furthermore these commute with the Casimir operator $\mathcal{C} = \text{Tr}(\mathcal{J}\mathcal{J}) = \text{Tr}(\mathcal{T}\mathcal{T})$. Say we have some maximally commuting set of current components $\mathcal{J}^1 \ldots \mathcal{J}^n$. Let us consider eigenstates of these operators and schematically denote the corresponding eigenvalue as i. An alternative maximally commuting set of operators in the quantum theory (2.5) is then $\mathcal{C}$, $\mathcal{J}^1 \ldots \mathcal{J}^n$ and $\mathcal{T}^1 \ldots \mathcal{T}^n$ so we find a Hilbert space $|R, i, j\rangle$.[7] Using the fact that on the one hand the currents act as differential operators on the basis $|g\rangle$ and on the other hand they are diagonal in the basis $|R, i, j\rangle$ one now immediately deduces by considering $\langle g| \mathcal{J}^a |R, i, j\rangle$ etcetera that indeed $\langle g|R, i, j\rangle$ needs to equal $R_{ij}(g)$ up to a proportionality constant. The proportionality constant is deduced by the orthonormality requirement and (2.6). This proves (2.8).

This structure matches with the known Hilbert space on an interval of BF theory, which is identical to the Hilbert space of 2d Yang-Mills on an interval and indeed spanned by either group elements $|g\rangle$ or states $|R, a, b\rangle$. To appreciate this one has to realize that we can obtain BF theory as the limit $e \to 0$ of 2d Yang-Mills:

$$S[\chi, A] = \int d^2x \, \text{Tr}\big(\chi F + e^2\chi^2\big). \tag{2.9}$$

For this reason, BF theory is often referred to as topological Yang-Mills or sometimes even weakly coupled Yang-Mills. The Hilbert space structure of 2d Yang-Mills is well understood and is precisely as claimed in the above. See for example [74, 78]. Anyway, the path integral of (2.5) gives:

$$Z(\beta) = \sum_R \dim R^2 \, e^{-\beta \mathcal{C}(R)}. \tag{2.10}$$

This answer can alternatively be obtained by directly doing the Gaussian path integral (2.5) a la Alekseev-Shatashvili [75, 76].

Before proceeding let us mention that the Hilbert space of 2d Yang-Mills on a circle

[7] The eigenvalues of $\mathcal{C}$ are one to one with irreps R.

consists of conjugacy class elements $|\lambda\rangle$ or of representations $|R\rangle$. The conjugacy classes are:

$$g \sim h \cdot g \cdot h^{-1} \qquad h \in G. \tag{2.11}$$

The associated wavefunctions are the finite characters of the group:

$$\langle \lambda | R \rangle = \chi_R(\lambda). \tag{2.12}$$

These characters are the traces of the representation matrices:

$$\chi_R(\lambda) = \sum_a R_{aa}(\lambda). \tag{2.13}$$

We note that the basis $|R\rangle$ decomposes into states $|R, a, a\rangle$ and similarly the conjugacy class elements $|\lambda\rangle$ decompose into all group elements $\left|h \cdot \lambda \cdot h^{-1}\right\rangle$ in its class. This will be important when we discuss factorization and entanglement in the next chapter.

In the remainder of this subsection we would like to understand what comes of operators in BF theory. How do these map to the dual 1d quantum mechanical system, if at all? There are three natural sets of operators in 2d Yang-Mills and hence also in 2d BF theory.

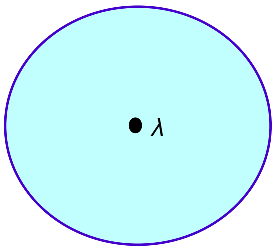
Punctures

There are operators which we will refer to as punctures:

$$\mathcal{P}_\lambda(x) = \mathrm{Tr}_\lambda\, e^{\chi(x)}. \tag{2.14}$$

As there is no dependence on the metric in the theory we can drop the dependence on x. We can represent the 2d BF disk path integral with such an insertion $P_\lambda(x)$ as:

$Z(\beta, \lambda) = \quad \beta$



. (2.15)

The dot represents the puncture $P_\lambda(x)$ inserted in the BF disk path integral with the aforementioned boundary conditions (2.4) and with coordinate length β. When we do the path integral over the bulk values of the field χ, we understand that the puncture acts as a point-like source for curvature F. Let's write this out in some more detail. See for example [64]. Within the BF path integral one has schematically :

$$\mathcal{P}_\lambda = \int d\omega e^{\mathrm{Tr}\left(\lambda \omega^{-1} \chi \omega\right)}. \tag{2.16}$$

Notably the trace is now in the "action". We can absorb the dependence on ω in the new "action" by an appropriate redefinition of F and χ with unit Jacobian [64]. We then have the following action:

$$\int_{\mathcal{M}} \text{Tr}(\chi F + \chi\lambda\delta(x)dx) - \frac{1}{2}\int_{\partial} \text{Tr}(\chi A). \tag{2.17}$$

Doing the path integral over χ now is indeed appreciated to introduce a source of curvature of Cartan algebra valued strength λ. Basically this is the weight vector associated with the irrep which we also labeled λ. By the divergence theorem we see that the path integral localizes on flat connections with a nontrivial holonomy λ:[8]

$$\int_{\mathcal{D}} F = \int_{\mathcal{C}} A = \beta\lambda. \tag{2.18}$$

The curve bounds the disk region here. The integration space can thus be parameterized as:

$$A = g^{-1}dg, \qquad g(\tau+\beta) = U_\lambda g(\tau), \qquad U_\lambda = e^{\beta\lambda}. \tag{2.19}$$

The path integral in the boundary quantum mechanics is thus a propagator on the group manifold from some base point $g(0)$ to some endpoint $g(\beta) = U_\lambda g(0)$. We can rewrite this as a usual Euclidean path integral with periodic boundary conditions by defining a new variable $g(\tau) = \Lambda(\tau)h(\tau)$ such that $h(\tau)$ is periodic $h(\tau+\beta) = h(\tau)$ and $\Lambda(\tau) = \exp(\tau\lambda)$ such that $\Lambda(\tau+\beta) = U_\lambda\Lambda(\tau)$. This means we have:

$$A = h^{-1}dh + h^{-1}\lambda h d\tau. \tag{2.20}$$

In the future we will not distinguish λ from U_λ. The context should clarify what we mean by it. More details on this construction can be found for example in [79, 80]. We end up with a Euclidean path integral over periodic paths $h(\tau)$ but with a modified action:

$$S[h,\lambda] = -\frac{1}{2}\int d\tau\, \text{Tr}\big(h^{-1}\partial_\tau h + h^{-1}\lambda h\big)^2. \tag{2.21}$$

This is not the action for a particle on a group manifold, but on a more general flag manifold. The best way to do this path integral is via the perspective of nontrivial propagation on the group manifold though [81]. Using the basis of wavefunctions (2.8) one basically just calculates:

$$Z(\beta,\lambda) = \langle g|\, e^{-\beta H}\, |\lambda\cdot g\rangle = \sum_{R,m,n} \psi^R_{mn}(g)\, \psi^R_{mn}(\lambda\cdot g)^* e^{-\beta\mathcal{C}(R)} = \sum_R \dim R\, \chi_R(\lambda) e^{-\beta\mathcal{C}_R}. \tag{2.22}$$

The corresponding bulk BF calculation will be explained in the next section.

Wilson loops

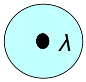
A second set of interesting operators in 2d Yang-Mills are closed Wilson loops, but in the topological limit these become trivial.[9] An exception to this is when Wilson lines

[8] The factor β here is just a useful convention.

[9] This is easy to check for the reader once we've spelled out the rules for calculating amplitudes in BF theory.

circle around the punctures. For example when a Wilson line in representation R circles a puncture in representation λ this results in an additional factor $\chi_R(\lambda)$ in the amplitude. We can understand this as $\mathcal{W}_R\,|\lambda\rangle = \chi_R(\lambda)\,|\lambda\rangle$ where $|\lambda\rangle$ is the conjugacy class state created by the puncture. For this, remember that a puncture fixes the holonomy around it via (2.18). Remember furthermore that in 2d Yang-Mills we can consider as Hilbert space of states on a circle the conjugacy class states $|\lambda\rangle$. One thus prepares such a state on a circular slice by inserting a puncture $P_\lambda(0)$ and evolving radially up to said slice:

$$|\lambda\rangle = \tag{2.23}$$



A Wilson loop in a certain representation is defined as:

$$\mathcal{W}_R(\mathcal{C}) = \mathrm{Tr}_R\left(\mathcal{P}\exp\left(-\int_{\mathcal{C}} A\right)\right) = \sum_m R_{mm}\left(\mathcal{P}\exp\left(-\int_{\mathcal{C}} A\right)\right). \tag{2.24}$$

This indeed acts diagonally on a holonomy eigenstate. By (2.18) we understand that a puncture creates such a holonomy eigenstate. The eigenvalue is recognized as the character. Wilson lines of this type will not play an important role in our story though.

Wilson lines

Finally, we can consider boundary anchored Wilson lines, these are representation matrices evaluated on the monodromy along a curve through the bulk connecting the boundary points:

$$\mathcal{W}_{R,a,b}(\tau_1,\tau_2) = R(g(\tau_1,\tau_2))_{ab}, \qquad g(\tau_1,\tau_2) = \mathcal{P}\exp\left(-\int_{\tau_1}^{\tau_2} A\right) \tag{2.25}$$

This operator doesn't depend on χ so the locus of the path integral is not affected by its insertion. Evaluating it on the flat connection $A = h^{-1}dh$ results in $g(\tau_1,\tau_2) = h(\tau_2)\cdot h^{-1}(\tau_1)$. We end up with the on shell evaluation:

$$\mathcal{W}_{R,a,b}(\tau_1,\tau_2) = R_{ab}(h(\tau_2)\cdot h^{-1}(\tau_1)) = \sum_c R_{ac}(h(\tau_2))\,R_{cb}(h^{-1}(\tau_1)). \tag{2.26}$$

This proves the equivalence of the BF path integral:

$$\int [\mathcal{D}\chi][\mathcal{D}A]\,\mathcal{W}_{R,a,b}(\tau_1,\tau_2)\dots e^{-S[\chi,A]}, \tag{2.27}$$

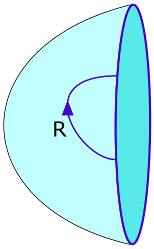
and the particle on a group correlator:

$$\sum_c \int [\mathcal{D}g]\,R_{ac}(g(\tau_2))\,R_{cb}(g^{-1}(\tau_1))\dots e^{-S[g]}. \tag{2.28}$$

In the next section we'll explicitly calculate and match both sides of this equality.

This duality between 2d BF theory and 1d quantum on a group is an example of holography, comparable to the duality of 2d Wess-Zumino-Witten boundary and 3d Chern-Simons theory or more generally the mathematically rigorous equivalence of 2d conformal field theories and 3d topological quantum field theories.[10] We would like to clarify this statement. Correlators in the boundary theory are obtained by path integrating out the bulk fields, and then integrating over boundary configurations. Schematically for example:

$$\int [\mathcal{D}g]$$



$$g \quad = \int [\mathcal{D}g]\, \psi_R[g]. \tag{2.29}$$

We can think about this as path integral preparing a wavefunction $\psi_{\mathcal{O}}[g]$ that depends on the inserted operators $\mathcal{O}$, much like we would think about the Hartle-Hawking wavefunction in cosmology. One then integrates over all possible configurations g on the Cauchy slice. The wavefunction $\psi_{\mathcal{O}}[g]$ is interpreted in the "dual" description as the exponential of some action with some possibly complicated operator insertions:[11]

$$\psi_{\mathcal{O}}[g] = \mathcal{O}[g] e^{-S[g]}. \tag{2.30}$$

It is reasonable to expect that ultimately every well understood holographic duality works like this, at least if we want it to be true that the holographically dual theory actually lives on the asymptotic boundaries of our spacetime. The field variables g here would more generally include the inequivalent asymptotic configurations of a bunch of bulk fields that satisfy some well-chosen boundary conditions. In these examples we only have asymptotic configurations of the connection A. Obviously the integration space of g as well as the explicit form of the wavefunctions depends on the choice of asymptotic boundary conditions. In the examples of 2d BF and 3d Chern-Simons those boundary conditions essentially specify all the moduli of the eventual dual correlator. For example the moduli can be the distance $\tau_1 - \tau_2$ between Wilson line endpoints and the total length β of the boundary for 2d BF. An other example is the conformal crossratio z and the complex structure q of the boundary in the case of a Wilson line network in 3d Chern-Simons ending on a torus boundary where the network is chosen such that it computes a conformal block.[12] In this sense we can alternatively think about the final answer of the computation as some wavefunction $\phi_{\mathcal{O}}[\mathcal{B}]$ where $\mathcal{B}$ labels the boundary conditions. The duality states that we can understand this functional either as some

[10] By rigorous it is meant that this equivalence holds true in a very precise mathematical sense. In other words it does not require the use of path integrals.

[11] The functional $\mathcal{O}[g]$ can depend in a complicated manner on the inserted operators $\mathcal{O}$.

[12] The relevant network is topologically what we expect of a conformal block. We have four endpoints touching the boundary and two trivalent vertices in the bulk with an intermediate Wilson line segment of which the label is also the label of the block.

bulk path integral with boundary conditions $\mathcal{B}$, or as a purely boundary path integral of which the action and explicit operator insertions depend in a possibly complicated manner on the boundary conditions. Schematically this would be:

$$\phi_{\mathcal{O}}[\mathcal{B}] = \int [\mathcal{D}g]\, \psi_{\mathcal{O}}[g, \mathcal{B}]. \tag{2.31}$$

What makes the 3d topological field theory or 2d conformal field theory examples so nice is that we can get immediate answers for $\phi_{\mathcal{O}}[\mathcal{B}]$ without using path integrals and only using the axioms of topological field theory respectively conformal field theory [82]. Of course we can have dualities between theories in several dimensions where one does not live on the boundary of the other, in which case the picture (2.29) is not relevant. One such example relevant in the context of this work is the matrix integral dual to JT gravity as a sum over Riemann surfaces [9]. This is essentially a 1d "quantum mechanical" theory dual to 2d JT gravity. It is not obviously holographically dual though. It does not obviously live on the boundary and the "fields" over which we integrate in the model have no obvious relation to boundary values of the metric or the dilaton in JT gravity.[13] Rather the fields are related to inherently bulk variables in JT, such as lengths of closed geodesic in the bulk.[14] In this sense, it seems plausible that we should rather be thinking about the 1d theory as describing the 2d topological field theory which emerges when we path integrate out the *boundary* fields. Not the other way around.

One particularly neat application of the duality between 3d topological field theories and 2d conformal field theories is the proof that AdS_3 gravity on a full torus is equivalent to two copies of a Virasoro coadjoint orbit. This is sometimes mistakenly referred to as Liouville theory [83]. The main purpose of this chapter is to show that similar reasoning can be used to proof the full equivalence of JT gravity on the disk topology with certain boundary conditions and Schwarzian quantum mechanics. The key to this is to identify the Schwarzian amplitudes as amplitudes of a particular coset of an "$\text{SL}(2,\mathbb{R})$" BF theory which describes JT gravity in its first order formulation. The dictionary includes in particular a map between bilocal boundary operators in the Schwarzian studied in [43] and massive particles propagating from boundary to boundary through the bulk of JT gravity. These correspond to boundary anchored Wilson lines in the BF theory language.

2.1.2 JT gravity as a BF theory

Let us now briefly discuss how that mapping works for JT gravity. In the remainder of this chapter we will then explicitly calculate the relevant bulk BF amplitudes both for general groups (and other structures) as well as for the particular coset of $\text{SL}^+(2,\mathbb{R})$ relevant to JT gravity. These will be matched to known amplitudes of respectively

[13] For example, it is not at all obvious that the theory reduces to 1d Schwarzian quantum mechanics in the "classical" limit.

[14] The energies E over which we integrate in the matrix model can be thought of as dual to the lengths b of the geodesics, in the sense that we could equivalently write down a matrix integral formulation where the "fields" are the lengths b via some Laplace transform.

quantum mechanics on the group manifold and the Schwarzian.
Consider now the action of JT gravity on a Riemann surface of Euler character χ:

$$S[g,\Phi] = -S_0\chi - \frac{1}{2}\int d^2x\sqrt{g}\,\Phi(R+2) - \int d\tau\sqrt{h}\,\Phi(K-1). \tag{2.32}$$

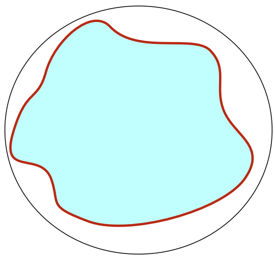
The first term represents the usual Einstein-Hilbert and Gibbons-Hawking contribution which is trivial in 2d. It will not play a role in this chapter, as we restrict the metrics to have the topology of a Euclidean disk. It is pivotal in chapter 3 though.
One now usually proceeds as follows. Just like any other theory, this action is to be supplemented with suitable boundary conditions. A natural set of boundary conditions for JT gravity fixes the boundary length as well as the dilaton at the boundary [21, 22, 23]:

$$\int d\tau\sqrt{h} = \frac{1}{\epsilon}\int du = \frac{\beta}{\epsilon}, \qquad \Phi|_\partial = \frac{1}{2\epsilon}. \tag{2.33}$$

The coordinate u is defined here as the proper time along the boundary trajectory, a trajectory which is not fixed in this version of the quantum theory. Path integrating over the bulk dilaton, the theory localizes on solutions which are locally AdS_2:

$$\int [\mathcal{D}g][\mathcal{D}\Phi]e^{-S[g,\Phi]} = \int [\mathcal{D}g]\,\delta(R+2)\,\exp\left(-\frac{1}{2\epsilon^2}\int_0^\beta du\,K + \frac{\beta}{2\epsilon^2}\right). \tag{2.34}$$

This means every configuration in the quantum theory is a patch of AdS_2, or a hyperbolic Riemann surface, whose boundary length is fixed to β/ϵ. We can represent this graphically as:

$$\int [\mathcal{D}g][\mathcal{D}\Phi]e^{-S[g,\Phi]} = \int [\mathcal{D}\theta]$$

θ . (2.35)

The black circle denotes the boundary of Euclidean AdS_2, or the real axis when considering the upper half plane. The length of the red curve is fixed to β/ϵ and lies very close to the black circle in reality. We see that the moduli of the solution space are the boundary trajectories, which we can parameterize for example by an angle function $\theta(u)$. The definition that u is proper time fixes a relation $r(u)$, so all freedom is indeed in $\theta(u)$. Defining a new function $f(u) = \tan\theta(u)/2$ it is then proven in [22] that the action reduces to that of Schwarzian quantum mechanics:

$$-\frac{1}{2\epsilon^2}\int_0^\beta du(K-1) = -\frac{1}{2}\int_0^\beta du\,S_f(u) = -S[f]. \tag{2.36}$$

Here we have defined the Schwarzian derivative:

$$S_f(u) = \frac{f'''(u)}{f'(u)} - \frac{3}{2}\frac{f''(u)^2}{f'(u)^2} = \left(\frac{f''(u)}{f'(u)}\right)' - \frac{1}{2}\left(\frac{f''(u)}{f'(u)}\right)^2. \tag{2.37}$$

In summary:

$$\int [\mathcal{D}g][\mathcal{D}\Phi]e^{-S[g,\Phi]} = \int [\mathcal{D}f]e^{-S[f]}. \tag{2.38}$$

This duality can again be considered a form of holography, in the sense (2.29). The integration measure is the natural one on the Schwarzian theory [83]:

$$[\mathcal{D}f] = \prod_u \frac{df(u)}{f'(u)} = \prod_u \frac{d\theta(u)}{\theta'(u)}. \tag{2.39}$$

It is far from straightforward to get this measure directly from the bulk JT gravity path integral though. The only way as far as we know to do this rigorously is explained in [9] and builds on the equivalence of JT gravity and Schwarzian quantum mechanics to an "$\mathrm{SL}(2,\mathbb{R})$" BF theory.[15] In short, one can prove that the path integral measure over metrics $[\mathcal{D}g]\,\delta(R+2)$ on closed manifolds implies the natural measure in the first order BF formulation. In the BF theory relevant to JT gravity this is the Weil-Petersson measure. There is no a priori second order gravitational prescription for how to define the path integral over metrics when there are dynamic boundaries, but we do have such a prescription in the first order formalism. Indeed, the BF theory comes with a single symplectic form that determines both the integration measure for the dynamical boundary degrees of freedom as well as the Weil-Petersson measure on closed manifold.[16] For a generic group this implies the Haar measure for the dynamical boundary field $g(\tau)$. We will see below that (2.39) indeed follows from the Haar measure on $\mathrm{SL}(2,\mathbb{R})$ once we impose the appropriate gravitational boundary conditions, which are slightly different from (2.4). Only with this understanding is it obvious that path integrals of JT gravity on a disk are precisely equivalent to Schwarzian path integrals.

First order formulation and quantum mechanics on a noncompact group

We would now like to point out that this duality between JT gravity and Schwarzian quantum mechanics is actually but an application of the duality between BF theory and quantum mechanics on a group manifold. The advantage of this identification is that it also immediately completely solves the theory, because correlators in BF theory are well-understood and will be discussed shortly. Another advantage is that it facilitates a generalization to other topologies, though there are some caveats there as we'll discuss in the next chapter. More in general it is fair to say that gauge theories are much less subtle then gravity, or at least we are more comfortable working with them as quantum theories, so there is in general much to be gained in rewriting a theory of quantum gravity as a gauge theory.

[15] We will shortly turn to addressing that equivalence.

[16] This measure is hence relevant on higher genus configurations, see chapter 3.

It has actually been known since the early days [68, 69, 70, 71] that the bulk action of JT gravity can be rewritten as an $\mathrm{SL}(2,\mathbb{R})$ BF theory. Before shortly reviewing that, let us refresh the statement that such an identification relies on the algebra only, and hence does not fix the particular exponentiation. In this context we will learn later on that we should be only integrating over a subregion of the $\mathrm{SL}(2,\mathbb{R})$ group manifold denoted as $\mathrm{SL}^+(2,\mathbb{R})$. This fact is closely related to the fact that we want to avoid integrating over singular metrics. This constrains the allowed conjugacy classes of group elements to hyperbolic ones, which is automatic in $\mathrm{SL}^+(2,\mathbb{R})$. We will explain this point in more detail further on.

Let us remember the first order formulation of JT gravity [71], see also [9, 61]. For this we will follow the literature and consider as $\mathrm{SL}(2,\mathbb{R})$ generators the Hermitian matrices:

$$2P_0 = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad 2P_1- = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 2P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.40}$$

These satisfy the $\mathrm{SL}(2,\mathbb{R})$ algebra in the form:

$$[P_0, P_1] = iP_2, \quad [P_2, P_0] = iP_1, \quad [P_1, P_2] = iP_0. \tag{2.41}$$

These matrices were chosen such that $2\,\mathrm{Tr}(P_a P_b) = \delta_{ab}$. The fundamental fields in the first order formulation of two dimensional gravity are the zweibein one form fields $e^\alpha = e^\alpha_\mu dx^\mu$ where α takes values 1 and 2, and the spin connection $\omega^\alpha{}_\beta = \epsilon^\alpha{}_\beta \omega$.[17] Furthermore $\omega = \omega_\mu dx^\mu$ is also a one form. The zweibein is related to the metric two form in the second order formalism as $g = \delta_{\alpha\beta} e^\alpha e^\beta = e \cdot e$ [84]. The equations of motion of two dimensional AdS_2 gravity include the no-torsion constraint:

$$T^\alpha = de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0. \tag{2.42}$$

Furthermore the scalar curvature is expressed as [84]:

$$dx\sqrt{g}\,R = \epsilon_{\alpha\beta} R^{\alpha\beta} = 2d\omega. \tag{2.43}$$

We furthermore have [84]:

$$dx\sqrt{g}\,2 = \epsilon_{\alpha\beta} e^\alpha \wedge e^\beta = 2e^1 \wedge e^2. \tag{2.44}$$

In total:

$$dx\sqrt{g}(R+2) = 2(d\omega + e^1 \wedge e^2). \tag{2.45}$$

The constraint (2.42) is naturally implemented in the path integral by introducing two Lagrange multiplier scalar fields Φ^α. Notably in the special case of JT gravity, there is a third Lagrange multiplier scalar field Φ enforcing that the spacetime is locally AdS_2. So we are led to consider as action for JT gravity in its first order formulation:

$$\int \Phi(d\omega + e^1 \wedge e^2) + \Phi^1(de^1 + \omega \wedge e^2) + \Phi^2(de^2 - \omega \wedge e^1) \tag{2.46}$$

[17] Here ϵ is the Levi-Cevita tensor in two dimensions with $\epsilon_{12} = 1$ and we have the flat Euclidean metric on the labels α etcetera.

Consider now an $\mathrm{SL}(2,\mathbb{R})$ algebra valued connection $A = A_a P_a$. We have three fields A_0, A_1 and A_2. Let us identify these with respectively the fields ω, e_1 and e_2. From an explicit computation of the associated field strength $F = dA + A^2$ using the algebra (2.41) one now finds:

$$F_0 = d\omega + e^1 \wedge e^2, \quad F_1 = de^1 + \omega \wedge e^2, \quad F_2 = de^2 + -\omega \wedge e^1. \tag{2.47}$$

It is furthermore convenient to package the fields Φ, Φ_1 and Φ_2 into an $\mathrm{SL}(2,\mathbb{R})$ algebra valued scalar χ as $\chi = \Phi P_0 + \Phi_1 P_1 + \Phi_2 P_2$. Taking into account the trace of the generators we see that we can rewrite the action of JT gravity in it's first order formulation as an $\mathrm{SL}(2,\mathbb{R})$ BF theory:

$$\int \mathrm{Tr}(\chi F). \tag{2.48}$$

Again, let us stress that this equivalence does not necessarily imply the full equivalence of JT gravity path integrals and $\mathrm{SL}(2,\mathbb{R})$ BF path integrals, on account of details of the path integration contour and of the measure in the path integral. One can prove however that the equivalence *does* hold at the quantum level, given that one restricts the $\mathrm{SL}(2,\mathbb{R})$ connections to hyperbolic conjugacy class elements and mods out by the mapping class group.[18] In terms of the measure one confirms that on closed manifolds, the integration measure on JT gravity is the natural Weil-Petersson measure on Riemann surfaces after we path integrate out the dilaton, which matches the measure that follows from the usual measure on BF theories [9].

Having reviewed the story in the "bulk" let us now point out that $\mathrm{SL}(2,\mathbb{R})$ BF theory on a disk with suitable "gravitational" boundary conditions matches JT gravity on the disk with boundary conditions (2.33). Given the discussion of the previous section we would initially be tempted to study quantum mechanics on $\mathrm{SL}(2,\mathbb{R})$, as this is the dual of $\mathrm{SL}(2,\mathbb{R})$ BF on a disk with boundary conditions (2.4):

$$S[g] = -\frac{1}{2}\int d\tau\, \mathrm{Tr}\big(g^{-1}\partial_\tau g\big)^2. \tag{2.49}$$

We can parameterize the $\mathrm{SL}(2,\mathbb{R})$ group manifold by Gaussian coordinates:

$$g = \begin{pmatrix} 1 & 0 \\ \gamma_- & 1 \end{pmatrix}\begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix}\begin{pmatrix} 1 & \gamma_+ \\ 0 & 1 \end{pmatrix} = e^{\gamma_- J_-}\, e^{\phi 2 J_0}\, e^{\gamma_+ J_+}. \tag{2.50}$$

Here we have introduced a different set of $\mathrm{SL}(2,\mathbb{R})$ generators J_-, J_+ and J_0 which will be used throughout:

$$2J_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.51}$$

They satisfy the algebra:

$$[J_\pm, J_0] = \pm J_\pm, \quad [J_+, J_-] = -2J_0. \tag{2.52}$$

[18] This is explained in more detail in chapter 3.

In this form, the trace in the action is just the usual trace on two-by-two matrices. From this one finds the algebraic components of the connection:

$$g^{-1}\partial_\tau g = \mathcal{J} = 2\mathcal{J}^0 J_0 + \mathcal{J}^- J_- + \mathcal{J}^+ J_+. \tag{2.53}$$

The current components $\mathcal{J}^a$ are:[19]

$$\mathcal{J}^0 = \phi' + \gamma_+\gamma_-' e^{-2\phi}, \quad , \mathcal{J}^- = \gamma_-' e^{-2\phi}, \quad \mathcal{J}^+ = \gamma_+' - 2\gamma_+\phi' - \gamma_+^2\gamma_-' e^{-2\phi}. \tag{2.54}$$

Writing out the action in terms of these Gauss coordinates one finds:

$$S[\phi, \gamma_-, \gamma_+] = -\int d\tau \left(\phi'^2 + \gamma_-'\gamma_+' e^{-2\phi}\right). \tag{2.55}$$

It is straightforward to quantize this theory, this is just an explicit example of the discussion below (2.8). The conjugate momenta are:[20]

$$\pi_\phi = 2\phi', \quad \pi_+ = \gamma_-' e^{-2\phi}, \quad \pi_- = \gamma_+' e^{-2\phi}. \tag{2.56}$$

The Hamiltonian can be found to equal the Lagrangian, conform with the fact that this is just free propagation on some background:

$$H = L = \frac{1}{4}\pi_\phi^2 + \pi_+\pi_- e^{2\phi}. \tag{2.57}$$

The theory enjoys a two-fold SL(2, $\mathbb{R}$) symmetry. Indeed, there are two sets of SL(2, $\mathbb{R}$) currents $\mathcal{J}^a$ and $\mathcal{T}^a$ in the quantum theory which commute with each other and each satisfy an SL(2, $\mathbb{R}$) charge algebra. Explicitly from the above we have for $\mathcal{J}^a$:

$$\mathcal{J}^0 = \frac{\pi_\phi}{2} + \pi_+\gamma_+, \quad \mathcal{J}^- = \pi_+, \quad \mathcal{J}^+ = \pi_- e^{2\phi} - \pi_\phi\gamma_+ - \pi_+\gamma_+^2. \tag{2.58}$$

The currents $\mathcal{T}^a$ are similar in spirit:

$$\mathcal{T}^0 = \frac{\pi_\phi}{2} + \gamma_-\pi_-, \quad \mathcal{T}^- = \pi_-, \quad \mathcal{T}^+ = \pi_- e^{2\phi} - \pi_\phi\gamma_- - \pi_-\gamma_-^2. \tag{2.59}$$

One finds indeed a type of SL(2, $\mathbb{R}$) charge algebra is implied for each of the currents upon canonical quantization. Perhaps surprisingly given the labels one should think about $\mathcal{J}^-$ as a generator τ_+ and about $\mathcal{J}^+$ as a generator τ_- in this SL(2, $\mathbb{R}$) charge algebra that is obtained from quantization.[21] This relation would be more natural if we would use generators J_1, J_2 and currents $\mathcal{J}^1$ and $\mathcal{J}^2$ with $J_\pm = J_1 \pm iJ_2$ instead, because in that case the metric on the Lie algebra would be diagonal. We note that the charges $\mathcal{T}$ and $\mathcal{J}$ are precisely the dimensional reduction of the two conserved currents in the SL(2, $\mathbb{R}$) Wess-Zumino-Witten model, see for example [1]. The Casimir for either current is by definition:

$$\mathcal{C} = \mathrm{Tr}(\mathcal{J}\mathcal{J}) = \mathrm{Tr}(\mathcal{T}\mathcal{T}) = H \tag{2.60}$$

[19] Here and in what follows, primes denote derivatives with respect to τ.

[20] Canonical quantization imposes $[\gamma_+, \pi_+] = i$ etcetera.

[21] The τ_a are like the J_a in (2.52).

Due to this double $\mathrm{SL}(2,\mathbb{R})$ symmetry, the spectrum of this model is organized in diagonal unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The restriction to hyperbolic conjugacy class elements restricts the allowed representations to just the continuous series for which $\mathcal{C}(k) = k^2$ roughly speaking. A maximal set of commuting observables is $\mathcal{C}$, $\mathcal{J}^a$ and $\mathcal{T}^b$ for specific choices of a and b. For example we could choose $\mathcal{C}$, $\mathcal{J}^0$ and $\mathcal{T}^0$. Denoting by $|s\rangle$ the states $\mathcal{J}^0 |s\rangle = s\,|s\rangle$ and mutatis mutandis for $|r\rangle$ and $\mathcal{T}^0$, the spectrum of the Hamiltonian are the states $|k,s,r\rangle$. Of course this is just the Peter-Weil theorem discussed earlier. The fine print of the associated wavefunctions and Plancherel decomposition depends on the full structure and not just on the algebra, so we'll postpone a detailed discussion.

Coset boundary conditions

Of course, this particle on $\mathrm{SL}(2,\mathbb{R})$ is not Schwarzian quantum mechanics, so clearly the naive BF boundary conditions do not correspond to the boundary conditions (2.33) in JT gravity. The correct boundary conditions to enforce in the BF description are instead coset boundary conditions:

$$A_\tau^a|_\partial = \chi^a|_\partial, \qquad A_\tau^-|_\partial = \mathcal{J}^- = 1. \tag{2.61}$$

The first equality is by definition of the current, the second defines a coset boundary condition. In the quantum theory this puts $\pi_+ |\psi\rangle = |\psi\rangle$. Notice that we cannot constrain other components $\mathcal{J}^a$ in the quantum theory, as these variables do not commute. Notice furthermore that the constraint $\pi_+ = 1$ does not modify the charge algebra of the $\mathcal{T}^a$ hence we still have one $\mathrm{SL}(2,\mathbb{R})$ symmetry. The resulting Hilbert space is schematically just $|k,s\rangle$ which implicitly has a fixed eigenvalue for $\mathcal{J}^-$. In terms of the field g, the constraint reads:

$$e^{2\phi}\gamma_-' = 1. \tag{2.62}$$

Imposing this constraint in the action we find:

$$S[\gamma_-,\gamma_+] = -\int du \left(\frac{1}{4}\frac{\gamma_-''^2}{\gamma_-'^2} + \gamma_+'\right) = \frac{1}{2}\int du\, S_{\gamma_-}(u). \tag{2.63}$$

The second term is a total derivative, so the action actually only depends on γ_-. Furthermore we see that up to again a total derivative, it becomes the Schwarzian action. The field γ_+ is clearly redundant in this model and to be modded out to avoid an overall infinity. One way to do this is to gauge-fix it.
The measure for quantum mechanics on $\mathrm{SL}(2,\mathbb{R})$ that follows from the BF path integral is the Haar measure, as discussed in the previous section, in this case:

$$[\mathcal{D}g] = \prod_u e^{-2\phi} d\phi d\gamma_- d\gamma_+. \tag{2.64}$$

Introducing a delta on the coset constraint, as well as a delta for the gauge fixing this becomes the usual measure on the Schwarzian theory [83]:

$$\prod_u e^{-2\phi} d\phi d\gamma_- d\gamma_+ \delta(e^{-2\phi}\gamma_-' - 1)\delta(\gamma_+ - \dots) = \prod_u \frac{d\gamma_-}{\gamma_-'}. \tag{2.65}$$

One advantage of working in the BF formulation is that we can actually derive this measure. We can write the coset boundary conditions in a more familiar form from the gravity point of view. One possible gauge choice is to fix γ_+ to:

$$\gamma_+ = -\phi' = \frac{1}{2}\frac{\gamma_-''}{\gamma_-'}. \tag{2.66}$$

This choice puts $\mathcal{J}^0 = 0$ in the path integral, but obviously this should not be considered a constraint on states in the physical Hilbert space which can't be eigenstates of both $\mathcal{J}^-$ and $\mathcal{J}^0$. In the Hilbert space language we just have that γ_+ is null on the physical Hilbert space so effectively $\langle k, s|\phi, \gamma_-, \gamma_+\rangle = \langle k, s|\phi, \gamma_-, 0\rangle$.[22] Furthermore it puts:

$$\mathcal{J}^+ = -\frac{1}{2}S_{\gamma_-}(u). \tag{2.67}$$

In other words, the flat connections at the boundary are gauge equivalent to:

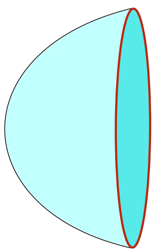
$$A_u|_\partial(u) = J^- - \frac{1}{2}S_{\gamma_-}(u)\, J^+ = \begin{pmatrix} 0 & -\frac{1}{2}S_{\gamma_-}(u) \\ 1 & 0 \end{pmatrix}. \tag{2.68}$$

This is the dimensional reduction of the gravitational boundary conditions on the flat Chern-Simons connections which reduces the dynamics to precisely that of AdS_3 gravity, for example on the holomorphic field:

$$A_z|_\partial(u) = iJ^- - \frac{1}{2}T(u)J^+ = \begin{pmatrix} 0 & -\frac{1}{2}T(u) \\ 1 & 0 \end{pmatrix}, \qquad A_{\bar{z}}|_\partial = 0. \tag{2.69}$$

It is possible to relate this behavior directly to the asymptotic behavior of the allowed metrics under the boundary conditions (2.33), see for example [9]. Of course this had to be the case, as the resulting dynamics is precisely equivalent. We will denote the coset boundary conditions by a red color. The JT gravity analogue of the picture (2.29), but now applied to the case with no operator insertions is:

$$\int [\mathcal{D}\gamma_-]$$



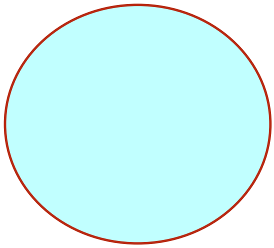
$$\gamma_- \quad = \int [\mathcal{D}\gamma_-] \exp\left(-\frac{1}{2}\int_\partial du S_{\gamma_-}(u)\right). \tag{2.70}$$

This Schwarzian path integral can be done explicitly [50, 9]. Note therefore that the field γ_- is periodic $\gamma_-(\tau+\beta) = \gamma_-(\tau)$. This is inherited from the periodicity of g in and

[22] This is not quite true, the more precise statement is that we don't need to integrate over γ_+ when inserting a complete set of group element states between physical states such as $|k, s\rangle$, we can fix γ_+ to one value and the answer of the resulting integrals doesn't depend on this value.

above (2.5). We will henceforth just represent this whole path integral and its answer by the following picture:

$$Z(\beta) = \quad \beta$$



$$= \int_0^\infty dE\, e^{-\beta E} \sinh 2\pi\sqrt{E}. \tag{2.71}$$

This picture looks a bit silly, but that's just because there are no operator insertions. Natural operators in JT gravity are those discussed for 2d BF theory in the previous section: boundary anchored Wilson lines and punctures. Let us briefly discuss how to interpret these operators from a second order gravitational point of view and how they map to operators in Schwarzian quantum mechanics in the sense of (5.13).

Punctures

Punctures are easily dealt with in the BF formulation, this is largely explained in [64], to which we refer for more details and more precise formulas.[23] Introducing an operator (2.14) in the $\mathrm{SL}(2,\mathbb{R})$ BF formulation results in a source of curvature. Remember now that λ is a weight vector in (2.14), which means that it is valued in the Cartan sub-algebra. Sticking to the notation of the first order formalism here, this means more precisely for $\mathrm{SL}(2,\mathbb{R})$:

$$F_0 = \lambda\delta(x), \quad F_1 = 0, \quad F_2 = 0. \tag{2.72}$$

In other words, we see that the puncture corresponds to insertion of a dilaton field in the JT gravity path integral:[24]

$$\mathcal{P}_\lambda = \exp\left(-\int dx\, \delta(x)\, \lambda\Phi\right). \tag{2.73}$$

Doing the dilaton path integral this is recognized as introducing a source of scalar curvature R. Anyway, the source (2.72) implies a nontrivial monodromy on the boundary flat connections, schematically:

$$g(\tau+\beta) = U_\lambda \cdot g(\tau). \tag{2.74}$$

This translates into a certain periodicity on the Schwarzian field $\gamma_-(\tau)$ after imposing the constraint equation (2.62) and the gauge fixing constraint (2.66):[25]

$$\gamma_-(\tau+\beta) = U_\lambda \cdot \gamma_-(\tau) = \frac{a\gamma_-(\tau)+b}{c\gamma_-(\tau)+d}, \quad U_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.75}$$

[23]We will be slightly schematic.
[24]Combine formulas (2.47), (2.46) and (2.17) to appreciate this.
[25]See [64] for a detailed calculation.

The monodromies are the exponentials of the weight vectors, as explained in the previous subsection, so very explicitly we have:

$$U_\lambda = e^{2\pi\lambda P_0} = \begin{pmatrix} \cos\pi\lambda & \sin\pi\lambda \\ -\sin\pi\lambda & \cos\pi\lambda \end{pmatrix}. \tag{2.76}$$

From this we read of the identification:

$$\gamma_-(\tau+\beta) = \frac{\gamma_-(\tau)+\tan\pi\lambda}{1-\gamma_-(\tau)\tan\pi\lambda}. \tag{2.77}$$

Much like around equation (2.19) it turns out to be convenient to transform to a new variable with trivial monodromy, but whose action is no longer the usual Schwarzian action. In this context of JT gravity on a disk, with a puncture inserted, it seems natural to rewrite everything in terms of the angular variable $\theta(\tau)$ that labels the boundary of the disk. Its corresponding identification is then obviously $\theta(\tau+\beta) = \theta(\tau) + 2\pi$. It is not too difficult to obtain from (2.77) the correct mapping between $\gamma_-(\tau)$ and $\theta(\tau)$:

$$\gamma_-(\tau) = \tan\lambda\theta(\tau)/2. \tag{2.78}$$

The usual JT disk with no insertion corresponds to $\lambda = 1$. This may seem counterintuitive because there is definitely no curvature source in that path integral. The puzzle is resolved because U_λ and $-U_\lambda$ clearly define the same identification in (2.75). In other words let us consider a dilaton source with strength $\lambda - 1$, which vanishes when $\lambda = 1$. This implies the monodromy:

$$U_\lambda = e^{-2\pi P_0} e^{2\pi\lambda P_0} = -\begin{pmatrix} \cos\pi\lambda & \sin\pi\lambda \\ -\sin\pi\lambda & \cos\pi\lambda \end{pmatrix}. \tag{2.79}$$

To this monodromy corresponds the identification (2.77) and the field redefinition (2.78). We notice that the path integral measure is unaffected by such a field redefinition, essentially because the Haar measure is invariant under left and right group multiplication.[26] Anyway, under the field redefinition (2.78) we have:

$$2S_{\gamma_-}(u) = -\frac{\gamma_-''^2}{\gamma_-'^2} = -\frac{\theta''^2}{\theta'^2} + \lambda^2\theta'^2 = 2S_\theta(u) + \lambda^2\theta'^2. \tag{2.80}$$

This action is sometimes referred to as a twisted Schwarzian action, basically because the theory is the classical limit of twisted Virasoro coadjoint orbits [64]. In summary we have the mapping of punctures in JT gravity to twisted Schwarzian path integrals:

$$\int [\mathcal{D}\theta] \exp\left(-\frac{1}{2}\int_\partial du S_\theta(u) + du\frac{\lambda^2\theta'^2}{2}\right). \tag{2.81}$$

It is straightforward to do this path integral, see for example [50, 9]. The result can be written as the Laplace transform of a cosh:

$$Z(\beta,\lambda) = \int_0^\infty dE\, e^{-\beta E}\frac{\cosh\pi\lambda\sqrt{E}}{\sqrt{E}}. \tag{2.82}$$

[26] In terms of the discussion around (2.19) we have $[\mathcal{D}g] = [\mathcal{D}h]$.

Of greater importance than the Schwarzian dual formulation, is the gravitational interpretation of the punctures. For real λ this is fairly straightforward to deduce from the sourced equations. The dilaton source with strength $\lambda-1$ results in a conical singularity of the same strength. For $\lambda < 1$ we have conical deficit geometries and for $\lambda > 1$ we have conical surplus geometries. This follows from integrating the scalar curvature over an infinitesimal disk surrounding the defect and using Gauss-Bonnet [64]. In this work though, we will rather be interested in purely imaginary values λ. Let us introduce a real parameter $b = i\lambda$. JT gravity disks with a puncture labeled by a real b inserted correspond to geometries that end in the bulk on a geodesic boundary, rather than a conical singularity. It is indeed well known that conical singularities and geodesic boundaries of Riemann surfaces are analytic continuations of one another to imaginary values. One neat way to understand this is to consider the bulk metrics that correspond to a certain reparameterization $\gamma_-(u)$. This relation is essentially fixed by the discussion around equation (2.33). Say we denote by τ the proper time on the boundary.[27] Construct a length coordinate z such that the metric in coordinates τ and z is conformally flat space. The conformal scaling factor follows from the fact that we should think about $\gamma_-(\tau)$ as a coordinate transformation on the boundary with the original metric being the Poincaré upper half plane one [21, 22, 23]:

$$ds^2(\gamma_-) = \frac{\gamma_-'(u)\gamma_-'(v)}{(\gamma_-(u)-\gamma_-(v))^2}du\,dv. \tag{2.83}$$

Here $u = t + z$ and $v = t - z$ and $\tau = it$. The validity of this formula hinges strongly on the fact that in (2.33) we are considering essentially infinite boundary lengths, this pushes the boundary of the patches of AdS_2 in the integral to the boundary of the Poincaré upper half plane. An in general nontrivial map $\gamma_-(\tau)$ remains though. We refer the reader to chapter 5 and to the original papers on this subject [21, 22, 23] for a more in depth explanation of this formula. Doing the field redefinition (2.78) this becomes:

$$ds^2(\theta) = \lambda^2 \frac{\theta'(u)\theta'(v)}{\sinh^2\frac{\lambda}{2}(\theta(u)-\theta(v))}du\,dv. \tag{2.84}$$

These are structurally equivalent to the metric related to the classical solution $\theta(\tau) = 2\pi\tau/\beta$, on which we will focus. After Wick rotation we have:

$$ds^2 = \left(\frac{2\pi\lambda}{\beta}\right)^2 \frac{dz^2+d\tau^2}{\sinh^2\frac{2\pi\lambda}{\beta}z}. \tag{2.85}$$

For real λ we can identify this as a conical defect geometry by going to polar coordinates as $\ln r = -\frac{2\pi\lambda}{\beta}z$. Near $z=\infty$ or $r=0$ we recognize a conical defect [64]:

$$ds^2 = dr^2 + \left(\frac{2\pi\lambda}{\beta}\right)^2 r^2 d\tau^2, \quad \tau \sim \tau+\beta. \tag{2.86}$$

[27] This was u previously, but here we want to reserve u for a lightcone coordinate.

Indeed, this is identical to:

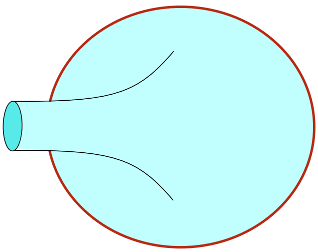
$$ds^2 = dr^2 + r^2 d\theta^2, \quad \theta \sim \theta + 2\pi\lambda. \tag{2.87}$$

This is a conical singularity unless $\lambda = 1$. The monodromies with real λ are known as *elliptic* monodromies. They clearly correspond to singular geometries, and hence when we will be considering the question of summing over bulk topologies in the next chapter, we will want to avoid flat $\mathrm{SL}(2,\mathbb{R})$ connections with elliptic monodromies in the path integration contour. Of more interest for what this work is concerned are the so called *hyperbolic* monodromies, which correspond to purely imaginary λ or positive real b. We can find the associated geometry by simply analytically continuing (2.85):

$$ds^2 = \left(\frac{2\pi b}{\beta}\right)^2 \frac{dz^2 + d\tau^2}{\sin^2 \frac{2\pi b}{\beta} z}. \tag{2.88}$$

This Riemann surface has a minimal geodesic along the identified τ direction at $z = \beta/4b$. The length of that geodesic is $2\pi b$. So when we are inserting a puncture labeled by b in a JT gravity path integral we are path integrating over annulus shaped Riemann surfaces with a geodesic boundary of length proportional to b that end also on the asymptotic boundary and we are also integrating over reparameterizations $\theta(u)$ of that asymptotic boundary. The result of this path integral is just the analytic continuation of (2.82). In the remainder we will represent this path integral and its answer in either the JT gravity or the Schwarzian calculation as:

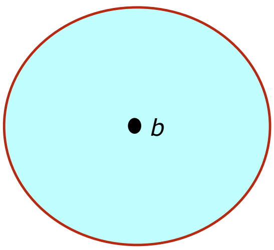
$Z(\beta, b) =$ b



$\beta \quad = \displaystyle\int_0^\infty dE\, e^{-\beta E}\, \frac{\cos \pi b\sqrt{E}}{\sqrt{E}}.$ (2.89)

Implicit in this picture is that we're summing over all Riemann surfaces that are topologically an annulus and satisfy both the fixed "asymptotic" length, and the geodesic boundary conditions. When we are more in an $\mathrm{SL}(2,\mathbb{R})$ BF mood, we can draw this equivalently as:

$Z(\beta, b) =$



β . (2.90)

At this point it is not meant to be obvious to the reader that the JT gravity calculation, which is the $\mathrm{SL}(2,\mathbb{R})$ BF calculation, gives precisely the answer (2.89) predicted from the Schwarzian path integral. This should be obvious by the time this chapter is finished though.

Wilson lines

Let us now investigate Wilson lines in the BF formulation of JT gravity. We are led to consider insertions of operators:

$$\mathcal{W}_{R,a,b}(\tau_1,\tau_2) = R(g(\tau_1,\tau_2))_{ab}, \qquad g(\tau_1,\tau_2) = \mathcal{P}\exp\left(-\int_{\tau_1}^{\tau_2} A\right) \tag{2.91}$$

Of course, the field $g(\tau)$ is now constrained by the gravitational boundary conditions. In particular we need to impose the constraint (2.62) and the gauge choice (2.66). In particular, say we parameterize g as:

$$g^{-1} = e^{i\gamma_L J^-} e^{2i\phi J^0} e^{i\gamma_R J^+} = \begin{pmatrix} 1 & 0 \\ \gamma_- & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & \gamma_+ \\ 0 & 1 \end{pmatrix}. \tag{2.92}$$

We then have:

$$e^{2\phi} = \gamma_-', \qquad \gamma_+ = -\frac{1}{2}\frac{\gamma_-''}{\gamma_-'}. \tag{2.93}$$

One further subtlety is the following. One endpoint of the Wilson line essentially intertwines between two states in the physical Hilbert space of the theory. Both theses states are eigenstates of $\mathcal{J}^-$ with eigenvalue 1. This forces us to only consider Wilson line endpoints that represent eigenstates of $\mathcal{J}^-$ with eigenvalue 0. Roughly speaking, we can only act internally in the physical Hilbert space if we don't change the $\mathcal{J}^-$ eigenvalue ergo if the operator itself has vanishing $\mathcal{J}^-$ charge. If this isn't clear intuitively, we'll have a more mathematical proof in section 2.2.3. We are interested in evaluating the Wilson line in the lowest weight representation of a discrete irrep $j = \ell$ of $\mathrm{SL}(2,\mathbb{R})$. This is a discrete representation, so the state with vanishing eigenvalue of $\mathcal{J}^-$ is just the lowest weight state. To proceed let us work with the Borel-Weil realization of $\mathrm{SL}(2,\mathbb{R})$ in which case we know the lowest weight states $|\ell,0\rangle$ to be of the form [85]:

$$\langle x|\ell,0\rangle = \frac{1}{x^{2\ell}}, \quad \langle \ell,0|x\rangle = \delta(x). \tag{2.94}$$

See also (2.200). The action of the generators exponentiates in this realization as explained in formula (2.200) further on. We are thus led to compute the following matrix element with the group elements action on the functions of x as in (2.200) and constrained as (2.93):

$$\langle \ell,0|\, g(\tau_2)g^{-1}(\tau_1)\, |\ell,0\rangle = \int dx\, \delta(x)(g(\tau_2)g^{-1}(\tau_1)\cdot x^{-2\ell}). \tag{2.95}$$

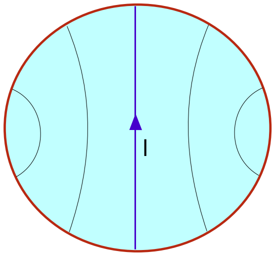
Explicitly using (2.200) we find:

$$g(\tau_2)g^{-1}(\tau_1)\cdot x^{-2\ell} = \frac{(\gamma_-'(\tau_1)\gamma_-'(\tau_2))^{\ell}}{\gamma_-'(\tau_2)x + (\gamma_-(\tau_1)-\gamma_-(\tau_2))^{2\ell}(1+x\gamma_+))^{2\ell}}. \tag{2.96}$$

Setting $x = 0$ we recover the Schwarzian bilocal operators:

$$\mathcal{W}_{\ell,0,0}(\tau_1,\tau_2) = \frac{\gamma_-'(\tau_1)^\ell \gamma_-'(\tau_2)^\ell}{(\gamma_-(\tau_1)-\gamma_-(\tau_2))^{2\ell}}. \tag{2.97}$$

The calculation is not affected by the presence of other Wilson lines. In the case of a single boundary anchored Wilson line, and in the spirit of (2.31), we can summarize the duality between the bulk JT gravity calculation and the Schwarzian calculation as:

$Z(\ell,\beta_1,\beta_2) = \beta_1$  β_2

$$= \int [\mathcal{D}\gamma_-] \frac{\gamma_-'(\tau_1)^\ell \gamma_-'(\tau_2)^\ell}{(\gamma_-(\tau_1)-\gamma_-(\tau_2))^{2\ell}} \exp\left(-\frac{1}{2}\int_0^\beta du S_{\gamma_-}(u)\right). \tag{2.98}$$

Here we have $\beta_1 = \tau_2 - \tau_1$ and $\beta_2 = \beta - \tau_2 + \tau_1$. Remarkably, Schwarzian correlators of this type have all been calculated exactly in [43]. The answers match perfectly with the answers we obtain from the direct first order JT gravity calculation, which will be presented in the remainder of this chapter. What this discussion does not address is the bulk JT gravity interpretation of such a Wilson line in the first order formulation. The answer is very intuitive [61]. Instead of repeating the proof given in [61], let us simply state the answer. The Wilson line corresponds to a massive quantum mechanical probe particle. In particular we have:

$$\mathcal{W}_{\ell,0,0}(\tau_1,\tau_2) = \int_{\tau_1,\tau_2} [\mathcal{D}x] \exp\left(-m(\ell)\int ds \left(g_{\mu\nu}(\gamma_-)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{1/2}\right). \tag{2.99}$$

The path integral is over all paths through the bulk of AdS_2 that start at the boundary point at τ_1 and end at the boundary at τ_2. Furthermore the metric depends on the Schwarzian wiggles as (2.83). Finally the mass of the probe particle is conform the usual holographic dictionary $m^2(\ell) = \ell(\ell-1)$. This completes our understanding of operators in the first order formulation of JT gravity and their 1d holographic duals in the sense of (2.29).

2.1.3 Plan for the rest of this chapter

This discussion has been rather lengthy, but captures the essence of this chapter. In the remaining sections we merely aim to check this correspondence between "SL$(2,\mathbb{R})$" BF with coset boundary conditions Schwarzian quantum mechanics. To do so we will claim a precise exponentiation of the SL$(2,\mathbb{R})$ algebra, replacing the vague "SL$(2,\mathbb{R})$"

BF theory by a more precise $\mathrm{SL}^+(2,\mathbb{R})$ BF theory. We then check if this theory is identical to Schwarzian mechanics by matching all known correlators on both sides of the duality. In the concluding remarks we provide additional physical motivation for an a priori consideration of $\mathrm{SL}^+(2,\mathbb{R})$ BF as first order formulation of JT gravity, without resorting to Schwarzian quantum mechanics. One of those is the fact that $\mathrm{SL}^+(2,\mathbb{R})$ BF has but hyperbolic monodromies, whilst generic connections in ordinary $\mathrm{SL}(2,\mathbb{R})$ BF can correspond to singular metrics with elliptic monodromy.

2.2 Exact correlation functions of BF theory

In this section we calculate correlation functions of boundary anchored Wilson lines in BF theory and map these to amplitudes of quantum mechanics on the group manifold. In particular we will consider BF theory on a Euclidean disk of circumference β.
As discussed in the previous section, the Hilbert space of BF theory on an interval is $|R,a,b\rangle$ whilst the Hilbert space on a circular slice is $|R\rangle$. The appropriate normalization of the wavefunctions $\langle g|R,a,b\rangle$ follows Schur's orthogonality relation:[28]

$$\int dg R_{i,ab}(g) R_{j,cd}(g^{-1}) = \frac{\delta_{ij}}{\dim R_i}\delta_{ad}\delta_{bc}. \tag{2.100}$$

The Hamiltonian of the theory is boundary supported and equals the Casimir of the algebra. This is so because:

$$A_\tau = (g^{-1}\partial_\tau g)^a J_a = \mathcal{J}^a J_a. \tag{2.101}$$

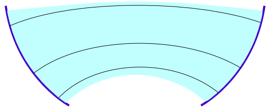
Canonical quantization implies that the currents $\mathcal{J}^a$ satisfy a $\mathfrak{g}$ algebra. The generators J_a satisfy a similar algebra, though as classical matrices instead of operators acting on a Hilbert space. For these theories it follows from canonical quantization that the Lagrangian equals the Hamiltonian which is indeed the Casimir:

$$H = \mathrm{Tr}\big(g^{-1}\partial_\tau g\big)^2 = \mathrm{Tr}(\mathcal{J}\mathcal{J}) = \mathcal{C}. \tag{2.102}$$

As explained in the previous section the wavefunctions $\langle g|R,a,b\rangle$ are eigenfunctions of the Casimir and therefore also eigenfunctions of the Hamiltonian with eigenvalue $\mathcal{C}(R)$. We note that for BF theories only the physical boundaries of the manifold come with a Hamiltonian weight, to be distinguished from cutting and gluing boundaries as discussed in chapter 3 and in the supplementary chapter B. In this section we will be calculating path integrals in BF theory in the Hamiltonian formulation, simply by inserting complete sets of states on intervals between operator insertions and by choosing appropriate initial and final states. Because we are in Euclidean signature, we can choose whichever family of "Cauchy slices" we like to. Say we have an interval the endpoints of which are on the boundary, and the family of Cauchy slices implies propagation along the boundary. We

[28] We note that the prefactor here is fixed by the normalization property of representation matrices $R_{ab}(1) = \delta_{ab}$ which follows from $R_{ab}(1)R_{bc}(1) = R_{ac}(1)$.

can picture this as:

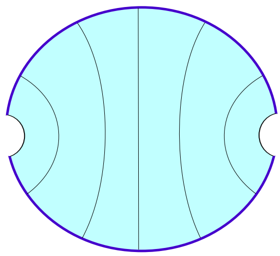
β_1  β_2 . (2.103)

The black slices denote “Cauchy slices”. We imagine that to a certain “timeflow” in the bulk, there corresponds a propagation of “time” β_1 on the boundary of the first endpoint of the interval, and a propagation “time” β_2 on the boundary of the second endpoint of the interval. If we take the state on the initial slice to be $|R, a, b\rangle$, then at the final slice we will have picked up a propagation factor:

$$e^{-(\beta_1+\beta_2)H} |R, a, b\rangle = e^{-(\beta_1+\beta_2)\mathcal{C}(R)} |R, a, b\rangle . \tag{2.104}$$

The weight depends on the total boundary length covered by the evolving Cauchy slices. The answer is by construction independent of the choice of slicing, so one just needs to remember that boundaries produce Hamiltonian weights corresponding to their lengths.

Imagine now a disk-shaped surface, and pick two points on the boundary. If we slightly thicken these points, this defines a semi-circular boundary segment. Let us choose the state on the first segment to be $|g\rangle$ and the state on the second segment to be $|h\rangle$. The corresponding amplitude can be pictured as:

$Z(g, h, \beta) =$ g  h . (2.105)

Here β denotes the total boundary lengths, and we have pictured again the Cauchy slices for clarification. In the Hamiltonian BF formulation we now just calculate this amplitude as:

$$Z(g, h, \beta) = \langle g| e^{-\beta H} |h\rangle . \tag{2.106}$$

We can either expand the boundary states in the representation basis as:[29]

$$|g\rangle = \sum_{R,a,b} \dim R^{1/2} R_{ab}(g) \, |R, a, b\rangle . \tag{2.108}$$

[29]We can check the normalization:

$$\langle h|g\rangle = \sum_R \dim R \, \chi_R(g \cdot h^{-1}) = \delta(g \cdot h^{-1} - 1). \tag{2.107}$$

This is essentially the definition of a Dirac delta on the group.

Or we simply insert a completeness relation in the basis $|R,a,b\rangle$. Anyway:

$$\begin{aligned} Z(\beta,g,h) &= \sum_{R,a,b} \dim \mathrm{R}\, R_{ab}(g) R_{ba}(h^{-1})\, e^{-\beta \mathcal{C}(R)} \\ &= \sum_{R} \dim R\, \chi_R(g\cdot h^{-1})\, e^{-\beta \mathcal{C}(R)}. \end{aligned} \tag{2.109}$$

It is intuitively clear that any non-identity g or h corresponds to some type of operator insertion on the BF boundary, comparable to the bulk punctures discussed previously. The bare disk amplitude corresponds to the case $g = h = 1$. This means there is no source of group elements on the boundary. We note that this boundary state $|1\rangle$ is closely related to the Ω-state in 2d Yang-Mills introduced by Gross and Taylor in [86], see also [87]. The latter is a state in the closed Hilbert space and is expanded in a basis of irreps as:

$$|\Omega\rangle = \sum_R \dim \mathrm{R}\; |R\rangle\,, \tag{2.110}$$

The former on the other hand resides in the open Hilbert space, and is expanded as:

$$|1\rangle = \sum_{R,a} \dim R^{1/2}\, |R,a,a\rangle\,. \tag{2.111}$$

This is the unique state for which the charge labels a match, so this boundary condition merely states that there are no charges inserted on the boundary. We can alternatively write this as:[30]

$$|1\rangle = \sum_{R,a} |R,a\rangle \otimes \langle R,a|\,. \tag{2.112}$$

Anyway, from the definition of the character (2.13) we have:

$$\chi_R(1) = \dim R. \tag{2.113}$$

Therefore the BF disk amplitude with no operator insertions $g = h = 1$ is just:

$$Z(\beta) = \sum_R \dim R^2\, e^{-\beta \mathcal{C}(R)}. \tag{2.114}$$

This is identical to the partition function of a particle on the group manifold, which is just calculated as:

$$Z(\beta) = \mathrm{Tr}\big(e^{-\beta H}\big) = \sum_{R,a,b} e^{-\beta \mathcal{C}(R)}. \tag{2.115}$$

The case with non-identity $g\cdot h^{-1}$ corresponds to a propagator on the group manifold, which indeed only depends on the relative positioning of the initial and final points, because the manifold is homogeneous.

[30] Here we define $R_{ab}(g) = \langle R,a|\, g\, |R,b\rangle$.

Before proceeding let us point out that there are two distinct calculations both on the BF side as on the particle on a group side of the duality which each give the answer:

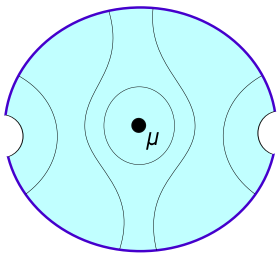
$$\sum_R \dim R\, \chi_R(\lambda)\, e^{-\beta \mathcal{C}(R)}. \tag{2.116}$$

The first one is the case where the BF theory has a nontrivial boundary state, which corresponds to a propagator on the group manifold. The action in this case is the common particle on a group, but the boundary conditions are now $g(\tau+\beta) = g(\tau)\cdot\lambda$. The second was discussed in the previous section and corresponds to the insertion of a bulk puncture in BF theory. This corresponds to a particle propagating on a background that is no longer simply the group, but rather some generic flag manifold. This results in a twisted particle on a group action (2.21) with periodic boundary conditions $g(\tau+\beta) = g(\tau)$. The equivalence of these configurations in terms of the particle on a group can be understood by absorbing the twist λ in (2.19) in the group element and replacing the integral over g with an integral over h where $h(\tau+\beta) = h(\tau)\cdot\lambda$ resulting in a common particle on a group but with twisted boundary conditions, as explained around (2.19). Finally, we can consider the BF setup with nontrivial initial and final states g respectively h *and* with a nontrivial puncture in the bulk associated with a holonomy μ. This results in the amplitude:

$$\sum_R \chi_R(\mu)\chi_R(g\cdot h^{-1})\, e^{-\beta \mathcal{C}(R)}. \tag{2.117}$$

On the particle on a group side of the duality this corresponds to a nontrivial propagator on a nontrivial flag manifold. This can not be rewritten as a complicated propagator on the ordinary group by field redefinition, testimony to the fact that (2.117) is fundamentally different from (2.116). For completeness we give the bulk BF computation of (2.117):

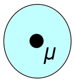
$Z(g,h,\mu,\beta) =$ g



h . (2.118)

This is also a useful exercise to become familiar with the machinery of bulk BF calculations. One can decompose the wavefunction on one of the "Cauchy slices" bending partly around the circular slice by writing the group element g_1 on such a slice as $g_1 = g_a\cdot g_b\cdot g_c$. One then associates g_a with the state on the interval running from the top boundary to the top of the circle, g_c with the state on the interval stretching from the bottom of the circular slice to the bottom boundary, and g_b with the state on the interval running along the half of the circular slice connecting the endpoints of the intervals associated with g_a and g_c. Furthermore we can decompose on the opposite slice $h_1 = h_a\cdot h_b\cdot h_c$. We have $h_a = g_a$ and $h_c = g_c$. Furthermore we have $\mu^{-1} = g_b\cdot h_b^{-1}$ which fixes $h_b = g_b\cdot\mu$. One now simply integrates over g_a, g_b, g_c, using no more than Schur's orthogonality relation (2.6). This calculation corresponds to inserting complete sets of states $|g_a\rangle$, $|g_b\rangle$, $|g_c\rangle$ and $|h_b\rangle$ on four intervals. This decomposes the amplitude (2.118) into three amplitudes which can be calculated using (2.109) and (2.23). The appropriate way to interpret the latter in this case is as a propagation amplitude in the closed Hilbert space. Because there is no Hamiltonian weight this just gives a delta:

$$Z(g_b, h_b, \mu) = \quad g_b$$



$$h_b \quad = \sum_R \chi_R(\mu)\chi_R(g_b \cdot h_b^{-1}) = \delta(\mu^{-1} - g_b \cdot h_b^{-1}). \quad (2.119)$$

The latter equality follows from (2.7) and (2.13). If the calculation of the other two amplitudes that contribute to (2.118) isn't clear, then certainly it should become clear after reading the next subsection.

2.2.1 Wilson lines

It is fairly straightforward at this point to calculate generic correlators of boundary anchored Wilson lines in BF theory. On a technical level we need several properties.

- The first is very basic: the definition of a representation:

$$R_{ab}(g \cdot h) = \sum_c R_{ac}(g)\, R_{cb}(h). \quad (2.120)$$

- The second is essentially the definition of a Clebsch-Gordan coefficient of $3j$ symbol:

$$\int dg R_{1,m_1 n_1}(g) R_{2,m_2 n_2}(g) R_{3,m_3 n_3}(g)^* = \begin{pmatrix} R_1 & R_2 & R_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \quad (2.121)$$

  Let us explain this formula. As explained in the introduction, the representation matrices form a basis for the space of square integrable wavefunctions on the group manifold. This in particular implies we can decompose products of representation matrices into sums of representation matrices, for example:

$$R_{i,ab}(g) R_{j,cd}(g) = \sum_{k,e,f} \dim R_k\, R_{k,ef}(g) \int dh\, R_{i,ab}(h)\, R_{j,cd}(h)\, R_{k,fe}(h^{-1}). \quad (2.122)$$

  The integral on the right hand side should be thought of as the expansion coefficient, calculated as always by an inner product on the relevant space, which here follows from (2.6). We can rewrite this using (2.8) as:

$$\langle k, e, f | i, a, b, j, c, d\rangle = \dim R_i^{1/2} \dim R_j^{1/2} \dim R_k^{1/2} \int dh\, R_{i,ab}(h)\, R_{j,cd}(h)\, R_{k,fe}(h^{-1}).$$

  The left hand side is by definition the product of two Clebsch-Gordan coefficients for coupling three states of the type $|j, m_j\rangle$. We prefer to work with the $3j$ symbols instead which are defined as rescalings of the Clebsch-Gordan coefficients by some dimension factors, and which have more symmetries. We have by definition:

$$\langle k, e, f | i, a, b, j, c, d \rangle = \dim R_i^{1/2} \dim R_j^{1/2} \dim R_k^{1/2} \begin{pmatrix} i & j & k \\ a & c & e \end{pmatrix} \begin{pmatrix} i & j & k \\ b & d & f \end{pmatrix}. \tag{2.123}$$

Therefore we find:

$$\int dh\, R_{i,ab}(h)\, R_{j,cd}(h)\, R_{k,fe}(h^{-1}) = \begin{pmatrix} i & j & k \\ a & c & e \end{pmatrix} \begin{pmatrix} i & j & k \\ b & d & f \end{pmatrix}. \tag{2.124}$$

Using the fact that representation matrices are unitary then proves (2.121). The $3j$-symbols satisfy the orthogonality property:

$$\sum_{a,c} \begin{pmatrix} i & j & k \\ a & c & e \end{pmatrix} \begin{pmatrix} i & j & l \\ a & c & f \end{pmatrix} = \frac{\delta_{kl}}{\dim R_k} \delta_{af} N_{ij}{}^{k}. \tag{2.125}$$

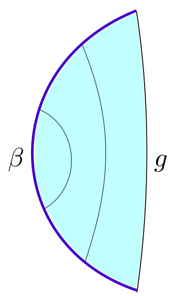
Summing this further results in:

$$\int dh \chi_i(h) \chi_j(h) \chi_k(h^{-1}) = N_{ij}{}^{k}. \tag{2.126}$$

The fusion coefficient N is zero or one. We will often leave it implicit as most answers include explicit $3j$ symbols which vanish whenever N vanishes, such that additional factors of N have no effect.

- The third property which we'll need is the amplitude of a BF disk with a boundary state $|g\rangle$ which we deduce from the discussion in the previous subsection as:

$Z(\beta, g) =$

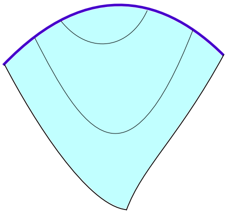


$$= \langle 1 | \, e^{-\beta H} \, | g \rangle$$

$$= \sum_{R,a} \dim R\, R_{aa}(g)\, e^{-\beta \mathcal{C}(R)}. \tag{2.127}$$

There are a labels associated with the endpoints of the "final" Cauchy slice, so at the intersection of the black and blue curves. The sum over a gives back the formula with the character. The same answer holds true when we consider an amplitude shaped like a piece of pie. Following (2.120) to decompose the wavefunction on the final slice we find:

$$Z(\beta, g\cdot h) = \quad g$$

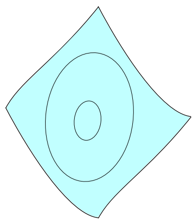


$$h \quad = \langle 1|\, e^{-\beta H}\, |g\cdot h\rangle \tag{2.128}$$

$$= \sum_{R,a,c} \dim R\, R_{ac}(g)\, R_{ca}(h)\, e^{-\beta \mathcal{C}(R)}.$$

There is a label a at the intersections of the blue and either of the black boundaries, and a label c is associated with the bottom point where the two black boundaries meet. Again doing the sum over a and c one recovers the character, but this form is more useful for the calculations. This trivially extends to more bulk corners. One just repeatedly uses (2.120). The case of two bulk corners is relevant for two of the three amplitudes that appear in the calculation of (2.118). A useful variant is to consider $\beta = 0$ corresponding to a piece of amplitude that does not touch the boundary. Say we have four boundary segments. We could calculate this as in the above using $\langle 1|g\cdot h\cdot f\cdot r\rangle$. Alternatively we may calculate this using the closed string states $|R\rangle$ and $|g\rangle$ where the latter are conjugacy class elements as in [88, 74]:

$$\langle 1|g\cdot h\cdot f\cdot r\rangle = \quad g\cdot h\cdot f\cdot r$$



$$. \tag{2.129}$$

$$= \sum_R \chi_R(1)\, \chi_R(g\cdot h\cdot f\cdot r)$$

$$= \sum_{R,a,b,c,d} \dim R\, R_{ab}(g)\, R_{bc}(h)\, R_{cd}(f)\, R_{da}(r).$$

There is one index label associated with each corner of the square boundary, but these are summed over. The state $|1\rangle$ implied in this calculation is precisely the Ω state of 2d Yang-Mills (2.110), and represents the lack of a puncture (2.14) in the interior. Of course this amplitude evaluates to just a delta, but for our purposes this form turns out to be more convenient.

- Finally we need the fact that a boundary anchored Wilson line is diagonal in the group element basis, as is the content of formula (2.25):

$$\langle h|\, \mathcal{W}_{i,a,b}\, |g\rangle = \delta(g\cdot h^{-1} - 1) R_{i,ab}(g). \tag{2.130}$$

A useful rewriting for what follows is:

$$\mathcal{W}_{i,a,b} = \int dg\, R_{i,ab}(g)\, |g\rangle \otimes \langle g|\,. \tag{2.131}$$

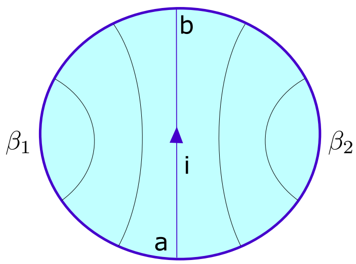
The definition (2.120) allows us to factorize Wilson lines into smaller segments of Wilson line:

$$\mathcal{W}_{i,a,b} = \sum_c \mathcal{W}_{i,a,c}\, \mathcal{W}_{i,c,b}. \tag{2.132}$$

It is then easy to calculate a generic amplitude. The boundary-anchored Wilson lines decompose the bulk into several regions bound by segments of Wilson lines. For each of the Wilson line segments, which begin and end either at the boundary or at a crossing of Wilson lines, one introduces a complete set of group elements states as in (2.131). For each region of the bulk, bounded by Wilson line segments and pieces of physical boundary, one then has an amplitude such as (2.128) where the boundary states are group element states. In the end we do the integral over the group elements. These all turn out to be of the type (2.121), resulting in various $3j$ symbols. Let us see how these calculations go for two example, and compare to a direct calculation in the boundary quantum mechanics.

One Wilson line

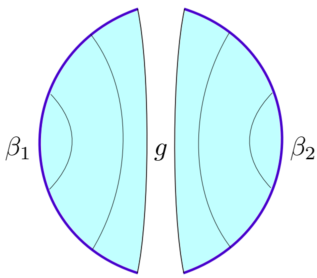
As a first example we can consider a single boundary anchored Wilson line. We have the following picture and corresponding Hamiltonian calculation:



$$= \langle 1|\, e^{-\beta_1 H} \mathcal{W}_{i,a,b} e^{-\beta_2 H}\, |1\rangle\,. \tag{2.133}$$

This decomposes using (2.131) as:

$$\int dg\, R_{i,ab}(g)$$



$$= \int dg\, Z(\beta_1, g)\, R_{i,ab}(g)\, Z(\beta_2, g). \tag{2.134}$$

Inserting (2.127) and doing the integral over the group elements using (2.121) one finds the answer:[31]

$$\sum_{R_1,n_1} \dim R_1\, e^{-\beta_1 \mathcal{C}(R_1)} \sum_{R_2,n_2} \dim R_2\, e^{-\beta_2 \mathcal{C}(R_2)} \begin{pmatrix} R_1 & R_i & R_2 \\ n_1 & a & n_2 \end{pmatrix} \begin{pmatrix} R_1 & R_i & R_2 \\ n_1 & b & n_2 \end{pmatrix}. \tag{2.136}$$

This matches the following particle on a group path integral:

$$\int [\mathcal{D}\chi][\mathcal{D}A]\, \mathcal{W}_{k,a,b}(\tau_1,\tau_2) e^{-S[\chi,A]} = \int [\mathcal{D}g]\, R_{k,ab}(g(\tau_2)\cdot g^{-1}(\tau_1))\, e^{-S[g]}. \tag{2.137}$$

The times are of course to be chosen such that $\tau_2 - \tau_1 = \beta_1$. Furthermore the group elements are periodic in the sense $g(\tau+\beta_1+\beta_2) = g(\tau)$, as there are no punctures in the bulk. The right hand side of the correlator (2.137) is just a quantum mechanics problem. We explained how quantum mechanics on a group works in the introduction, so a straightforward calculation gives:

$$\begin{aligned} &\int [\mathcal{D}g]\, R_{k,ab}(g(\tau_2)\cdot g^{-1}(\tau_1))\, e^{-S[g]} \\ &\quad = \sum_c \int [\mathcal{D}g]\, R_{k,ac}(g(\tau_2))\, R_{k,cb}(g^{-1}(\tau_1))\, e^{-S[g]} \\ &\quad = \sum_c \operatorname{Tr}\big(e^{-\beta_1 H}\, \mathcal{W}_{k,ac}\, e^{-\beta_2 H}\, \mathcal{W}_{k,bc}\big) \\ &\quad = \sum_c \int dg \int dh\, \langle h|\, e^{-\beta_1 H}\, |g\rangle\, R_{k,ac}(g)\, \langle g|\, e^{-\beta_2 H}\, |h\rangle\, R_{k,bc}(h). \end{aligned} \tag{2.138}$$

The latter is just an application of (2.131). We then insert complete sets of states in the Hilbert space of this quantum mechanical model:

$$\langle h|\, e^{-\beta_1 H}\, |g\rangle = \sum_{j,p,q} \dim R_j\, R_{j,pq}(h^{-1})\, R_{i,pq}(g)\, e^{-\beta_1 \mathcal{C}(R_j)} \tag{2.139}$$

$$\langle g|\, e^{-\beta_2 H}\, |h\rangle = \sum_{i,d,e} \dim R_i\, R_{i,de}(g^{-1})\, R_{i,de}(h)\, e^{-\beta_2 \mathcal{C}(R_i)}. \tag{2.140}$$

The integral over g gets two $3j$ symbols via (2.121) and likewise for the integral over h. Two of those four $3j$ symbols can be removed by summing over c, e, q:

$$\sum_{c,e,q} \begin{pmatrix} i & k & j \\ e & c & q \end{pmatrix} \begin{pmatrix} i & k & j \\ e & c & q \end{pmatrix} = N_{ikj}. \tag{2.141}$$

[31]This further simplifies to:

$$\sum_{i,j} \frac{\dim R_i \dim R_j}{\dim R_k}\, e^{-\beta_1 \mathcal{C}(R_i)}\, e^{-\beta_2 \mathcal{C}(R_j)}. \tag{2.135}$$

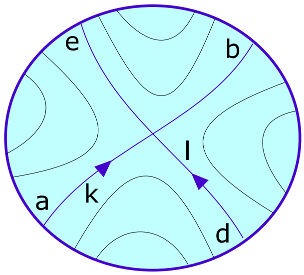
No such simplification occurs for more complicated diagrams though, so one shouldn't look into this too much.

Within the remaining sum the fusion coefficient is already implied by the remaining $3j$ symbols which vanish unless the fusion coefficient is one. In the end, we recover (4.3). This is an as explicit check on the duality (2.137) as it gets. We calculate without too much ado both path integrals and the results match.
The direct Hilbert space calculations for the correlators of quantum mechanics on the group trivially extend to generic amplitudes. One can check explicitly that they match all BF Wilson line calculations. We leave the check in the second example as an exercise to the reader. We would like to point out that despite appearances the 1d quantum mechanics calculation is not *manifestly* identical to the bulk 2d BF calculation. Indeed, one could try to read the former calculations as manifestly identical to the 2d BF calculation when we choose "radial" Cauchy slices in the bulk. However those are not an acceptable family of bulk Cauchy slices, as there is a singular point where they all meet.

Crossing Wilson lines

As a second and final example we can consider a scenario where Wilson lines cross in the bulk, for example:

$$\int [\mathcal{D}\chi][\mathcal{D}A]\, \mathcal{W}_{k,a,b}(\tau_1,\tau_3)\, \mathcal{W}_{l,d,e}(\tau_2,\tau_4)\, e^{-S[\chi,A]} =$$



. (2.142)

We've omitted denoting the boundary lengths to avoid cluttering. One now splits the Wilson lines at the crossings using (2.132) and writes each Wilson line segment as (2.131). The amplitude decomposes into four pie-shaped amplitudes of the type (2.128) with fixed group elements on their boundaries because basically we just cut the 2d BF amplitude on the Wilson lines by using (2.131). There are four distinct group elements to be integrated over, each has as "kernel" a representation matrix, representing the piece of Wilson line between the two boundaries of the pieces of pie we are gluing together following (2.131). We end up with:

$$\sum_{c,p} \int dg \int dh \int df \int dr\, Z(\beta_1, g, h)\, R_{k,ac}(h)\, Z(\beta_2, h, f)\, R_{l,dp}(f)$$
$$Z(\beta_3, f, r)\, R_{k,cb}(r)\, Z(\beta_4, r, g)\, R_{l,pe}(h). \qquad (2.143)$$

We leave it to the reader to sketch the associated picture conform (2.134). The labels c and p are from decomposing the Wilson lines as (2.132). Shipping in the answers for (2.128) we see that each of the group element integrals result in two $3j$ symbols. Each such $3j$ symbol is associated with the endpoint of a Wilson line segment. Four of the total of eight $3j$ symbols are associated with the crossing in the picture (2.142), the others are associated with the four points where a Wilson touches the physical boundary. An important point is that we can take the sum over the six labels associated with the bulk crossing.[32] The result of summing the four $3j$ symbols over these six labels is a $6j$ symbol:

$$\begin{aligned} &\begin{Bmatrix} R_1 & R_2 & k \\ R_3 & R_4 & l \end{Bmatrix} \\ &= \sum_{c,p,m_1,m_2,m_3,m_4} \begin{pmatrix} R_1 & R_2 & k \\ m_1 & m_2 & c \end{pmatrix} \begin{pmatrix} R_2 & R_3 & l \\ m_2 & m_3 & p \end{pmatrix} \begin{pmatrix} R_3 & R_4 & k \\ m_3 & m_4 & c \end{pmatrix} \begin{pmatrix} R_4 & R_1 & l \\ m_1 & m_2 & p \end{pmatrix}. \end{aligned} \tag{2.144}$$

One immediately understands that such a $6j$ symbol appears generically at every bulk crossing. This result is actually well-known in 2d Yang-Mills [74] and since the argument doesn't depend on Hamiltonian weights, it holds up equally well in BF. We end up with a lengthy though highly structured answer.[33] An important point is that due to this structure, the calculation extends trivially to generic correlators, in the sense that we could even write down a set of rules that immediately gives the answer to any diagram, see [2]. We will refrain from doing so here. Let us just mention that the rules are as we would expect them to be for a topological field theory. In particular they can be though of as the classical limits of the "rules" that define a three dimensional topological field theory. In that context, such rules are simply the axioms of a modular tensor category, and the $6j$ symbol that appears here is related to the braiding property. For more on this see for example [89, 82]. One checks that with these rules, the Wilson lines can be freely deformed and moved through each other in the bulk, as it should be in this two

[32] Two of the labels c and p are associated with the Wilson lines, the others are the labels associated with the bulk corner in (2.128), one for each of the pieces of pie.

[33] In particular:

$$\begin{aligned} &\prod_i \left(\sum_{R_i,n_i} \dim R_i \, e^{-\beta_i \mathcal{C}(R_i)} \right) \begin{Bmatrix} R_B & R_1 & R_4 \\ R_A & R_3 & R_2 \end{Bmatrix} \\ &\begin{pmatrix} R_1 & k & R_2 \\ n_1 & a & n_2 \end{pmatrix} \begin{pmatrix} R_1 & l & R_4 \\ n_1 & e & n_4 \end{pmatrix} \begin{pmatrix} R_2 & l & R_3 \\ n_2 & d & n_3 \end{pmatrix} \begin{pmatrix} R_3 & k & R_4 \\ n_3 & d & n_4 \end{pmatrix}. \end{aligned} \tag{2.145}$$

dimensional example.[34] Indeed, we can't have knots in two dimensions.
For the discussion on JT gravity it might be convenient to keep some of the general properties in mind though. It is in particular convenient to remember that each disk-shaped region comes with a sum over representations R and is weighed by $\dim R$. Furthermore each Wilson line crossing the boundary contributes a $3j$ symbol.

2.2.2 Punctures

It is straightforward to extend this discussion to including the insertion of puncture operators (2.14) in the path integral. For this, one has to realize that a puncture creates a conjugacy class state on a tiny circle surrounding it:

$$\mathcal{P}_\lambda \left|1\right\rangle = Tr_\lambda e^{\chi(0)} \left|1\right\rangle = \left|\lambda\right\rangle . \tag{2.149}$$

Here $\left|1\right\rangle$ is the Ω state (2.110). This was proven in the introduction. Given this understanding (2.149) one immediately writes down a generic amplitude including punctures. It is just a matter of inserting complete sets of states $\left|\lambda\right\rangle$ on circular slices and sets $\left|g\right\rangle$ on interval slices. In this way we can decompose the most generic amplitude into products of elementary amplitudes such as (2.129) and (2.128). One then does the gluing integrals using (2.6), and in the end we are left with one or more sums over representations with a certain weight.
We can streamline dealing with punctures as follows. Suppose that after cutting on all Wilson lines as in for example (2.142) we focus on one disk-shaped region. This topological disk region has some boundary conditions which we'll abbreviate by $\ldots$, these could be fixed length or fixed group element for example. Furthermore in general this region could contain a set of bulk punctures with labels $\lambda_1 \ldots \lambda_n$. Now introduce a complete set of states $\left|\mu\right\rangle$ on a circular slice surrounding all punctures. Using the state operator correspondence (2.23) this can and should be interpreted as *inserting a complete set of punctures*. This decomposes the amplitude into two pieces. The first is the same disk region as before, but now with a single bulk puncture μ. The second is a sphere with $n+1$ punctures. Let's explicitly take three punctures:

$$Z(\ldots \lambda_1, \lambda_2, \lambda_3) = \sum_\mu Z(\ldots \mu)\, Z(\mu, \lambda_1, \lambda_2, \lambda_3). \tag{2.150}$$

[34] More explicitly, we can effectively undo the double crossing of two Wilson lines by using:

$$\dim R \sum_X \dim X \begin{Bmatrix} R_A & R_1 & R_3 \\ R_B & R_2 & X \end{Bmatrix} \begin{Bmatrix} R_A & R_1 & R_4 \\ R_B & R_2 & X \end{Bmatrix} = \delta_{R_3 R_4}. \tag{2.146}$$

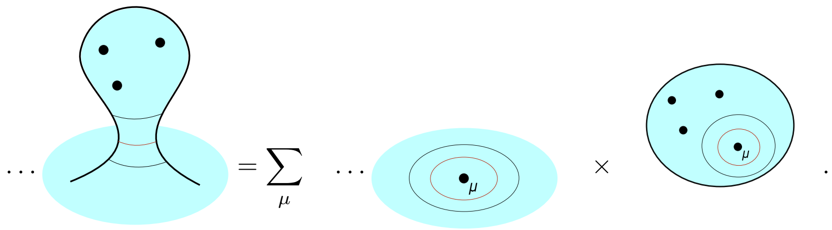
Similarly, a self-crossing of a Wilson line can be effectively undone using:

$$\sum_X \dim X \begin{Bmatrix} R_A & R_1 & R_2 \\ R_A & R_1 & X \end{Bmatrix} = 1. \tag{2.147}$$

Furthermore, a single bulk Wilson loop is found to produce just an overall degeneracy factor $\dim R$. To appreciate this one uses (2.126) and furthermore:

$$\sum_{R_i} \dim R_i\, N_{ijk} = \dim R_j \dim R_k. \tag{2.148}$$

Graphically this looks like we are cutting the manifold by introducing a complete set of punctures:



(2.151)

We have pictured some of the circular Cauchy slices to which we could locally apply this argument. Here we imagine introducing a complete set of states $|\mu\rangle$ on the red slice on the left. Via the state operator correspondence (2.23) we arrive at the picture on the right. We will phrase this calculation in the language of the supplementary chapter B in chapter 3 where the focus will be on manifolds with non-disk topology. Anyway, the first amplitude is easily calculated, using the same type of calculation as used around (2.118). Schematically we have an answer of the type:

$$Z(\ldots 1) = \sum_R \ldots \dim R \quad , \qquad Z(\ldots \lambda) = \sum_R \ldots \chi_R(\lambda). \tag{2.152}$$

The second part of the amplitude represents BF theory on a sphere with four punctures. It is straightforward to calculate its answer as explained above by evolving different sets of Cauchy slices on the surface and gluing them together with a completeness relation where the slices meet. The calculation is identical in spirit to that for 2d Yang-Mills explained in detail in [74]. The answer is:

$$Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_R \dim R^{-2}\, \chi_R(\lambda_1)\chi_R(\lambda_2)\chi_R(\lambda_3)\chi_R(\lambda_4). \tag{2.153}$$

This generalizes immediately to any number of punctures. Let us point out that this particular BF calculation doesn't involve an asymptotic boundary. Therefore path integrating out the dilaton leaves no action. The path integral is essentially just counting flat connections:

$$Z(\lambda_1 \ldots \lambda_n) = \int_{\lambda_1 \ldots \lambda_n} [\mathcal{D}g]. \tag{2.154}$$

This formula is very much to be read between quotes. The integration space are all non-equivalent flat connections with the correct holonomies around each of the punctures. So we are calculating volumes of the moduli spaces of flat G connections here. For more on this we kindly refer the reader to [74], though we will return to it in chapter 3. One important point which we would already like to bring to the attention of the reader is that it turns out the Hilbert space inspired "cutting rule" (2.150) is somewhat naive. In particular it is only true in a handwaving sense that $|\lambda\rangle$ are complete sets of states. In the more precise calculations one finds a nontrivial measure in the sum over λ, which as we'll see becomes an integral for non-compact groups. In general this measure would just include a factor Vol(G) independent of λ. In this case we just get overall powers of Vol(G) in formulas such as (2.153), see for example formula (4.72) in [74] for a precise answer. These don't seem so harmful. However as it turns out, in the BF theory relevant to JT gravity, there is actually nontrivial dependence on λ introduced. It is next to impossible to get this additional dependence right on the nose by thinking about complete sets of states. Fortunately it does follow unambiguously from the path integral. We will elaborate on this in chapter 3.

Back to business. Keeping in mind the discussion of the introduction on punctures in JT gravity, formula (2.152) is suddenly extremely natural. Indeed, we learned that in JT gravity a puncture with label b introduces a geodesic boundary of length b to the Riemann surface. Furthermore as it turns out, on closed manifolds, JT gravity reduces to an integral over the moduli space of inequivalent Riemann surfaces in the spirit of (2.154). Formula (2.152) then states nothing but the fact that we can decompose a complicated Riemann surface into simpler Riemann surfaces by cutting it on some closed geodesic. The different values for the geodesic lengths $b_1 \ldots b_n$ indeed label inequivalent Riemann surfaces. For much more on this see chapter 3.

2.2.3 Extensions

This then completes the solution of BF theory on a disk for Lie groups and with boundary conditions (2.4). In the remainder of this section we will extend that discussion to include a BF description of quantum mechanics on coset spaces and on non-compact groups. As we learned in the introduction, this will be relevant for JT gravity, because Schwarzian quantum mechanics is quantum mechanics on a coset of "SL$(2,\mathbb{R})$" and because the latter is non-compact.

Cosets

The boundary condition (2.4) are in a sense the least restrictive option. More restrictive boundary conditions can be obtained by restricting one or more components of the gauge field on the boundary to vanish:

$$A_t^b|_{\text{bdy}} = \chi^b|_{\text{bdy}} = 0. \tag{2.155}$$

Here the label b denotes a subset of generators. The result is obviously a constrained particle on a group action, as (2.155) implies relations between the coordinates on the original group manifold such that the particle now propagates on a submanifold of G. When the associated generators J^b span a sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ the particle propagates on the right coset G/H.[35] To appreciate this we can consider the Peter-Weyl theorem for functions on the right coset G/H. Functions on G/H are functions on G which are right invariant under H in the sense that $\psi(g) = \psi(g \cdot h)$. They are spanned by the following matrix elements:

$$R_{a0}(g) = \langle R, a| \, g \, |R, 0\rangle = \langle R, a| \, g \cdot h \, |R, 0\rangle \, . \tag{2.156}$$

[35] The extreme case of $H = G$ destroys all boundary dynamics, the theory is then completely topological.

The states $|R,0\rangle$ are defined to be invariant under H:

$$h\,|R,0\rangle = |R,0\rangle\,, \qquad h \in H. \tag{2.157}$$

This in turn implies that they are annihilated by all the generators in $\mathfrak{h}$:

$$\mathcal{J}^b\,|R,0\rangle = 0. \tag{2.158}$$

This is indeed precisely the content of the boundary condition (2.155). For homogeneous spaces to which we restrict from hereon there is only one such basis vector per irrep [90]. The Hilbert space of quantum mechanics on G/H is thus spanned by an orthogonal basis of so called *spherical* functions:

$$\phi^R_a(g) = \langle g|R,a,0\rangle = \dim R^{1/2} R_{a0}(g). \tag{2.159}$$

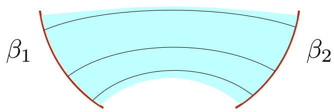
The dual of this discussion in the bulk BF picture is that when we consider the Hilbert space on an interval where both endpoints of the interval touch the boundary, then the boundary conditions $\mathcal{J}^b = 0$ constrain the states on this interval to be the so-called *zonal* spherical functions:

$$\phi^R(g) = \langle g|R,0,0\rangle = \dim R^{1/2} R_{00}(g). \tag{2.160}$$

To appreciate the relation with the previous discussion, notice that the group element on the interval is the holonomy of the connection from the starting point of the line to the endpoint of the line. Therefore it evaluates to $g = g_1^{-1}\cdot g_2$ with g_1 and g_2 the location of the boundary particle on the group manifold at two distinct times. Because the particle is actually travelling on a coset, we have $g_1 \sim g_1\cdot h_1$ and $g_2 \sim g_2 \cdot h_2$ for independent h_1 and h_2. The matrix element on the interval should respect this symmetry, therefore we need:[36]

$$R_{00}(g) = R_{00}(h_1\cdot g\cdot h_2). \tag{2.161}$$

This is the content of (2.160). We can picture a family of Cauchy slices on which the states are $|R,0,0\rangle$ as the analogue of (2.103):



. (2.162)

Notice in particular we have given the coset boundary a distinct red color, which we'll use throughout this work. Notice that coset boundary conditions do not constrain the labels of matrix elements associated with bulk vertices. Such labels were for example important to find the $6j$ symbol earlier. Indeed, we have for example:

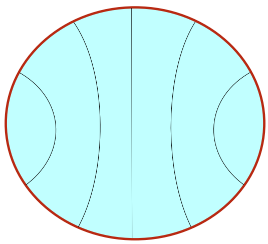
$$R_{00}(g_1\cdot g_2) = \sum_a R_{0a}(g_1)\, R_{a0}(g_2). \tag{2.163}$$

[36]Of course $h_1^{-1} \in H$ so we can drop the inversion if we need a property to hold for all group elements.

So in particular wavefunctions on intervals with only one endpoint on the physical boundary are only constrained to be zonal spherical functions, and wavefunctions on intervals with no endpoint on the physical boundary are not affected by the boundary conditions at all.
This had to be the case. We could hardly refer to (2.155) as boundary conditions if they would restrict the "Hilbert space" of the theory on a slice that doesn't touch the boundary, for example some circle in the bulk. Therefore in particular the Hilbert space on bulk circles persists to consist of irreps $|R\rangle$ and conjugacy class elements $|\lambda\rangle$ of the group G.[37] If we now think back to the calculations in section 2.2.1, we immediately learn that essentially on account of (2.163) bulk crossings of Wilson lines will still be weighed with $6j$ symbols. As an illuminating example of a quantum particle on a coset manifold one can consider the sphere $S^2 = SU(2)/U(1)$. In this case, the full matrix element is the Wigner D-function, the spherical functions are the standard spherical harmonics, and the zonal spherical function is the Legendre function.

To clarify the ramifications of coset boundary conditions in BF theory let us explain in a bit more detail the computation of the partition function and the single boundary anchored Wilson line. The partition function is calculated in BF language conform (2.105):

$$Z(\beta) = \langle 1| \, e^{-\beta H} \, |1\rangle = \quad \beta \tag{2.165}$$



Notice the red color again denoting that the Hilbert space on the depicted slices is constrained to be $|R, 0, 0\rangle$. The result is:

$$Z(\beta) = \sum_\mu \phi^\mu(1)\phi^\mu(1)\, e^{-\beta \mathcal{C}(\mu)} = \sum_\mu \dim \mu \, e^{-\beta \mathcal{C}(\mu)}. \tag{2.166}$$

In comparison to the answer for the BF theory with unconstrained boundary conditions (2.114) notice the different power of the $\dim \mu$. The corresponding calculation in quantum mechanics on a coset is:

$$Z(\beta) = \mathrm{Tr}\big(e^{-\beta H}\big) = \sum_{R,a} e^{-\beta \mathcal{C}(R)}. \tag{2.167}$$

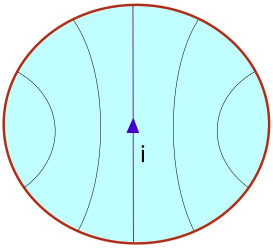
Indeed, the Hilbert space is now $|R, a, 0\rangle$ with notably one less label than the Hilbert space $|R, a, b\rangle$ for quantum mechanics on the group. The correlator of a single boundary

[37] For example one finds that the path integral on a disk-shaped part of the bulk with boundary state $|g\rangle$ is still:

$$\sum_\mu \dim \mu \chi_\mu(g). \tag{2.164}$$

anchored Wilson line is computed as:

$$\langle 1| \, e^{-\beta_1 H} \mathcal{W}_i e^{-\beta_2 H} \, |1\rangle = \beta_1$$



$$\beta_2 \quad . \tag{2.168}$$

Notice that as compared to (2.133), the Wilson line now carries only the label of the representation. It corresponds to the following Wilson line in the group:

$$\mathcal{W}_i = \mathcal{W}_{i,0,0}. \tag{2.169}$$

The insertion of any other Wilson line gives a vanishing result. To appreciate this, notice that the calculation of a more generic Wilson line involves the integral:

$$\int dg \, R_{\mu_1,00}(g) R_{i,ab}(g) R_{\mu_2,00}(g). \tag{2.170}$$

Consider now:

$$\langle \mu_1, 0| \, \mathcal{J}^b \, |i, a, \mu_2, 0\rangle \, , \qquad |i, a, \mu_2, 0\rangle = |i, a\rangle \otimes |\mu_2, 0\rangle \, . \tag{2.171}$$

Working on the bra the operator gives zero via the boundary conditions. Working on the ket though the operator given a nonzero answer unless $a = 0$. Therefore:

$$\langle \mu_1, 0|i, a, \mu_2, 0\rangle = 0, \qquad a \neq 0. \tag{2.172}$$

This in turn implies that the corresponding Clebsch-Gordan coefficient or $3j$ symbol vanishes:

$$\begin{pmatrix} \mu_1 & i & \mu_2 \\ 0 & a & 0 \end{pmatrix} = 0, \qquad a \neq 0. \tag{2.173}$$

Therefore, the integral (2.170) vanishes except for the Wilson line (2.169). This same argument proves more in general that $3j$ symbols respect current conservation. In a coset theory this severely restricts the possible interactions or the possible Wilson lines. Decomposing $\mathcal{W}_i$ as in (2.131) and via a similar intermediate step as in (2.134) one arrives at the answer:

$$\langle 1| \, e^{-\beta_1 H} \mathcal{W}_i e^{-\beta_2 H} \, |1\rangle = \sum_{\mu_1} \dim \mu_1 \, e^{-\beta_1 \mathcal{C}(\mu_1)} \sum_{\mu_2} \dim \mu_2 \, e^{-\beta_2 \mathcal{C}(\mu_2)} \begin{pmatrix} \mu_1 & i & \mu_2 \\ 0 & 0 & 0 \end{pmatrix}^2 . \tag{2.174}$$

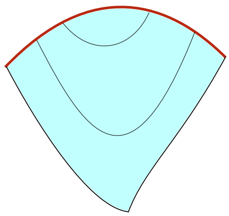
A calculation in quantum mechanics on the coset using the Hilbert space $|R, a, 0\rangle$ as in (2.138) is readily checked to reproduce this answer. Notice in particular as compared to

(2.136) that the powers of the dimensions are identical, but there is no sum over labels. This is because for the coset we have:

$$Z(\beta, g) = \sum_{\mu} \dim_{\mu} \, R_{\mu,00}(g)\, e^{-\beta \mathcal{C}(\mu)}. \tag{2.175}$$

This should be compared to (2.127). Its generalization (2.128) follows in a straightforward manner:

$$Z(\beta, g_1 \cdot g_2) = \quad g_1$$



$$g_2 \quad = \sum_{\mu,a} \dim \mu \, R_{\mu,0a}(g_1)\, R_{\mu,a0}(g_2)\, e^{-\beta \mathcal{C}(\mu)}. \tag{2.176}$$

Generalizing the arbitrary number of Wilson lines is straightforward. Furthermore the treatment of bulk punctures is not affected, as the coset only affects the behavior at the boundary.

All would agree at this point that the BF calculations we've discussed so far are ridiculously easy, to the point that one might wonder why this chapter is so long. This is precisely one of the points we are trying to make. This ridiculous simplicity of BF calculations is one of the main advantages of thinking about JT gravity in a BF language.

Noncompact groups

As a further extension which will turn out to be useful for JT gravity, we would like to understand how these 2d BF calculations extends to noncompact groups. As always we have access to a Plancherel decomposition:

$$f(g) = \sum_{k,a,b} f_{k,ab}\, R_{k,ab}(g), \quad \forall f \in L^2(G). \tag{2.177}$$

This is essentially just stating a complete set of solutions to the Laplace equation on a group manifold. As compared to compact groups, the difference is that we can have continuous series of irreps as well as infinite dimensional representations. Think for example of the continuous momentum label for a particle on the real line versus the discretized momentum for a particle living on a circle. Let us focus on such continuous irrep series for now. In this case Schur's orthogonality relation (2.100) becomes:

$$\int dg\, R^{k}_{ab}(g) R^{k'}_{cd}(g)^* = \frac{\delta(k-k')}{\rho(k)} \delta_{ac}\delta_{bd}. \tag{2.178}$$

This defines the so called Plancherel measure $\rho(k)$ which takes over the role of $\dim k$. An orthogonal basis of wavefunctions is then:

$$\psi^{k}_{sr}(g) = \langle g | k, s, r \rangle = \rho(k)^{1/2} R_{k,sr}(g). \tag{2.179}$$

From this knowledge we can immediately calculate the propagator on the group manifold over a distance λ, assuming we only have these continuous irreps in the Plancherel decomposition [90]:

$$\langle g| e^{-\beta H} |g \cdot \lambda\rangle = \int ds \int dr \int dk\, \phi^k_{sr}(g)\phi^k_{sr}(g\cdot\lambda)^*\, e^{-\beta \mathcal{C}(k)} = \int dk\, \rho(k)\, \chi_k(\lambda)\, e^{-\beta \mathcal{C}_k}. \tag{2.180}$$

The latter equality holds by definition of the character as trace of a representation matrix:

$$\chi_k(g) = \int ds R_{k,ss}(g). \tag{2.181}$$

The bra and ket here are not to be confused with the bra and ket used for example in (2.128). In the language we are using here, formula (2.128) would be $\langle g| e^{-\beta H} |g\rangle$. This is conform our earlier statement that the 1d quantum mechanics calculation looks like it's associated with a bulk BF calculation with a radial slicing. Having understood the propagator, the question arises what is the thermal partition function. To understand the answer we will deduce that the following identity holds for noncompact groups as well, modulo certain volume factors which we leave implicit here for reader comfort:[38]

$$\chi_k(1) = \dim k = \rho(k). \tag{2.182}$$

We can prove this relation by resorting to cosets of noncompact groups for which the partition functions is well-understood and described in much detail in [90]. Assuming again only continuous irreps contribute, the propagator on G/H is the generalization of (2.166):[39]

$$Z_{G/H}(\beta) = \int dk\, \rho_G(k)\, e^{-\beta \mathcal{C}(k)}. \tag{2.183}$$

Notably this includes one copy of the Plancherel measure on the group G. Let's consider some examples.

- The partition function on $H_3^+ = SL(2,\mathbb{C})/SU(2)$ is [91]:

$$Z_{H_3^+}(\beta) = \int ds\, s^2\, e^{-\beta \mathcal{C}(s)}. \tag{2.184}$$

  We recognize the $SL(2,\mathbb{C})$ Plancherel measure $\rho(s) = s^2$.

- The partition function on the Hyperbolic plane $H_2^+ = \mathrm{SL}(2,\mathbb{R})/U(1)$ is:

$$Z_{H_2^+}(\beta) = \int ds\, s \tanh(\pi s)\, e^{-\beta \mathcal{C}(s)}. \tag{2.185}$$

  One recognizes the $\mathrm{SL}(2,\mathbb{R})$ Plancherel measure $\rho(s) = s\tanh(\pi s)$. Notice that there are no contributions from the discrete representations of $\mathrm{SL}(2,\mathbb{R})$. This is

[38] See [2] for more details.
[39] This is true in the same sense that $R_{k,00}(1) = 1$ is true,up to some volume factors.

because the spectrum of the Laplacian on H_2^+ doesn't contain discrete solutions. This is as good a time as any to pause and give some details about the Plancherel decomposition of $\mathrm{SL}(2,\mathbb{R})$. The generators have been introduced in (2.52). Notice that these are neither Hermitian nor anti-Hermitian generators. For example:

$$J_+^\dagger = J_-. \tag{2.186}$$

Let us label eigenvalues of the Casimir as $\mathcal{C}(j) = -j(j+1)$. States are labeled by Casimir eigenvalues and eigenvalues of one generators $J_a\,|j,m\rangle = m\,|j,m\rangle$ as this makes up one maximally commuting algebra of observables. Different choices of generator J_a correspond to different bases referred to as *elliptic* when we diagonalize J_2, *hyperbolic* when we diagonalize J_0 and *parabolic* when we diagonalize J_+ or J_-.[40] The Plancherel decomposition of $f(g)$ with $G \in \mathrm{SL}(2,\mathbb{R})$ can then be proven to include so called continuous series irreps $j = is - 1/2$ with Plancherel measure and Casimir:

$$\rho(s) = s\tanh \pi s, \qquad \mathcal{C}(s) = s^2 + \frac{1}{4}. \tag{2.188}$$

Furthermore it includes both the so called highest and lowest weight principal discrete series irreps with $j = -\ell$ and ℓ half integer:

$$\rho(\ell) = \ell + \frac{1}{2}, \qquad \mathcal{C}(\ell) = \ell(1-\ell). \tag{2.189}$$

It is actually rather straightforward to derive the Plancherel measure (2.188) for $\mathrm{SL}(2,\mathbb{R})$ and the corresponding matrix elements. We have done so explicitly and ab initio in [1] but will not repeat that calculation here as it is a bit technical.

A peculiar example of a coset is the group itself.[41] The Plancherel measure for a direct product group is just the product of the Plancherel measures of the constituents. Only the diagonal representations survive in the coset construction of the group G resulting in:

$$Z_G(\beta) = \int dk\, \rho(k)^2\, e^{-\beta(k)}. \tag{2.190}$$

This completes the proof of (2.182). Of course $3j$ symbols and $6j$ symbols also exist for noncompact groups. The takeaway is that we *can* modify the BF calculations for compact groups to noncompact groups with the understanding that $\dim k = \rho(k)$.

2.3 Exact correlation functions of JT gravity

The Schwarzian correlation functions by the arguments of the introduction are identical to the correlation functions of some 2d "$\mathrm{SL}(2,\mathbb{R})$" BF theory with coset boundary

[40] We have:

$$J_\pm = J_1 \pm i J_2. \tag{2.187}$$

[41] We have $G = (G\times G)/G_{\text{diag}}$.

conditions. Indeed, one deduces from [43] that the correlators of Schwarzian quantum mechanics are structurally as expected of a 2d BF theory with coset boundary conditions, given that we associate bilocal operators in Schwarzian quantum mechanics to boundary anchored Wilson lines in a 2d BF theory via (2.97) and that we associate twisted boundary conditions to bulk punctures as in the discussion below (2.72).

Doing this, and following the rules spelled out in [43] we find that structurally Schwarzian quantum mechanics is identical to a 2d BF theory for a noncompact structure with the following specifications. The Plancherel measure and Casimir are:

$$\rho(k) = k \sinh 2\pi k, \qquad \mathcal{C}(k) = \frac{1}{4} + k^2. \tag{2.191}$$

The constant in the Casimir can be thought of as definition of the vacuum energy. It is meaningless but useful for comparison to (2.188). At the intersection of two bulk Wilson lines with labels ℓ_1 and ℓ_2 one finds, translating Schwarzian amplitudes to would-be bulk 2d BF amplitudes, an $\mathrm{SL}(2,\mathbb{R})$ $6j$ symbol with four labels referring to a continuous representation and two labels referring to a discrete representation of the type (2.189):

$$\begin{Bmatrix} k_1 & \ell_1 & k_3 \\ k_2 & \ell_2 & k_4 \end{Bmatrix}. \tag{2.192}$$

This fact was noticed already in [43] where this $6j$ symbol was obtained as the classical limit of a Virasoro Fusion matrix. This is conform our earlier remark that 2d BF theory is the classical limit of conformal field theory in the sense of [89, 82]. In the discussion on BF theory for cosets we learned to expect constrained $3j$ symbols to be associated with every intersection of a Wilson line with the boundary. Tracing back to the Schwarzian amplitudes [43] we expect to identify the following function as a $3j$ symbol of "$\mathrm{SL}(2,\mathbb{R})$":[42]

$$\begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2 = \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}. \tag{2.194}$$

The labels in this formula are implied by the coset boundary conditions (2.62). Crucially, the fact that we should beyond any doubt be expecting a coset from the Schwarzian point of view with Plancherel measure (2.191) follows from the study of higher point functions in [43]. In particular, the Schwarzian partition function is [50, 43]:

$$Z(\beta) = \int dk\, k \sinh 2\pi k\, e^{-\beta k^2}. \tag{2.195}$$

This is identical to (2.71) with the substitution $E = k^2$. This should be compared to the partition function of a coset 2d BF disk (2.166) and of a regular 2d BF disk (2.114).

[42] We define

$$\Gamma(\ell \pm ik_1 \pm ik_2) = \Gamma(\ell + ik_1 + ik_2)\Gamma(\ell + ik_1 - ik_2)\Gamma(\ell - ik_1 + ik_2)\Gamma(\ell - ik_1 - ik_2). \tag{2.193}$$

Furthermore it follows from the calculations in [43] that the answer for a disk-shaped region in the bulk with boundary holonomy λ is:

$$\langle 1|\lambda\rangle = \int dk\, k \sinh 2\pi k\, \chi_k(\lambda). \tag{2.196}$$

This should be compared to formula (2.129) earlier. These two comparisons uniquely fix the fact that we should be looking for a coset BF theory rather than a regular BF theory, via the different powers of the dimensions in (2.166) and (2.114). More in particular they uniquely fix the associated Plancherel measure and Casimir to (2.191).

This might look like somewhat of a puzzle [55]. We have what looks like the structure of an $\mathrm{SL}(2,\mathbb{R})$ BF theory in terms of $6j$ symbols, representations and all that, but the Plancherel measure (2.191) doesn't match the Plancherel measure (2.188) of $\mathrm{SL}(2,\mathbb{R})$. One might naively still have hope that we could maybe absorb the difference in some rescaling of $3j$ and $6j$ symbols. However, more pressingly, $\mathrm{SL}(2,\mathbb{R})$ also has discrete series representations showing up in its spectrum or Plancherel decomposition. The Schwarzian theory and JT gravity certainly don't have that property. The resolution of this tension is a fact often alluded to in the introduction. The Casimir, the action, the $6j$ symbols and all that are all essentially "local" properties of the group. They are fixed by the algebra. They do not however fix the exponentiation. The path integral depends explicitly on this exponentiation via the contour of integration for "group" valued fields g. We could for example constrain g to a certain path of $\mathrm{SL}(2,\mathbb{R})$, or conversely we could consider g to take values in the universal cover of $\mathrm{SL}(2,\mathbb{R})$ as was for example considered in [61, 56], effectively considering a larger integration space than just the group. We propose that the relevant structure to consider for JT gravity is a subsemigroup known as $\mathrm{SL}^+(2,\mathbb{R})$. The semigroup $\mathrm{SL}^+(2,\mathbb{R})$ can be defined as the set of *positive* $\mathrm{SL}(2,\mathbb{R})$ matrices with the usual matrix operations:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \quad a, b, c, d > 0. \tag{2.197}$$

The key point is that this subsemigroup *does* have the correct Plancherel decomposition (2.191) expected for JT gravity, and crucially furthermore its Plancherel decomposition or spectrum doesn't include discrete states. There are several other reasons why a priori one might like this structure. One pressing reason is that the "quantization" of $\mathrm{SL}^+(2,\mathbb{R})$ governs the dynamics of AdS_3 gravity. In other words, the conformal field theory known as Liouville theory whose classical limit in the sense of [89, 82] is precisely JT gravity, is governed by axioms of a three dimensional topological field theory of which the modular tensor category is the "quantization" of $\mathrm{SL}^+(2,\mathbb{R})$, not $\mathrm{SL}(2,\mathbb{R})$ [92, 93, 94]. A further strong motivation is that $\mathrm{SL}^+(2,\mathbb{R})$ doesn't have elliptic connections in its contour, so we are automatically only allowing smooth metrics. This is closely related to the fact that we don't have discrete representations in the spectrum. We elaborate on this in an "apologia for the subsemigroup" in section 2.4.

In the remainder of this section we explain the computation of 2d BF amplitudes for

the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ with coset boundary conditions determined by (2.62).
In **section 2.3.1** we present some details of the relevant representation theory. This is a more technical section which can be skipped on a first reading. Nevertheless it is important to understand the details of the computations in section 2.3.3.
In **section 2.3.2** we check that it even makes sense in the first place to study a 2d BF theory associated with a subsemigroup. This is not a priori obvious because we don't have the structure of a whole group. We end up being saved by the prefix "sub". This section is rather technical as well, and can also be skipped on a first reading.
In **section 2.3.3** we explain the calculations of amplitudes in 2d $\mathrm{SL}^+(2,\mathbb{R})$ BF with gravitational coset boundary conditions and recover the aforementioned structure, in particular (2.194). On a technical level, the group theoretic calculations actually become manifestly identical to the calculations of [57], with the group integrals playing the role of integrating over boundary-to-boundary geodesic lengths, which are indeed known to be a complete set of states for JT gravity on an interval connecting two boundaries [54].[43] In the case of crossing Wilson lines, the basis isn't as simple anymore. This is why [57] isn't able to compute such correlators. It would be interesting to have a geometric interpretation of the corresponding group-theoretic calculations which we are presenting here.
In **section 2.3.4** we do some explicit computations of a couple of amplitudes in JT gravity. These examples are meant to illuminate the more technical discussion in the preceding sections. The reader who is not interested in the technical details can just skip to this section. The takeaway is that we recover all Schwarzian amplitudes but from a bulk JT gravity calculation.

2.3.1 Some representation theory

Let us briefly discuss the Plancherel decomposition of $\mathrm{SL}^+(2,\mathbb{R})$ and its matrix elements. The representation theory of $\mathrm{SL}^+(2,\mathbb{R})$ is closely related to that of $\mathrm{SL}(2,\mathbb{R})$ itself and large parts of it can be found in the available literature [95, 96, 97] on which this section is based. In spite of the lack of an inverse, hence *semi*group, it is possible to set up a meaningful representation theory for subsemigroups in the sense:

$$R(g_1 \cdot g_2) = R(g_1) \cdot R(g_2). \tag{2.198}$$

The generators of the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ are the same as those for $\mathrm{SL}(2,\mathbb{R})$. A basis for the Hilbert space of $\mathrm{SL}^+(2,\mathbb{R})$ is obtained by diagonalizing the operators J_0 and C. We end up with states $|js\rangle$ where the eigenvalues of J_0 is s and the eigenvalue of C is $-j(j+1)$. We can associate to these states a basis of square integrable wavefunctions on the positive real axis:

$$\langle x|js\rangle = f^j_s(x), \qquad x > 0. \tag{2.199}$$

[43]This is only true though if the global topology of interest is that of a disk. As explained in detail in chapter 3 it is for example plain wrong to think boundary-to-boundary geodesic lengths span the Hilbert space of JT gravity on an annulus. Large diff invariance will not have any of it.

The action of subsemigroup elements on these wavefunctions is defined as:

$$\langle x|js\rangle \to \langle x|\,g\,|js\rangle = (g\cdot f^j_s)(x) = |bx+d|^{2j} f^j_s\left(\frac{ax+c}{bx+d}\right). \tag{2.200}$$

We see that for positive group elements this is indeed an internal operation in the set of functions on the positive real axis. Infinitesimally using $g = 1 + i\epsilon^a J_a$ we find that the current algebra works in this basis of functions as:

$$J_- = \partial_x, \quad J_0 = -x\partial_x + j, \quad J_+ = -x^2\partial_x + 2jx. \tag{2.201}$$

One immediately checks that indeed $\mathcal{C} = -j(j+1)$ on any function of x. Notice that in the case of $\text{SL}^+(2,\mathbb{R})$ we don't really get to choose which generator J^a we are diagonalizing if we want to obtain a basis of states $|js\rangle$. Indeed, only the operator J_0 can be found to be self-adjoint on the positive real line. This is for example clearly not the case for $J_- = \partial_x$ which is self-adjoint on the real line but not on the positive real axis. This doesn't mean we can't consider eigenstates $|jm\rangle$ of for example J_+ when we are talking coset constraints. It just means that $|jm\rangle$ are not a basis. They are not necessarily orthogonal nor complete. For our discussion that is a detail modulo for the fact that the reader should keep in mind that all intermediate labels associated with bulk points will refer to eigenstates of J_0. For example the sum over labels a in (2.163) would necessarily be a sum over hyperbolic labels s here.
Back to setting up representation theory. Representation matrices can essentially be defined as the Fourier components of transformed functions:

$$R_{j,s_1s_2}(g) = \langle js_1|\,g\,|js_2\rangle = \int dx\,\langle js_1|x\rangle\,\langle x|\,g\,|js_2\rangle = \int dx\, f^j_s(x)^*\,(g\cdot f^j_s)(x). \tag{2.202}$$

More in general one defines:

$$\langle js_1|\,g\,|is_2\rangle = 0, \qquad j\neq i. \tag{2.203}$$

We can check that this now indeed defines a representation by inserting a complete set of states $|is_a\rangle$ in:[44]

$$\begin{aligned} R_{j,s_1s_2}(g_1\cdot g_2) = \langle js_1|\,g_1\cdot g_2\,|js_2\rangle &= \int ds_a \int di\;\langle js_1|\,g_1\,|is_a\rangle\,\langle is_a|\,g_2\,|js_2\rangle \\ &= \int ds_a\, R_{j,s_1s_a}(g_1)\,R_{j,s_as_2}(g_2). \end{aligned} \tag{2.204}$$

Because $g_1\cdot g_2$ is positive if g_1 and g_2 are positive, this defines representations for the subsemigroup $\text{SL}^+(2,\mathbb{R})$. Let us now explicitly calculate these representation matrix elements in the hyperbolic basis. The matrix elements of $\text{SL}^+(2,\mathbb{R})$ in the hyperbolic basis where we diagonalize J_0 are closely related to those of $\text{SL}(2,\mathbb{R})$. This is because we

[44]Furthermore one needs to check that $R(g_1\cdot g_2) = R(g)$ with $g = g_1\cdot g_2$ or, in other words, that group composition works properly on the functions (2.200), which it does.

can obtain the Borel-Weil realization (2.201) of the full group $\mathrm{SL}(2,\mathbb{R})$ by considering a basis of square integrable wavefunctions on the whole real axis:

$$\langle x|js\rangle = f_s^j(x). \tag{2.205}$$

These can obviously be decomposed into the direct product space of square integrable wavefunctions on the negative real axis and positive real axis, much like wavefunctions of a scalar field in Minkowski space can be expanded in left and right Rindler wavefunctions via the Bogoliubov transformations. So we have for example:

$$f_s^j(x) = f_s^{j+}(x) + f_s^{j-}(x). \tag{2.206}$$

The former have support on the positive real axis, the latter on the negative real axis. Very explicitly we have conform (2.201) and diagonalizing J_0:

$$f_s^{j+}(x) = \langle x|js\rangle = \frac{1}{\sqrt{2\pi}} x^{is-1/2}\theta(x), \qquad \langle js|x\rangle = \frac{1}{\sqrt{2\pi}} x^{-is-1/2}\theta(x). \tag{2.207}$$

Here s is related to the J_0 eigenvalue via (2.201). These are indeed a basis on the positive real axis:

$$\int_0^{+\infty} \frac{dx}{x}\, x^{is_1} x^{-is_2} = \delta(s_1 - s_2) \tag{2.208}$$

$$\int_{-\infty}^{+\infty} ds\, x^{is-1/2} y^{-is-1/2} = \delta(x-y). \tag{2.209}$$

The basis on the negative real axis is constructed completely analogously. For $\mathrm{SL}(2,\mathbb{R})$ the group elements work just as in (2.200). The difference with $\mathrm{SL}^+(2,\mathbb{R})$ is that because g isn't positive, a function with only support on the positive real axis is mapped to a function with support in both regions. Therefore in general we have four matrix elements to calculate:

$$K^{\pm\pm}_{j,s_1 s_2}(g) = \int_{-\infty}^{+\infty} dx\, f_{s_1}^{j\pm}(x)\,(g\cdot f_{s_2}^{j\pm})(x). \tag{2.210}$$

The result can be arranged into a matrix of representation matrices:

$$K_{s_1 s_2}(g) = \begin{pmatrix} K^{++}_{s_1 s_2}(g) & K^{+-}_{s_1 s_2}(g) \\ K^{-+}_{s_1 s_2}(g) & K^{--}_{s_1 s_2}(g) \end{pmatrix}. \tag{2.211}$$

The entries of this matrix are well known [95, 96].[45] This matrix composes under group transformations using matrix multiplication. Specifying now to $\mathrm{SL}^+(2,\mathbb{R})$ it is obvious from (2.210) that the matrix elements in the hyperbolic basis of $\mathrm{SL}^+(2,\mathbb{R})$ are just $K^{++}_{s_1 s_2}(g)$ and that $K^{+-}_{s_1 s_2}(g) = 0$ for positive group elements. In particular for such elements the $\mathrm{SL}(2,\mathbb{R})$ group composition directly implies that the matrices $K^{++}_{s_1 s_2}(g)$

[45] Also the q-deformed variants of $K^{++}_{s_1 s_2}(g)$ are known [94] which reduce to the answers of [95, 96].

furnish a representation in the sense that $K_{++}(g_1 \cdot g_2) = K_{++}(g_1) \cdot K_{++}(g_2)$. We can just compute this matrix element ab initio:

$$K^{++}_{s_1 s_2}(g) = \langle s_1 | \, g \, | s_2 \rangle = \int_0^{+\infty} dx x^{-is_1-1/2} (g \cdot x^{is_2 - 1/2}). \tag{2.212}$$

The problem can be chopped into pieces by considering the Gauss decomposition of $\text{SL}^+(2,\mathbb{R})$:

$$g = e^{\gamma_- J_-} e^{2\phi J_0} e^{\gamma_+ J_+} = \begin{pmatrix} 1 & 0 \\ \gamma_- & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & \gamma_+ \\ 0 & 1 \end{pmatrix}, \quad \gamma_-, \gamma_+ > 0. \tag{2.213}$$

It suffices to calculate the representation matrix element for each of these three matrices in the decomposition. The generic answer then follows from the group composition property. Specifying to the continuous series irreps $j = -1/2 + ik$ we find [96]:

$$\begin{aligned} K_{++}(s_1, s_2; \phi) &= e^{2i(k-s_2)\phi} \delta(s_1 - s_2) \\ K_{++}(s_1, s_2; \gamma_-) &= \frac{1}{2\pi} \frac{\Gamma(-is_1 + 1/2)\Gamma(is_1 - is_2)}{\Gamma(-is_2 + 1/2)} \gamma_-^{is_2 - is_1} \\ K_{++}(s_1, s_2; \gamma_+) &= \frac{1}{2\pi} \frac{\Gamma(is_2 - is_1)\Gamma(is_1 + 1/2 - 2ik)}{\Gamma(is_2 + 1/2 - 2ik)} \gamma_+^{is_1 - is_2}. \end{aligned} \tag{2.214}$$

The orthogonal wavefunctions are then obtained as:

$$\psi^k_{s_1 s_2}(g) = (k \sinh 2\pi k)^{1/2} \, K^{++}_{k, s_1 s_2}(g). \tag{2.215}$$

From this one furthermore deduces the Plancherel measure on $\text{SL}^+(2,\mathbb{R})$:

$$\rho(k) = k \sinh 2\pi k. \tag{2.216}$$

It is common to write the Plancherel decomposition as:

$$L^2(\text{SL}^+(2,\mathbb{R})) \simeq \int_0^\infty dk \, (k \sinh 2\pi k) \, P_k \otimes P_k. \tag{2.217}$$

Here P_k labels the principal continuous irreps of $\text{SL}(2,\mathbb{R})$. We note that this is the classical limit of the Plancherel decomposition of $L^2(\text{SL}_q^+(2,\mathbb{R}))$. The latter was conjectured to hold in [92, 93] but proven only later in the Mathematics literature [94]. In comparison, we repeat the Plancherel decomposition of $\text{SL}(2,\mathbb{R})$:

$$L^2(\text{SL}(2,\mathbb{R})) \simeq \int_0^\infty dk \, (k \tanh \pi k) \, P_k \otimes P_k \quad \oplus \quad \sum_{\ell=0}^{\infty} (\ell + 1/2) \, P_\ell \otimes P_\ell. \tag{2.218}$$

2.3.2 Quantum mechanics on a subsemigroup

It is not a priori obvious that BF theory even makes sense for a subsemigroup, as the inverse of a group element is not in the subsemigroup. Let us make sure that quantum mechanics on $\mathrm{SL}^+(2,\mathbb{R})$ is mathematically consistent. We have in (2.215) a basis of wavefunctions on $\mathrm{SL}^+(2,\mathbb{R})$, so we can immediately write down the propagator on $\mathrm{SL}^+(2,\mathbb{R})$ from g to $\lambda \cdot g$ with both g and λ positive:

$$\langle g| e^{-\beta H} |\lambda \cdot g\rangle = \int ds_1 \int ds_2 \int dk\, k \sinh 2\pi k\, e^{-\beta \mathcal{C}(k)}\, K^{++}_{s_1 s_2}(\lambda \cdot g) K^{++}_{s_1 s_2}(g)^*. \quad (2.219)$$

This is conform (2.180). The matrix elements as calculated in (2.214) can be checked to be unitary:[46]

$$K^{++}_{s_2 s_1}(g)^* = K^{++}_{s_1 s_2}(g)^{-1} \quad (2.220)$$

For g positive we can use the fact that $K^{+-}_{s_1 s_2}(g) = 0$ and $\mathrm{SL}(2,\mathbb{R})$ group composition of the matrix $K_{s_1 s_2}(g)$ to find the inverse of the matrix $K^{++}_{s_1 s_2}(g)$:

$$K^{++}_{s_1 s_2}(g)^{-1} = K^{++}_{s_1 s_2}(g^{-1}), \quad g > 0. \quad (2.221)$$

This only makes sense because $\mathrm{SL}^+(2,\mathbb{R})$ is not just a semigroup, but is actually part of some group which is here $\mathrm{SL}(2,\mathbb{R})$. Without this embedding we couldn't make sense of g^{-1}. Elements of $\mathrm{SL}^+(2,\mathbb{R})$ do have an inverse, but it lies outside of $\mathrm{SL}^+(2,\mathbb{R})$. This is ultimately why BF theory for the subsemigroup works just like for a group. We note that (2.221) is part of a more general identity that holds for any g in the group and h in the subsemigroup:

$$K^{++}(h) \cdot K^{++}(g) = K^{++}(h \cdot g), \quad h > 0. \quad (2.222)$$

Combining unitarity (2.220) and the specific inverse matrix (2.221) we find:

$$K^{++}_{s_1 s_2}(g)^* = K^{++}_{s_2 s_1}(g^{-1}), \quad g > 0. \quad (2.223)$$

Combining formulas (2.222) and (2.223) we then obtain:

$$\int ds_2\, K^{++}_{s_1 s_2}(\lambda \cdot g)\, K^{++}_{s_1 s_2}(g)^* = K^{++}_{s_1 s_1}(\lambda). \quad (2.224)$$

Notice that it's important (2.222) holds for all group elements, so in particular also for the inverse of a positive element as is required here. Taking the integral over s_1 in (2.219) one finds the character of the subsemigroup by definition:

$$\langle g| e^{-\beta H} |\lambda \cdot g\rangle = \int dk\, k \sinh 2\pi k\, \chi^+_k(\lambda)\, e^{-\beta \mathcal{C}_k}. \quad (2.225)$$

The character of the subsemigroup is essentially identical to the character of the group itself.[47] The characters of $\mathrm{SL}(2,\mathbb{R})$ given our choice of normalization are:

$$\chi_k(\lambda) = \cos \pi k \lambda. \quad (2.228)$$

[46] See for example [2].

[47] Indeed we have $\chi_k(\lambda) = \mathrm{Tr}\, K^{++}(\lambda) + \mathrm{Tr}\, K^{--}(\lambda)$. Using formulas (9) and (10) on page 358 of [95] one finds $K^{--}(\lambda) = K^{++}(\lambda)$ and hence $\chi_k(\lambda) = 2\chi^+_k(\lambda)$. In fact we can use [95] to prove a more

This formula deserves some extra attention. When one usually thinks about finite characters, one imagines using the integration measure on the space of conjugacy class elements as inferred from the Haar measure. This choice of measure was taken for example in [98]. For SL$(2,\mathbb{R})$ this measure is $d\lambda \sinh^2 \lambda$. The resulting characters, orthogonal with respect to these measures, are in the case of SL$(2,\mathbb{R})$:

$$\chi_k(\lambda) = \frac{\cos \pi k \lambda}{\sinh \lambda}. \tag{2.229}$$

The point made in [74] is however, that this integration measure is a choice, and depending on the situation a different normalization might be required. In this context it is more natural to choose the measure obtained naturally by thinking about 2d BF theory as the classical limit of 3d Chern-Simons theory. We understand that in this case the measure on the space of conjugacy class elements is essentially the flat one. The appropriate characters are those where we essentially drop the denominators of (2.229), resulting in (2.228). In fact this is not yet the full story, as discussed below (2.154), but let us postpone that discussion to the chapter 3.
Returning to our initial problem, we find for the propagator of our theory:

$$Z(\beta,\lambda) = \int dk\, k \sinh 2\pi k\, \cos 2\pi k\, e^{-\beta k^2}. \tag{2.230}$$

Similarly the partition function is:

$$Z(\beta) = \int_0^{+\infty} dk\, (k \sinh 2\pi k)^2\, e^{-\beta k^2}. \tag{2.231}$$

Proceeding to solve the 2d SL$^+(2,\mathbb{R})$ BF theory one could include boundary anchored Wilson lines. Via the logic of around equation (2.134) one is ultimately led to calculate integrals of the type:

$$\int dg\, K^{++}_{\mu_1,s_1 s_4}(g) K^{++}_{\mu_2,s_2 s_5}(g) K^{++}_{\mu_3,s_3 s_6}(g)^* = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ s_4 & s_5 & s_6 \end{pmatrix}. \tag{2.232}$$

The equality is guaranteed by the property (2.223) resulting in the defining integral for a product of $3j$ symbols (2.124).
Mathematical consistency of the 2d BF Hamiltonian calculations requires some further explaining. In particular it requires that we can write each disk-shaped region as a

---

generic property. For generic positive g we have:

$$\chi_\mu(g) = \chi^+_\mu(g) + \chi^+_\mu(e \cdot g \cdot e). \tag{2.226}$$

Here $e = \text{diag}(-1,1)$. The action of e on wavefunctions $f^{\mu+}_s(x)$ is $e \cdot f^{\mu+}_s(x) = f^{\mu+}_s(-x)$ effectively mapping the left and right axis or K^{++} and K^{--}. Explicitly for the relevant wavefunctions we have $\langle -x|s\rangle = e^{\pi s}\langle x|s\rangle$ and $\langle s|-x\rangle = e^{-\pi s}\langle s|x\rangle$. We then find:

$$\chi^+_\mu(e \cdot g \cdot e) = \chi^+_\mu(g), \qquad \chi_\mu(g) = 2\chi_\mu(g). \tag{2.227}$$

propagator from one positive group element to another.[48] Positivity of a group element along a certain line requires the choice of an orientation on this line. This is accomplished by choosing a set of oriented Cauchy surfaces within the disk, which immediately also provides a consistent calculation.

2.3.3 Gravitational coset

To recover the Schwarzian amplitudes from a bulk 2d $\text{SL}^+(2,\mathbb{R})$ BF calculation, all we need to do is implement the coset boundary conditions (2.62) on physical states:

$$J_+ \left|\psi\right\rangle = \left|\psi\right\rangle . \tag{2.233}$$

Notice that we are diagonalizing J_+. Indeed the constraint (2.61) constrains the current component $\mathcal{J}^-$ and as pointed out below (2.58) this current component becomes the analogue of an $\text{SL}(2,\mathbb{R})$ generator J_+ upon canonical quantization. Let us denote the corresponding physical spectrum of states as $|k, 1_+\rangle$. The corresponding wavefunction on the positive real axis analogous to (3.113) is readily obtained from the Borel-Weil realization (2.201) as:

$$\langle x|k, 1_+\rangle = e^{-1/x} x^{2ik-1}. \tag{2.234}$$

Consider furthermore the state:

$$J_- \left|k, 1_-\right\rangle = \left|k, 1_-\right\rangle , \qquad \langle x|k, 1_-\rangle = e^{-x}. \tag{2.235}$$

Eigenstates of J_- and J_+ can be transformed into each other by acting with the group element:

$$\omega = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right). \tag{2.236}$$

To understand this, consider (2.200) and write down the corresponding eigenfunctions. From the discussion on cosets we understand that the BF states on an interval with both endpoints on the boundary are $|k, 1_+, 1_+\rangle$. We will therefore be interested in computing matrix elements of the type $R_{k,1_+1_+}(g)$. Unlike the wavefunctions (2.159) discussed in the coset section, these do not transform invariantly under $g \to g \cdot h$ with h an exponential of J_+. They rather transform covariantly. In particular this means that if we parameterize g using the Gauss decomposition (2.213) then we don't need to bother with the dependence on γ_+. Indeed, we can use the fact that the wavefunction transforms covariantly under right multiplication by an exponential of J_+ to distill an overall factor $e^{-\gamma_+}$. Following steps similar to those that led to (2.170), we understand that these matrix elements will feature in the calculation of $3j$ symbols of the type:

$$\int dg\, R_{k_1,1_+1_+}(g)\, R_{\ell,0_+0_+}(g)\, R_{k_2,1_+1_+}(g)^* = \begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2 . \tag{2.237}$$

These are the $3j$ symbols alluded to in (2.194). To actually calculate this integral it is convenient to shift $g \to \omega \cdot g$. This is a symmetry of the integral, as the Haar measure is

[48]Otherwise $\text{SL}(2,\mathbb{R})$ representation theory is required contradictory to the ansatz that a consistent truncation to $\text{SL}^+(2,\mathbb{R})$ BF theory exists.

invariant under such shifts. One set of the labels is then transformed into a set of labels associated with an eigenstate of J_-:

$$\int dg\, R_{k_1,1_-1_+}(g)\, R_{\ell,0_-0_+}(g)\, R_{k_2,1_-1_+}(g)^* = \begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2 . \tag{2.238}$$

The advantage is that the resulting matrix elements are more easily computed, see for example [97, 96]. This is obvious when considering again the Gauss decomposition (2.213). All three matrix elements in the integral (2.238) are now such that they transform covariantly under left multiplication by an exponential of J_- and under right multiplication by an exponential of J_+. This then simplifies the computations significantly. For example:

$$R_{k,1_-1_+}(g) = R_{k,1_-1_+}(\phi) e^{-\gamma_+} e^{-\gamma_-}. \tag{2.239}$$

The remaining integral can be done in a straightforward manner. Working with the diagonal group element $e^{2\phi J_0}$ on the wavefunction $\langle x|k, 1_+\rangle$ and computing the representation matrix element as in (2.202) we find:

$$R_{k,1_-1_+}(\phi) = e^{\phi} e^{-2ik\phi} \int_0^{\infty} dx\, x^{2ik-1} e^{-x} e^{-\frac{e^{2\phi}}{x}} = e^{\phi} K_{2ik}(e^{\phi}). \tag{2.240}$$

The latter equality follows from the integral representation of the modified Bessel function of the second kind:

$$\int_0^{\infty} dx\, x^{2ik-1} e^{-\nu x} e^{-\frac{\lambda}{x}} = \left(\frac{\lambda}{\nu}\right)^{ik} K_{2ik}(\sqrt{\lambda\nu}), \quad \lambda,\, \nu > 0. \tag{2.241}$$

The integral over γ_- and γ_- in (2.238) just gives a constant, because the integration range is from 0 to ∞ for both variables in the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$. This means we can essentially neglect the dependence on γ_- and γ_+ and think about normalized coset wavefunctions on the interval slices as:

$$\psi^k(\phi) = (k \sinh 2\pi k)^{1/2} e^{\phi} K_{2ik}(e^{\phi}). \tag{2.242}$$

Indeed, the Haar measure essentially reduces to $e^{-2\phi} d\phi$ and we have:

$$\int_{-\infty}^{+\infty} d\phi e^{-2\phi}\, R_{k_1,i_-i_+}(\phi) R_{k_2,i_-i_+}(\phi)^* = \frac{1}{k_1 \sinh 2\pi k_1} \delta(k_1 - k_2). \tag{2.243}$$

This is the expected normalization for wavefunctions or matrix elements in a BF theory with Plancherel measure (2.191). In fact this computation can be considered an independent check of the statement that the Plancherel measure on $\mathrm{SL}^+(2,\mathbb{R})$ is $\rho(k) = k \sinh 2\pi k$. All that's left to do in order to compute the $3j$ symbols via (2.238) is to obtain an expression for $R_{\ell,0_-0_+}(\phi)$. Obviously this matrix element is independent of γ_- and γ_+ because in this case we have an invariant under left multiplication by a function of J_- and under right group multiplication with a function of J_+. As it turns out [96], the discrete series matrix elements of interest are represented not by modified Bessel functions of the second kind, but by modified Bessel functions of the first kind with the same arguments. For example:

$$R_{\ell,m_-m_+}(\phi) = e^{\phi} I_{2\ell-1}(me^{\phi}), \qquad m > 0. \tag{2.244}$$

These are eigenfunctions of the same differential equation as (2.242) and thus also eigenfunctions of the SL$(2,\mathbb{R})$ Casimir. In our case we need to take a limit of this modified Bessel function. Up to a constant prefactor we find:

$$R_{\ell,0_-0_+}(\phi) = e^{2\ell\phi}. \tag{2.245}$$

The resulting integral (2.238) can be done directly using formula (6.576) of [99]:

$$\int_{-\infty}^{+\infty} d\phi\, e^{2\ell\phi} K_{2ik_1}(e^{\phi}) K_{2ik_2}(e^{\phi}) = \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}. \tag{2.246}$$

This formula, as well as formula (2.242) should be compared to equivalent formulas that were derived later in [57] using a completely different perspective on JT gravity as the quantum mechanics of a particle moving on AdS_2 in an infinite external magnetic field. This proves that both formalisms are structurally completely identical. More importantly, combining with (2.238) we find the relevant $3j$ symbols associated with Wilson line endpoints on the boundary in JT gravity as:

$$\begin{pmatrix} k_1 & \ell & k_2 \\ 1 & 0 & 1 \end{pmatrix}^2 = \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}. \tag{2.247}$$

This exactly matches our prediction from Schwarzian quantum mechanics (2.194). Together with the identification of the Plancherel measure of $\text{SL}^+(2,\mathbb{R})$ as (2.191) and with the knowledge that the $6j$ symbol of $\text{SL}^+(2,\mathbb{R})$ appears when trying to interpret the Schwarzian amplitudes of [43] as bulk BF amplitudes, this completes the identification of JT gravity as a 2d $\text{SL}^+(2,\mathbb{R})$ BF theory with coset boundary conditions (2.62).

2.3.4 Some explicit amplitudes

All this has been rather technical. We would like to end this section on a happy note by just writing down some explicit amplitudes for JT gravity in the spirit of the coset calculations.

Partition function

The disk diagram is computed precisely as in the usual coset picture (2.165):

$$Z(\beta) = \langle 1 | e^{-\beta H} | 1 \rangle = \int dk\, k \sinh 2\pi k\, R_{k,1_+1_+}(1)\, R_{k,1_+1_+}(1)\, e^{-\beta C(k)}. \tag{2.248}$$

Here we used the basis of states $|k, 1_+, 1_+\rangle$ as appropriate for the coset theory (2.233). To evaluate this it is convenient to resort once again the property:

$$R_{k,1_+1_+}(1) = R_{k,1_-1_+}(\omega). \tag{2.249}$$

We have an explicit expression for $R_{k,1_-1_+}(g)$ in terms of the Gauss parameterization, so we just need to find the Gaussian coordinates of ω. One checks that ω corresponds with the point $\gamma_+ = e^{\phi}$, $\gamma_- = e^{-\phi}$ and $\phi = \infty$. The dependence on γ_+ and γ_- is thus cancelled in (2.239) and we end up with:

$$Z(\beta) = \int dk\, \psi^k(\infty)\, \psi^k(\infty)\, e^{-\beta\mathcal{C}(k)}. \tag{2.250}$$

For large values of its argument the modified Bessel function of the second kind goes to a constant independent of the frequency. Therefore, up to an overall constant we have:

$$\psi^k(\infty) = (k \sinh 2\pi k)^{1/2}. \tag{2.251}$$

This is to be expected, it is saying that modulo a constant the representation matrix in the parabolic basis gives the natural unit answer when evaluated on the identity:

$$R_{k,1_+1_+}(1) = 1. \tag{2.252}$$

We end up with:

$$Z(\beta) = \int_0^\infty dk\, k \sinh 2\pi k\, e^{-\beta k^2}. \tag{2.253}$$

This indeed matches precisely with the corresponding direct calculation of the Schwarzian path integral (2.71). Let us reemphasize that the Schwarzian spectral density $\rho(k) = k \sinh 2\pi k$ differs fundamentally from the $\mathrm{SL}(2,\mathbb{R})$ Plancherel measure $\rho(k) = k \tanh(\pi k)$ and that this is explained in bulk first order formalism by the fact that the relevant structure is the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ and not the $\mathrm{SL}(2,\mathbb{R})$ group manifold itself. We note that the $\mathrm{SL}^+(2,\mathbb{R})$ Plancherel measure naturally arises as the classical limit of the so called quantum dimension in the modular tensor category that underlies Liouville theory, Virasoro conformal field theory and AdS_3 gravity where this dimension is determined by vacuum S-matrix elements:

$$\dim s = S_0{}^s = \sinh 2\pi b s \sinh 2\pi b^{-1} s. \tag{2.254}$$

The classical limit takes b to zero whilst zooming in on low energies $s = bk$. We indeed recover the dimensions of $\mathrm{SL}^+(2,\mathbb{R})$ representations.

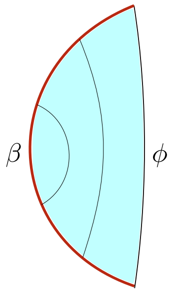
One Wilson line

The computation of a single boundary anchored Wilson line follows along the same lines of (2.174) and (2.134). Whilst in danger of slightly repeating ourselves, let us take the time to spell out this computation. We have learned from the discussion of around (2.245) that the analogue of the Wilson line decomposition (2.131) for JT gravity is:

$$\mathcal{W}_\ell = \int_{-\infty}^{+\infty} d\phi e^{2\phi}\, \left|\phi\right\rangle \otimes \left\langle\phi\right| \mathcal{W}_\ell(\phi), \qquad \mathcal{W}_\ell(\phi) = e^{2\ell\phi}. \tag{2.255}$$

This is with the understanding that we have implicitly shifted $g \to \omega \cdot g$ in all integrals of the type (2.237). By consequence all wavefunctions are effectively of the mixed parabolic type as in (2.238). Let us now calculate the amplitude of a disk with one boundary segments fixed to some length β and with the other segment defining a boundary state $\omega \cdot g$. As explained between (2.239) and (2.242) the dependence of g on γ_+ and γ_- doesn't do very interesting things in this setup. We can essentially drop it. We find the analogue of (2.127):

$Z(\beta, \phi) =$



$$= \langle 1|\, e^{-\beta H}\, |\omega \cdot g\rangle = \int_0^\infty dk\, \psi_k(\infty)\, \psi_k(\phi)\, e^{-\beta k^2}$$

$$= \int_0^\infty dk\, k \sinh 2\pi k\, e^{\phi} K_{2ik}(e^{\phi}). \tag{2.256}$$

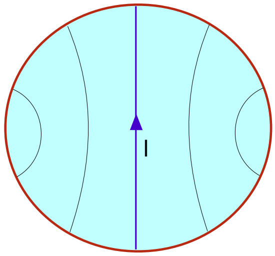
This answer matches precisely with the computation in [57, 18] of a JT gravity disk with one fixed length boundary and one geodesic boundary of regularized length $L = -2\phi$ connecting the endpoints of the fixed length boundary. Though the perspective on JT gravity bulk amplitudes in [57] is different, we see that structurally it is equivalent to the perspective of JT gravity as a 2d $\mathrm{SL}^+(2,\mathbb{R})$ BF theory with coset boundary conditions (2.233). One advantage of the formulation of [57] is that in a theory of quantum gravity is is much more intuitive to have a Hilbert space of states with a direct gravitational interpretation. Moreover the Wilson line operator is in that language immediately identified as the amplitude of a massive particle propagating through the bulk as discussed in (2.99). Indeed we now have:

$$\mathcal{W}_\ell = \int_{-\infty}^{+\infty} dL e^{-L}\, |L\rangle \otimes \langle L|\, \mathcal{W}_\ell(L), \qquad \mathcal{W}_\ell(L) = e^{-\ell L}. \tag{2.257}$$

This is indeed just the boundary to boundary propagator along a geodesic of regularized length L for an operator of boundary conformal weight ℓ, see [18] for more on this. As we briefly discuss in the next example though, this nice intuitive gravitational language doesn't quite do it anymore when we consider crossing Wilson lines. Semiclassically this corresponds to two particles propagating to the bulk and interacting via a gravitational shockwave [100]. It would be very interesting to understand how the corresponding quantum amplitudes are described using natural geometric variables. Here we will just describe it in the 2d $\mathrm{SL}^+(2,\mathbb{R})$ BF language which has no trouble whatsoever with Wilson line crossings. Such more complicated configurations, as well as the higher configurations discussed in chapter 3 are where the BF formulation truly earns its stripes.

Returning to the problem at hand, we can now apply the discussion of around (2.133), (2.134) and (2.174) to calculate the correlator of the single boundary anchored Wilson line:

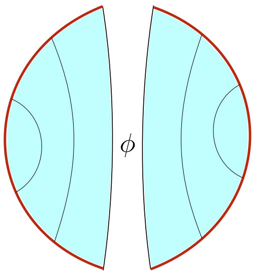
$$\langle 1| e^{-\beta_1 H} \mathcal{W}_\ell e^{-\beta_2 H} |1\rangle = \beta_1$$



$$\beta_2 \quad . \tag{2.258}$$

This decomposes using (2.255) as:

$$\int d\phi e^{-2\phi}\, e^{2\ell\phi} \quad \beta_1$$



$$\beta_2 \; = \int d\phi e^{-2\phi}\, Z(\beta_1,\phi)\, \mathcal{W}_\ell(\phi)\, Z(\beta_2,\phi).$$

One sees that the integral over ϕ reduces precisely to (2.246), with two Bessel functions extracted from the partition function (2.256). An integral over two spectra k_1 and k_2 from both partition functions (2.256) remains. The final result for the single boundary anchored Wilson line in JT gravity is:

$$\int_0^\infty dk_1\, k_1 \sinh 2\pi k_1\, e^{-\beta_1 k_1^2} \int_0^\infty dk_2\, k_2 \sinh 2\pi k_2\, e^{-\beta_2 k_2^2}\, \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}. \tag{2.259}$$

This matches precisely the answer for Schwarzian mechanics of [43], which is obtained by solving the path integral (2.98).

Crossing Wilson lines

We will not do in full the calculation with crossed Wilson lines, which is straightforward to obtain from the analogous calculation around (2.142). Let us just note a few important points. In particular we learned around (2.163) that the Hilbert space of 2d BF on an interval cares less about the the coset constraints when only one endpoint of the interval is on the boundary, and it doesn't care about these constraints at all when neither endpoint is on the boundary. We decompose for example:

$$R_{k,1_-1_+}(g_1 \cdot g_2) = \int_{-\infty}^{+\infty} ds\, R_{k,1_-s}(g_1)\, R_{k,s1_+}(g_2). \tag{2.260}$$

Say we parameterize g_1 by ϕ_1, γ_{+1} and γ_{-1}. In the first matrix element on the right hand side we can extract the dependence on γ_{-1} following the logic of around (2.239). Similarly for the second matrix element we can extract the dependence on γ_{+2}. Integrating over the corresponding dependence produces overall constants as around (2.242). Crucially though we are left with a wavefunction of ϕ_1 and γ_{+1} for the first matrix element for example. So schematically the wavefunctions on a half-line are:

$$\psi_s^k(\phi,\gamma). \tag{2.261}$$

This function can in principle be calculated explicitly using (2.214). We will refrain from doing so, because we know from first principles in the sense of (2.121) that the corresponding group integrals will produce $3j$ symbols. These can be re-packaged into $6j$ symbols associated with each bulk crossing of Wilson lines as explained around (2.144). The $6j$ symbols of $\mathrm{SL}(2,\mathbb{R})$ are well documented, and they appear automatically in this calculation. Crucially, the same conclusion was reached starting out from Schwarzian quantum mechanics in [43]. This completes the proof that 2d $\mathrm{SL}^+(2,\mathbb{R})$ BF theory with coset boundary conditions (2.233) is completely quantum mechanically equivalent to Schwarzian quantum mechanics and hence is the precise first order formulation of JT gravity.

More in general on intervals where neither endpoint touches the boundary, all three coordinates will give nontrivial dependence, so we should be interested in wavefunctions of the type:

$$\psi_{s_1 s_2}^k(\phi,\gamma_+,\gamma_-). \tag{2.262}$$

As explained around (2.256) there is a nice intuitive gravitational interpretation of the wavefunctions on an interval with both endpoints touching the boundary, where the group element eigenstates $|\phi\rangle$ are related to eigenstates of regularized geodesic lengths between both boundaries $|L\rangle$. What is lacking to complete the story of [57] is a nice intuitive geometric interpretation of states $|\phi,\gamma\rangle$ relevant to intervals ending at some bulk vertex, or more generally states $|\phi,\gamma_+,\gamma_-\rangle$ relevant to intervals between bulk vertices. This is why it does not seem to be straightforward a priori to obtain $6j$ symbols following the story of [57]. The group theoretical interpretation presented in this chapter on the other hand, gives a borderline trivial calculation of all exact correlators. The downside is that we are not immediately gaining any real gravitational intuition. It would therefore be valuable to have a gravitational interpretation of generic group element states.

2.4 Concluding remarks

We end this chapter with some concluding remarks. In the first part of this section we present some further intuitive motivation why it makes sense to consider the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$ as associated with the first order formulation of JT gravity. In the second section we briefly touch on how this entire chapter can be considered the classical limit of a well known story involving three dimensional Chern-Simons theory, 2d Wess-Zumino-Witten models and 2d Virasoro coadjoint orbit action.

2.4.1 An apologia for the subsemigroup

We remind the reader that the structure $\text{SL}^+(2,\mathbb{R})$ is a subsemigroup, consisting of $\text{SL}(2,\mathbb{R})$ matrices with all positive entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \quad a, b, c, d > 0. \tag{2.263}$$

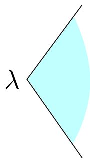
Let us present three quick reasons why its a good thing to associate this subsemigroup to the first order formulation of JT gravity. In particular we will discuss three reasons which do not refer to boundary Schwarzian quantum mechanics. The most important argument of them all is a very pragmatic one and has hopefully been appreciated by now. Considering $\text{SL}^+(2,\mathbb{R})$ is correct because it gives the correct answers. A rigorous though intensive way to prove that two theories are identical is to proof that all correlators match. We have thus rigorously proven that Schwarzian quantum mechanics is identical to a 2d $\text{SL}^+(2,\mathbb{R})$ BF theory with coset boundary conditions. Therefore the latter is also precisely identical to JT gravity on a Euclidean disk with boundary conditions (2.33). A related though fundamentally different 2d BF interpretation of Schwarzian quantum mechanics and JT gravity on a disk exists. It is associated with the universal cover of $\text{SL}(2,\mathbb{R})$ [61]. In that model one has to impose some rather nontrivial constraints on this structure, arguably about as nontrivial as constraining to positive group elements. Putting aside the discussion on naturalness of both constructions, they both give the same answers. This proves that the 2d $\text{SL}^+(2,\mathbb{R})$ BF theory with coset boundary conditions discussed in this chapter is actually equivalent to the 2d universal cover of $\text{SL}(\tilde{2},\mathbb{R})$ BF theory with certain constraints argued for in [61]. It would be interesting to understand why this is the case from the path integral point of view, but crucially there does not *need* to be path integral motivation of this duality. The duality has already been established by matching all correlators. Back to some other motivation for the subsemigroup now.

Plancherel measure and black hole spectrum

First a quick argument why we like a spectral density for JT gravity that goes like $\rho(E) = \sinh 2\pi\sqrt{E}$ which doesn't build on the fact that it's dual to the Schwarzian, which has this spectrum. This type of spectrum should be compared to the type of continuum spectrum (2.188) obtained from a 2d $\text{SL}(2,\mathbb{R})$ BF theory which would be $\rho(E) = \tanh \pi\sqrt{E}$. The former has a Cardy rise at large energies, consistent with the semiclassical Bekenstein-Hawking entropy formula. The latter doesn't. So an $\text{SL}(2,\mathbb{R})$ BF theory will not result in a correct Euclidean calculation of the black hole entropy as in[55]. There simply are not enough black hole configurations in the theory, if any.

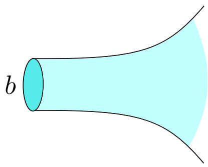
Hyperbolic geometry

A further argument in favor of $\text{SL}^+(2,\mathbb{R})$ can be made by thinking about more complicated geometries, as we'll do in chapter 3. As explained in section 2.1 the wavefunctions of BF theory on a circle are characters of all irreducible representations. The dual bases are all unitary irreps in the Plancherel decomposition, or alternatively all conjugacy class element states $|\lambda\rangle$. For $\mathrm{SL}(2,\mathbb{R})$ the conjugacy classes come in three types. We have elliptic classes for which $|\mathrm{Tr}(\lambda)| < 2$, parabolic classes for which $|\mathrm{Tr}(\lambda)| = 2$ and hyperbolic classes for which $|\mathrm{Tr}(\lambda)| > 2$. As explained by the state-operator correspondence (2.23), we can locally on a manifold replace a boundary Cauchy slice with state $|\lambda\rangle$ by a smooth 2d BF disk ending on a puncture with label λ. See also (2.151). The gravitational interpretation of a 2d BF disk ending on a puncture was discussed around (2.85) and (2.89). An elliptic puncture corresponds to a Riemann surface ending in a conical singularity or cusp, schematically:



. (2.264)

On the other hand, hyperbolic punctures correspond to Riemann surfaces ending on a geodesics boundary of some fixed length:



. (2.265)

The former are singular metrics, which are to be avoided in a quantum gravity path integral. It is indeed well known that the moduli space of flat $\mathrm{SL}(2,\mathbb{R})$ connections related to gravity is the hyperbolic component where we only have hyperbolic holonomies [65, 101, 102]. So it is clear that when we aim to describe JT gravity as a type 2d $\mathrm{SL}(2,\mathbb{R})$ BF theory, we somehow should constrain the group elements to be hyperbolic $|\mathrm{Tr}(g)| > 2$. See in particular [65] but also the discussion below (2.154) and chapter 3. One major a posteriori motivation to consider restricting to positive group elements is that such positive group elements can only be hyperbolic. Indeed, because the matrix is positive we have $b, c > 0$. The determinant constraint then implies $a > 1/d$. We find $\mathrm{Tr}(g) = a + d > a + 1/a > 2$. We could thus consider the restriction to positive group elements as following from a restriction to hyperbolic Riemann surfaces, which is just the statement that we need to exclude singular metrics in quantum gravity.

Virasoro conformal field theory

JT gravity can be defined as the classical limit of a type of 3d $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory, which in its turn is intimately related to AdS_3 gravity. See for example [83, 84]. We will have more to say on this correspondence to "classical" Chern-Simons in the next section. For now let us just point out that this type of $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons is ruled by (or defined by) the same modular tensor category which defines 2d Virasoro conformal field theory, think the Virasoro modular bootstrap or Liouville conformal field theory. This category is well known to be associated with the quantum group $\mathrm{SL}_q^+(2,\mathbb{R})$. This was discussed in great detail in [92, 93]. One intuitive way to appreciate this is that complete sets of states in the Virasoro bootstrap are continuum Verma modules. This is in a way one to one with constraining to quantum hyperbolic geometry much like in JT we are constraining to hyperbolic geometry. It suggests to consider $\mathrm{SL}_q^+(2,\mathbb{R})$. Without the plus we should be summing over discrete intermediates as well, roughly speaking. Anyway the key point we want to make here is that the classical limit of this category reproduces the representation theory of $\mathrm{SL}^+(2,\mathbb{R})$. The plus restriction is thus not just a peculiarity of AdS_2 quantum gravity, it is actually a well known though presumably not widely appreciated feature of AdS_3 quantum gravity as well. Let us note two striking features.

- In discussing the harmonic analysis on the quantum group $\mathrm{SL}_q^+(2,\mathbb{R})$ Ponsot and Teschner write down the following Plancherel decomposition:

$$L^2(\mathrm{SL}_q^+(2,\mathbb{R})) \simeq \int_\oplus d\mu(s)\, \mathcal{P}_s \otimes \mathcal{P}_s, \quad d\mu(s) = ds\, \sinh(2\pi b s) \sinh\big(2\pi b^{-1} s\big). \tag{2.266}$$

  This is as announced in (2.254). In fact the above formula was postulated in [92, 93] and proven only later in the mathematics literature [94]. Here $\mathcal{P}_s$ are the self-dual representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$. In the classical limit where we take b to zero and $s = bk$ this manifestly reduces to the Plancherel measure of $\mathrm{SL}^+(2,\mathbb{R})$. The classical limit of (2.266) gives manifestly the Plancherel decomposition of $\mathrm{SL}^+(2,\mathbb{R})$ (2.217). Remember that this just states that the matrix elements (2.215) are complete for square integrable functions on $\mathrm{SL}^+(2,\mathbb{R})$. As a consistency check on the limiting procedure from (2.266) to (2.217) recall the gravitational wavefunction:

$$R^k(\phi) = e^\phi K_{2ik}(e^\phi). \tag{2.267}$$

  This was calculated as the mixed parabolic matrix element of the Cartan element ϕ. In the mathematics literature, this is known as the Whittaker function [103, 104, 105, 106]. This JT Whittaker function indeed matches perfectly with the classical limit of the Whittaker function of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ calculated in [107].

- The precise expression for the $6j$ symbols (2.192) relevant to JT gravity was written down in [43]. This massively complicated function can be obtained as precisely the classical limit of the fusion matrices of Virasoro conformal blocks. These are by definition the $6j$ symbols of the quantum group $\mathrm{SL}_q^+(2,\mathbb{R})$. As a consistency check consider the orthogonality relation of the quantum $6j$ symbols [108]:

$$\int d\mu(s) \begin{Bmatrix} K_1 & L_1 & s \\ K_2 & L_2 & p \end{Bmatrix}_q \begin{Bmatrix} K_1 & L_1 & s \\ K_2 & L_2 & r \end{Bmatrix}_q = \frac{\delta(p-r)}{S_0{}^p}. \tag{2.268}$$

  In the classical limit this reduces to:

$$\int dp\, p \sinh 2\pi p \begin{Bmatrix} k_1 & l_1 & p \\ k_2 & l_2 & q \end{Bmatrix} \begin{Bmatrix} k_1 & l_1 & p \\ k_2 & l_2 & r \end{Bmatrix} = \frac{\delta(q-r)}{q \sinh 2\pi q}. \tag{2.269}$$

Within JT gravity, gravitational Wilson lines can be uncrossed in the bulk at no cost. This can be proven directly in the path integral before initiating an explicit calculation. The above formula expresses precisely this operation after doing the calculation BF style. It can only be used if we work with a BF theory of which the Plancherel decomposition is precisely (2.217) though. So in a sense the precise form of the $6j$ symbols in [43] are enough to conclude we should be looking for an $\mathrm{SL}^+(2,\mathbb{R})$ BF theory and certainly not for an $\mathrm{SL}(2,\mathbb{R})$ BF theory.

2.4.2 Relation to AdS$_3$ gravity and Chern-Simons theory

To close off this chapter let us briefly touch on some aspects of the duality between 3d Chern-Simons theory with a toric boundary and 2d conformal field theory living on that boundary which are closely related to the discussion we've presented in this chapter. A statement that is true in general, but which we won't explain in too much detail, is that we can understand every facet of 3d Chern-Simons theory on $\Sigma_{g,n}(\mathcal{B}_1 \dots \mathcal{B}_n) \times S_1$ as the "quantization" of 2d BF theory on $\Sigma_{g,n}(\mathcal{B}_1 \dots \mathcal{B}_n)$ where the latter is a genus g Riemann surface with n boundaries with boundary conditions $\mathcal{B}_1 \dots \mathcal{B}_n$. For example, formula (2.154) is a phase space integral. The corresponding formula for 3d Chern-Simons is a phase space thermal path integral with natural action and vanishing Hamiltonian, which calculates essentially by definition the dimension of the Hilbert space of the quantized moduli space of flat connections, whose classical counterpart is integrated over in (2.154). This link between 3d Chern-Simons and 2d BF is key in several seminal works by Witten [109, 74]. For more information on what follows we kindly refer the reader to [79, 80, 89, 82]. This is meant more as a bird's eye perspective than a complete story. We note that this section is not essential to follow the remainder of this work.

Chern-Simons theory and coadjoint orbits

The holographic duality for 3d Chern-Simons theory via the path integral works identical to that for 2d BF theory and can be pictured as (2.29). We need to identify the relevant fields g. Let's focus here on a three manifold that's topologically a solid doughnut $\Sigma_{0,1}(\mathcal{B}) \times S_1$. The argument is trivially extended to more general three manifolds of the type $\Sigma_{g,n}(\mathcal{B}_1 \dots \mathcal{B}_n) \times S_1$. When we don't have operators ending on the boundary, the boundary conditions $\mathcal{B}$ boil down to a choice of complex structure on the boundary surface, and to whether or not we impose coset constraints. The former is the parameter q or τ known from torus zero point conformal blocks or characters in conformal field theory. Excluding twists this boils down to specifying the ratio β of the lengths of the two cycles of the torus.[49] Leaving the coset choice implicit we can write the corresponding amplitudes as:

$$Z_{g,n}(\beta_1 \dots \beta_n). \tag{2.270}$$

[49]See for example [110, 111].

Let's very explicitly return to the solid doughnut. The action is:[50]

$$S[A] = \int d^3x \epsilon^{\mu\nu\sigma} \operatorname{Tr}\left(A_\mu \partial_\nu A_\sigma + \frac{2}{3} A_\mu A_\nu A_\sigma \right) \tag{2.271}$$

$$= \int dt dr d\phi \operatorname{Tr}(A_r \partial_t A_\phi - A_\phi \partial_t A_r + 2 A_t F_{\phi r}) - \int_\partial dt d\phi \operatorname{Tr}(A_t A_\phi). \tag{2.272}$$

We parameterize the three manifold in such a way that time runs along the S_1 and ϕ runs along the orthogonal boundary circle, which is contractible in the bulk. Variation results in the boundary condition $A_\phi|_\partial \sim A_t|_\partial$. Rescaling the coordinates is a symmetry of the problem hence we can bring the proportionality factor to ± 1. Changing sign corresponds to changing orientation and with our ordering of the coordinates, only the $+$ sign leads to a positive Hamiltonian. Therefore we need to choose the boundary conditions:

$$(A_t - A_\phi)|_{\text{bdy}} = A_{\bar{z}}|_{\text{bdy}} = 0. \tag{2.273}$$

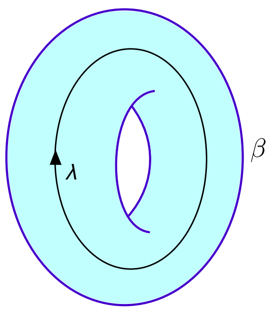
Path integration over the Lagrange multiplier A_t localizes on the constraint of vanishing spatial field strength. The locus is:

$$A_\phi = g^{-1} \partial_\phi g \tag{2.274}$$

$$A_r = g^{-1} \partial_r g. \tag{2.275}$$

Here g is a group valued field on the doughnut. It is in general twisted in the ϕ direction as $g(\phi + 2\pi) = U_\lambda g(\phi)$. Much as was discussed in BF theory around (2.19) and (2.15), any nontrivial twisting λ corresponds to the insertion of a "defect" in the Chern-Simons path integral. In the quantum theory it turns out that (2.16) is replaced by a Wilson line treading the S_1 direction [79, 109]. This is just the BF configuration (2.15) propagated along the S_1. Using the same conventions are were used in the BF calculations, we can picture the associated Wilson line amplitude in 3d Chern-Simons theory as:

$Z_{0,1}(\beta, \lambda) =$



. (2.276)

Anyway, it turns out the action (2.272) on this locus can be rewritten as a total derivative, and we are left with just a boundary term. By consequence bulk values of g are

[50] The background-dependence is only in the orientation of the chosen coordinate axes which we choose $\epsilon^{tr\phi} = 1$.

redundant and only its boundary profile is a physical degree of freedom. This is a manifestation of the picture (2.29).[51] The path integral over A is thus reduced to a path integral over boundary configurations g in this simple topology. Making the substitution $g(\phi) \to \Lambda(\phi)g(\phi)$ with $\Lambda(\phi+2\pi)^{-1}\Lambda(\phi) = U_\lambda$ we can untwist $g(\phi)$. Using partial integration combined with the boundary conditions (2.273) we find that the Chern-Simons action (2.272) becomes a right-moving affine coadjoint orbit action:

$$S[g,\lambda] = k\int_0^{2\pi k^{-1}} dt \int_0^\beta d\phi\, \mathrm{Tr}\big((g^{-1}\partial_\phi g + \lambda)g^{-1}\partial_t g - (g^{-1}\partial_\phi g + \lambda)^2\big). \tag{2.277}$$

The path integral is over fields periodic in the ϕ direction. Sometimes this theory is referred to as a chiral Wess-Zumino-Witten model. Doing so is about equally sensible as referring to an up quark as a chiral hydrogen atom. Only a particular combination of these coadjoint orbits gives an actual Wess-Zumino-Witten model. A plethora of other interesting configurations exist and from the Chern-Simons point of view one is not more special than another.

The path integral over the coadjoint orbit is well known to give a chiral character $\chi_\lambda(S\cdot\beta)$ of the relevant affine Lie algebra. The S matrix working on the complex structure just denotes how we've chosen the lengths of our cycles versus what we've chosen as time coordinate. This duality between 3d Chern-Simons theory on a doughnut and a 2d coadjoint orbit model on the boundary of the doughnut is the most primitive example of a completely general duality between 3d topological field theory ribbon graphs in some three manifold with a boundary, and 2d conformal field theory correlators on the boundary of said manifold [109, 82, 89]. Here because we are working with Chern-Simons which is very hands-on, we see very explicitly how certain boundary conditions give rise to a certain 2d conformal field theory. For example we could choose more restrictive boundary conditions than (2.273) resulting in characters of the affine Lie algebra associated with a coset G/H. These boundary conditions don't change the modular tensor category though, which is completely determined by an underlying 3d topological field theory. This is why for example the fusion matrices of Liouville theory, which is a particular combination of Virasoro coadjoint orbits, are those of an affine Lie algebra associated with $\mathrm{SL}(2,\mathbb{R})$.

Classical limit of Chern-Simons

Affine coadjoint orbits have been referred to as the "quantization" of quantum mechanics on the group, pardon the possible confusion, see for example [44, 83]. In particular we can think about the action for the twisted particle on a group as the evaluation of the Hamiltonian of (2.277) on static configurations. Schematically we can think of the coadjoint orbit as a phase space path integral:

$$\int [\mathcal{D}g][\mathcal{D}\pi_g]\exp\bigg(-\int_\partial \mathrm{Tr}(\partial_t g\pi_g) + H(g,\pi_g)\bigg). \tag{2.278}$$

[51] Moreover there is a global G redundancy in (2.274), (2.275) under $g \to Vg$ with V constant. This results in the equivalence $U \sim V^{-1}UV$ hence the space of all inequivalent holonomies U is isomorphic to the space of conjugacy class elements λ.

The quantum mechanics on the group corresponds to a classical plain old phase space integral, where we've localized on static configurations:

$$\int dg\, d\pi_g \exp\left(- \int_\partial H(g, \pi_g) \right). \tag{2.279}$$

Indeed, the classical solutions of the coadjoint orbit field theory are static group elements $g(\phi)$. How this limit on the path integral precisely works is explained nicely in [44]. Consider now very explicitly the classical limit of the vacuum character. We can decompose the vacuum character as:

$$\chi_0(S\cdot\beta) = \sum_\mu {S_0}^\mu \chi_\mu(\beta), \quad \chi_\mu(\beta) = \text{Tr}_\mu\, e^{-\beta H}, \quad H\,|\mu, n\rangle = k(\mathcal{C}(\mu)+n)\,|R, n\rangle\,. \tag{2.280}$$

For the latter characters and quantization one should imagine time to flow along the ϕ direction, as it does in the associated BF configuration. In the classical limit, one now takes $k \to \infty$. This effectively scales out all descendants. Furthermore, the sum gets dominated by light representations such that $k\,\mathcal{C}(\mu)$ is finite. In this limit, we have [74, 88]:

$$\lim_{k\to\infty} {S_0}^\mu = \dim\mu, \qquad \lim_{k\to\infty} {S_\lambda}^\mu = \chi_\mu(\lambda). \tag{2.281}$$

Here it is understood that the integrable reps are scaled in an appropriate manner. Therefore, the classical limit of the vacuum character is:

$$\lim_{k\to\infty} \chi_0(S\cdot\beta) = \sum_\mu \dim\mu^2\, e^{-\beta\mathcal{C}(\mu)}. \tag{2.282}$$

The classical limit of a nontrivial character is:

$$\lim_{k\to\infty} \chi_\lambda(S\cdot\beta) = \sum_\mu \dim\mu\, \chi_\mu(\lambda)\, e^{-\beta\mathcal{C}(\mu)}. \tag{2.283}$$

We can understand these results alternatively as the dimensional reduction of the Wilson line in the Chern-Simons torus, along the circle which the Wilson line winds around as in (2.276). More in particular, the dimensional reduction describes the classical solution space of that Chern-Simons setup consisting of static configuration. At any time slice this look precisely like the BF configuration (2.15). The integral over the moduli space of classical solutions boils down to a particle on a group path integral. We can understand this directly from the coadjoint orbit path integral (2.277). We localize on static configurations by taking k to infinity because the circumference of the time direction in the thermal path integral is secretly $2\pi k^{-1}$. This can be understood because in this context $k = 1/\hbar$ [44].[52] This immediately takes the action (2.277) to (2.21). This is in more detail how we go from a phase space path integral (2.278) to a phase space integral (2.279).

[52]This circumference should not be confused with the β in the above character formulas, which here is associated with the circumference in the ϕ direction.

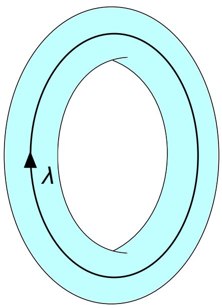
We note that we could in principle also implement the dimensional reduction of the coadjoint orbit action along the ϕ direction. The result is:

$$S[g,\lambda] \to \int_0^\beta dt\, \mathrm{Tr}(\lambda g^{-1}\partial_t g - \lambda^2). \tag{2.284}$$

The associated familiar path integral computes just a single character of the group. The term $\mathrm{Tr}\lambda^2$ fixes the Casimir as Hamiltonian in the finite character:

$$\chi_\lambda(e^{-\beta\mathcal{C}}) = \mathrm{Tr}_\lambda\, e^{-\beta\mathcal{C}} = \dim \lambda e^{-\beta\mathcal{C}(\lambda)}. \tag{2.285}$$

The perhaps more common characters $\chi_\lambda(\mu)$ have some generator of the algebra as "Hamiltonian". The exponentiation of these "Hamiltonians" gives generic conjugacy class elements μ. The difference with the twisted particle on a group path integral is which circumference of the torus is small. Conversely one of the directions is made long. If we choose the Cauchy slices orthogonal to the long direction, then descendants in the Hilbert space constructed on that Cauchy slice are projected out. Conversely, because the circumference of the Cauchy slice is tiny, there is a large mass gap to descendants which correspond to Fourier modes on the small boundary on the Cauchy slice. This is much like the reason why we don't see stringy excitations when the compact dimensions are small. This is part of the story of how 2d conformal field theory goes to classical representation theory [89, 82].
It is straightforward, and we feel good fun, to extend this story to include multiple punctures or Wilson lines. This requires a modification of (2.150) and related formulas. The key to this was explained in Witten's seminal work [109] and lays the ground work for more abstract definitions of 3d topological quantum field theories. Basically what we need is that in 3d Chern-Simons there is a complete set of states on any Cauchy slice shaped as a torus. The states $|\lambda\rangle$ are one to one with *integrable* representations. They are the primaries at a certain level k in the corresponding 2d conformal field theory. The number such states $|\lambda\rangle$ is strictly less [110] than the number of conjugacy class element states in the classical theory $k=\infty$ which corresponds to the Hilbert space of 2d BF on a circle. The state-operator correspondence (2.23) in BF theory which was instrumental in calculating (2.150) and others, has an analogue in 3d Chern-Simons [109]. Instead of a complete set of defects as punctures, we have a complete set of defects as Wilson lines. We end up with the following state-operator correspondence:

$|\lambda\rangle =$ 

. (2.286)

This gives us the machinery to cut and glue three manifolds in 3d Chern-Simons as in (2.150). In particular we can cut amplitudes on torus shaped surfaces, and insert complete sets of the above tori with Wilson lines. This results in a new three manifold, possibly disconnected, with two additional Wilson lines inserted, and with a sum over the identical labels of these Wilson lines. This is essentially the surgery Witten talks about in [109]. As the focus of this work is not on 3d Chern-Simons we will not give all the details, but using basically exactly the same steps used to obtain (2.153) one proves for example a variant of the Verlinde fusion rules [112]:

$$Z_{0,4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\sum_\mu \frac{{S_{\lambda_1}}^\mu {S_{\lambda_2}}^\mu {S_{\lambda_3}}^\mu {S_{\lambda_4}}^\mu}{{S_0}^\mu {S_0}^\mu} \tag{2.287}$$

The trick is to rewrite the amplitude on $S_2\times S_1$ with four parallel Wilson lines in terms of amplitudes on S_3 with six Wilson lines. The two new Wilson lines have label μ. One of them circles around the four initial Wilson lines, the other one is decoupled and produces just a factor ${S_0}^\mu$. This amplitude is easily calculated using the axioms of topological field theory. Summing over μ effectively changes the topology back to $S_2\times S_1$. This formula is only obviously the Verlinde formula when we notice that path integrals of 3d Chern-Simons on a closed surface with some punctures times a circle are phase space path integrals with vanishing Hamiltonian, so this just computes dimensions of Hilbert spaces of Chern-Simons on said closed surface with punctures. In this case the closed surface can be denoted as $\Sigma_{0,4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. The dimension of this Hilbert space is then the number of four point conformal blocks on the sphere, because we can imagine the duality between 3d Chern-Simons on some three manifold ending on $\Sigma_{0,4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ with four ribbons ending on the four punctures, and a particular 2d conformal field theory four point function, which is known to decompose indeed into conformal blocks. Each conformal block is like an "orthonormal wavefunction" so can be thought of as associated with some basis state. The number such states is then by definition the fusion coefficient and is furthermore then also clearly the answer returned by the Chern-Simons path integral on $\Sigma_{0,4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\times S_1$, because the blocks are orthonormal wavefunctions. This completes the proof [109]. Anyway, the point we are trying to make is that taking the classical limit, these 3d Chern-Simons topological calculations all reduce trivially to the bulk 2d BF topological calculations as discussed around (2.150).

Classical limit of vacuum Virasoro coadjoint orbit

These similarities between 3d Chern-Simons and 2d BF carry over to similarities between AdS_3 gravity and JT gravity. Let us not go into too much detail here, especially regarding the story including multiple punctures and higher genus Riemann surfaces which is more subtle. There are subtle aspects already in the classical case of JT gravity discussed in chapter 3, which do not get less subtle when "quantizing" to AdS_3 gravity. The details of how the story works for AdS_3 gravity are forthcoming [45], but will not be communicated in this work the purpose of which it to review, not to expand.
We can say a little bit about the simplest example though, JT gravity on a disk or

correspondingly AdS_3 gravity on a doughnut. It is well-known that the action of AdS_3 gravity can be written as two copies of a 3d $SL(2,\mathbb{R})$ Chern-Simons action. There are ambiguities in the exponentiation as for BF, but let's not go there in this work. In general we've learned that 3d $SL(2,\mathbb{R})$ Chern-Simons on a doughnut is dual to a vacuum $SL(2,\mathbb{R})$ coadjoint orbit model. There are no Wilson lines inserted unless we would want to compute a correlator in AdS_3 gravity. In total we'll have two such coadjoint orbits. If we feel like being naive about zero modes, or twists of the flat connections, we could interpret this as defining a full-fledged 2d $SL(2,\mathbb{R})$ Wess-Zumino-Witten model. If we parameterize the metric as $ds^2 = d\phi^2 + e^{-2\phi} d\gamma_- d\gamma_+$ this corresponds to the following action [113, 114, 115]:

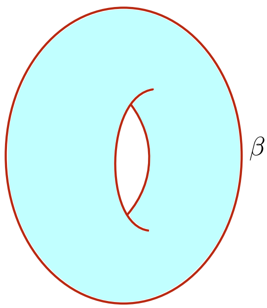
$$\int dz\, d\bar{z} \left(\partial\phi\bar{\partial}\phi + \bar{\partial}\gamma_L \partial\gamma_R e^{-2\phi} \right). \tag{2.288}$$

One obtains Liouville theory from $SL(2,\mathbb{R})$ Wess-Zumino-Witten by means of a Drinfeld-Sokolov Hamiltonian reduction [84, 116, 97, 117, 118]. This is just a fancy way to say that we are considering a Wess-Zumino-Witten model of a coset space. Liouville is essentially a collection of Virasoro coadjoint orbit models. Dealing with the zero modes properly for AdS_3 gravity, we don't find Liouville theory but rather two vacuum Virasoro coadjoint orbits as in [83]. The coset constraints are the immediate generalization of those for JT gravity (2.61):

$$\mathcal{J}^- = 1. \tag{2.289}$$

One can equivalently understand on the level of Hilbert space why we go from $SL(2,\mathbb{R})$ Verma modules to Virasoro Verma modules. For more on this see [1]. Denoting coset boundaries in red again, we could picture the amplitude in $SL(2,\mathbb{R})$ Chern-Simons as:

$Z(\beta) =$



$$= \chi_0(S\cdot\beta)$$

$$= \int_0^\infty ds\, \sinh 2\pi b s \sinh 2\pi b^{-1} s\, \chi_s(\beta). \tag{2.290}$$

The AdS_3 gravity "partition function" is then just the square of this answer [83]. Even a blind man could see that the above formula is structurally identical to the JT gravity answer (2.71). In particular we immediately recognize (2.71) in the above when we take b to zero. This deep structural similarity is trivially extended to include punctures of Wilson lines conform (2.89). One finds:

$$Z(\beta, b) = \chi_b(S\cdot\beta) = \int_0^\infty ds\, \cos\pi b s\, \chi_s(\beta). \tag{2.291}$$

Here we introduced the Virasoro S matrix elements ${S_b}^s = \cos \pi bs$. Anyway one sees that this story is the immediate generalization of the story which we've presented for JT gravity, or conversely the story in this chapter is nothing but the classical limit of the AdS_3 gravity story.

3 Baby universes and random matrices

In this chapter we present the solution of JT gravity on higher genus Riemann surfaces. The following is based largely on material scattered throughout three publications [2, 3, 4] by the author in collaboration with Thomas Mertens and Henri Verschelde. A more coherent story on JT gravity as a sum over topologies can be found in [9], on which we draw here. In fact, significant portions of this chapter are based on that paper by other authors. Working in a quickly evolving field it is impossible to see the works by the author and collaborators as logically independent of the other evolutions in the field. Therefore we are urged to summarize at times work by others such as [22, 64, 61, 9].

3.1 Introduction

In the introduction of chapter 2 we emphasized the importance of specifying the contour in the path integral over metrics in JT gravity. There are two aspects of that story which we would like to deal with in this chapter.

Baby universes

The first aspect is the effect of allowing topology changing Euclidean amplitudes in the theory. Let us explain the conceptual issue in a bit more detail. Remember the action of JT gravity:

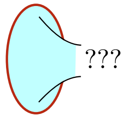
$$S[g,\Phi] = -S_0\chi - \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\Phi(R+2) - \int_{\partial\mathcal{M}} dt\sqrt{h}\Phi(K-1). \tag{3.1}$$

The Euler character χ comes from the Einstein-Hilbert term in 2d.[1] As we learned in chapter 2, integrating out Φ localizes the metrics g on hyperbolic Riemann surfaces with wiggly boundaries. There are patches of the Poincaré disk, with asymptotically AdS_2 boundary conditions. The latter boil down to fixing the total length of the asymptotic boundary to β/ϵ, and the boundary value of the dilaton to $1/2\epsilon$ as in (2.33). Due to the localization we are essentially just counting hyperbolic Riemann surfaces. Specifying the

[1] For a 2d manifold of genus g with b boundaries we have $\chi = 2 - 2g - b$.

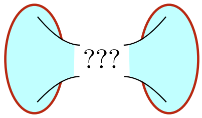
path integration contour of g is thus equivalent to specifying which surfaces to count. For example for the JT gravity partition function, which corresponds to the insertion of a single holographic boundary in the path integral, we have schematically:

$$Z(\beta) = \int dE\, e^{-\beta E}\, \rho(E) = \quad \beta \tag{3.2}$$



We could also consider configurations with multiple wiggly boundaries. For example we could consider a correlation function $Z(\beta_1, \beta_2)$ with two asymptotic boundaries of respective lengths β_1/ϵ and β_2/ϵ:

$$Z(\beta_1, \beta_2) = \int dE_1\, e^{-\beta_1 E_1} \int dE_2\, e^{-\beta_2 E_2}\, \rho(E_1, E_2) = \beta_1 \quad \beta_2 \tag{3.3}$$



Different definitions of ??? can lead to structurally very different theories. In chapter 2 we restricted ??? to disk topologies:

$$\int_{\text{disks}} [\mathcal{D}g]\, (\ldots). \tag{3.4}$$

In this chapter we would like to consider the effect of summing over all hyperbolic Riemann surfaces that end on the specified boundaries following [9]:

$$\int_{\text{all } \chi} [\mathcal{D}g]\, (\ldots). \tag{3.5}$$

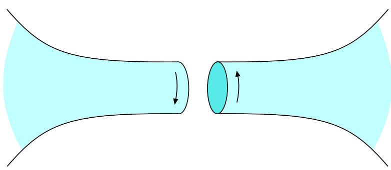
Such amplitudes include the possibility of topology changing dynamics in quantum gravity. Baby universes can now be emitted from and reabsorbed into the parent universe. The question arises whether this is in some sense an improvement or whether we should be avoiding such contributions in quantum gravity in general. We will argue that the former is the case. Including the sum over topologies turns out to reveal a fundamental aspect of JT quantum gravity, namely its quantum chaotic nature [7]. Quantum black holes are generically expected to be quantum chaotic systems. This means the level spacing statistics of nearby discrete energy levels in the spectrum of black hole microstates is not dictated by the Poisson distribution, but rather by a more correct variant of the Wigner surmise [119]. When considering suitable averaged quantities in such a system, such as an operator expectation value averaged over some energy bin, late time averages, or even ensemble averages over different such quantum chaotic system, one in general expects a description of the associated physics as random matrix theory [17]. Now, as it turns out, when we include a sum over topology changing amplitudes in JT gravity observables, then JT gravity becomes exactly dual to a random matrix theory [9]. The resulting physics is as expected on general grounds of ensemble averages of single realizations of quantum gravity. This is a vast improvement on the situation encountered in chapter 2. Schwarzian quantum mechanics does not know about the Wigner surmise and the associated eigenvalue repulsion. It is too simple a description to capture traces of the fundamental discreteness of JT quantum gravity. In this chapter we will not focus too much on the physical implications of summing over topologies though, as much of that is the topic of chapters 4 and 5.

Modding out by the mapping class group

The second aspect of the path integration contour is associated with the relation between the first order and the second order formulation of gravity. It is clear that we have much better analytic control of what we are doing when everything is phrased in a first order formulation. In the case of JT gravity we are just calculating SL$(2,\mathbb{R})$ BF path integrals with suitable boundary conditions. Indeed. Consider for example a closed genus g two manifold Σ. The BF path integral reduces to a path integral over the moduli space of flat SL$(2,\mathbb{R})$ connections. As explained in the previous chapter, we will want to limit ourselves to hyperbolic monodromies b if we want to avoid singular metrics. One point in the resulting moduli space is then specified by choosing a complete set of $3g-3$ independent closed geodesics on Σ and specifying the monodromy b and a so called twist τ on each of these geodesics. The naive answer for the BF path integral is then just the volume of the moduli space of flat hyperbolic SL$(2,\mathbb{R})$ connections:

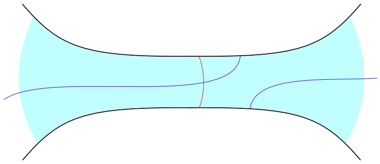
$$Z_{g,0} \overset{?}{=} \text{Vol}(\mathcal{T}_{g,0}) = \prod_{i=1}^{3g-3} \left(\int_0^\infty db_i \int_{-\infty}^{+\infty} d\tau_i \right) = \prod_{i=1}^{3g-3} \left(\infty \int_0^\infty db_i \right). \tag{3.6}$$

We will explain in detail why this is the case in the remainder of this chapter. Good references in this regard are [74, 120, 65, 9, 62, 18]. Anyway, the above result is manifestly infinite. Therefore this naive path integral contour for SL$(2,\mathbb{R})$ BF does not result in a sensible theory of quantum gravity when summing over topologies. This contour is implementing precisely classical Teichmuller theory. The relevant volumes are the volumes of Teichmuller space which are indeed well known to be infinite for higher genus Riemann surfaces. We would like a more sensible contour which results in finite amplitudes. The resolution in this case of two dimensional geometry is well known. We are to mod out the so called mapping class group [120]. This takes us from the moduli space of flat SL$(2,\mathbb{R})$ connections to the moduli space of Riemann surfaces. In string theory for example this corresponds to integrating the moduli over one fundamental domain $\mathcal{F}$ rather than over the entire range specified in (3.6). Locally around each closed geodesic γ_i this means that the range of the twists is constrained to run from 0 to b. Indeed. In a gravitational context the twists can be thought of literally as twisting two Riemann surfaces relative to each other before attaching them together on some geodesic:

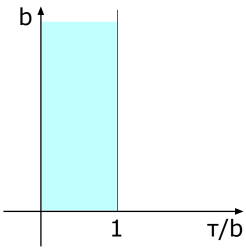


. (3.7)

The twist then naturally lives on $U(1)_b$. The question in 2d BF theory is whether or not we consider flat $\mathrm{SL}(2,\mathbb{R})$ connections associated with a full twist as identical or not. Teichmuller theory corresponds to considering the twists to live in the universal cover of $U(1)_b$. A neat way to imagine Teichmuller configurations is as dressed Riemann surfaces, with a so called Moore Seiberg graph. Roughly speaking locally around each closed geodesic the Riemann surface is an annulus. The Moore Seiberg graph is just a line running along each annulus, which can twist around it. If we picture the geodesic γ by a red line and the Moore Seiberg graph by a blue line we have for example:

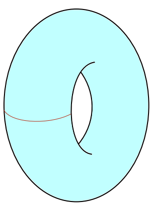
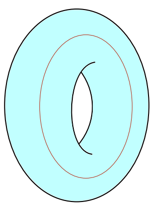

. (3.8)

Large twists wind the line around the annulus. The above picture is an example of precisely that phenomenon. Here the twist τ is slightly larger than the length b of γ. Clearly the additional dressing with these lines breaks the invariance of Riemann surfaces under the mapping class group. The resulting moduli space of dressed Riemann surfaces is called Teichmuller space. Let us for future convenience and following the literature denote the moduli space of Riemann surfaces of genus g with n boundaries as $\mathcal{M}_{g,n}$. Constraining the twists to run from 0 to b boils down to modding out the generalization of modular T transforms. To divide by the full mapping class group though we have to account for both the T *and* S transforms. To understand how we might proceed, consider the example of the torus. Based on the gravitational discussion thus far, we would specify inequivalent tori by coordinates (b,τ) which take values in a strip $\mathcal{S}$:

$\mathcal{S} =$


. (3.9)

But the strip is still vastly overcounting. We would like to mod by the whole mapping class group at once. Let us explain how to achieve that. An important point is that we could a priori choose either one out of an infinite number of closed geodesics γ on the torus to specify our coordinates. For example we could pick either one of the usual A or B cycle:

γ_A

, γ_B

. (3.10)

Other geodesics have more involved winding, and are equally sensible candidates. A technical trick is that we can write the relevant Teichmuller space as follows:

$$\mathcal{T}_{1,0} = \sum_{\gamma} \mathcal{F}_{\gamma}. \tag{3.11}$$

Here γ is a short way to write all the modular transformations, obtained by acting with combinations of the familiar T and S modular transforms on the moduli (b, τ).[2] Indeed. The modular transformations map on a generic Riemann surface a complete set of non intersecting closed geodesics γ_i to a different set such non intersecting closed geodesics. Therefore summing over the modular transformations is equivalent to summing over all possible sets of closed geodesics.[3] It is well known how the modular transforms act on the coordinates (b, τ). See for example [111]. Say we are given two coordinates (b_1, τ_1) and (b_2, τ_2) which are mapped into each other under some γ_{12}. We can choose to think about the coordinates (b_1, τ_1) as associated with a torus where we specified the length and twist around the geodesic γ_1. Similarly we can choose to think about the coordinates (b_2, τ_2) as associated with a torus where we specified the length and twist around the geodesic γ_2. By definition there is now a modular transformation labelled as γ_{12} which maps these tori onto each other, so that they are identical as Riemann surfaces. Clearly we would hence be overcounting by integrating over all (b, τ) in $\mathcal{T}_{1,0}$ which in particular would include (b_1, τ_1) and (b_2, τ_2). Following the same argument one establishes that really we should only be integrating over either one of the fundamental domains F_γ. All tori in for example $\mathcal{F}_1$ correspond to Riemann surfaces with an equivalent representative in $\mathcal{F}_2$. On a more technical level we could arrive at the same result without necessarily thinking about equivalent gravitational configurations. We are tasked to mod out by the mapping class in (3.6). This is immediately achieved using (3.11):

$$\mathrm{Vol}(\mathcal{M}_{1,0}) = \frac{\mathrm{Vol}(\mathcal{T}_{1,0})}{\mathrm{M}(\gamma)} = \frac{\sum_\gamma \mathrm{Vol}(\mathcal{F}_\gamma)}{\mathrm{M}(\gamma)} = \mathrm{Vol}(\mathcal{F}). \tag{3.12}$$

Here we introduced the notation $\mathrm{M}(\gamma)$ as the sum over all mappings of the geodesic γ. In the final equality we used the fact that all the fundamental domains have the same volume. For generic Riemann surface it is difficult to determine a fundamental domain $\mathcal{F}$ explicitly, and therefore it is also difficult to calculate amplitudes such as:

$$Z_{g,0} = \mathrm{Vol}(\mathcal{M}_{g,0}). \tag{3.13}$$

This requires a precise specification of the integration range $\mathcal{F}$. This was de facto achieved by Mirzakhani [122, 123] who spells out recursive formulas for these volumes of the moduli space of Riemann surfaces. The trick (3.12) looks rather trivial. One point worth making though is that Mirzakhani's recursion relation can be thought of as just a slightly more complicated version of (3.12). This is explained beautifully in an appendix of [65]. We will present an intuitive version of their argument further on in section 3.3.

[2]On the torus it is more common to use (τ_1, τ_2) as labels for these coordinates.

[3]One imagines fixing some basic set of geodesics. Then any other set γ_i specifies a unique modular transform.

In the following sections we will explain all these comments about Teichmuller space and the space of Riemann surfaces in more detail. The remainder of this chapter is structured as follows.
In **section 3.2** we discuss the JT gravity path integral on the annulus. We start by giving a detailed account of the annulus path integral for 2d BF theories which has very similar properties. We then present the JT gravity story with focus on the integration space and the intricacies of dealing with the mapping class group. Finally we comment on the factorization debate.
In **section 3.3** we solve JT gravity for metrics of an arbitrary but fixed topology and eventually sum over all such topologies. This corresponds to allowing the spontaneous emission and reabsorption of baby universes from our parent universe in quantum gravity. A significant amount of focus is on the difficulties presented by the mapping class group modding. We note that in line with the rest of this work we will often sacrifice mathematical rigor or even physical rigor for intuitive arguments, especially when it comes to modding out the mapping class group.
In **section 3.4** we conclude with an apologia for summing over baby universes in any model of quantum gravity and comment on the Hilbert space of JT gravity.

3.2 Annulus amplitude

The annulus amplitude it the simplest example of the exchange of a baby universe between two asymptotic boundaries. Alternatively one might say that is is the simplest example of a Euclidean wormhole connecting two otherwise independent observers.

3.2.1 Cutting and gluing in BF theory

It will turn out advantageous to keep working in the BF formulation of JT gravity. In the spirit of one problem at a time let us start with investigating the annulus path integral in ordinary 2d BF theory for some compact Lie group such as $SU(2)$.

Path integral of BF theory on an annulus

Consider BF theory on an annulus with boundaries ∂_1 and ∂_2:

$$S[\chi, A] = \int_{\mathcal{A}} \mathrm{Tr}(\chi F) - \frac{1}{2}\int_{\partial_1} \mathrm{Tr}(\chi A) - \frac{1}{2}\int_{\partial_2} \mathrm{Tr}(\chi A). \tag{3.14}$$

The boundary conditions are as in (2.4):

$$A_\tau|_{\partial 1} = \chi|_{\partial 1}, \qquad A_\tau|_{\partial 2} = \chi|_{\partial 2}. \tag{3.15}$$

We would like to do the associated path integral and recognize a prescription for gluing. Doing the path integral over χ localizes on flat connections:

$$A = g^{-1}dg. \tag{3.16}$$

Let us think about the possible winding of g. Winding corresponds to a holonomy of A around the neck γ of the annulus:

$$\int_\gamma A = \lambda. \tag{3.17}$$

Such Wilson lines are certainly physical data not to be modded out. Rather they are part of the physical phase space, there is hence an integral over such monodromies. These correspond to a winding condition on g in the sense that:

$$g(\tau+\beta) = U_\lambda \cdot g(\tau). \tag{3.18}$$

We can absorb this winding condition in a linear field $\Lambda(\tau)$ by replacing $g(\tau)$ with $\Lambda(\tau)\cdot g(\tau)$ One now has:

$$A = g^{-1}dg + g^{-1}\lambda g\, d\tau, \qquad g(\tau+\beta) = g(\tau). \tag{3.19}$$

Left multiplication by a constant does not change the connection, if and only if that constant commutes with λ. Therefore such constant shifts are redundant:

$$g \sim \tau_+ \cdot g, \qquad \lambda\cdot\tau_+ = \tau_+\cdot\lambda. \tag{3.20}$$

In writing this we have already imagined that we are integrating over just a single λ for each conjugacy class. If this was not the case then τ_+ would be group valued. Indeed $h^{-1}\cdot\lambda\cdot h$ is in the same conjugacy class as λ. We know that only the conjugacy class of the monodromy is physical information, not the monodromy itself. This is because the monodromy is changed by a conjugation by choosing a different starting point on γ. This translates into the fact that characters $\chi_R(\lambda)$ are conjugacy class functions. Anyway. The elements τ_+ that commute with a single λ span the so called centralizer of λ. For hyperbolic monodromies b of $\mathrm{SL}(2,\mathbb{R})$ this centralizer is a circle $U(1)$. For the vacuum class the centralizer is the full group $\mathrm{SL}(2,\mathbb{R})$. Anyway, the action of the theory thus reduces to a twisted quantum mechanics on the group on either boundary:

$$S[g,\lambda] = -\frac{1}{2}\int_{\partial_1} d\tau\, \mathrm{Tr}\big(g^{-1}\partial_\tau g + g^{-1}\lambda g\big)^2 - \frac{1}{2}\int_{\partial_2} d\tau\, \mathrm{Tr}\big(g^{-1}\partial_\tau g + g^{-1}\lambda g\big)^2. \tag{3.21}$$

Small values of g are redundant, and so are the above mentioned constant shifts. The physical variables are the boundary values of g and the monodromies around closed geodesics. Let us denote:

$$g_1 = g|_{\partial_1}, \qquad g_2 = g|_{\partial_2}. \tag{3.22}$$

The constant redundant shifts now map to diagonal shifts:

$$(g_1, g_2) \sim (h_+\cdot g_1, h_+\cdot g_2). \tag{3.23}$$

Notice that for each λ in principle one has to make a case by case study of which τ_+ feature here. The path integrals of twisted quantum mechanics on a group such as those discussed in chapter 2 are usually over group elements valued on the circle modulo constants in the centralizer. We can denote this as:

$$\frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)}. \tag{3.24}$$

The loop group of untwisted fields in fact does not depend on λ. However, the corresponding action does depend on it, making this notation useful. The current state of affairs is that the integration space for our annulus path integral is:

$$\int_{\mathrm{C}(G)} d\lambda \, \frac{\mathrm{LG}(\lambda) \times \mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)}. \tag{3.25}$$

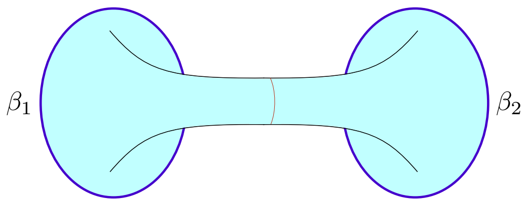
The integral is over conjugacy class elements. So at this point we couldn't immediately do the path integrals as we did in chapter 2. We can deal with this as follows. Let us imagine parameterizing the boundary group elements as $\tau_1 \cdot g_1$ and $\tau_2 \cdot g_2$ with τ_1 and τ_2 both constants in the centralizer and g_1 and g_2 fields with no constant component in the centralizer. In other words we imagine the redundancy (3.23) to make identifications in the coordinates τ_1 and τ_2 but not in g_1 and g_2. This means the path integration space for g_1 at fixed λ is precisely (3.24). Given τ_1 and τ_2 we can uniquely define two constant fields in the centralize τ_+ and τ_- such that $\tau_+ \cdot \tau_- = \tau_1$ and $\tau_+ \cdot \tau_-^{-1} = \tau_2$. Via (3.23) we learn that the zero mode τ_+ is redundant, but the zero mode τ_- is physical. We end up with the following integration space:

$$\int_{\mathrm{C}(G)} d\lambda \int_{\mathrm{C}(\lambda)} d\tau \, \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)} \times \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)} \tag{3.26}$$

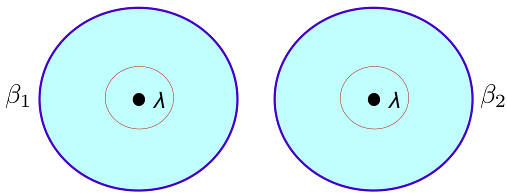
Now the path integrals over g_1 and g_1 *can* be performed independently. They are by definition identical to the calculations of the punctured BF disks with boundary conditions (2.4) discussed in chapter 2. We find the annulus amplitude in BF theory:

$$Z_{0,2}(\beta_1, \beta_2) = \int_{\mathrm{C}(G)} d\lambda \int_{\mathrm{C}(\lambda)} d\tau \, Z_{0,1}(\beta_1, \lambda) \, Z_{0,1}(\beta_2, \lambda). \tag{3.27}$$

Graphically if we remember the pictures from chapter 2 this becomes:



$$= \int_{\mathrm{C}(G)} d\lambda \int_{\mathrm{C}(\lambda)} d\tau \quad \tag{3.28}$$



.

This is the simplest example of cutting a BF amplitude into simpler amplitudes by introducing a complete set of punctures $|\lambda\rangle$. For usual Lie groups such as $SU(2)$ and $SU(3)$ the integral over the centralizer gives just a constant volume factor independent of λ. Let us denote this as $\mathrm{Vol}(T)$. Using the explicit formulas for $Z(\beta, \lambda)$ and character orthonormality one then finds:

$$Z_{0,2}(\beta_1, \beta_2) = \mathrm{Vol}(T) \sum_R \dim R^2 \, e^{-(\beta_1+\beta_2)\mathcal{C}(R)}. \tag{3.29}$$

We have implicitly absorbed any potential other normalization constants associated with the measure on the space of conjugacy class elements in what we mean by $\mathrm{Vol}(T)$.

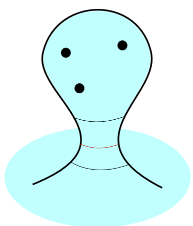
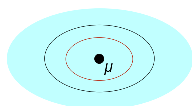
This prefactor may seem innocent. Actually though it is quite puzzling when we think back to the discussion around (2.150). The Hilbert space type computations would seem to give a very precise answer for the path integrals of BF theory which do not have this prefactor. This type of effect is well known and discussed extensively in [74]. As it turns out there are certain normalization ambiguities in the Hilbert space type calculations which cannot be resolved internally. The take away is that in order to get a precise answer for the BF path integral you should actually do the BF path integral. In this sense the current calculation is much more rigorous than the one presented around (2.150). It is not yet entirely rigorous though in that we have not derived the precise integration measure on the conjugacy class elements λ and the twists τ. The precise measure on the moduli space of flat connections in BF theories is well known though, see for example [9, 65]. We note that the most rigorous way to go about BF path integrals is via a combinatoric method based on so called torsions. Those calculations are too mathematical for what this work is concerned though, we are trying to keep it light. One should just keep in mind that everything we are saying is based on rigorous derivations which are explained in much more detail in several papers by Witten [74, 65].

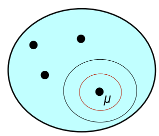
Local derivation of gluing formulas

Formula (3.27) suggests the more precise statement replacing (2.150) is the following:

$$Z(\ldots \lambda_1, \lambda_2, \lambda_3) = \int_{\mathrm{C}(G)} d\mu \int_{\mathrm{C}(\mu)} d\tau \, Z(\ldots \mu) \, Z(\mu, \lambda_1, \lambda_2, \lambda_3). \tag{3.30}$$

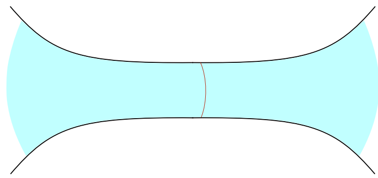
With some goodwill this could still be interpreted as introducing a complete set of punctures but with a nontrivial measure:

$\ldots$  $= \int_{\mathrm{C}(G)} d\mu \int_{\mathrm{C}(\mu)} d\tau \quad \ldots$ 

$\times$ 

$.$ (3.31)

With the purpose in mind of understanding JT gravity on higher genus Riemann surfaces we would like to prove this gluing formula from a local manipulation of the BF path integral. To obtain such a formula we can follow the logic of the supplementary chapter B. The idea is, just as in that chapter, to simply insert a functional delta constraint on the cutting surface. This formulation will also be very useful for our comments on factorization further on where we'll want to consider a path integral version of the trace over the extended Hilbert space associated with one side of the cutting surface. Anyway. Imagine we are supposed to calculate the BF path integral on some genus g Riemann surface with n boundaries. Choose a complete set of $3g-3+n$ non-intersecting geodesics γ_i on the surface. Locally around each of the geodesics γ we have an annulus shaped region:

$\dots$  $\dots$. (3.32)

The geodesic γ on which we cut is represented by a thin red line as before.[4] We can start decomposing the BF path integral by inserting a functional delta on the geodesic:

$$\int \frac{[\mathcal{D}A]}{\text{Vol}(G)}[\mathcal{D}\chi]\, e^{-S[A,\chi]}\dots = \frac{1}{\text{Vol}(G_\partial)}\int \frac{[\mathcal{D}A_1]}{\text{Vol}(G_1)}[\mathcal{D}\chi_1]e^{-S[A_1,\chi_1]}\dots \int \frac{[\mathcal{D}A_2]}{\text{Vol}(G_2)}[\mathcal{D}\chi_2]e^{-S[A_2,\chi_2]}\dots\delta(A_1|_\partial - A_2|_\partial). \quad (3.33)$$

The dots represent whatever is going on away from the cutting surface and do not really play a role here. One might be tempted to introduce furthermore a functional delta on the values of the field χ on the cutting surface. That would be overkill as the field χ should not be considered fundamental. It is just a dumb Lagrange multiplier forcing us on flat connections. This can be appreciated from the fact that 2d BF theory is just the weak coupling limit of 2d Yang-Mills [124]. Schematically:

$$\int [\mathcal{D}A][\mathcal{D}\chi]\exp\left(-\int_\mathcal{M}\text{Tr}(\chi F) + e^2\,\text{Tr}\big(\chi^2\big)dx\right) = \int [\mathcal{D}A]\exp\left(-\frac{1}{e^2}\int_\mathcal{M}\text{tr}(F\wedge F)\right).$$

We learned from [6] or chapter B how to properly cut and glue in 2d Yang-Mills. The current discussion is but an application of that story, so clearly there should be no additional delta on the field χ in (3.33). One now proceeds by introducing a Lagrange multiplier charge field $\mathcal{Q}$ on the cutting surface:

$$\delta(A_1|_\partial - A_2|_\partial) = \int [\mathcal{D}\mathcal{Q}]\exp\left(-\int_\partial \text{Tr}(\mathcal{Q}A_1 - \mathcal{Q}A_2)\right). \quad (3.34)$$

Doing the path integral over χ_1 and χ_2 localizes on flat connections in each subregion. Each boundary component of the subregion is weighed with a certain action. For the external boundary regions this is an action of the type:

$$S[g] = -\frac{1}{2}\int_\partial d\tau\,\text{Tr}\big(g^{-1}\partial_\tau g\big)^2 \quad (3.35)$$

[4]This convention will be followed throughout this chapter.

This may in general by supplemented with for example coset constraints. On the cutting boundary clearly we can not be imposing coset constraints. Anyway. Evaluating the associated action on flat connections we find:

$$S[\mathcal{Q}, g] = \int_\partial \mathrm{Tr}\big(\mathcal{Q} g^{-1} dg\big). \tag{3.36}$$

We can understand this as a limit of the phase space path integral for the particle on a group when we take the Hamiltonian to zero:

$$\int_\partial \mathrm{Tr}\big(\mathcal{Q} g^{-1} dg\big) + e^2\, \mathrm{Tr}\big(\mathcal{Q}^2\big) d\tau = \frac{1}{e^2} \int_\partial d\tau\, \mathrm{Tr}\big(g^{-1}\partial_\tau g\big)^2 \tag{3.37}$$

We can absorb the prefactor e^{-2} into the inverse "temperature" of the boundary. Say initially the τ coordinate runs from 0 to β. We can transform to a new time coordinate that runs from 0 to $e^2\beta$ which effectively removes the prefactor in the above action. The result is the partition function of quantum mechanics on a group but where the particle propagates over a very short thermal "time". Let us split up the gauge connections on either side of the boundary as:

$$A_1 = g_1^{-1} dg_2 + g_1^{-1}\lambda_1 g_1\, d\tau, \qquad A_2 = g_2^{-1} dg_2 + g_2^{-1}\lambda_2 g_2\, d\tau. \tag{3.38}$$

The fields g_1 and g_2 are here understood to be periodic around the geodesic γ. In general on the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ we would have:

$$A_1 = g_1^{-1} dg_1, \qquad A_1 = g_2^{-1} dg_2. \tag{3.39}$$

Locally close to every geodesic say on $\mathcal{M}_1$ we could transform to variables where g_1 is untwisted. Regardless of that comment we are left locally with the path integral:

$$\frac{1}{\mathrm{Vol}(G_\partial)} \int [\mathcal{D}g_1] \int_{\mathrm{C}(G)} d\lambda_1 \dots \int [\mathcal{D}g_2] \int_{\mathrm{C}(G)} d\lambda_2 \dots \tag{3.40}$$
$$\int [\mathcal{D}\mathcal{Q}] \exp\Big(- \int_\partial \mathrm{Tr}\big(\mathcal{Q} g_1^{-1} dg_1 - \mathcal{Q} g_2^{-1} dg_2\big) + \mathrm{Tr}\big(\mathcal{Q} g_1^{-1}\lambda_1 g_1 - \mathcal{Q} g_2^{-1}\lambda_2 g_2\big) d\tau \Big).$$

All dependence on bulk values of g_1 and g_2 is redundant and cancelled by the factors $\mathrm{Vol}(G_1)$ and $\mathrm{Vol}(G_2)$ in (3.33). We can deal with this path integral locally as follows. Let us imagine for each fixed g_1 and g_2 replacing $g_1^{-1}\lambda_1 g_1$ by $g_2^{-1}\lambda_1 g_2$. We can do this because we can choose essentially any set of conjugacy class elements to represent the holonomies when we are integrating over $\mathrm{C}(G)$. We just need to stick to a particular choice. Let us furthermore parameterize g_2 as $g_2 = g_- g_1$. Finally let us replace $\mathcal{Q}$ by $g_1 \mathcal{Q} g_1^{-1}$. This greatly simplifies the above path integral. In fact all dependence on g_1 has dropped out. We can use the definition:

$$\int \frac{[\mathcal{D}g_1]}{\mathrm{Vol}(G_\partial)} = 1. \tag{3.41}$$

We are then left with:

$$\int_{\mathrm{C}(G)} d\lambda_1 \ldots \int_{\mathrm{C}(G)} d\lambda_2 \ldots \qquad (3.42)$$
$$\int [\mathcal{D}g_-][\mathcal{D}\mathcal{Q}] \exp\Big(- \int_\partial \mathrm{Tr}\big(\mathcal{Q} g_-^{-1} dg_-\big) + \mathrm{Tr}\big(\mathcal{Q} g_-^{-1}(\lambda_1 - \lambda_2) g_-\big) d\tau \Big).$$

The path integral on the second line is of the type discussed by Alekseev and Shatashvilli in [75, 76]. It is essentially the path integral of quantum mechanics on the group but with the Cartan element associated with the holonomy $\lambda_1 - \lambda_2$ playing the role of a "Hamiltonian". The result of Alekseev and Shatashvilli is:

$$\int [\mathcal{D}g][\mathcal{D}\mathcal{Q}] \exp\Big(- \int_\partial \mathrm{Tr}\big(\mathcal{Q} g^{-1} dg\big) + \mathrm{Tr}\big(\mathcal{Q} g^{-1}\lambda g\big) d\tau \Big) = \sum_R \dim R \chi_R(\lambda) = \delta(\lambda).$$

The first equality is a brute force calculation. The second equality is the definition of the delta on a group. Inserting this into (3.42) and remembering from the discussion below (3.25) that there is an additional integral over twists that emerges from the original integration over g_2 which has been left implicit in (3.42) we find:[5]

$$\int_{\mathrm{C}(G)} d\lambda_1 \ldots \int_{\mathrm{C}(G)} d\lambda_2 \ldots \delta(\lambda_1 - \lambda_2) \int_{\mathrm{C}(\lambda_2)} d\tau. \qquad (3.43)$$

This was to prove (3.30). We can apply this procedure locally around every closed geodesic γ_i. This decomposes the amplitude into simpler and simpler pieces. The ... end up meaning either one of two possibilities. One type of amplitude that can remain is that of a three holed sphere in BF theory with three punctures λ_i, λ_j and λ_k. This is the number of inequivalent flat G connections on a three holed sphere with holonomy fixed around three points. This number is denoted as $N_{\lambda_i \lambda_j \lambda_k}$. For hyperbolic $\mathrm{SL}(2,\mathbb{R})$ conjugacy class elements it is always one. The second possibility is that the ... represent the path integral of BF on a punctured disk with puncture λ_i and an external boundary with some boundary Hamiltonian and boundary conditions a la (2.4). Let us take their lengths or couplings to be β_i.[6] The ... then evaluate to $Z(\beta_i, \lambda_i)$. We can now calculate the most generic path integral in BF theory. For example for the three holed sphere amplitude we find:

$$\begin{aligned} Z_{0,3}(\beta_1, \beta_2, \beta_3) &= \prod_{i=1}^3 \int_{\mathrm{C}(G)} d\lambda_i \int_{\mathrm{C}(\lambda_i)} d\tau_i \, Z(\beta_i, \lambda_i) \, N_{\lambda_1 \lambda_2 \lambda_3} \\ &= \mathrm{Vol}(T)^3 \prod_{i=1}^3 \int_{\mathrm{C}(G)} d\lambda_i \, Z(\beta_i, \lambda_i) \, N_{\lambda_1 \lambda_2 \lambda_3}. \end{aligned} \qquad (3.44)$$

The appearance of the factor $\mathrm{Vol}(T)^3$ is essentially the only difference with the discussion of around (2.153). On a general genus g surface with n boundaries its power would be

[5] The other twist integral was cancelled in the equality (3.41). The twist factor emerges because the path integrals a la Alekseev and Shatashvilli are over g_1 and g_2 in (3.24) implicitly.

[6] The choice of coupling of the boundary Hamiltonian can be absorbed into this.

$3g - 3 + n$. This conclusion is certainly not new. This formula was basically written down already in [74]. We hope though that this detailed derivation will make the step to JT gravity more easily digestible.

3.2.2 The annulus

Let us now repeat this analysis for JT gravity, in particular the annulus computation. We will be able to do this in a swift manner as this is essentially an application of the BF computation with coset boundary constraint. We can then focus most of our attention on local gluing formulas and the role of the mapping class group.

Path integral of JT gravity on an annulus

As explained in section 2.1.2 JT gravity is essentially an $\mathrm{SL}(2,\mathbb{R})$ BF theory with coset boundary conditions which has been limited to hyperbolic conjugacy class elements only. The coset boundary conditions are (2.61). For the JT gravity path integral on a fixed annulus topology we are then led to a BF path integral with action:

$$S[\chi, A] = \int_{\mathcal{A}} \mathrm{Tr}(\chi F) - \frac{1}{2}\int_{\partial_1} \mathrm{Tr}(\chi A) - \frac{1}{2}\int_{\partial_2} \mathrm{Tr}(\chi A). \tag{3.45}$$

The boundary conditions are:

$$A_\tau|_{\partial_1} = \chi|_{\partial_1}, \quad A_\tau^-|_{\partial_1} = \mathcal{J}_1^- = 1, \quad A_\tau|_{\partial_2} = \chi|_{\partial_2}, \quad A_\tau^-|_{\partial_2} = \mathcal{J}_2^- = 1. \tag{3.46}$$

The path integral over χ localizes on flat connections of which only the boundary values $g|_{\partial_1} = g_1$ and $g|_{\partial_2} = g_2$ as well as the monodromy of A around the neck of the annulus γ are physical. The latter are constrained to hyperbolic conjugacy class elements. The coset constraints constrain the dynamics on each boundary to be an independent Schwarzian quantum mechanics with fields γ_{-1} and γ_{-2}. Let us furthermore for convenience fix the redundant fields γ_{+1} and γ_{+2} as (2.66). From (2.77) we remember that a hyperbolic monodromy b of the connection A implies the following type of "periodic" identifications for γ_{-1} and γ_{-2}:

$$\gamma_{-1}(\tau+\beta) = \frac{\gamma_{-1}(\tau) + \tanh \pi b}{1 - \gamma_{-1}(\tau)\tanh \pi b}, \qquad \gamma_{-2}(\tau+\beta) = \frac{\gamma_{-2}(\tau) + \tanh \pi b}{1 - \gamma_{-2}(\tau)\tanh \pi b}. \tag{3.47}$$

Forget for a moment about these identifications. Writing $A = g^{-1}dg$ introduces a redundancy in description associated with left multiplication of g by a constant h in $\mathrm{SL}(2,\mathbb{R})$. This translates on the boundaries to a diagonal redundancy in g_1 and g_2 much like we had in (3.23) for BF theory with no coset constraint. One deduces the corresponding diagonal redundancy in γ_{-1} and γ_{-2} in the same way as one deduces (2.75) from (2.74). Explicitly we have as in (3.23):

$$(g_1, g_2) \sim (h \cdot g_1, h \cdot g_2)). \tag{3.48}$$

This translates into:

$$(\gamma_{-1}, \gamma_{-2}) \sim (h \cdot \gamma_{-1}, h \cdot \gamma_{-2}) \qquad h \cdot \gamma_{-1} = \frac{a\gamma_{-1} + b}{c\gamma_{-1} + d}, \qquad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.49}$$

Within a class of connections A with fixed monodromy though, not all of these fields h are redundant. Say we write [64]:

$$U_b = e^{2\pi b P}, \qquad U_b = \begin{pmatrix} \cosh \pi b & \sinh \pi b \\ \sinh \pi b & \cosh \pi b \end{pmatrix}. \tag{3.50}$$

The constraint is then:

$$P \cdot h = h \cdot P. \tag{3.51}$$

This follows from the discussion above and below (3.20). Notice that to write this one has to first fix a specific monodromy matrix U_b in each orbit, as explained below (3.20). Rather than solving this equation we will look for a more straightforward manner to understand what is the precise redundancy. For this we transform to a different set of coordinates. Let us introduce conform (2.78) two field variables $b_1(\tau)$ and $b_2(\tau)$ as:

$$\gamma_{-1}(\tau) = \tanh \pi b_1(\tau), \qquad \gamma_{-2}(\tau) = \tanh \pi b_2(\tau). \tag{3.52}$$

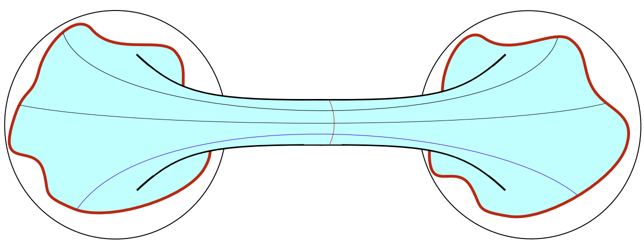
In terms of these new fields the monodromies become simple:

$$b_1(\tau + \beta) = b_1(\tau) + b, \qquad b_2(\tau + \beta) = b_2(\tau) + b. \tag{3.53}$$

Here we imagine two boundaries of identical lengths β. One immediately modifies the discussion to generic β_1 and β_2. For this one needs to rescale the boundary proper times as respectively:

$$\tau_1 = \frac{\beta_1}{\beta}\tau, \qquad \tau_2 = \frac{\beta_2}{\beta}\tau. \tag{3.54}$$

This results in a prefactor β/β_1 as de facto coupling for the action in (3.56) associated to ∂_1 and a coupling β/β_1 for the action in (3.56) associated to ∂_2. This does not make the path integrals more complicated. Anyway. These coordinates b_1 and b_2 are intuitively related to reparameterizations of proper lengths along the geodesic γ of length b. More in particular we could imagine a smooth frame periodic in b that interpolates between the proper length on the geodesic and the coordinates b_1 and b_2 on the asymptotic boundaries. Graphically:

$b_1(\tau)$  $b_2(\tau)$. (3.55)

We pictured three lines of fixed τ. One of these is pictured in blue and represents a choice of the line $\tau = 0$. Of course these fields $b_1(\tau)$ and $b_2(\tau)$ also take values on the circle of length b much like $\theta(\tau)$ in (2.78) takes values on the circle of length 2π. The constraint (3.53) is just the statement that the fields wind precisely once when we go around the circle. The corresponding action is immediately obtained from the discussion below (2.78):

$$S[b_1, b_2] = \int_{\partial_1} d\tau \left(-\frac{1}{4}\frac{b_1''^2}{b_1'^2} - \pi^2 b_1'^2 \right) + \int_{\partial_2} d\tau \left(-\frac{1}{4}\frac{b_2''^2}{b_2'^2} - \pi^2 b_2'^2 \right). \tag{3.56}$$

Here the integrals range from 0 to β. The symmetry of either action is $U(1)_b$ which shifts for example $b_1(\tau) \to b_1(\tau) + \tau_1$. Here τ_1 takes values on the circle $U(1)_b$ of length b due to the the fact that $b_1(\tau)$ lives on the same circle. For the usual Schwarzian action (2.63) corresponding to the vacuum puncture, the symmetry is the whole of $\mathrm{SL}(2,\mathbb{R})$ because the Schwarzian derivative is invariant under $\mathrm{SL}(2,\mathbb{R})$. This is not the case for the second term in (2.81) resulting in this smaller symmetry group. It should then be obvious that the precise form of the redundancies on the annulus combining (3.49) and (3.51) is:

$$(b_1, b_2) \sim (b_1 + \tau_+, b_2 + \tau_+), \qquad \tau_+ \in U(1)_b = S_1(b). \tag{3.57}$$

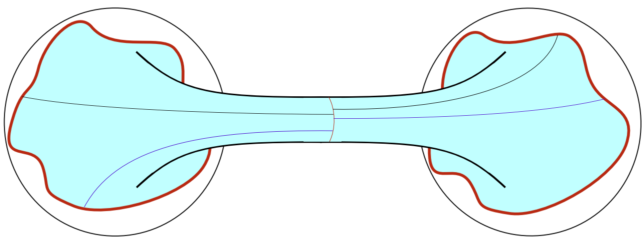
This is just saying that it is physically irrelevant where precisely on the annulus (3.55) we choose the line $\tau = 0$. In other words it is translating the rotation symmetry of the annulus. Much like below (3.24) the integration space for the path integral is thus at the moment:

$$\int_0^\infty db\, \frac{\mathrm{Diff}(b) \times \mathrm{Diff}(b)}{S_1(b)}. \tag{3.58}$$

Indeed the loop group in question becomes diffeomorphisms of the circle $S(b)$ of length b. Proceeding now as below (3.23) we can isolate the constant zero mode of $b_1(\tau)$ and $b_2(\tau)$. Let us call these τ_1 and τ_2. The new fields $b_1(\tau)$ and $b_2(\tau)$ are now each understood to live in the loop group modulo constant zero modes:

$$\frac{\mathrm{Diff}(b)}{S_1(b)} \tag{3.59}$$

Denote $\tau_1 = \tau_+ + \tau_1$ and $\tau_2 = \tau_+ - \tau_-$. We learn that the diagonal part τ_+ is redundant but the off diagonal zero mode τ_- is found to be physical. It corresponds to the possibility of choosing a different origin for the time coordinate τ on either part of the geodesic γ. Only the difference between these two origins τ_- is physical. In the same spirit as (3.55) we can picture what a configuration with nonzero τ_- looks like:

$b_1(\tau)$  $b_2(\tau)$. (3.60)

The offset between the lines of fixed τ_1 and the lines of fixed τ_2 at the geodesic γ represents τ_-. Indeed it is well known that we get a different Riemann surface by twisting two Riemann surfaces relative to each other on some closed geodesic. The question now is what range to associate to τ_-. Clearly when we are interested in counting Riemann surfaces we should take τ_1 to take values in $S(b)$. Indeed, we get the same configuration in (3.60) when one part of the annulus is rotated over a length b. A priori though one might be interested in allowing nontrivial winding in τ_-. As flat $\mathrm{SL}(2,\mathbb{R})$ connections, configurations with different winding are indeed not necessarily equivalent. They correspond to the same Riemann surface though. The difference is precisely the difference between Teichmuller space and the moduli space of Riemann surfaces for this configuration. In the latter case we identify configurations identical up to these so called Dehn twists. They correspond to modular T transformations, which is indeed the modular group of the annulus. We can summarize this discussion as follows. In Teichmuller theory one is led to the following integration space:

$$\mathcal{T}_{0,2}(\beta_1,\beta_2)=\int_0^\infty db\int_{-\infty}^{+\infty}d\tau\frac{\mathrm{Diff}(b)}{S_1(b)}\times\frac{\mathrm{Diff}(b)}{S_1(b)}. \tag{3.61}$$

In gravity on the other hand where we mod out by the modular group we are led to the integration space:

$$\mathcal{M}_{0,2}(\beta_1,\beta_2)=\int_0^\infty db\int_0^b d\tau\frac{\mathrm{Diff}(b)}{S_1(b)}\times\frac{\mathrm{Diff}(b)}{S_1(b)}. \tag{3.62}$$

Proceeding in either manner the path integrals over b_1 and b_2 can now be performed. The path integral with action (3.56) and integration domain (3.59) is indeed precisely the one discussed in chapter 2 with answer (2.89).[7] We find the annulus amplitude in JT gravity [8, 9]:

$$Z_{0,2}(\beta_1,\beta_2)=\int_0^\infty db\, b\, Z_{0,1}(\beta_1,b)Z_{0,1}(\beta_2,b)=\frac{1}{2\pi}\frac{\sqrt{\beta_1}\sqrt{\beta_2}}{\beta_1+\beta_2}. \tag{3.63}$$

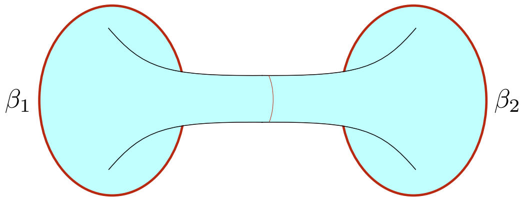
Here:

$$Z_{0,1}(\beta_1,b)=\int_0^\infty dE\, e^{-\beta_1 E}\,\frac{\cos\pi b\sqrt{E}}{\sqrt{E}}=\pi^{1/2}\beta_1^{-1/2}\exp\left(-\frac{\pi^2b^2}{4\beta_1}\right). \tag{3.64}$$

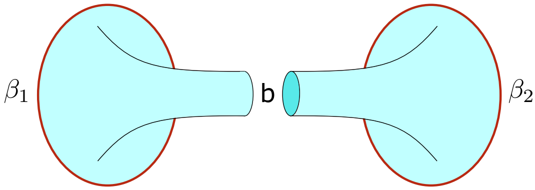
This is the JT gravity path integral on a "trumpet" as explained around (2.89). Therefore formula (3.63) should be read as proving how to glue two JT gravity trumpets (2.89)

[7] In that discussion we were implicit about modding by overall constants in the centralizer of the conjugacy class elements.

into an annulus:



$$= \int_0^\infty db \int_0^b d\tau$$



. (3.65)

The relative twisting of the surfaces around the geodesic boundary γ was left implicit in this picture. In writing (3.63) we derived the appropriate integration spaces but we have been slightly cavalier with the path integration measures on those spaces. One can prove very rigorously though that the natural path integration measure of $\mathrm{SL}(2,\mathbb{R})$ BF theory implies precisely the Schwarzian measure (2.39) for the boundary fluctuation fields b_1 and b_2 as well as the flat measure $db\, d\tau$ for the integration over conjugacy class elements and twists [9]. The latter is the so called Weil Petersson measure relevant to integration over the moduli space of Riemann surfaces.

Tensions between modular invariance and Hilbert space interpretation

Suppose now we would be interested in doing the path integral in Teichmuller theory instead. The integral over τ then gives an overall infinite prefactor:

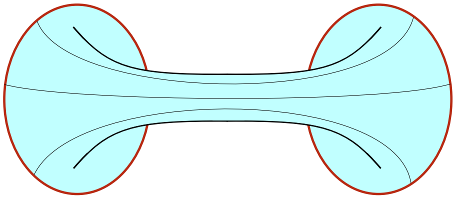
$$\begin{aligned} Z_{0,2}(\beta_1,\beta_2) &= \infty \int_0^\infty db\, Z_{0,1}(\beta_1,b) Z_{0,1}(\beta_2,b) \\ &= \infty \int_0^\infty dE\, e^{-(\beta_1+\beta_2)E}\, \frac{1}{\sqrt{E}} = \frac{\infty}{\beta_1+\beta_2}. \end{aligned} \quad (3.66)$$

This penultimate formula is structurally identical to the usual BF answer (3.29) when we remember the continuous series irreps in question are labeled as $\mathcal{C}(k) = E(k) = k^2$. The relevant formula for $\mathrm{Vol}(T)$ is here indeed ∞:

$$Z_{0,2}(\beta_1,\beta_2) = \mathrm{Vol}(T) \int_0^\infty dk\, e^{-(\beta_1+\beta_2)k^2}. \quad (3.67)$$

Of course ∞ is not a sensible answer, so we see already at the level of the annulus that it would be a bad idea to be working within Teichmuller theory. On higher genus surfaces the problem only worsens.

We would like to note that if we forget about this prefactor ∞ for a while, that the Teichmuller answer looks like it could have been obtained by a "Hilbert space type" calculation a la chapter 2. Indeed, if we would blindly apply the logic of section 2.3.3 then we would be led to the following Hilbert space calculation:[8]

$$Z_{0,2}(\beta_1,\beta_2) \stackrel{?}{=} \beta_1$$  $$\beta_2 \quad . \tag{3.68}$$

We imagine evolving around the annulus using these Cauchy slices. A complete set of states on each Cauchy slice are $|k,1_+,1_+\rangle$ with inner product:

$$\langle k_1,1_+,1_+|k_2,1_+,1_+\rangle = \delta(k_1-k_2). \tag{3.69}$$

This follows from the discussion above (2.237). One finds with a suitably defined trace on a continuous Hilbert space:

$$Z_{0,2}(\beta_1,\beta_2) \stackrel{?}{=} \int_0^\infty dk\, e^{-(\beta_1+\beta_2)k^2}. \tag{3.70}$$

Up to the factor Vol(T) this matches the exact Teichmuller calculation. The conclusion is that with some goodwill and a grain of salt one could say that there is a Hilbert space type calculation underlying the Teichmuller calculations. This is in precisely the same sense as we would say there is a Hilbert space calculation underlying the BF calculations a la (3.44). The rationale being that one could handwavingly associate the extra prefactors with some normalization ambiguities of the states. Let us now make an important point. Regardless of whether one could make such a type of handwaving argument precise in Teichmuller theory, there is zero hope of doing so in actual JT gravity. The answer (3.63) looks in no way as if it could be obtained by evolving states over the annulus as in (3.68). A more precise statement is the following. The BF path integral on the annulus does not *technically* admit a Hilbert space interpretation of the type that we would naively associate to two copies of quantum mechanics on the group due to the additional integral over twists. We would have such an interpretation if the integration space would be:

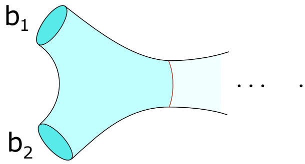
$$\int_{\mathrm{C}(G)} d\lambda\, \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)} \times \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)}. \tag{3.71}$$

One might hope to interpret the additional Vol(T) as some overall additional degeneracy but that is obscured by the fact it is not an integer. Whereas the BF path integral does not *technically* allow for a Hilbert space interpretation, the JT gravity path integral *manifestly* does not allow such an interpretation. The twist integral for gravity depends explicitly on the conjugacy class element. Therefore we do not even land on a real multiple of the integration space (3.71). In this sense this apparent harmless technicality

[8]It might not be the best choice of words to refer to calculations of this type as Hilbert space calculation, but it is what we are sticking to.

should be considered a symptom of a serious disease.

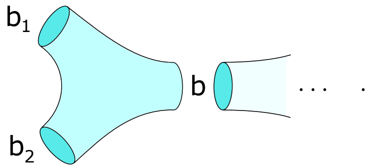
The situation is even more dramatic on more complicated geometries. We can follow the logic of the proof of (3.30) to try and deduce a local cutting and gluing formula for JT gravity. To understand the intricacies it is convenient to start with the Teichmuller formulation and then try to enforce the modding by the mapping class group. Let us imagine we start out with some genus g Riemann surface with n boundary geodesics. Consider the scenario where two boundary geodesics of lengths b_1 and b_2 and some closed geodesic γ are connected by a three holed sphere:



(3.72)

The dots represent the remainder of the surface. Following the application of (3.30) to Teichmuller theory we would conclude that this can be decomposed by effectively cutting on γ as:

$$\int_0^\infty db\,\infty$$



(3.73)

The formal infinity is due to the twist integral. In formulas:

$$Z_{g,n}(b_1 \dots b_n) = \int_0^\infty db\,\infty\, Z_{0,3}(b_1, b_2, b)\, Z_{g,n}(b, b_3 \dots b_n). \tag{3.74}$$

For JT gravity the answer for the three holed sphere is one [65]:

$$Z_{0,3}(b_1, b_2, b_3) = N_{b_1 b_2 b_3} = 1. \tag{3.75}$$

There is precisely one inequivalent flat $\mathrm{SL}(2,\mathbb{R})$ connection on the three holed sphere with hyperbolic monodromy around each hole [65]. Proceeding like this by cutting off one after the other three holed sphere by cutting the surface on each of a complete set of $3g-3+n$ closed geodesics γ_i, one finds a strongly divergent answer for any higher genus amplitude in Teichmuller theory. The sensible thing to do is mod out by the mapping class group. As a first step one could start by modding out by the analogue of the modular T transformations of the torus. This means we mod by Dehn twists around each of the geodesics, thus limiting the integration range of all the twists τ_i to a circle of circumference the length b_i of the geodesic in question. One would then find:

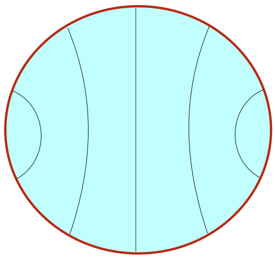
$$Z_{g,n}(b_1 \dots b_n) \overset{?}{=} \int_0^\infty db\, b\, Z_{g,n}(b, b_3 \dots b_n) \overset{?}{=} \dots \overset{?}{=} \prod_{i=1}^{3g-3+n} \int_0^\infty db_i\, b_i. \tag{3.76}$$

This is still infinite. That was to be expected. At this point we are still integrating over the analogue of the strip $\mathcal{S}$ for the torus. Modding by the full modular group results in an integral of the b_i and τ_i of some fundamental domain $\mathcal{F}$ of the surface. Unfortunately that is a very hard thing to do in general. Of course the procedure we have laid out was not particularly modular invariant to begin with. We chose one particular closed geodesic γ on which to cut. A more invariant thing to do would be to sum over all possible geodesics that bound a three holed sphere, together with the geodesics of lengths b_1 and b_2. All of these give an identical contribution because the remaining surface is topologically identical in each case. One would mod out by this overall infinity as it is part of the mapping class group of the initial surface. So this does not help our cause. Fortunately there is a trick to obtain the correct answer in general in a recursive manner [122, 123]. In the next section we will present the reader with some insight in this trick and try to explain in simple terms how one can in principle go about modding out by the mapping class group.

3.2.3 Some comments on the factorization debate

Before proceeding to explain the modding by the mapping class group on more complicated topologies, let us pause briefly and discuss an application of the annulus calculation. We would like to present some comments and criticism on the factorization debate that has been going on in the context of JT gravity [2, 54, 55, 125]. The question of factorization is actually precisely the question we have been trying to answer for gauge theories in the supplementary chapters A and B. Consider the JT gravity calculation of the thermal partition function when we ignore higher genus corrections:

$$Z(\beta) = \quad \beta$$



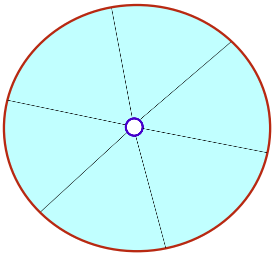
$$= \int_0^\infty dk\, k \sinh 2\pi k\, e^{-\beta k^2}. \qquad (3.77)$$

From this function one could calculate the thermal entropy of our theory via the usual replica trick. Ignoring again higher genus corrections and replica wormholes one can follow the calculation of [55]. One finds the analogue of the Bekenstein Hawking entropy for JT gravity:

$$S(\beta) = (1 - \beta\partial_\beta) \ln Z(\beta) \approx \frac{1}{\beta} + \dots \qquad (3.78)$$

The dots represent corrections away from the semiclassical saddle. We are not being precise with overall constants but [55] was. The conclusion is that the first term is precisely what one would call $A/4G$ in JT gravity. This calculation should be interpreted as being on the same footing as the Kabat calculation of the thermal partition function of electromagnetism in [126, 127, 128] with contact term contribution and all that, see

chapter A.
The central point in the Euclidean disk is the analogue of the Rindler horizon in Lorentzian signature. The question of factorization is the following. Does there exist an inherently single sided computation that reproduces the thermal calculation? To put it more precisely, imagine we are quantizing some theory in a half space such as a Rindler wedge. This requires a choice of boundary conditions on the boundary of the half space, which is the horizon in Rindler coordinates. The path integral of the quantum fields in Rindler depends in a sensitive manner on the choice of such boundary conditions. In general we expect there to be at most one choice of boundary conditions on the horizon which correctly reproduces the thermal path integral calculation (3.77). If there is such a choice of boundary conditions $\mathcal{B}$ then we have schematically:

$Z(\beta) \overset{?}{=}$ β  $\overset{?}{=} \int_0^\infty dk\, k \sinh 2\pi k\, e^{-\beta k^2}.$ (3.79)

The boundary conditions $\mathcal{B}$ on the Euclidean horizon were pictured by a blue color here and one imagines an exact match to the smooth path integral when the size of the blue circle shrinks to zero. The upshot of trying to do this is that one would hope this second picture has a Hilbert space interpretation with the slices as shown in (3.79). If this turns out to be the case then we have succeeded in diagonalizing the modular Hamiltonian K of the theory on a half space. In other words we would have the density matrix ρ for the theory on a half space.[9] One could then calculate the von Neumann entropy of this density matrix. By construction of (3.79) it would match the thermal calculation $S(\beta)$. The latter though contains the analogue of the famous Bekenstein Hawking entropy in JT gravity [55]. This means if this calculation can be carried through all the way to the end, then we have a Hilbert space interpretation of the Bekenstein Hawking entropy in JT gravity. By definition we would have obtained the microstates in this model of quantum gravity. That is what is at stake here.

In the supplementary chapter B we propose a generic prescription that results in a well defined singe sided theory. The prescription was put to the test in a variety of situations. In particular it was checked that the resulting single sided calculation of the partition function matches the thermal path integral where the horizon is just a smooth point. The idea is summarized in (B.8) and (B.9). One effectively cuts the theory on some surface by first introducing a functional delta on the gauge fields at the surface as in (3.33). One then writes this out as:

$$\delta(A_1|_\partial - A_2|_\partial) = \int [\mathcal{D}\mathcal{Q}] \exp\Big(-\int_\partial \mathrm{Tr}(\mathcal{Q}A_1 - \mathcal{Q}A_2)\Big). \tag{3.80}$$

[9] This is related to K as $K = -\ln \rho$.

We can further rewrite this as:

$$\int [\mathcal{D}\mathcal{Q}_1] \exp\Big(-\int_\partial \operatorname{Tr}(\mathcal{Q}_1 A_1)\Big) \int [\mathcal{D}\mathcal{Q}_2] \exp\Big(-\int_\partial \operatorname{Tr}(\mathcal{Q}_2 A_2)\Big)\, \delta(\mathcal{Q}_1+\mathcal{Q}_2). \tag{3.81}$$

The charges here represent boundary conditions for the single sided path integrals. Gluing the theories together corresponds to the so called physical Hilbert space where the net charge on the cutting surface vanishes. Schematically:

$$\mathcal{H} \sim \int [\mathcal{D}\mathcal{Q}_1] \cdots \int [\mathcal{D}\mathcal{Q}_2] \ldots \frac{\delta(\mathcal{Q}_1+\mathcal{Q}_2)}{\operatorname{Vol}(G_\partial)}. \tag{3.82}$$

On the other hand there is a so called extended Hilbert space where the charges do not necessarily match. This corresponds to the path integral without the functional delta on the charges. The resulting path integral factorizes completely:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \sim \int [\mathcal{D}\mathcal{Q}_1] \cdots \int [\mathcal{D}\mathcal{Q}_2] \ldots . \tag{3.83}$$

One of these decoupled path integrals was first claimed and then checked to result in the correct single sided partition function a la (3.79) for a wide range of gauge theories in [6]. In this section we will apply this reasoning to 2d BF theories and JT gravity. As it turns out this prescription results in precisely an annulus path integral in both case. One of the boundaries has a finite length β and in the case of JT gravity is supplemented with coset boundary constraints (2.61) resulting in Schwarzian boundary dynamics. The other boundary though does not have coset boundary conditions, consistent with the fact that the metric is fluctuating freely in the bulk of (3.77). In particular at the Euclidean horizon it would be very unnatural to impose something like asymptotically AdS_2 constraints on the metric fluctuations. Furthermore the renormalized length of this second boundary is found to be taken to zero, consistent with the fact that we have a tiny circle around the Euclidean horizon. The conclusion is that the picture (3.79) is not half bad. The blue color was used in chapter 2 to denote the lack of coset boundary conditions. The question then becomes whether or not the resulting single sided path integral matches (3.77) and whether or not it has a Hilbert space interpretation. Given the discussion around (3.71) the conclusion is very unsurprising. With some goodwill the construction could be claimed to work for usual BF theories as well as in Teichmuller theory. This is especially so because the prescription of chapter B is slightly ambiguous in terms of fixing an overall constant. The result then *does* have the expected Hilbert space interpretation and one might argue these theories factorize. This is definitely not the case for JT gravity though. The annulus calculation is not precisely identical to (3.63) due to the lack of coset constraints on the inner boundary, but the conclusions remains. The required invariance under large diffeomorphisms forces us to mod out by the mapping class group. There is a nontrivial mapping class group on the annulus (3.79) with no analogue on the smooth disk (3.77). We cannot imagine how any boundary condition on the blue circle could correct the answer on the annulus in such a way that it would end up cancelling this effect of modding by the mapping class group. This is ultimately why we do not see there can be hope of genuine factorization in JT gravity.

In summary we would not necessarily say that a priori the diagonal modding in (3.25) removes the hope of factorization, because overall constants may not be very important. Certainly we would not say that the modding in (3.24) is bad for factorization a priori. But we would say that modding out by the mapping class group spells trouble.

Before getting starting let us give two popular and closely related a priori point of criticism of trying this calculation in the first place. The first [50, 54] is that Schwarzian quantum mechanics is not a good representative of models we expect to be dual to quantum gravity in general. The reason is that its spectrum is continuous. In general however, in any number of dimensions, the boundary theory is some field theory on a compact space. consequently one expects in general a discrete spectrum for any sensible theory of quantum gravity. This culminates into important general expectations about physics in quantum gravity such as the erratic behavior of late time correlators [14, 7]. More importantly one expects a discrete spectrum with a particular type of energy level statistics. Roughly speaking, the levels can never be very close together. This is the telltale of quantum chaos [17]. We expect black holes in quantum gravity. Black holes are quantum chaotic, therefore the dual theory should be a quantum chaotic model. This discreteness and level statistics which we believe to hold universally in any model of quantum gravity, are not captured by Schwarzian quantum mechanics. For more on this see chapters 4 and 5. Therefore even if we would find dual microstates, they would not have any of these universal properties. One could then fairly question how universal the conclusions about their bulk gravitational interpretation would be. The answer is they would not be universal at all. The second point of criticism is that in quantum gravity in general we expect that topology changing processes are important. In this sense the topologically trivial version of JT gravity discussed in chapter 2 is a priori already not a great model of quantum gravity. It would be exponentially more interesting to understand a potential factorization in a model with topology change. The problem is unfortunately also orders of magnitude more difficult as compared to the current one. For example modular invariance suggests that an invariantly defined cutting surface should probably not have a fixed topology. Rather we might need to allow any type of topology. The work on 2d Yang Mills by mainly Donnelly and Wong [87, 129, 130] might be interesting in this context. It is important to realize that these bad features of Schwarzian quantum mechanics are actually one to one with not summing over topologies in the bulk. As should hopefully be clear by the end of this work we *can* get the type of spectrum we universally expect from quantum gravity in a version of JT gravity. The trick is precisely to sum over all topology changing processes and furthermore to allow for absorption and emission of baby universes at so called eigenbranes hovering in the bulk. It would be extremely interesting to understand any sense of factorization in that more realistic model of quantum gravity. We comment on this in chapters 4, 5 as well as in the discussion chapter 6. See also [10, 19]. Even with all these caveats we still feel that it is a sensible first step to try and understand factorization in the topologically trivial version of JT gravity.

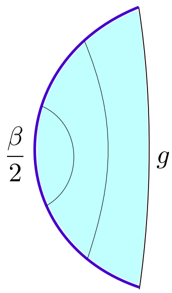
Suggestion of an extended Hilbert space from the thermofield double

Before turning to the actual single sided path integral calculation it is instructive to explain the type of Hilbert space interpretation of the single sided amplitude (3.79) one might hope to find. For this we could turn to the mathematical factorization of the Hilbert space of BF theory on an interval between the two asymptotic boundaries as implicit in (2.120) and (2.128). The JT gravity analogues are a suitable modification of (2.163) but with interior labels of the type that appeared in (2.202). This is summarized in (2.260) and the discussion around it:

$$R_{k,1_+1_+}(g_1 \cdot g_2) = \int_{-\infty}^{+\infty} ds\, R_{k,1_+s}(g_1)\, R_{k,s1_+}(g_2). \tag{3.84}$$

We can use this to write for example the thermofield double state of 2d BF theory in a way that suggests a Hilbert space for the single sided calculation 3.79. We have:

$\langle \mathrm{HH}|g\rangle =$



. (3.85)

This evaluates according to (2.127) to :

$$\langle \mathrm{HH}|g\rangle = \sum_{R,a} \dim R\, R_{aa}(g)\, e^{-\frac{\beta}{2}\mathcal{C}_R}. \tag{3.86}$$

In terms of the Hilbert space $|R,a,b\rangle$ associated with 2d BF on an interval we write:

$$|\mathrm{HH}\rangle = \sum_{R,a} \dim R^{1/2}\, e^{-\frac{\beta}{2}\mathcal{C}_R}\, |R,a,a\rangle\,. \tag{3.87}$$

Consider now the wavefunction $\langle g_1 \cdot g_2|R,a,a\rangle$ in combination with (2.120). Taking into account the normalization of the wavefunctions (2.8) we find the mathematical factorization:

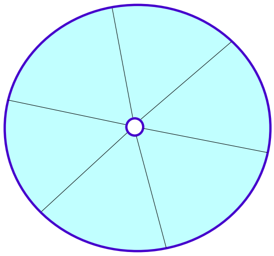
$$\langle g_1 \cdot g_2|R,a,a\rangle = \sum_{b} \dim R^{-1/2}\, \langle g_1|R,a,b\rangle\, \langle g_2|R,b,a\rangle\,. \tag{3.88}$$

Using this, we can write the thermofield double in a manner that is suggestive:

$$|\mathrm{TFD}\rangle = \sum_{R,a,b} e^{-\frac{\beta}{2}\mathcal{C}_R}\, |R,a,b\rangle \otimes |R,a,b\rangle\,. \tag{3.89}$$

It suggests that the single sided Hilbert space sought after in the BF analogue of (3.79) are the states $|R, a, b\rangle$. We might indeed try to associate this to the path integral of 2d BF theory on a Euclidean annulus, the analogue of (3.68) but with blue boundaries and β_1 taken to zero. We might thus hope that the following is true:

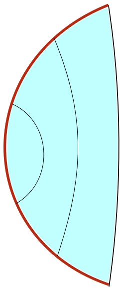
$$Z(\beta) \stackrel{?}{=} \quad \beta$$



$$. \qquad (3.90)$$

We can get a very similar suggestion in JT gravity using (3.84). We have a mild rewriting of (2.256):

$$\langle \text{HH}|g\rangle = \quad \frac{\beta}{2}$$



$$g \quad = \int_0^\infty dk\, k \sinh 2\pi k\, R_{k,1_+1_+}(g)\, e^{-\beta k^2}. \qquad (3.91)$$

We don't care about the actual formula for the representation matrices here, so we don't have to resort to the logic around (2.256) to obtain manageable formulas. From this one reads off:

$$|\text{HH}\rangle = \int_0^\infty dk\, \dim k^{1/2}\, e^{-\beta k^2}\, |k, 1_+, 1_+\rangle\,. \qquad (3.92)$$

In writing this we used (2.182) and (2.191). From (3.84) we find the analogue of (3.88):

$$|k, 1_+, 1_+\rangle = \dim k^{-1/2} \int_{-\infty}^{+\infty} ds\, |k, 1_+, s\rangle \otimes |k, 1_+, s\rangle\,. \qquad (3.93)$$

The thermofield double or Hartle Hawking state can now again be rewritten in a suggestive manner:

$$|\text{HH}\rangle = \int_0^{+\infty} dk \int_{-\infty}^{+\infty} ds\, e^{-\frac{\beta}{2}k^2}\, |k, s, 1_+\rangle \otimes |k, s, 1 - +\rangle\,. \qquad (3.94)$$

This is conform the intuition one might get from looking at (3.79). One might expect to find a single sided Hilbert space of the form $|k, s, 1_+\rangle$, that is, if there exists a single sided Hilbert space. One label is unconstrained because we are not imposing asymptotic coset constraints at the interior boundary. Indeed it is clear that there is no hope of ever recovering the smooth disk (3.77) if we would constrain metric fluctuations on the cutting surface in the single sided theory. This makes sense. One can indeed match $Z(\beta)$ precisely to a thermal trace in a tentative single sided spectrum $|k, s, 1_+\rangle$. The point which we would like to argue in what follows though is that it is highly likely that there is not one possible choice of boundary conditions so that a single sided JT gravity calculation actually ends up having this precise Hilbert space interpretation. The reason is, as we are in danger of repeating ourselves, that the single sided computation necessarily introduces a tiny additional boundary. The resulting JT gravity computation has a larger mapping class group to mod out than does the original disk (3.77) making it highly unlikely that the two calculations could ever match. In principle we cannot exclude the possibility that there might exists such a match. It is just that one: we do not see any reason why there would *need* to be a matching single sided calculation, and two: that all evidence suggests there is none.

Edge dynamics from the BF path integral

Imagine an annulus with boundary lengths β_1 and β_2 with some 2d BF theory on it and with asymptotic boundary conditions (2.4). Imagine trying to cut it on some non contractable curve γ into two regions which are each topologically an annulus. Such an annulus has one asymptotic boundary and one "cutting" boundary. We now want to imagine associating edge degrees of freedom to this cutting boundary in the sense that we want to understand the path integral analogue of the extended Hilbert space construction $\mathcal{H}_1 \otimes \mathcal{H}_2$. Via the logic of (3.33), (3.80) and (3.83) we are led to the following path integral for either one of the annuli which we could associate to for example $\mathcal{H}_1$:

$$\mathcal{H}_1 \sim \int \frac{[\mathcal{D}A_1]}{\text{Vol}(G_1)}[\mathcal{D}\chi_1] \exp\left(-S[A_1, \chi_1] - \frac{1}{2}\int_{\partial_A} \text{Tr}(\chi A)\right) \int [\mathcal{D}\mathcal{Q}_1] \exp\left(-\int_{\partial_C} \text{Tr}(\mathcal{Q}_1 A_1)\right). \tag{3.95}$$

We can write this as in (B.10):

$$\mathcal{H}_1 \sim \int [\mathcal{D}\mathcal{Q}_1] Z_{0,2}(\mathcal{Q}_1, \beta_1) \tag{3.96}$$

The field $\mathcal{Q}_1$ represents a boundary charge on the cutting surface. The prescription from chapter B is to integrate over the boundary charges $\mathcal{Q}_1$ and $\mathcal{Q}_2$ independently for the extended Hilbert space computation but to include a functional delta on zero total charge as in (3.83) to recover the glued configuration. The question is what the physical content of the single sided theory (3.95) is. What do we get from actually doing this path integral over boundary charges? Notice that we have a boundary term of the type (3.14) on the asymptotic boundary ∂_A with boundary conditions (2.4). Furthermore we have a boundary term of the type in (3.81) on the cutting boundary ∂_C. Let us make a quick side step. It might not be clear a priori how to interpret the factor $\text{Vol}(G_\partial)^{-1}$

in (3.33) in this context. Another option might be to multiply the above path integral by $\mathrm{Vol}(G_\partial)^{-1/2}$. We will stick to the logic of chapter B though where this factor was interpreted as part of the gluing prescription (B.11) to be compared to the single sided path integral (B.13). This ends up giving sensible results with no overall infinities anywhere. In particular it ends up giving the correct answer for electromagnetism in Rindler space which is our premier point of comparison. We interpret all this as evidence that we should indeed understand the factor $\mathrm{Vol}(G_\partial)^{-1}$ in this way and consider (3.95). Moving on. Let us drop the label referring to the specific subregion. We see that other than the difference in precise boundary actions, the path integral over A and χ in (3.95) is structurally identical to the usual BF path integral on the annulus as discussed below (3.14). We can make the resemblance more obvious when doing the path integral over χ and localizing on flat connections. Let us label the boundary values of the gauge field as $g|_{\partial_A}$ as g_A and similarly $g|_{\partial_C}$ as g_C. We find the action:

$$S[g_A, g_C, \mathcal{Q}] = -\frac{1}{2}\int_{\partial_A} d\tau\, \mathrm{Tr}\big(g_A^{-1}\partial_\tau g_A\big)^2 - \int_{\partial_C} d\tau\, \mathrm{Tr}\big(\mathcal{Q}\, g_C^{-1}\partial_\tau g_C\big). \tag{3.97}$$

The second term is in fact also an action for quantum mechanics on the group but with vanishing Hamiltonian as explained around (3.37). The relation could be made more explicit if we would write the path integral for g_A also as a phase space path integral by introducing a charge $\mathcal{Q}_A$. Anyway. The integration contour for the fields g_A and g_C as deduced from (3.95) is precisely identical to that in (3.21):

$$\int_{\mathrm{C}(G)} d\lambda\, \frac{\mathrm{LG}(\lambda)\times \mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)}. \tag{3.98}$$

Proceeding as below (3.25) we can extract variables g_A and g_C with no zero modes. We then have the integration space:

$$\int_{\mathrm{C}(G)} d\lambda \int_{\mathrm{C}(\lambda)} d\tau\, \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)} \times \frac{\mathrm{LG}(\lambda)}{\mathrm{C}(\lambda)} \tag{3.99}$$

We can now just immediately do the path integral (3.95). For example by remembering that there is effectively no Hamiltonian for g_C or by just doing the phase space path integral over g_C and $\mathcal{Q}$ at fixed monodromy λ a la Alekseev and Shatashvilli one finds it contributes $Z(0,\lambda)$. The total path integral (3.95) gives:

$$\begin{aligned}\int [\mathcal{D}\mathcal{Q}] Z_{0,2}(\mathcal{Q},\beta) &= \int_{\mathrm{C}(G)} d\lambda \int_{\mathrm{C}(\lambda)} d\tau\, Z_{0,1}(0,\lambda)\, Z_{0,1}(\beta,\lambda)\\ &= Z_{0,2}(0,\beta) = \mathrm{Vol}(T)\sum_R \dim R^2\, e^{-\beta \mathcal{C}(R)}\\ &= \mathrm{Vol}(T)\, Z(\beta). \end{aligned} \tag{3.100}$$

In the second equality we recognize the answer for the 2d BF annulus path integral (3.27). The left hand side is our proposal for a single sided theory in BF theory. Up to the factor $\mathrm{Vol}(T)$ it does have a Hilbert space interpretation starring precisely the annulus states $|R,a,b\rangle$ predicted from (3.89). We find very precisely that our single sided theory brings to life the picture on the right hand side of (3.90). Indeed we find effectively an annulus in BF theory where one of the lengths is taken to zero. In this calculation it might be sensible not to be too bothered with the relative overall factor $\mathrm{Vol}(T)$ that *technically* spoils the equality of the single sided computation and the smooth Euclidean disk calculation (3.90). This equality does not hold in the same sense that *technically* the BF annulus path integral (3.27) does not have a Hilbert space interpretation. Let us choose to ignore this overall constant here and claim factorization does work for the usual BF theories. The situation for JT gravity though is exponentially worse.

Edge dynamics from the JT path integral

The calculation for JT gravity is very similar, much like the calculation of the annulus in JT gravity around (3.56) parallels that of the annulus in 2d BF theory around (3.21). The only thing that really changes is that now we have coset constraints. It is important to realize that we only have such coset constraints on the asymptotic boundary ∂_A. Indeed, at no point in introducing a functional delta on the cutting surface, writing it like (3.80) and path integrating out χ do we need to constrain the values of the connection on the cutting surface $A|_{\partial_A}$ in any way. As a result we will be left with a Schwarzian action on ∂_A and the action of quantum mechanics on $\mathrm{SL}(2,\mathbb{R})$ on ∂_C though again with vanishing Hamiltonian. For fixed hyperbolic monodromy b we have:

$$S[b_A, g_C, Q] = \int_{\partial_A} d\tau \left(-\frac{1}{4}\frac{b_A''^2}{b_A'^2} - \pi^2 b_A'^2 \right) - \frac{1}{2}\int_{\partial_C} d\tau\, \mathrm{Tr}\big(\mathcal{Q}\, g_C^{-1}\partial_\tau g_C\big) \tag{3.101}$$

The monodromy constraints on g_C are the same as those on the field b_A:

$$g_C(\tau+\beta) = U_b \cdot g_C(\tau). \tag{3.102}$$

The integration space is now:

$$\int_0^\infty db\, \frac{\mathrm{LG}(b)\times \mathrm{Diff}(b)}{S_1(b)}. \tag{3.103}$$

Writing this out and modding by the mapping class group that exists on this annulus shaped surface we end up with the analogue of (3.62) but with one of the boundary fields now integrated over the loop group instead of just reparameterizations:

$$\int_0^\infty db \int_0^b d\tau \frac{\mathrm{LG}(b)}{S_1(b)} \times \frac{\mathrm{Diff}(b)}{S_1(b)}. \tag{3.104}$$

It is straightforward to do the corresponding path integral. Before doing so let us note that the result is manifestly what we would mean when drawing the picture in (3.79). For example the path integral over g_C and $\mathcal{Q}$ for fixed monodromy b was actually calculated in (2.230). One finds:

$$Z_{0,1}(0,b) = \int_0^\infty dE\, \sinh 2\pi\sqrt{E}\, \cos \pi b\sqrt{E}. \tag{3.105}$$

The total answer for the single sided computation is:

$$\int [\mathcal{D}\mathcal{Q}] Z_{0,2}(\mathcal{Q},\beta) = \int_0^\infty dE_1\, \sinh 2\pi\sqrt{E_1} \int_0^\infty dE_2\, e^{-\beta E_2} \frac{1}{\sqrt{E_2}} \int_0^\infty db\, b\, \cos\pi b\sqrt{E_1}\, \cos\pi b\sqrt{E_2}. \tag{3.106}$$

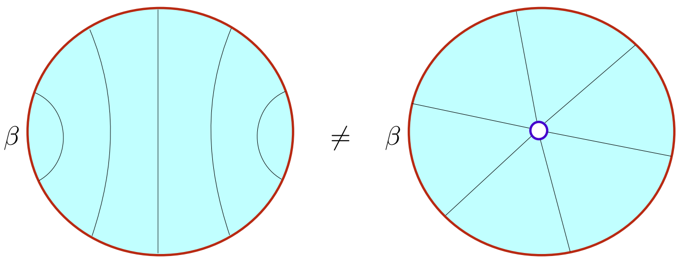
We could try to do this integral, but it certainly does not look even remotely close to $Z(\beta)$ in (3.77). We can understand that this is entirely due to the finite range of the twist measure in gravity. This results in the factor b in the above integral. Let us be naive for a while and work in Teichmuller theory. Now the twist integral in (3.104) would range over the entire real axis. We would find:

$$\int [\mathcal{D}\mathcal{Q}] Z_{0,2}(\mathcal{Q},\beta) \stackrel{?}{=} \int_0^\infty dE_1\, \sinh 2\pi\sqrt{E_1} \int_0^\infty dE_2\, e^{-\beta E_2} \frac{1}{\sqrt{E_2}} \infty \int_0^\infty db\, \cos\pi b\sqrt{E_1}\, \cos\pi b\sqrt{E_2}. \tag{3.107}$$

These integrals we can all do easily. One finds:

$$\int [\mathcal{D}\mathcal{Q}] Z_{0,2}(\mathcal{Q},\beta) \stackrel{?}{=} \infty \int_0^\infty dE\, \sinh 2\pi\sqrt{E}\, e^{-\beta E} = \infty\, Z(\beta).$$

We see that the Teichmuller calculation has a Hilbert space interpretation and factorizes in precisely the same sense that ordinary BF calculations would. They do not factorize *technically* but the difference is just an overall ,albeit infinite, prefactor. The mapping class group as announced completely obliterated this glimmer of hope. Much like the usual annulus amplitude in JT gravity (3.63) is not remotely close to having a Hilbert space interpretation,[10] the single sided annulus computation (3.107) is not even remotely close to matching the thermal disk answer or to having a Hilbert space interpretation. We can summarize this discussion as:



. (3.108)

And it is not a close call. As discussed already in the beginning of this section, the mapping class group spells trouble for any naive generalization of BF type Hilbert space reasoning to JT gravity. The sole exception is when we are on the disk where the mapping class group acts trivially. The single sided computation however, is essentially by definition a calculation on the annulus.

[10] This was discussed around (3.71).

3.3 Baby universes

Having warmed up with the annulus computation, let us now do the calculation of the JT gravity path integral for *any* fixed topology and finally let us sum over topologies. Of all the sections in this work, this one is most heavily based on work by others. In particular we refer the reader to [9, 65, 18] though parts of this were also presented in [2, 3].

3.3.1 Other topologies

Let us now return to the discussion of around (3.76). We would like to understand how to appropriately mod by the full mapping class group on higher genus Riemann surfaces. Historically has proven to be a very difficult problem to solve, but eventually Mirzakhani cracked it [122, 123]. Explaining the whole solution is certainly feasible but doing so here would take us too far. Readers interested in the whole argument are kindly referred to [65] where this is explained really in layman terms. Our goal here is to explain one particular aspect to the reader which we feel is important to appreciate. We would like to explain that given one technical identity known as the sum rule, it is straightforward to mod by the mapping class group using a trick not unsimilar to that in (3.12). As we have hammered intensively on the importance of the mapping class group we strongly feel that this part of the story is one that cannot be left out of any self consistent discussion. The following largely follows arguments of [65] which we have rephrased in a more convenient manner. Before getting started let us emphasize that we will *not* be seeking after mathematical rigor. For example we will be making a bunch of rather vague statements about the relations between the mapping class group on surfaces of different topology. All of these are intuitively obvious when we remember that roughly speaking the mapping class group is just mapping different sets of geodesics into each other. Technically though they would require a more careful mathematical proof which we will not attempt to give. The purpose of this section is to give the reader intuitive insight, and not at all to dazzle with complicated mathematics.

Setting up a recursion formula

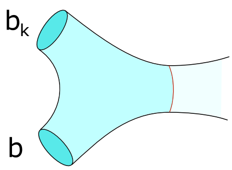
The calculation which we would like to present is a recursive way to calculate the JT gravity path integral on a genus g Riemann surface with n hyperbolic punctures:

$$Z_{g,n}(b_1 \dots b_n). \tag{3.109}$$

The punctures ofcourse represent geodesic boundaries. The number of independent non intersecting geodesics on such a surface is $N = 3g - 3 + n$. For the recursive calculation we imagine that we know all amplitudes up to a certain value of N. Let us call surfaces with lower values of N "simpler" than those with higher values of N. For example a three holed sphere is simpler than a four holed sphere. We imagine that we could write the answer as:

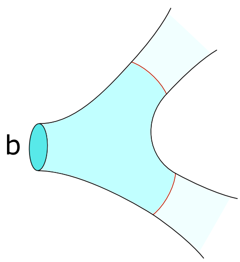
$$Z_{g,n}(b_1 \dots b_n) = \int \prod_{i=1}^{N} da_i \, a_i \, f(b_1 \dots b_n, a_1 \dots a_N) = \int \mathrm{Vol}(\Sigma). \tag{3.110}$$

For example for a four holed sphere we have a single integral over b_5 and a function $f(b_1, b_2, b_3, b_4, b_5)$ where the first four label the boundary lengths. The measure function could be roughly thought of as coming from integrating the twists over a fundamental domain. In other words we assume to have found a way to correctly mod by the mapping class group on these "simple" surfaces. We now want to calculate the answer for a surface Σ that is slightly less simple. To do so, pick one boundary circle of Λ and label its lengths as b. Now imagine all three holed spheres Λ with three geodesic boundaries and the following properties. They are entirely contained in Λ and one of their geodesic boundaries is our "chosen" geodesic γ with length b. We can divide these into two classes.
One class if where a second geodesic boundary of Λ is an existing geodesic boundary of Σ say γ_k with length b_k.[11] We can picture the embedding in Σ as:



. (3.111)

Let us denote the remainder of the surface, which has been given a lighter color, by Σ_k. For each k there is actually an infinite number such three holed spheres, obtained by mapping the third geodesic γ' (on which the three holed sphere ends) around under the mapping class group of Σ. Let us label all such three holed spheres for a fixed k as Λ_k. A second class is where the two geodesic boundaries of Λ other than γ lie entirely in Σ. Let us refer to these as γ' and γ''. We can picture the embedding in Σ as:



. (3.112)

By drawing some punctures one can convince oneself of the fact that for fixed γ and Σ, the surface Σ_j obtained by cutting Λ off Σ can have various topologies. For example Σ_j can be connected or disconnected. Let us label such topologies by j.[12] For each

[11] If Σ has n boundaries then k can take $n-1$ values.
[12] So j captures the genus and number of boundaries of Σ_j. In terms of disconnected pieces we can have a bunch of different "values" for j.

fixed j there is again an infinite number of three holed spheres Λ_j resulting in the same topology for the remaining surface Σ_j obtained by acting with a subset of the mapping class group of Σ which maps around γ' and γ''.
In fact when we think about it, we can understand that the mapping class group of Σ for any fixed j or k consists of the mapping class group of Σ_j or Σ_k composed with the transformations that map either γ' respectively γ' and γ'' around whilst maintaining the topology of the pieces when we cut on respectively γ' respectively γ' and γ''. In other words we have the mapping class group on say Σ_j as well as mappings that map different Λ_j into each other. Again this should be intuitively obvious and we are not attempting a rigorous proof. Schematically:

$$\mathrm{MCG}(\Sigma) = \mathrm{MCG}(\Sigma_j)\,\mathrm{M}(\Lambda_j). \tag{3.113}$$

The latter denotes the number of all mappings of different Λ_j for fixed j into each other. There is no sum over j in this identity. We are being extremely cavalier in terms of notation and do not distinguish the volume of the mapping class group from the group itself. A similar expression holds for each k. We want to find an expression for the integral over the moduli space of Σ where we have divided by the entire mapping class group. For each fixed j or k we have:

$$\int \mathrm{Vol}(\Sigma) = \int \frac{1}{\mathrm{M}(\Lambda_j)} db'\, b'\, db''\, b''\, \mathrm{Vol}(\Sigma_j) = \int \frac{1}{\mathrm{M}(\Lambda_k)} db'\, b'\, \mathrm{Vol}(\Sigma_k). \tag{3.114}$$

Indeed. Each of the surfaces Σ_j and Σ_k is simpler than Σ. By assumption of our recursive proof we have found a way to divide out the first factor in (3.113), resulting in a certain volume form such as (3.110) for each Σ_j and Σ_k. The integral of each such volume form is furthermore assumed to be finite.[13] Therefore we only need to mod the remainder by the second factor in (3.113). Furthermore we learned in (3.76) that not properly modding by the mapping class group (as it has been presented here) results in integration over the strip $\mathcal{S}$ for each geodesic. We have in this scenario either the geodesic γ' or the geodesics γ' and γ'' for which this would be the naive integration domain. Actually the equality (3.114) is an equality on volume forms that holds prior to integration. For the following technical trick let us instead consider an equivalent equality:

$$b \int \mathrm{Vol}(\Sigma) = \int \frac{b}{\mathrm{M}(\Lambda_j)} db'\, b'\, db''\, b''\, \mathrm{Vol}(\Sigma_j) = \int \frac{b}{\mathrm{M}(\Lambda_k)} db'\, b'\, \mathrm{Vol}(\Sigma_k). \tag{3.115}$$

Remember that b is the length of an exterior boundary γ so we are not integrating over it. Again this is an equality on volume forms.

Technical trick to divide by the mapping class group

As it stands we would just be dividing infinity by infinity when we would do the integral over b' and b''. It does not seem as if we are getting any wiser. Here comes the

[13] One can check at the end that this is true.

technical trick though. Let us imagine that we would have an identity:

$$b \stackrel{?}{=} \sum_{\Lambda_j} f(b, b', b'', \Lambda_j). \tag{3.116}$$

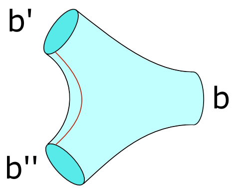
This seems reasonable. Let us now imagine that the function that enters here is actually independent of the particular Λ_j for any fixed j. This is starting to sound less likely. Have faith though for now. If this were to be the case in some miraculous way, then we could drop the infinite sum in nominator and denominator of (3.115). We would find:

$$b \int \mathrm{Vol}(\Sigma) \stackrel{?}{=} \int b'\, b'\, db''\, b''\, f(b, b', b'')\, \mathrm{Vol}(\Sigma_j). \tag{3.117}$$

This we could imagine would give a finite result upon integration because we would have succeeded in dividing precisely by the infinite mapping class groups worth of degrees of freedom. As it turns out we do not precisely have an identity of the type (3.116). What we do have is an identity of the type:

$$b = \sum_j \sum_{\Lambda_j} f_2(b, b', b'') + \sum_k \sum_{\lambda_k} f_1(b, b', b_k). \tag{3.118}$$

This is a half miracle. It is known as the Mirzakhani sum rule [122, 123]. There is actually a nice geometric derivation on this identity that is explained in detail in [65]. We will skip it here because it is not absolutely essential in the JT gravity story.[14] Let us note though that the geometric derivation also immediately provides an answer for the functions in question. The key is more than anything to appreciate that these only depend on the three boundary lengths of the three holed sphere with which they are associated. This is because three lengths b, b' and b'' of boundary geodesics γ, γ' and γ'' completely determine the geometric properties of a three holed sphere bound by γ, γ' and γ'' in hyperbolic geometry. The technical part of the computation therefore boils down to simply calculating properties of some fixed three holed sphere. As it turns out [65] the functions in question are related to the orthogonal geodesic distance *between* each of the three geodesics. So for example one is led to calculate $d(\gamma, \gamma')$ in a three holed sphere bound by γ, γ' and γ'' and this ends up giving us the functions in question. One such length for example is measured along the red geodesic curve here:



. (3.119)

Anyway. For the purpose of this argument is does not matter how one derives this sum rule. We can just take is as a mathematical identity and use it to write:

$$b \int \mathrm{Vol}(\Sigma) = \sum_j \int \sum_{\Lambda_j} f_2(b, b', b'')\, \mathrm{Vol}(\Sigma) + \sum_k \int \sum_{\lambda_k} f_1(b, b', b_k)\, \mathrm{Vol}(\Sigma). \tag{3.120}$$

[14] We try to limit excursions to a minimum.

For each separate term in the sum one can now replace Vol(Σ) by a matching decomposition as in (3.114). In each separate term, say the term associated with Σ_j, the sum over the Λ_j cancels between the nominator and the denominator. We are left with:

$$b\int \mathrm{Vol}(\Sigma) = \sum_j \int f_2(b,b',b'')\, db'\, b'\, db''\, b''\, \mathrm{Vol}(\Sigma_j) + \sum_k \int f_1(b,b',b_k)\, db'\, b'\, \mathrm{Vol}(\Sigma_k). \tag{3.121}$$

Translated back to the amplitudes this becomes a recursion relation that writes out the JT path integral on Σ into simpler JT path integrals. Schematically:

$$\begin{aligned} b\, Z_\Sigma(b,\{b_i\}) = \sum_j \int_0^\infty db'\, b' \int_0^\infty db''\, b''\, f_2(b,b',b'')\, Z_{\Sigma_j}(b,b',\{b_i\}) \\ + \sum_k \int_0^\infty db'\, b'\, f_1(b,b',b_k)\, Z_{\Sigma_k}(b',\{b_i\}/b_k). \end{aligned} \tag{3.122}$$

This is the Mirzakhani recursion relation [122, 123]. The JT gravity path integrals on a genus g Riemann surface with n boundaries are in that context referred to as so called Weil-Petersson volumes:[15]

$$Z_{g,n}(b_1\ldots b_n) = V_{g,n}(b_1\ldots b_n). \tag{3.123}$$

The integration measure on the moduli space on flat $\mathrm{SL}(2,\mathbb{R})$ connections is similarly referred to as the Weil-Petersson measure:

$$\prod_{i=1}^{3g-3+n} db_i\, d\tau_i. \tag{3.124}$$

Using the recursion relation (3.122) with the precise answers for the functions filled in it is not to difficult to find explicit answers for these Weil-Petersson volumes. They are polynomials in b_1^2 etcetera of some maximal degree. We consider it to be too much information to write down precise answers. The interested reader is referred to for example [122, 123, 9, 65]. One thing worth mentioning is that the seed for this recursion relation is essentially the answer for the annulus and the three holed sphere:

$$V_{0,2}(b_1,b_2) = \frac{1}{b_1}\delta(b_1-b_2), \qquad V_{0,3}(b_1,b_2,b_3) = 1. \tag{3.125}$$

The latter is true because there is only one and precisely one inequivalent flat hyperbolic $\mathrm{SL}(2,\mathbb{R})$ connection on the three holed sphere for each b_1, b_2 and b_3. The former has implicitly been derived in section 3.2.2. In fact it is very much a choice to refer to this as a Weil-Petersson volume, but not a significant one.

Path integral of JT gravity on any topology

[15] For recent discussions see for example [131, 122, 123, 120, 9].

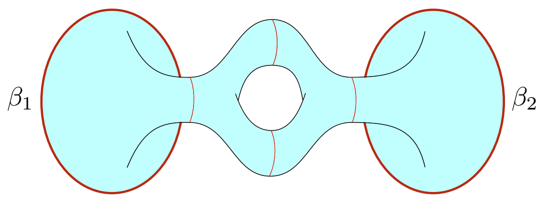
Armed with these expressions we can write down the amplitude for JT gravity an any topology with any number of asymptotic boundaries following [9]. We can go about this by combining the logic that led to (3.62) and the one that led to (3.76). Say we are given a genus g Riemann surface with n boundaries of lengths $\beta_1 \dots \beta_n$. Let us denote the JT gravity amplitude as:

$$Z_{g,n}(\beta_1 \dots \beta_n). \tag{3.126}$$

We could alternatively for example consider fixed microcanonical energy boundaries $E_1 \dots E_n$ by doing several inverse Laplace transforms of (3.126). Choose now a complete set γ_i of non intersecting geodesics on this surface. Clearly n of these will be homologous to the boundary components. Let us denote these γ_k and denote the other geodesics by γ_j. Let us now apply the logic that led to (3.76) and do a local cutting of the amplitude on each of the geodesics γ_k and γ_j. Let us furthermore already mod out by the modular T transforms associated with each of the geodesics. This constrains the twist integrals for each geodesic to the strip $\mathcal{S}$ as in (3.76). As discussed around (3.27) and proven below (3.30) the precise form of the annulus amplitude can be considered an application of this cutting rule by cutting on the single geodesic γ around the neck of the annulus. The JT path integral for each of the trumpet shaped diagrams was already done in (2.89). In case of the JT gravity annulus amplitudes this has culminated in the identities (3.63) and (3.65). Doing the cutting on all γ_k and γ_j on our higher genus Riemann surface we see that we obtain a single JT path integral on a trumpet for each asymptotic region. Furthermore we have a bunch of three holed spheres whose JT gravity path integral gives one and then an integral over the whole strip $\mathcal{S}$ for each of the moduli associated with the geodesics γ_k and γ_j. This still needs to be modded out by the relevant mapping class group MCG(Σ). We formally end up with:

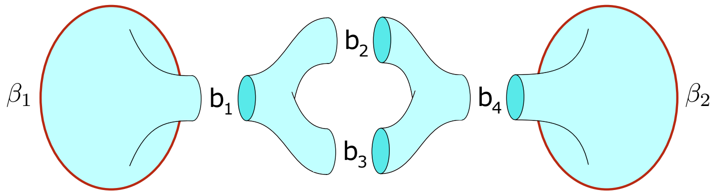
$$Z_{g,n}(\beta_1 \dots \beta_n) = \prod_k \int_0^\infty db_k\, b_k\, Z(\beta_k, b_k)\, \frac{1}{\text{MCG}(\Sigma)} \prod_j \int_0^\infty da_j\, a_j. \tag{3.127}$$

As in (3.110) we have opted to label the interior geodesic lengths as a_j and the exterior ones as b_k. An example might be useful to clarify this. Consider therefore a genus one connected contribution to the JT gravity path integral with two asymptotic boundaries:

$Z_{1,2}(\beta_1, \beta_2) =$  . (3.128)

We have already highlighted the four geodesics on which we will cut. Remember that this choice is physically irrelevant. The precise geometrical translation of (3.127) in this case becomes:

$$Z_{1,2}(\beta_1, \beta_2) = \quad \frac{1}{\text{MCG}(\Sigma)} \prod_{i=1}^{4} \int_0^\infty db_i \, b_i \tag{3.129}$$



.

Each of the four individual amplitudes has a precise JT gravity answer which we have explained previously. Back to the general case (3.127) now. We learned in the previous paragraph how to go about the modding by the mapping class group. Retracing that logic we observe that the final two factors of (3.127) combine to define the JT gravity amplitude on a genus g surface with n geodesic boundaries which evaluates to the Weil-Petersson volume in question, by definition:

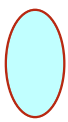
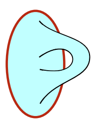
$$\frac{1}{\text{MCG}(\Sigma)} \prod_j \int_0^\infty da_j \, a_j = V_{g,n}(b_1 \dots b_n). \tag{3.130}$$

Using now furthermore the explicit answer (3.64) for $Z_{0,1}(\beta_k, b_k)$ we find [9]:

$$Z_{g,n}(\beta_1 \dots \beta_n) = \pi^{n/2} \beta_1^{-1/2} \dots \beta_n^{-1/2} \int_{-\infty}^{+\infty} db_1^2 \, \exp\left(-\frac{\pi^2 b_1^2}{4\beta_1}\right) \dots$$
$$\dots \int_{-\infty}^{+\infty} db_n^2 \, \exp\left(-\frac{\pi^2 b_n^2}{4\beta_n}\right) V_{g,n}(b_1 \dots b_n). \tag{3.131}$$

Since the Weil-Petersson formulas are known polynomials in b_1^2 etcetera it is now just a matter of doing basic Gaussian integrals to find an explicit answer for any such JT gravity amplitude. What we would like to do now is to take the sum over all g for a fixed number of boundaries n and define this sum to be the answer for the JT gravity path integral with a number of asymptotic boundaries as announced in (3.5). For example we would like to define the partition function as:

$$Z(\beta) = \sum_{g=0}^{\infty} e^{-2gS_0} Z_{g,1}(\beta)$$

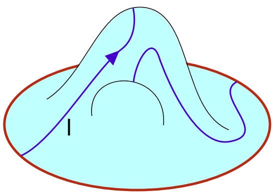
$= \quad \beta$  $\quad + \quad \beta$  $\quad + \dots \qquad (3.132)$

The powers of e^{-S_0} are due to the Einstein-Hilbert term in the action (3.1) which does not feature in the JT gravity calculation on a fixed topology. They are implicit in the pictures. As it turns out though this series is asymptotic. This can be understood because the Weil-Petersson volumes for large g and fixed n behave asymptotically as $2g!$ the sum of which indeed does not converge [9]. To make sense of this theory we require a nonperturbative definition. Before discussing that briefly, let us contemplate how we would go about calculating correlators on higher genus topologies.

3.3.2 Correlation functions

We would like to understand the fate of correlation functions in JT gravity, first on a fixed topology but eventually as sum over topology. We will not try to go into too much detail here, nor will we attempt a rigorous general proof of the formulas. For more information the reader is referred to [3, 18, 121]. Our goal here is mainly to present some intuition about how to deal with the boundary two point function in JT gravity given its importance in the following chapters. As explained around (2.99) the boundary two point function corresponds to the path integral of a massive quantum mechanical probe particle travelling through the bulk geometry with the initial and final conditions that it starts and ends at some point infinitesimally close to the asymptotic boundary. On the disk topology and in the first order BF formalism this translates into a gravitational Wilson line stretching between the boundary points. Imagine now that the topology in the bulk is more complicated. In general we have a disk with any number of handles on it. The particle can now follow topologically different trajectories whilst travelling between the boundary points. For example it could wind around a handle:

$$Z_{1,1}(\beta_1,\beta_2,\ell) \supset \quad \beta_1$$



β_2 . (3.133)

For each fixed topology of the path the quantum particle path integral reduces to a gravitational Wilson line with the same topology as that of the path. This can be understood almost immediately when we think of the JT gravity path integral as integrating over boundary to boundary geodesic lengths L along each Wilson line as in (2.257). It is almost manifest in this formulation that the particle path integral will evaluate to $e^{-\ell L}$ when we fix the proper length L of its trajectory. For more on this see [57, 61]. Anyway. For each fixed topology of the surface Σ on which the particle travels we now have a total JT gravity amplitude that decomposes into a sum over Wilson line amplitudes for each topology. Schematically:

$$Z_{1,1}(\beta_1,\beta_2,\ell) = \sum_{\gamma} Z_{1,1}(\beta_1,\beta_2,\ell,\gamma). \tag{3.134}$$

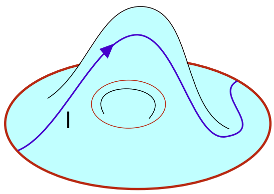
Here the topologies are one to one with the different geodesics γ between the endpoints.

More fun with the mapping class group

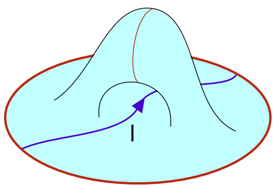
Notice now that we can identify this summing over γ as part of summing over the action of the mapping class group of Σ. Let us denote the surface obtained by cutting Σ on γ as Σ'. The topology of Σ' does not depend on the choice of γ.[16] We then have schematically:

$$\mathrm{MCG}(\Sigma) = \mathrm{M}(\gamma)\,\mathrm{MCG}(\Sigma'). \tag{3.135}$$

Imagine now that for a fixed surface we start to apply the rationale of the previous section. We choose a bunch of closed geodesics γ_i on which to cut, resulting eventually in simple amplitudes all of which we could calculate as an integral over the modular strip $\mathcal{S}$ for the moduli associated with each of the geodesics on which we cut. One is furthermore being asked to divide by the mapping class group of Σ. Notice now that the JT gravity amplitudes for each of the Wilson line topologies γ will end up giving exactly the same answer. The trick is to choose different sets of geodesics γ_i for each Wilson line trajectory. This can be done in such a way that *after* the cutting, the resulting amplitudes are manifestly identical. A simple example [18] is to consider on the one hand:

$Z_{1,1}(\beta_1, \beta_2, \ell) \subset \quad \beta_1$  $\beta_2 \quad .$ (3.136)

We imagine that as a first step in the cutting process we cut on the red line in this picture. Compare this to the first step in the cutting process of:

$Z_{1,1}(\beta_1, \beta_2, \ell) \supset \quad \beta_1$  $\beta_2 \quad .$ (3.137)

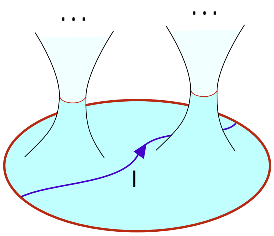
The resulting amplitudes after making this "first cut" on the red geodesic are *manifestly* identical. One can convince oneself that this holds for any topology of Σ and for any topology of γ.[17] This equality means that upon doing our calculation and writing it in a form similar to (3.127) we will end up with an overall sum over γ generating a formally infinite factor $\mathrm{M}(\gamma)$. Furthermore we are still supposed to divide by $\mathrm{MCG}(\Sigma)$. We can cancel the common factor $\mathrm{M}(\gamma)$ and only mod by $\mathrm{MCG}(\Sigma')$. This proves it is completely equivalent to either pick any fixed embedding of the Wilson line γ and only mod by the mapping class group of Σ/γ, or to sum over all embeddings γ and mod by the whole mapping class group of Σ. We believe this to be a statement that should again be intuitively clear to the reader. For a slightly more detailed proof of this fact

[16] This is actually not entirely true. Imagining it to be true just makes life easier for now. We will be more precise around (3.139).

[17] This may involve some mental gymnastics.

we refer to [18].
The point is that the procedure where we choose one embedding γ and mod by the mapping class group of Σ/γ is far easier to implement in practice. One proceeds as follows for example, for the case where we have just the single boundary circle, by choosing the following embedding γ:

$$Z_{g,1}(\beta_1, \beta_2, \ell) = \quad \beta_1$$



$$\beta_2 \quad . \tag{3.138}$$

We imagine making the "first cuts" on the red circles. The dots represent the surface Σ'. This can be either connected or disconnected. Let us now come clean. We have secretly been cutting some corners lately. When we think about it more carefully, it becomes obvious that the sum over γ and the modding by the mapping class group will actually result in a sum over *all* possible topologies of Σ' consistent with cutting a boundary to boundary line out of Σ. We could label the topologically different surfaces as Σ_j. A more precise statement of (3.135) is:

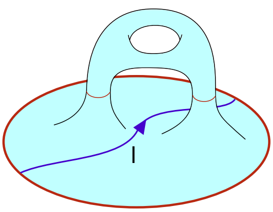
$$\mathrm{MCG}(\Sigma) = \mathrm{M}(\gamma_j)\,\mathrm{MCG}(\Sigma_j). \tag{3.139}$$

There is no sum over j implied. Furthermore we have:

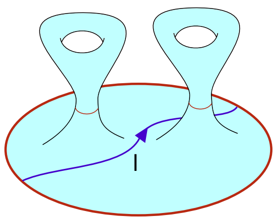
$$\mathrm{M}(\gamma) = \sum_j \mathrm{M}(\gamma_j). \tag{3.140}$$

Obviously now the Wilson line amplitudes more precisely only give an equivalent answer for embeddings γ_j which are in the same class j. The sum over all geodesics γ then reduces to a sum over all topologies Σ_j after cancelling common factors of $\mathrm{M}(\gamma_j)$ in the sum over geodesics and in the relevant mapping class group. For example we have at genus two:

$$Z_{2,1}(\beta_1, \beta_2, \ell) = \quad \beta_1$$



$$\beta_2 \quad + \quad \beta_1$$



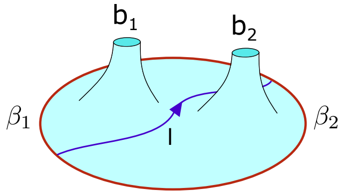
$$\beta_2 \quad . \tag{3.141}$$

We opted to leave this extra complication out of our earlier argument in the spirit of "one problem at a time". Both (3.138) and (3.141) should be understood as implying a modding by only the mapping class group of Σ_j for each fixed j. We can now proceed with the calculation of (3.138). Cutting on the red circles first and assigning to these the lengths b_1 and b_2 and then cutting the surface Σ_j into three holed spheres with length parameters a_i we find for each fixed j:

$$\int_0^\infty db_1\, b_1 \int_0^\infty db_2\, b_2\, Z_{0,3}(\beta_1, \beta_1, \ell, b_1, b_2)\, \frac{1}{\mathrm{MCG}(\Sigma_j)} \prod_i \int_0^\infty da_i\, a_i. \tag{3.142}$$

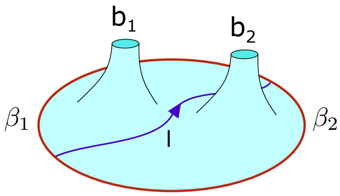
Here we introduced a shorthand notation for the following JT gravity amplitude where there is implicitly *no* modding by any mapping class group whatsoever:

$$Z_{0,3}(\beta_1, \beta_1, \ell, b_1, b_2) =$$



$$. \tag{3.143}$$

The fact that there is *no* modding by any mapping class group in this picture implies that we can calculate it naively just like we would proceed in any BF theory for a compact group when we would have a correlator on the disk with one boundary anchored Wilson line and a puncture inserted on either side of the Wilson line. One immediately uses the techniques of chapter 2 to find the answer:



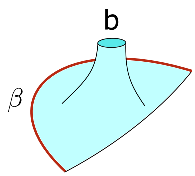
$$= \int_0^\infty dE_1\, e^{-\beta_1 E_1}\, \frac{\cos \pi b_1 \sqrt{E_1}}{\sqrt{E_1}} \int_0^\infty dE_2\, e^{-\beta_2 E_2}\, \frac{\cos \pi b_2 \sqrt{E_2}}{\sqrt{E_2}}\, |\mathcal{O}_{\ell, E_1 E_2}|^2. \tag{3.144}$$

Here we introduced the notation:

$$|\mathcal{O}_{\ell, E_1 E_2}|^2 = \frac{\Gamma(\ell \pm i\sqrt{E_1} \pm i\sqrt{E_2})}{\Gamma(2\ell)}. \tag{3.145}$$

These are the familiar vertices (2.194) associated with Wilson line endpoints as explained in chapter 2. Furthermore one recognizes in (3.144) familiar factors for disk topologies with a puncture and with a piece of fixed length boundary (2.89). This should be considered a generalization of (2.256):

$$Z_{0,2}(\beta, b, \phi) =$$



$$\phi = \int_0^\infty dk\, \cos \pi b k\, e^{\phi} K_{2ik}(e^{\phi}). \tag{3.146}$$

Integrating over ϕ with the Wilson line as in (2.245) and (2.246) then indeed results in (3.144). Consider now the final two factors in (3.142). Remember that the lengths a_i are associated solely with cutting up Σ_j into hyperbolic three holed spheres. By definition modding this collection of modular strip $\mathcal{S}$ volume integrals by the mapping class group of Σ_j results precisely in the Weil-Petersson volume of Σ_j as is the content of formula (3.130). For example in the case that the surface is connected we have:

$$\frac{1}{\mathrm{MCG}(\Sigma_c)}\prod_i \int_0^\infty da_i\, a_i = V_{g-1,2}(b_1,b_2). \tag{3.147}$$

The disconnected configurations j can be labeled with an integer h denoting the genus of one of the disconnected surfaces. In this case the final two factors in (3.142) actually should be read as the product of two terms each of which gives a Weil-Petersson volume:

$$\frac{1}{\mathrm{MCG}(\Sigma_h)}\prod_i \int_0^\infty da_i\, a_i \,\frac{1}{\mathrm{MCG}(\Sigma_{g-h})}\prod_j \int_0^\infty dc_j\, c_j = V_{h,1}(b_1)\, V_{g-h,1}(b_2). \tag{3.148}$$

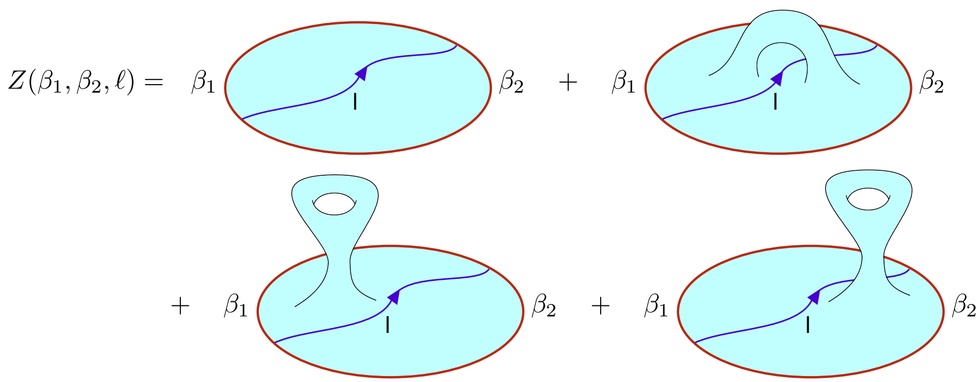
This can be combined with (3.144) and (3.142) to give a precise answer for the JT gravity boundary two point function on a surface of fixed genus g with circular boundary (3.138). It is not particularly illuminating to write down though.

Summing over topologies

In quantum gravity we would like to sum over all topologies as in (3.132):

$$Z(\beta_1,\beta_2,\ell) = \sum_{g=0}^{\infty} e^{-2gS_0}\, Z_{g,1}(\beta_1,\beta_2,\ell). \tag{3.149}$$

The powers of e^{-S_0} are due to the Einstein-Hilbert term in the action (3.1) which does not feature in the JT gravity calculation on a fixed topology. This sum turns out to have a much nicer structure as compared to the structure of each term in the sum at itself. In particular one immediately notices that every possible topology of the surfaces Σ_j with two boundaries of lengths b_1 and b_2 features exactly once in this formula. A genus expansion thus looks like:



$$+ \quad \ldots \quad . \tag{3.150}$$

We can do this sum exactly using the results of the previous paragraph. Let us suggestively write the answer as:

$$Z(\beta_1, \beta_2, \ell) = e^{-S_0} \int_0^\infty dE_1\, e^{-\beta_{E1}} \int_0^\infty dE_2\, e^{-\beta_2 E_2}\, \rho(E_1, E_2)\, |\mathcal{O}_{\ell, E_1 E_2}|^2. \tag{3.151}$$

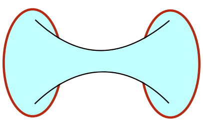
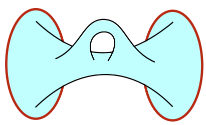
Here we have defined:

$$\begin{aligned} \rho(E_1, E_2) = \int_0^\infty db_1\, b_1 \frac{\cos \pi b_1 \sqrt{E_1}}{\sqrt{E_1}} \int_0^\infty db_2\, b_2 \frac{\cos \pi b_2 \sqrt{E_2}}{\sqrt{E_2}} \\ \sum_{g=0}^\infty e^{-S_0} e^{-2gS_0} V_{g,2}(b_1, b_2) \\ + \sum_{g_1=0}^\infty e^{-2g_1 S_0} V_{g_1,1}(b_1) \sum_{g_2=0}^\infty e^{-2g_2 S_0} V_{g_2,1}(b_2). \end{aligned} \tag{3.152}$$

Notice the powers of e^{S_0} here. The first term represents the connected contribution. This starts with the second contribution on the first line of (3.150). The second term factorizes into a genus expansion for every topological disk region in the lowest order diagram. The other three terms pictured in (3.150) are of this type, with no Riemann surface connecting one side of the Wilson line to the other. Note that we have been slightly imprecise regarding the terms where either of the values g_1 or g_2 vanishes. In that case one should not use the Weil-Petersson volumes at all since we now have that the topology of Σ on one side of the Wilson line is trivial. One then just uses the usual formula (2.256) for these special cases. Formula (3.152) might not look like a particularly elegant expression, but really it is. To appreciate that fact, consider for a short while the JT gravity amplitude with two boundaries where we sum over Riemann surfaces which stretch between the two boundaries (3.3):

$$\begin{aligned} Z(\beta_1, \beta_2) &= \sum_{g=1}^\infty e^{-S_0} e^{-2gS_0} Z_{g,2}(\beta_1, \beta_2) \\ &\quad + \sum_{g_1=1}^\infty e^{-2g_1} Z_{g_1,1}(\beta_1) \sum_{g_2=1}^\infty e^{-2g_2} Z_{g_2,1}(\beta_2) \\ &= Z(\beta_1, \beta_2)_{\text{conn}} + Z(\beta_1) Z(\beta_2). \end{aligned} \tag{3.153}$$

The first term represent the connected geometries with a topological expansion of the type:

$Z(\beta_1, \beta_2)_{\text{conn}} = \beta_1$  $\beta_2 \; + \; \beta_1$  $\beta_2 + \ldots$ (3.154)

The second term in (3.153) consists of factorized contributions where each disk shaped region can develop handles as in (3.132). By now the reader should be familiar with how to compute each contribution to this path integral. Going through the gears one eventually finds (3.3):

$$Z(\beta_1,\beta_2)=\int_0^\infty dE_1\,e^{-\beta_1 E_1}\int_0^\infty dE_2\,e^{-\beta_2 E_2}\,\rho(E_1,E_2). \tag{3.155}$$

Here the integration kernel is precisely the same (3.152) as for the two point function. This is a strong result in the sense that if we were to obtain an exact answer for this two boundary correlator as sum over geometries, then we would also immediately have found an exact answer for the two point function. One just takes the double inverse Laplace transform to obtain $\rho(E_1,E_2)$ and inserts this in (3.151).

Other correlators

We will not redo this derivation for a more generic correlator such as a boundary four point function. This would be a bookkeeping exercise, rather than a calculation which would actually teach us something new. It should not be too hard to prove that the following holds in general. Suppose we have a bunch of boundary operators on a circle. Connect these via Wilson lines in the circle like we would do if we would be making a poor man's tennis racket. The correlator in JT gravity as sum over Riemann surfaces is obtained by summing over all possible geometries that end on this poor man's tennis racket. Given this fact, it is obvious that it's always possible to rewrite theses sums over Riemann surfaces in terms of some more generic function $\rho(E_1\dots E_n)$ which is the multiple inverse Laplace transform of the multi boundary correlator $Z(\beta_1\dots\beta_n)$ in JT gravity as sum over topologies. All we need to completely solve the model are then the universal vertices (3.145), the group theoretic kernel (2.192) that shows up at the crossing of every two "strings" of the tennis racket, and an exact answer for $\rho(E_1\dots E_n)$. The problem with obtaining the latter is as mentioned around (3.132) that the associated genus expansion is an asymptotic series. We require a nonperturbative definition of JT gravity to make sense of this. This was provided in [9] to which we turn next.

Simplification at late time and low temperature

Before doing so let us make a funny remark which enables us to make even more analytically tractable claims on the behavior of the two point function in a certain parametric regime. Imagine that we consider (3.151) with $\beta_1\gg 1$ and $\beta_2\gg 1$. Upon going to Lorentzian signature this corresponds to considering late times $t\gg 1$ and low temperatures $\beta\gg 1$. Late time physics in holography is a very interesting subject, not in the least due to Maldacena's formulation of the information paradox [14]. We will have much more to say about this "paradox" in chapters 4 and 5. Anyway. In this parametric regime the exponentials want to force the integrals to be dominated by the regions where $E_1\ll 1$ and $E_2\ll 1$. This is not suppressed by the shape of $\rho(E_1,E_2)$. One checks that all contributions behave as power laws for low energies. In fact all higher genus contributions behave as power laws but with negative powers, so they would rather favor this dominance at low energies. This isn't suppressed by the vertices (3.145) either. Now, because the integral is dominated by low energies we can use the method of Laplace to Taylor expand the vertices at low energies without changing the answer:

$$|\mathcal{O}_{\ell,E_1E_2}|^2 \approx \frac{\Gamma(\ell)}{\Gamma(2\ell)}, \qquad E_1, E_2 \ll 1. \tag{3.156}$$

We can pull this constant out of the integral to find:

$$Z(\beta_1,\beta_2,\ell) \approx e^{-S_0}\frac{\Gamma(\ell)}{\Gamma(2\ell)}\,Z(\beta_1,\beta_2). \tag{3.157}$$

This means that effectively the Wilson line pinches of the surfaces on which it travels. Indeed, the genus expansion in (3.150) then becomes manifestly identical to that in (3.153). The power of e^{-S_0} can be understood due to the different Euler characteristic associated with the contribution on either side. This is powerful in the sense that it means we can focus all our attention on $Z(\beta_1,\beta_2)$ if our only purpose is to understand the late time features of the holographic two point function. This quantity is known as the spectral form factors in the quantum chaos literature, see references in [7]. The spectral form factor was indeed introduced with the hope that it would be a sensible probe of late time holographic correlators.

3.3.3 Random matrices

At this point one might argue we have not really made a big improvement by summing over topologies. We went from a well defined theory with finite answers to a theory with ill defined answers. Indeed, as it stands, essentially each of the sums over genus we would be interested in computing, is an asymptotic series. This calls for a nonperturbative definition of JT gravity that goes beyond the genus expansion. For string theory exists a famous conjecture that M theory has a nonperturbative definition as a matrix model [132]. As it turns out we can also give a matrix model nonperturbative definition of JT gravity.

For background on the type of matrix models discussed in this work see [133, 119]. The partition function of an ensemble of $L\times L$ Hermitian matrices with bare potential $V(M)$ is defined as:

$$\mathcal{Z}_L = \int DM e^{-L\,\mathrm{Tr}\,V(M)}. \tag{3.158}$$

A more convenient way to write this is in terms of the eigenvalues λ_i of the matrices:

$$\mathcal{Z}_L = \int_{\mathcal{C}}\prod_{i=1}^{L}\left(d\lambda_i\,e^{-LV(\lambda_i)}\right)\Delta(\lambda_1,\dots,\lambda_L), \qquad \Delta(\lambda_1,\dots,\lambda_L) = \prod_{i<j}^{L}(\lambda_i-\lambda_j)^2. \tag{3.159}$$

Here $\Delta(\lambda_1,\dots,\lambda_L)$ is the Vandermonde determinant, accounting for eigenvalue repulsion. Typical observables in the matrix model are products of the spectral density $\rho(E)$

or the macroscopic loop operator $Z(\beta)$:

$$\rho(E) = \sum_{i=1}^{L} \delta(E - \lambda_i), \qquad Z(\beta) = \sum_{i=1}^{L} e^{-\beta\lambda_i}. \tag{3.160}$$

Correlators are calculated as ensemble averages, for example:

$$\langle Z(\beta) \rangle = \frac{1}{\mathcal{Z}_L} \int d\lambda_1 \ldots e^{-LV(\lambda_1\ldots)} \Delta(\lambda_1 \ldots) \sum_{i=1}^{L} e^{-\beta\lambda_i}. \tag{3.161}$$

The two-level spectral density is defined as $\rho(E_1, E_2) = \rho(E_1)\rho(E_2)$. Ensemble averaging leads to correlation as connected contributions, for example for the two-loop operator defined as $Z(\beta_1, \beta_2) = Z(\beta_1)Z(\beta_2)$:

$$\langle Z(\beta_1, \beta_2) \rangle = \langle Z(\beta_1) \rangle \langle Z(\beta_2) \rangle + \langle Z(\beta_1, \beta_2) \rangle_{\text{conn}}. \tag{3.162}$$

The point is that these types of matrix integral correlators have a Taylor expansion in $1/L$. This is essentially a genus expansion when we draw the associated Feynman diagrams. There furthermore exists a recursion relation between contributions to some correlator at different genus. One can now prove [134, 135] that for a particular matrix integral which we will discuss in more detail in chapter 4, the recursion relations of the matrix integral exactly match Mirzakhani's recursion relation for Weil-Petersson formulas (3.122). Combined with the fact that Weil-Petersson volumes are basically the correlators of JT gravity (3.123) this led the authors of [9] to find that the genus expansion of for example $Z(\beta)$ and $Z(\beta_1, \beta_2)$ in JT gravity is exactly identical the the genus expansion of the corresponding matrix integral observables such as $\langle Z(\beta) \rangle$ and $\langle Z(\beta_1, \beta_2) \rangle$. For the matrix integral observables though we *do* get a finite answer. For one there is a rich literature on nonperturbative effects in matrix integrals. Equally important is that that in certain regimes such as at low energies, the matrix integral of interest to JT gravity is exactly solvable. This is important because low energies is where the perturbative contributions became important and there was dire need for a nonperturbative answer. Via the matrix integral we get such an answer. So that is the lesson here. The sum over topologies in JT gravity is the perturbative expansion of a matrix integral. We are then urged to give a nonperturbative definition of JT gravity as precisely that matrix integral. This gives us powerful tools to calculate important regimes in for example $\rho(E_1)$ and $\rho(E_1, E_2)$ as we'll discuss in detail in chapter 4. Perhaps equally important as the technical tools provided by the identification of JT gravity as a matrix integral, are the conceptual problems this identification both poses and solves. Let us turn to this next in the concluding remarks.

3.4 Concluding remarks

We present two concluding remarks. The first aims to point out why including baby universes is a tremendous improvement as compared to the topologically trivial version of

JT gravity. The second concerns comments about conceptual difficulties associated with Hilbert spaces in JT gravity as well as potential avenues to move forward on this matter.

We should always be including baby universes

The conclusion of the last section is that JT gravity as a sum over all surfaces that end on certain boundary conditions has a nonperturbative definition as a matrix integral. This is huge for the following reason. As explained in the introduction black holes are quantum chaotic systems. Furthermore certain averaged properties of quantum chaotic systems have essentially by definition [17] an effective description in random matrix theory. This means suitable averages of certain observables in *any* realistic theory of quantum gravity are expected to have an effective description as random matrix theory. In JT gravity as sum over geometries this could not be more explicit. The entire model *is* a random matrix theory. In this sense JT gravity has been promoted from arguably a fairly poor model of quantum gravity with not so much expected universal lessons on quantum gravity in sight to an incredibly powerful model to probe the effects of random matrix universality on bulk quantum gravity.[18] It may be too early to call, but the general lesson here seems to be that we might need to include a sum over topologies in general in quantum gravity if we want to see the underlying random matrix statistics. Certainly there is evidence this is the case for AdS_3 gravity [45]. You might not care that much about random matrix statistics. That is your good right. But if you care about quantum gravity at all then you should definitely care about discreteness. We expect on general grounds that any realistic model of quantum gravity comes with a discrete set of microstates such that it is holographically dual to some finite dimensional quantum mechanical system [14]. The latter evolves in a unitary manner. We would not go as far as stating that discreteness is one to one with a unitary theory, but it is clear beyond doubt that the two concepts are linked. The point is now that if you have a discrete quantum chaotic system then by definition you will have random matrix statistics. So in a sense even if you do not care about random matrix statistics, you *should* be slightly worried when you do not see it in your model of quantum gravity. Actually in this context of unitarity there is much more general evidence that we *need* to include a sum over topologies in bulk quantum gravity. That is unless if you prefer a model of quantum gravity in which black holes do not evaporate in a unitary manner. Indeed, including this sum in an appropriate manner in *all* Euclidean calculation can be shown quite generally to explain the Page curve of black hole evaporation from the bulk gravitational point of view [10, 11].
This raises the following puzzle though. Clearly this version of JT gravity in which we sum over topologies is a much more realistic model of quantum gravity than the version in which we do not, at least for the questions we are trying to answer. In one version we have random matrix statistics which can be interpreted as traces of discreteness whilst in the other version we do not. The problem is though that certainly this is not a unitary theory. Rather it is essentially by definition an *ensemble average* of unitary theories.

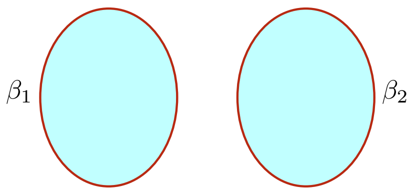
[18] "Fairly poor" obviously is a strongly biased statement affected by the questions we are interested in answering. Others may find the Schwarzian model very rich if they aim to probe other questions.

The ensemble averaging does not preserve unitarity. We would like to understand the gravitational interpretation of this ensemble averaging. More importantly once we understand it, we would like then to *not* average in our gravitational theory. The result ought to be a version of bulk quantum gravity with a discrete quantum chaotic dual. This is what we expect generic models of quantum gravity to look like. Therefore we might hope for ever more fundamental and universal lessons if we are able to realize a JT gravity version of such a discrete theory of quantum gravity. This is what we will aim for in the next chapter. To remove any possible confusion, let us remind the reader that even a single discrete quantum chaotic system has random matrix statistics although there is nothing "random" about the system itself. The point is that we have a large number of eigenvalues and we can check statistical properties of those eigenvalues, for example their average spacing. This is where "random" matrix statistics comes in [119, 17].

Comments on the Hilbert space of JT gravity

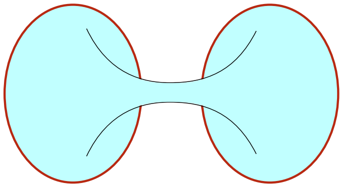
Large diffeomorphism invariance, as we learned in the main text, spoils any straightforward Hilbert space interpretation of JT gravity amplitudes that one might hope to get from the BF picture. The situation is already dramatic on a single fixed topology. Modular invariance is already difficult to take into account when we try to decompose the amplitude in smaller amplitudes. Let alone we could find a sensible result simply be evolving Cauchy slices over the surface. The problem is that there is no straightforward modular invariant way of specifying a set of Cauchy slices. This becomes even more dramatic when we sum over typologies. Now the would-be Cauchy slice does not even have a fixed topology. It is clear that the sum over topologies calls for a fundamentally "new" way of thinking about the Hilbert space in a theory of quantum gravity.

The way to proceed is probably by thinking about states as defining a set of boundary conditions. This would be conform with how we think about states in a topological field theory on some surface. The difference with the topological field theory would be that in quantum gravity we want to sum over all geometries ending on this surface. We can then define inner products on all states a la Witten [136].[19] Schematically $\langle \mathcal{B}_1 | \mathcal{B}_2 \rangle$ is calculated by summing over all geometries that end on the union of the boundaries with boundary conditions $\mathcal{B}_1$ in the infinite past and $\mathcal{B}_2$ in the infinite future. For example we could imagine in the dS_2 version of JT gravity with the time flowing to the right [63] to compute the inner product:

$\langle \beta_1 | \beta_2 \rangle =$ 

---

[19]For recent discussion on such inner products see also [63, 19].

$+ \quad \beta_1$  $\beta_2 \quad + \dots \quad .$ (3.163)

If the resulting inner product space is positive semi definite, we could build a Hilbert space by modding out by null states. It is not difficult to construct such a Hilbert space for JT gravity combining the logic of [4, 19]. We will not do so in detail here but we can't resist commenting on it in the following chapters.

Let us here instead focus on a rather different puzzle. Suppose we are given a discrete quantum chaotic system with a gravitational dual. Suppose the states of this system are labeled by only energies so we would have $|\lambda_i\rangle$. There is no sense in which holography demands that these states are also the Hilbert space of our bulk model of quantum gravity. What is demanded rather is something along the lines of (2.31). There needs to be a gravitational calculation with certain boundary conditions such that the answer of that calculation is exactly identical to the answer of a boundary calculation. For example say we are calculating the expectation value of some operator $\mathcal{O}$ in the dual quantum theory, which we imagine includes the density matrix:

$$\mathrm{Tr}(\mathcal{O}) = \sum_{i=1}^{L} \langle\lambda_i|\,\mathcal{O}\,|\lambda_i\rangle\,. \tag{3.164}$$

It is a realistic possibility, strengthened by our current understanding of JT gravity and AdS_3 gravity, that in some sense all the states $|\lambda_i\rangle$ map to some complicated boundary condition for the gravitational calculation:

$$|\mathcal{B}\rangle = |\lambda_1 \dots \lambda_L\rangle \tag{3.165}$$

Let us denote the no boundary state by $|\mathrm{HH}\rangle$. One could then imagine there is some operator insertion $\mathcal{O}$ in the bulk gravitational theory such that the following holds true:

$$\langle\mathrm{HH}|\,\mathcal{O}\,|\lambda_1 \dots \lambda_L\rangle = \sum_{i=1}^{L} \langle\lambda_i|\,\mathcal{O}\,|\lambda_i\rangle\,. \tag{3.166}$$

If this would be the case, then the Hilbert space of the gravitational theory would essentially be one to one with the set of all possible boundary conditions on all possible topologies. This is nothing like the Hilbert space of the dual discrete system. That claim in itself is by no means in contradiction with the holographic duality. Let us emphasize again that there is no a priori reason why those Hilbert spaces *need* to be the same. We feel it is an important open question to understand the Hilbert space of JT gravity. The work of [4, 19] shows how this could naturally be achieved in a cosmological dS_2 context where we have future and past Cauchy slices that look like a set of closed boundaries. Of course it would also work fine in a Euclidean AdS_2 context where we can choose our Cauchy slices any way we want. It is not clear though how any of this would translate to a Lorentzian AdS_2 context. One of the problems is that the Witten way [136] of constructing a Hilbert space in quantum gravity requires two "Cauchy slices" at infinity on which to define states and on which we would like a dual observer to be able to act. In dS_2 this is natural in the sense that the dual observer computes S matrix elements. The typical Lorentzian AdS_2 setup does not look like providing such Cauchy slices on which the dual observer acts. Potentially we could define an initial state via some Euclidean path integral preparation. We could even define the Euclidean evolution to come with the spontaneous nucleation and annihilation of pairs of wiggly boundaries, on which one might imagine an asymptotic observer might act. Still it's totally unclear to us at the moment how the eventual dictionary would work. For example on which of the infinite number of boundaries would the dual observer act in the sense of the extrapolate dictionary? It is probably advisable if one is interested in finding an answer to this question, to start by understanding the physics in the dS_2 context in detail first.

4 Eigenbranes and discreteness

In this chapter we present a work by the author in collaboration with Thomas Mertens and Henri Verschelde [4] on signatures of discreteness in JT gravity. In particular we specify a bulk theory of quantum gravity with a discrete spectrum. The following discussion builds heavily on the exact quantization of JT gravity on the disk discussed in chapter 2. Even more so it builds on the exact amplitudes of JT gravity on higher genus Riemann surfaces discussed in chapter 3 and on the nonperturbative completion of the associated genus expansion as a matrix integral [9].

4.1 Introduction

In finite-volume holography, there is a deep tension between discreteness of the boundary theory, and quasi-normal decay in the holographic bulk. This tension is one version of the information paradox, due to Maldacena [14]. A bulk quantum gravity explanation for this behavior remains to some degree an open question, though important steps have been taken in [7, 8, 9]. More in general it is not yet obvious how semiclassical gravitational arguments are to be augmented in a way that resolves Hawking's information paradox. A couple of years ago we would have in fact said it is totally unclear how that happens. Recently though there has been very exciting progress on a bulk quantum gravitational interpretation of Hawking's information paradox [137, 138, 139, 140, 10, 11, 19]. This addresses Hawking's information paradox in terms of the Euclidean path integral. Neither of these papers works in Lorentzian signature though. It is not obvious how their arguments are to be modified to resolve Hawking's information paradox completely, which is inherently to be done in Lorentzian signature.
In this chapter we will address Maldacena's version of an information paradox within pure JT gravity and via the Euclidean path integral. In recent years, most activity concerning JT gravity has been in a version of the theory that is dual to Schwarzian quantum mechanics, which comes with a topologically trivial bulk. The Schwarzian has a continuous spectrum as we emphasized in chapter 2, so this version of JT gravity has fairly little to do with holographic discreteness.
More recently though there has been interest in a version of JT gravity that includes a sum over higher genus topologies in the bulk [9]. We've discussed this partially in chapter 3. Such higher genus contributions represent spacetimes with Euclidean wormholes connecting them, which if sliced appropriately, look like baby universes splitting off from

the parent universes. This model can be defined non-perturbatively as a double-scaled matrix integral. It is therefore an ensemble average over discrete systems. Due to the averaging, the spectrum remains continuous. Nonetheless the averaging does not eradicate all traces of discreteness. In particular late-time holographic correlators do not generically decay to zero in this version of JT gravity [9, 3, 18, 121].
Motivated by these developments, we want to point out that a further alternative version of JT gravity which resembles the behavior of a single discrete holographic system can be defined. In particular it captures the behavior of the spectral form factor of a discrete system for all times, including the late-time erratic oscillations [7]. The new feature is to include a set of fixed energy boundaries in the gravitational path integral on which Riemann surfaces can end, to be distinguished from the asymptotic boundaries. Each energy label corresponds to the energy of a state in the discrete system we aim to emulate. These boundaries, which we will refer to as "eigenbranes", are related to unmarked FZZT boundaries and correspond to fixed eigenvalues in the matrix ensemble of [9].

This chapter is structured as follows.
In **section 4.1** we first introduce a simple probe of discreteness. Using the freedom to choose the contour of the path integral over metrics in quantum gravity, we then discuss three possible definitions of JT gravity. One has only disk topologies, one has all topologies and is completed as an ensemble of Hermitian random matrices. The final one includes eigenbranes and is dual to an ensemble of Hermitian random matrices with a large number of eigenvalues fixed.
In **section 4.2** we review and discuss calculation techniques for spectral densities in matrix integrals, or multi-boundary correlators in gravity. Some of the more technical material is exiled to the supplementary chapter 3.
In **section 4.3** we consider an ensemble of random matrices with certain eigenvalues kept fixed, and derive its perturbative interpretation as a gravity path integral with surfaces ending also on a number of fixed-energy boundaries. We prove the extent to which the resulting version of JT gravity resembles a discrete system. In particular, we show that the spectrum essentially reduces to a series of delta functions, and we show how different asymptotic boundaries essentially disconnect. The key to this is to include the gravitational equivalent of the off-diagonal terms in the discrete correlators such as $Z(\beta_1)Z(\beta_2)$ as suggested in [10]. Where diagonal terms $Z(\beta_1+\beta_2)$ correspond to Euclidean wormholes connecting the different asymptotic boundaries, and are captured by the averaged matrix integral description of [9], the off-diagonal terms correspond to spacetimes where the asymptotic boundaries are connected to the eigenbranes via Euclidean wormholes and can only be captured by fixing eigenvalues in the matrix ensemble. Summing over all contributions we recover the trivial factorizing structure of discrete correlators such as $Z(\beta_1)Z(\beta_2)$ from a bulk gravity calculation.
In **section 4.4** we briefly discuss a gravitational interpretation of the delta functions as due to boundary mergers [141].

4.1.1 Diagnosing discreteness

Consider a particular discrete maximally chaotic quantum mechanical system with an L-dimensional discrete Hilbert space:

$$\rho(E) = \sum_{i=1}^{L} \delta(E - \lambda_i). \tag{4.1}$$

We choose the model in such a way that its coarse-grained spectrum matches the JT spectrum on the disk up to some large energy cutoff $\Lambda \gg 1$ [27, 22, 7, 50]:

$$\rho_{\text{coarse}}(E) = \frac{e^{S_0}}{4\pi^2} \sinh 2\pi\sqrt{E} = \rho_0(E), \qquad E < \Lambda. \tag{4.2}$$

The system is taken to be quantum chaotic, because we aim for it to be dual to quantum black holes [7, 142, 143, 144]. Specifying further to a system without time-reversal invariance, this implies its local level statistics should be those of a random matrix taken from an appropriately rescaled Gaussian unitary ensemble (GUE) [119]. At low energies $E < \Lambda$ or late times we expect that this discrete quantum mechanical system has an effective pure JT gravity bulk description.[1] In the remainder of this work we collect evidence in favor of this. More particularly we will study an ensemble average of discrete systems (4.1), but with only the UV degrees of freedom considered as random variables. The resulting part-discrete-part-continuous system is found to have a pure JT gravity dual at low energies $E < \Lambda$. We learn that the continuum density is negligible in the region $E < \Lambda$ due to eigenvalue repulsion in the associated matrix ensemble. Therefore by construction we end up with an effective bulk theory of quantum gravity with a discrete spectrum.

We will be interested in thermal correlation functions, for example the two-sided two-point function:

$$\text{Tr}\big[e^{-\beta H}\mathcal{O}(t)\, e^{-\beta H}\mathcal{O}(0)\big] = \int_{-\infty}^{+\infty} dE_1\, \rho(E_1)\, e^{-(\beta+it)E_1} \int_{-\infty}^{+\infty} dE_2\, \rho(E_2)\, e^{-(\beta-it)E_2}\, |\mathcal{O}_{E_1E_2}|^2. \tag{4.3}$$

At early times, the Fourier transform is insensitive to the fine structure of the spectrum, and coarse-grains. As a consequence, the thermal two-point function will be well approximated by a JT disk calculation. The late-time Fourier transform on the other hand is highly sensitive to the fine structure. Therefore late-time correlators are in general suitable probes of discreteness [14]. In particular we will be interested in the simplest such probe, a local version of the two-point function where we integrate both E_1 and E_2 over only a narrow energy interval bin(E). We take $1/\rho_0(E) \ll |\text{bin}(E)| \ll 1$. This is small enough so that $\rho_0(E)$ is approximately constant, and large enough so that it

[1]The late-time behavior of chaotic systems was recently studied in [121], where it was found that late-time correlation functions factorize into a purely spectral quantity governing the time-dependence, and an operator-dependent prefactor. This spectral quantity is probe-independent and expected to have a pure gravity bulk description.

contains a large number of eigenvalues. Because the system (4.1) is quantum chaotic, we know from the eigenvalue thermalization hypothesis that the vertices $|\mathcal{O}_{E_1E_2}|^2$ are also relatively varying in this bin [15, 16, 7, 18]. Therefore they essentially “come out” of the integral over the bin:

$$|\mathcal{O}_{EE}|^2 \int_{\text{bin}(E)} dE_1\, \rho(E_1)\, e^{-(\beta+it)E_1} \int_{\text{bin}(E)} dE_2\, \rho(E_2)\, e^{-(\beta-it)E_2}. \tag{4.4}$$

We are interested only in the time dependence of this object, so we can remove the constant matrix element and take $\beta = 0$. We are left with a local version of the spectral form factor [8]:

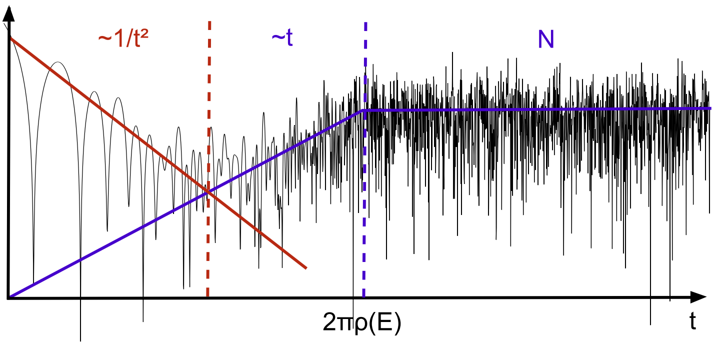
$$S_E(t) = \int_{\text{bin}(E)} dE_1\, \rho(E_1)\, e^{-itE_1} \int_{\text{bin}(E)} dE_2\, \rho(E_2)\, e^{itE_2}. \tag{4.5}$$

Labeling the eigenvalues of (4.1) within bin(E) as $\lambda_1 \ldots \lambda_N$, with $1 \ll N \ll L$, we get:

$$S_E(t) = \sum_{i,j=1}^{N} \cos t(\lambda_i - \lambda_j) = N + \sum_{i\neq j}^{N} \cos t(\lambda_i - \lambda_j). \tag{4.6}$$

A log-log plot for a representative sample gives:[2]

$$S_E(t) = \tag{4.7}$$



As compared to the usual spectral form factor, there are additional low-frequency oscillations, but the general shape is the same [7, 145]. In particular it has the same ramp and plateau structure which includes erratic oscillations and an “average” plateau height of N. We will reproduce these erratic oscillations via a bulk JT gravity calculation in section **??**.

4.1.2 Models of JT gravity

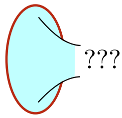
For convenience let us remind the reader of some of the properties of JT gravity discussed in chapter 2 and chapter 3. The action of the dilaton gravity model is:

$$S[g,\Phi] = -S_0\chi - \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\Phi(R+2) - \int_{\partial\mathcal{M}} dt\sqrt{h}\Phi(K-1). \tag{4.8}$$

[2]We took 128 consecutive eigenvalues of one large matrix drawn from a GUE far from the edge.

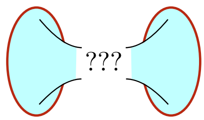
The Euler character χ comes from the Einstein-Hilbert term in 2d.[3] The quantity S_0 corresponds to the extremal entropy, and is an external free parameter from the 2d gravity point of view. As we learned in chapters 2 and 3, integrating out Φ localizes the metrics g on hyperbolic Riemann surfaces, or patches of the Poincaré disk with asymptotically NAdS_2 boundary conditions. This boils down to fixing the total length of the asymptotic boundary to β/ϵ and the boundary value of the dilaton to $1/2\epsilon$ [21, 22, 23]. Because of the localization we are essentially counting hyperbolic Riemann surfaces, so specifying the path integration space of g is equivalent to specifying which surfaces to count. For the JT gravity partition function, which corresponds to the insertion of a single holographic boundary in the path integral, we have schematically:

$$Z(\beta) = \int dE\, e^{-\beta E}\, \rho(E) = \beta \tag{4.9}$$



The spectral form factor is related in JT gravity to a correlation function $Z(\beta_1, \beta_2)$ with two asymptotic boundaries of respective lengths β_1/ϵ and β_2/ϵ:

$$Z(\beta_1, \beta_2) = \int dE_1\, e^{-\beta_1 E_1} \int dE_2\, e^{-\beta_2 E_2}\, \rho(E_1, E_2) = \beta_1 \quad \cdots \quad \beta_2 \tag{4.10}$$



From this, one calculates the local spectral form factor (4.5) in gravity as:

$$S_E(t) = \int_{\text{bin(E)}} dE_1\, e^{itE_1} \int_{\text{bin(E)}} dE_2\, e^{-itE_2} \rho(E_1, E_2). \tag{4.11}$$

In the remainder of this section we highlight how different definitions of ??? can lead to structurally very different theories, using the spectral form factor (4.11) as probe. The different models we will discuss can be specified by the integration space in the path integral over metrics:

$$\int_{\text{disks}} [\mathcal{D}g]\,(\dots) \quad , \qquad \int_{\text{all}\ \chi} [\mathcal{D}g]\,(\dots) \quad , \qquad \int_{\lambda_1 \dots \lambda_n\ \text{all}\ \chi} [\mathcal{D}g]\,(\dots) \quad . \tag{4.12}$$

In particular, we want to point out that the last definition which takes the energies $\lambda_1 \dots \lambda_n$ as input, resembles the discrete system with spectrum (4.1).

Version 1. Euclidean disks

The simplest definition is to restrict ??? to Riemann surfaces which are topologically disks. This was discussed in chapter 2. The spectrum and correlation functions of JT gravity on the disk have been extensively studied in recent years, resulting in several

[3] For a 2d manifold of genus g with b boundaries we have $\chi = 2 - 2g - b$.

complementary perspectives [48, 49, 43, 44, 100, 1, 56, 57, 61]. The spectrum of this model was found to be continuous [27, 22, 7, 50]:

$$\rho_0(E) = \frac{e^{S_0}}{4\pi^2} \sinh 2\pi\sqrt{E}. \tag{4.13}$$

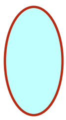
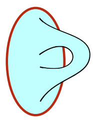
Unlike in other chapters we will here follow this more precise normalization or $\rho_0(E)$. For disk topologies, the gravitational spectral form factor factorizes because $\rho(E_1, E_2)$ factorizes into $\rho_0(E_1)\rho_0(E_2)$. We find:

$$S_E(t) = \rho_0(E)^2 \int_{\text{bin(E)}} dE_1\, e^{itE_1} \int_{\text{bin(E)}} dE_2\, e^{-itE_2} = \frac{4\rho_0(E)^2}{t^2} \sin^2 \frac{\text{bin}(E)}{2} t. \tag{4.14}$$

This dependence correctly captures the part of the curve (4.7) before the dotted red line, including the relatively slow oscillations. At later times, the Fourier transform is able to distinguish the coarse-grained disk spectrum (4.13) from the discrete spectrum (4.1).

Version 2. Baby universes and random matrices

A second possible definition of JT gravity is to allow for Riemann surfaces of arbitrary genus to end on the asymptotic boundaries. This version was introduced and discussed in [9], see also chapter 3.[4] The JT gravity partition function now has a genus expansion:

$$Z_{\text{JT}}(\beta) = \int dE\, e^{-\beta E} \rho(E) = \beta$$  $+ \; \beta$  $+ \ldots$ (4.15)

The same is true for the spectral density, and all other observables:

$$\langle \rho(E) \rangle = \rho_0(E) + \sum_{g=1}^{\infty} e^{-2gS_0} \rho_g(E). \tag{4.16}$$

We will refer to (4.15) as the “perturbative” definition of JT gravity.[5] It is very feasible to calculate each term in this series, as we briefly review in section 4.2. See also chapter 3. The resulting perturbative series turns out to be asymptotic, and hence requires a non-perturbative definition. We can define JT gravity non-perturbatively as a double-scaled matrix integral with genus zero spectral density (4.13) [9].[6] In the weakly coupled regime $e^{S_0} \gg 1$, it turns out that one can essentially neglect all perturbative contributions and

[4] Aspects of JT gravity on higher genus Riemann surfaces were also discussed in [2, 3, 59, 62, 65, 18].

[5] It is perturbative in the string coupling e^{-S_0} but non-perturbative in the Newton constant $G \sim 1/S_0$.

[6] see also [146, 120].

that the leading correction is nonperturbative in the string coupling:[7]

$$\langle\rho(E)\rangle = \rho_0(E) + \rho_{\text{nonp}}(E), \qquad \rho_{\text{nonp}}(E) = -\frac{1}{4\pi E}\cos\left(2\pi\int_0^E dM\rho_0(M)\right). \quad (4.17)$$

Such oscillatory non-perturbative contributions in general are not required to have, and in most cases indeed do not have a geometrical interpretation as counting Riemann surfaces.[8]

We remind the reader that the partition function of an ensemble of $L\times L$ Hermitian matrices with bare potential $V(M)$ is defined as:

$$\mathcal{Z}_L = \int DMe^{-L\,\text{Tr}\,V(M)}. \quad (4.18)$$

A more convenient way to write this is in terms of the eigenvalues λ_i of the matrices:[9]

$$\mathcal{Z}_L = \int_{\mathcal{C}}\prod_{i=1}^{L}\left(d\lambda_i\, e^{-LV(\lambda_i)}\right)\Delta(\lambda_1,\dots,\lambda_L), \qquad \Delta(\lambda_1,\dots,\lambda_L) = \prod_{i<j}^{L}(\lambda_i-\lambda_j)^2. \quad (4.19)$$

Here $\Delta(\lambda_1,\dots,\lambda_L)$ is the Vandermonde determinant, accounting for eigenvalue repulsion. An intuitive way to think about such matrix integrals is as the steady state of the Brownian motion of L charged particles in an external potential $V(x)$, a so-called Dyson gas [147, 148, 149, 150, 151, 152]. The Vandermonde determinant then represents the electrostatic repulsion. Typical observables in the matrix model are products of the spectral density $\rho(E)$ or the "macroscopic loop" operator $Z(\beta)$:

$$\rho(E) = \sum_{i=1}^{L}\delta(E-\lambda_i), \qquad Z(\beta) = \sum_{i=1}^{L} e^{-\beta\lambda_i}. \quad (4.20)$$

Correlators are calculated as ensemble averages, for example:[10]

$$\langle Z(\beta)\rangle = \frac{1}{\mathcal{Z}_L}\int d\lambda_1\dots e^{-LV(\lambda_1\dots)}\Delta(\lambda_1\dots)\sum_{i=1}^{L} e^{-\beta\lambda_i}. \quad (4.21)$$

[7]This is only true when $E \gg e^{-2S_0/3}$. We will operate under this assumption throughout the main parts of this chapter. Otherwise, we have to resort to an exact analysis of the Airy model, as we do in the supplementary chapter 4.5.1.

[8]The nonperturbative contribution in the forbidden region $E<0$ has a leading order interpretation as counting Riemann surfaces that end on the appropriate boundaries, stretching between asymptotic boundaries and ZZ branes in the bulk [9]. These are to be distinguished from the FZZT branes discussed in the remainder of this chapter. Roughly speaking a ZZ brane corresponds to an FZZT brane pair $\psi^2(E)$ evaluated on its "saddle point" in the forbidden region $E=-1/4$.

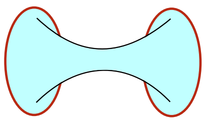
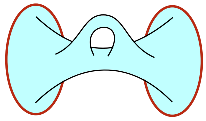
[9]Any contour $\mathcal{C}$ represents a definition of a matrix model. In case of JT gravity, for stability reasons the contour cannot be chosen to extend along the real energy axis all the way up to $-\infty$ [9]. The part of the contour that deviates from the negative real axis is not important for the content of this chapter though, so we will drop the subscript in the remainder. See the supplementary section 4.5.1 for fixed eigenvalues in the forbidden region.

[10]They are normalized such that $\langle 1\rangle = 1$.

The two-level spectral density is defined as $\rho(E_1, E_2) = \rho(E_1)\rho(E_2)$. Ensemble averaging leads to correlation as connected contributions, for example for the two-loop operator defined as $Z(\beta_1, \beta_2) = Z(\beta_1)Z(\beta_2)$:

$$\langle Z(\beta_1, \beta_2)\rangle = \langle Z(\beta_1)\rangle \langle Z(\beta_2)\rangle + \langle Z(\beta_1, \beta_2)\rangle_{\text{conn}} \,. \tag{4.22}$$

Comparing to the perturbative JT gravity definition of $Z(\beta_1, \beta_2)$ in (4.10), which counts all Riemann surfaces that end on the two asymptotic boundaries, one sees that connected correlators correspond to connected geometries:

$$\langle Z(\beta_1, \beta_2)\rangle_{\text{conn}} = \beta_1$$

$$\beta_2 + \beta_1$$

$$\beta_2 + \dots \tag{4.23}$$

Again it is not hard to calculate both perturbative and nonperturbative contributions to $\langle\rho(E_1)\rho(E_2)\rangle$). See section 4.2, the supplementary 4.5 and [9]. The only significant perturbative contributions are due to the disconnected disks ending on each of the boundaries and due to the annulus connecting the two boundaries. There are also significant nonperturbative contributions:[11]

$$\begin{aligned}\langle\rho(E_1)\rho(E_2)\rangle = \delta(E_1 - E_2)\rho_0(E_1) + \rho_0(E_1)\rho_0(E_2) \\ - \frac{1}{2\pi^2}\frac{1}{(E_1 - E_2)^2} + \rho_{\text{nonp}}(E_1, E_2),\end{aligned} \tag{4.24}$$

with

$$\rho_{\text{nonp}}(E_1, E_2) = \frac{1}{2\pi^2}\frac{1}{(E_1 - E_2)^2}\cos\left(2\pi\int_{E_1}^{E_2} dM\,\rho_0(M)\right). \tag{4.25}$$

The second contribution here is of the same oscillatory type as (4.17). Unlike that contribution though, it isn't particularly small when the two energies are close together due to the multiplicative pole. Therefore this contribution cannot be neglected in our analysis. The spectral form factor is calculated as in (4.11). The $\rho_0(E_1)\rho_0(E_2)$ contribution in (4.24) gives the power-law decay (4.14). The Dirac delta yields the constant plateau contribution N. The other contributions add up to the sine kernel (4.48) which gives a variant of the ramp at late times:[12]

$$S_E(t) \supset N - N\text{Ramp}\left(\frac{t}{2\pi\rho(E)}\right), \qquad \text{Ramp}(x) = (1 - |x|)\theta(1 - |x|). \tag{4.27}$$

[11] In addition to these contributions we find a generalization of the wiggles (4.17) in the supplementary section 4.5. Such wiggles are genuinely small corrections though, unlike (4.25) and are not relevant to our discussion.

[12] In more detail, the contribution of the sine kernel is:

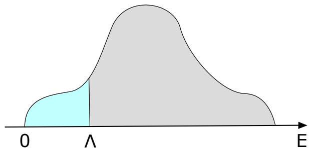
$$S_E(t) \supset -N\int d\tau\left(\frac{1}{\pi}\frac{N}{2\rho(E)}\text{sinc}^2\frac{N}{2\rho(E)}\tau\right)\text{Ramp}\left(\frac{t-\tau}{2\pi\rho(E)}\right), \qquad \text{sinc}(x) = \frac{\sin(x)}{x}. \tag{4.26}$$

This is a low-frequency filtered version of the usual ramp. Qualitatively, at $t \ll 2\pi\rho(E)$ there will be significant smoothening of the onset of the usual ramp. In the regime of interest where $N \gg 1$, the kernel acts as a Dirac-function and one obtains the linear ramp with plateau time $t = 2\pi\rho(E)$.

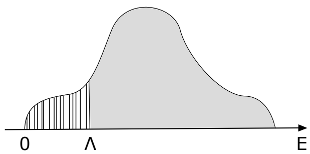
This resulting function follows (4.7) before the dotted red line, and the blue curve at late times.

Version 3. Eigenbranes and discreteness

The version of JT gravity discussed above is a double-scaled matrix integral. This means we take $L \to \infty$ and simultaneously zoom in on a region $E < \Lambda$ near the edge of the spectrum, keeping the total average number of eigenvalues $1 \ll N \ll L$ in this region fixed. We can visualize this as:

$\langle \rho(E) \rangle =$  (4.28)

The blue region represents JT gravity with spectral density (4.17). By definition, our discrete system (4.1) can be thought of as a single Hamiltonian M of such a matrix ensemble. Let us denote its lowest N eigenvalues by $\lambda_1 \dots \lambda_N$. We expect that the IR behavior of our system (4.1) is accurately described by a modified matrix ensemble where N eigenvalues are kept fixed to $\lambda_1 \dots \lambda_N$. This corresponds to a Dyson gas of charged particles equilibrating in an external potential around N static point charges. These fixed charges repel the charged gas, resulting in a void. We expect the spectral density of this new ensemble to essentially follow the spectrum of the discrete system (4.1) for $E < \Lambda$ and that of the original ensemble (4.28) for $E > \Lambda$:

$\langle \rho(E) \rangle =$  (4.29)

In the remainder of this work, we will make this picture precise and pinpoint its JT gravity interpretation. In particular, we will see that each eigenvalue λ corresponds to a fixed energy boundary with label λ hovering in the Euclidean bulk. The contour (4.12) in the gravitational path integral is hence over all Riemann surfaces that end on the union of the asymptotic boundaries and on N fixed energy boundaries with labels $\lambda_1 \dots \lambda_N$, as shown in (4.55). This version of JT gravity is able to capture the IR discreteness of (4.1). In particular we will recover the spectral form factor (4.7) including erratic oscillations.

Let us note that it is not clear whether in this picture smooth geometry in the bulk is in jeopardy or not. On the one hand one might imagine that smooth geometry is provided by our ignorance of the UV part of the system (4.1), which corresponds to the $L \gg N$ eigenvalues that remain in the continuum of the matrix integral. On the other hand it is probably so that there is backreaction in the calculations of the supplementary section 4.5 when we fix a large number of eigenvalues. We still expect the picture (4.29) to be correct, but we might no longer be able to give a precise geometric interpretation of all relevant terms in the brane calculation. It would be interesting to understand this better.

4.2 Multi-boundary correlators

This section prepares for section 4.3 where we will encounter multi-spectral density correlators $\langle\rho(E_1)\ldots\rho(E_n)\rangle$. We discuss an efficient way to calculate all significant perturbative and nonperturbative contributions for $e^{S_0}\gg 1$ based on [153].

Genus expansion

In JT gravity it is natural to consider fixed length boundary conditions, as discussed around (4.9) and in previous chapters. These correspond to the insertion of "macroscopic loop" operators in the matrix integral, and are the Laplace transforms of the multi-spectral densities:

$$Z(\beta_1\ldots\beta_n)=\int_{\mathcal{C}}d\lambda_1\,e^{-\beta_1\lambda_1}\cdots\int_{\mathcal{C}}d\lambda_n\,e^{-\beta_n\lambda_n}\,\langle\rho(\lambda_1)\ldots\rho(\lambda_n)\rangle\,. \tag{4.30}$$

This relation is an efficient tool to calculate the perturbative contributions to $\rho(E_1\ldots E_n)$ [9]. For example as explained in chapter 3, the genus g contribution to the n-loop correlation function is:

$$e^{(2-2g-n)S_0}\int_0^\infty db_1\,b_1\,Z(\beta_1,b_1)\cdots\int_0^\infty db_n\,b_n\,Z(\beta_n,b_n)\,V_{g,n}(b_1\ldots b_n). \tag{4.31}$$

Here $V_{g,n}(b_1\ldots b_n)$ is the volume of the moduli space of the n-holed sphere with g handles. This Weil-Petersson volume is a polynomial in b_1^2, b_2^2 etcetera and is easily calculated recursively [122, 123]. The proof of that recursion relation is somewhat less trivial though we tried to paint an intuitive picture in chapter 3. The single twisted Schwarzian partition function $Z(\beta,b)$ is just a Gaussian (3.64).[13]. The Gaussian integrals yield polynomials in β_1, β_2 etcetera multiplied by $(\beta_1\ldots\beta_n)^{1/2}$. Inverse Laplace transforming then gives us the spectral densities, which are polynomials in $1/E_1$, $1/E_2$ etcetera multiplied by $(E_1\ldots E_n)^{-3/2}$. The only exceptions to this polynomial behavior are the disk and annulus topologies for which the Weil-Petersson volumes $V_{0,1}(b_1)$ and $V_{0,2}(b_1,b_2)$ are technically undefined.[14] The disk density of states is (4.13). The annulus amplitude is (3.63):

$$\int_0^\infty db\,b\,Z(\beta_1,b)\,Z(\beta_2,b). \tag{4.32}$$

Its contribution to the two-level spectral density is:

$$\rho(E_1,E_2)\supset-\frac{1}{4\pi^2}\frac{E_1+E_2}{\sqrt{E_1}\sqrt{E_2}(E_1-E_2)^2}\approx-\frac{1}{2\pi^2}\frac{1}{(E_1-E_2)^2}. \tag{4.33}$$

[13]See also [50, 9] for the evaluation and interpretation of the twisted Schwarzian partition function.
[14]One could take $V_{0,2}(b_1,b_2)=b_1^{-1}\delta(b_1-b_2)$.

Here we approximated the answer for $|E_1 - E_2| \ll 1$. We see that with the exception of the disks and annuli, all the perturbative contributions are small corrections as long as we stay far enough from the spectral edge.[15] The annulus contribution itself can become large and comparable to the size of the disk contribution $\rho_0(E_1)\rho_0(E_2)$ for $|E_1 - E_2| \sim 1/\rho_0(E)$, the typical eigenvalue spacing in the ensemble. For much larger separations it is negligible. As $\rho_0(E) \sim e^{S_0}$ and we are steering clear of the spectral edge, the only significant contributions from the annulus arise well within the range $|E_1 - E_2| \ll 1$. This validates using the second equality in (4.33) throughout this chapter. In conclusion, when away from the spectral edge and in the regime $e^{S_0} \gg 1$, all perturbative contributions to the spectral densities are negligible except for those associated with the disk and annuli topologies.
As the inverse Laplace transforms of fixed length correlators in JT gravity, the spectral densities $\rho(E_1 \dots E_n)$ correspond to imposing certain fixed energy boundary conditions at the n boundaries of the Riemann surfaces. These boundary conditions are closely related to the FZZT boundary conditions in Liouville theory [154].[16]

Exact answer

On top of the perturbative contributions discussed in the previous subsection, an exact matrix integral analysis reveals nonperturbative contributions to $\rho(E_1 \dots E_2)$. We will now discuss an efficient way to calculate all significant contributions in the regime $e^{S_0} \gg 1$, perturbative and non-perturbative, with more details in the supplementary section 4.5.[17] Consider brane operators in the matrix ensemble defined as:

$$\psi(E) = e^{-\frac{LV(E)}{2}} \prod_{i=1}^{L} (E - \lambda_i). \tag{4.34}$$

We can use this function to extract and write out the dependence on λ_1 of the Vandermonde determinant in (4.19) as:

$$e^{-L\sum_{i=1}^{L} V(\lambda_i)} \Delta(\lambda_1 \dots) = \psi^2(\lambda_1) e^{-L\sum_{i=2}^{L} V(\lambda_i)} \Delta(\lambda_2 \dots). \tag{4.35}$$

More generally we have:

$$e^{-L\sum_{i=1}^{L} V(\lambda_i)} \Delta(\lambda_1 \dots) = \Delta(\lambda_1 \dots \lambda_n) \psi^2(\lambda_1) \dots \psi^2(\lambda_n)\, e^{-L\sum_{i=n+1}^{L} V(\lambda_i)} \Delta(\lambda_{n+1} \dots).$$

This essentially decomposes the measure of the matrix ensemble (4.19) as:

$$d\mu(\lambda_1 \dots) = d\lambda_1 \dots d\lambda_n \Delta(\lambda_1 \dots \lambda_n) \psi^2(\lambda_1) \dots \psi^2(\lambda_n)\, d\mu(\lambda_{n+1} \dots). \tag{4.36}$$

[15] We should take $E \gg e^{-2S_0/3}$, which ensures the topological suppression prevails over the poles in the spectral densities for small energies, the weakest and hence most important of which is $(E_1 \dots E_2)^{-3/2}$. This is confirmed from the exact analysis near the spectral edge in appendix 4.5.1 where $E \gg e^{-2S_0/3}$ is the condition for higher loop contributions to the Airy function to become negligible [153].

[16] See for example [9, 46].

[17] For an alternative calculation of nonperturbative effects which is in spirit very closely related to the one presented here we refer to [9]. The value of what follows is in part that on a technical level our calculations are arguably significantly "easier" than those of [9] and it's significantly simpler to see the expected structure appear.

For the branes that feature in this formula, the product in (4.34) is over the $L-n$ remaining eigenvalues. This basic formula allows us to extract exact formulas for spectral densities, which we can easily calculate exactly. Let us demonstrate this case-by-case.

- **1 eigenvalue.** We can use the symmetries of the ensemble, and the property (4.36) to rewrite (4.21) as:

$$\begin{aligned}\langle Z(\beta)\rangle &= \frac{L}{\mathcal{Z}_L}\int d\lambda_1\, e^{-\beta\lambda_1}\int d\lambda_2\ldots e^{-LV(\lambda_1\ldots)}\Delta(\lambda_1\ldots)\\ &= \frac{L\mathcal{Z}_{L-1}}{\mathcal{Z}_L}\int d\lambda_1\, e^{-\beta_1\lambda_1}\left\langle\psi^2(\lambda_1)\right\rangle_{L-1}. \end{aligned} \tag{4.37}$$

From this, we read off the spectral density [9]:[18]

$$\langle\rho(E)\rangle = \frac{L\mathcal{Z}_{L-1}}{\mathcal{Z}_L}\left\langle\psi^2(E)\right\rangle_{L-1}. \tag{4.38}$$

Both sides of this equality can be calculated independently in JT gravity. In the supplementary section 4.5 we calculate the double brane correlator using techniques of [153]. The spectral density was calculated using related techniques but via a different computation in [9]. We find:

$$\langle\rho(E)\rangle = \frac{1}{2\pi}\left\langle\psi^2(E)\right\rangle_{L-1}. \tag{4.39}$$

Comparison gives us a recursion relation for the matrix integral partition function at large L:[19]

$$\mathcal{Z}_L \approx 2\pi L\mathcal{Z}_{L-1}. \tag{4.40}$$

This will enable us to eliminate any dependence on $\mathcal{Z}_L$ from the calculations that follow.

- **2 eigenvalues.** The 2-boundary correlator decomposes as:

$$\begin{aligned}&\langle Z(\beta_1)Z(\beta_2)\rangle\\ &= \frac{1}{\mathcal{Z}_L}\int d\lambda_1\ldots e^{-LV(\lambda_1\ldots)}\Delta(\lambda_1\ldots)\sum_{i=1}^{L}e^{-\beta_1\lambda_i}\sum_{j=1}^{L}e^{-\beta_2\lambda_j}\\ &= \frac{L}{\mathcal{Z}_L}\int d\lambda_1\, e^{-(\beta_1+\beta_2)\lambda_1}\int d\lambda_2\ldots e^{-LV(\lambda_1\ldots)}\Delta(\lambda_1\ldots)\\ &\quad+\frac{L(L-1)}{\mathcal{Z}_L}\int d\lambda_1\, e^{-\beta_1\lambda_1}\int d\lambda_2\, e^{-\beta_2\lambda_2}\int d\lambda_3\ldots e^{-LV(\lambda_1\ldots)}\Delta(\lambda_1\ldots)\\ &= \frac{L\mathcal{Z}_{L-1}}{\mathcal{Z}_L}\int d\lambda_1\, e^{-(\beta_1+\beta_2)\lambda_1}\left\langle\psi^2(\lambda_1)\right\rangle_{L-1}\\ &\quad+\frac{L(L-1)\mathcal{Z}_{L-2}}{\mathcal{Z}_L}\int d\lambda_1\, e^{-\beta_1\lambda_1}\int d\lambda_2\, e^{-\beta_2\lambda_2}\,(\lambda_1-\lambda_2)^2\left\langle\psi^2(\lambda_1)\psi^2(\lambda_2)\right\rangle_{L-2}.\end{aligned} \tag{4.41}$$

[18] All averaged quantities are L independent for $L \gg 1$.

[19] This recursion relation holds for all double-scaled matrix models. It also holds for the CUE ensemble exactly, see e.g. [155].

Using the recursive formula (4.40), we end up with:

$$\langle \rho(E_1)\rho(E_2)\rangle = \frac{1}{(2\pi)^2}(E_1 - E_2)^2 \left\langle \psi^2(E_1)\psi^2(E_2)\right\rangle_{L-2} + \delta(E_1 - E_2)\frac{1}{2\pi}\left\langle \psi^2(E_1)\right\rangle_{L-1}. \tag{4.42}$$

These types of formulas are well-known in the random matrix literature. In fact they are referred to simply as the *correlation functions* [119]:[20,21]

$$R(E_1 \dots E_n) = \frac{1}{(2\pi)^n}\Delta(E_1 \dots E_n)\left\langle \psi^2(E_1)\dots\psi^2(E_n)\right\rangle_{L-n}. \tag{4.43}$$

These are smooth functions. Notice that the operators in (4.43) represent the repulsive force exerted by a set of charges charges at $E_1 \dots E_2$ on the remainder of the Dyson gas.

- **3 eigenvalues.** An equally easy calculation holds for the 3-level spectral density. We find:

$$\begin{aligned}\langle \rho(E_1)\rho(E_2)\rho(E_3)\rangle =& R(E_1, E_2, E_3) + \delta(E_1 - E_2)R(E_1, E_3)\\ &+ \delta(E_1 - E_3)R(E_2, E_3) + \delta(E_2 - E_3)R(E_1, E_2)\\ &+ \delta(E_1 - E_2)\delta(E_2 - E_3)R(E_1).\end{aligned} \tag{4.44}$$

This is readily generalized to any number of boundaries.

The delta functions that appear in expressions of this type are contact terms. Whereas a geometric interpretation of the correlation functions $R(E_1 \dots E_n)$ is obvious from the discussion at the beginning of this section, the interpretation of these terms is somewhat obscure. For example the calculation of [9] suggests these are nonperturbative corrections which then by definition don't need to have an interpretation in terms of Riemann surfaces. However in these calculations they are more like the usual contact terms in quantum field theory, as they appear to be associated with touching or merging boundaries of Riemann surfaces. We will come back to these mergers in the concluding remarks of section 4.4. It is then not clear whether we should have allowed these mergers in the perturbative JT gravity path integral (4.12) from the get-go or whether they represent nonperturbative corrections to (4.12). From a matrix integral point of view we would prefer the former, but holography might prefer the latter picture.
It is convenient to extract from the correlation functions $R(E_1 \dots E_n)$ the fully connected contribution $T(E_1, \dots E_n)$ known as the cluster function. The remaining disconnected

[20] The constant in formula (6.1.1) of [119] is $1/\mathcal{Z}_L$. The average in (6.1.2) generates a factor $\mathcal{Z}_{L-n}$, the recursion relation (4.40) removes the combinatorial prefactors.

[21] Brane operators in the matrix integral are closely related to exponentiated spacetimes attached to a brane, see the supplementary section 4.5. In this sense, formulas of the type (4.42) are quite surprising since they say that a brane pair correlator actually corresponds to a single (fixed energy) boundary in gravity. This may be related to the comments on large diffeomorphisms invariance in [19].

pieces are then products of cluster functions at lower values of n. For example [119]:[22]

$$\begin{aligned} R(E) =& T(E), \\ R(E_1, E_2) =& -T(E_1, E_2) + T(E_1)T(E_2), \\ R(E_1, E_2, E_3) =& T(E_1, E_2, E_3) - T(E_1)T(E_2, E_3) - T(E_2)T(E_1, E_3) \\ & - T(E_3)T(E_1, E_2) + T(E_1)T(E_2)T(E_3). \end{aligned} \tag{4.45}$$

Following the logic of around (4.23) one deduces that the clusters $T(E_1, \ldots E_n)$ correspond to the nonperturbative completion of the gravitational genus expansion starting with the n-holed sphere.[23] The cluster functions have the property that they vanish when the spacing of two of its arguments is large compared to the average eigenvalue spacing. This means the only significant contributions of the cluster functions to the correlation functions $R(E_1 \ldots E_n)$ are when $|E_i - E_j| \ll 1$ for all energies in a cluster. The perturbative disk and annuli contributions discussed at the beginning of this section are part of the terms $T(E_i)$ respectively $T(E_i, E_j)$ that contribute a generic correlator $R(E_1 \ldots E_n)$. As mentioned earlier, these are the only significant perturbative contributions to $R(E_1 \ldots E_n)$ away from the spectral edge.

An exact calculation of the correlators $R(E_1 \ldots E_n)$ in JT gravity reveals a set of nonperturbative contributions similar to those in (4.25). An efficient way to calculate these exactly in JT gravity is via formula (4.43). We do so in the supplementary section 4.5 in detail for $R(E_1)$ and $R(E_1, E_2)$ and discuss certain aspects of the calculation for $R(E_1, E_2, E_3)$. The general trend is the appearance of significant non-perturbative contributions to $R(E_1 \ldots E_n)$ of the type:

$$\frac{\exp\Big(\pm i\pi \int_{E_i}^{E_j} dM\, \rho(M)\Big)}{(E_i - E_j)}. \tag{4.47}$$

It is convenient to extract from these calculations the cluster functions which, as explained above, can be evaluated for all intends and purposes at $|E_i - E_j| \ll 1$. We find:

$$\begin{aligned} T(E) &= \rho(E) \\ T(E_1, E_2) &= \rho(E_1)\rho(E_2)\,\mathrm{sinc}^2 \rho(E_1)(E_1 - E_2) = S(E_1, E_2)^2. \end{aligned} \tag{4.48}$$

The sine kernel $S(E_1, E_2)$ also appears in higher clusters, for example:

$$T(E_1, E_2, E_3) = 2\, S(E_1, E_2)\, S(E_2, E_3)\, S(E_3, E_1). \tag{4.49}$$

[22]The minus signs are convention [119].

[23]The more precise version of this statement follows from the decomposition of the correlators $\langle \rho(E_1) \ldots \rho(E_n) \rangle$ into cluster functions, including contact terms. The resulting cluster functions correspond precisely to the nonperturbative completion of the n-holed sphere genus expansion, which will generate the same contact terms. For example, the three-holed sphere with all corrections gives:

$$\begin{aligned} \langle \rho(E_1)\rho(E_2)\rho(E_2) \rangle_{\text{conn}} =& T(E_1, E_2, E_3) - \delta(E_1 - E_2)T(E_1, E_3) - \delta(E_1 - E_3)T(E_1, E_2) \\ & - \delta(E_2 - E_3)T(E_1, E_3) + \delta(E_1 - E_2)\delta(E_1 - E_3)T(E_1). \end{aligned} \tag{4.46}$$

This follows directly from (4.44), but also follows intuitively from the discussion on merging boundaries in section 4.4.

This is not very surprising. It is a widely held conjecture [119] for any Hermitian matrix ensemble that cluster functions are exactly equal to the universal GUE cluster functions when their arguments are close together $|E_i - E_j| \ll 1$. The latter are known exactly [119] and feature only the sine kernel. In the brane calculations these arise due to the contributions of the type (4.47). The calculations of the supplementary section 4.5 merely reassure us that this conjecture is true in JT gravity. We are then free to ship in the GUE clusters to calculate $\langle \rho(E_1) \ldots \rho(E_n) \rangle$ in JT gravity.

4.3 Fixing eigenvalues or introducing boundaries

In this section we investigate a matrix ensemble with a series of eigenvalues fixed to consecutive ones of (4.1) and specify the integration space in the JT gravity path integral over metrics (4.12) associated with this ensemble. The specific contour follows from formula (4.43) combined with (4.36). Each fixed eigenvalue corresponds to an additional fixed-energy boundary in the bulk on which Riemann surfaces can end. A matrix ensemble with n eigenvalues fixed to $\lambda_1 \ldots \lambda_n$ (assumed all different) is obtained from the original ensemble (4.19) by including appropriate deltas in the measure:[24]

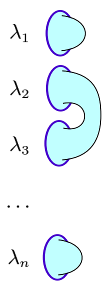
$$d\mu(\kappa_1 \ldots \kappa_L) \prod_{i=1}^{L} \delta(\kappa_i - \lambda_i). \tag{4.50}$$

The partition function replacing (4.19) is then:

$$\begin{aligned} \mathcal{Z}_{L,\lambda_1 \ldots \lambda_n} &= \int d\lambda_{n+1} \ldots d\lambda_L \, e^{-LV(\lambda_1 \ldots)} \, \Delta(\lambda_1 \ldots) \\ &= \mathcal{Z}_{L-n} \Delta(\lambda_1 \ldots \lambda_n) \left\langle \psi^2(\lambda_1) \ldots \psi^2(\lambda_n) \right\rangle_{L-n} \\ &= (2\pi)^n \mathcal{Z}_{L-n} \left\langle \rho(\lambda_1) \ldots \rho(\lambda_n) \right\rangle_L . \end{aligned} \tag{4.51}$$

Here we used (4.36) and the generalization of (4.44) to n boundaries. Notice that the contact terms vanish because the eigenvalues of (4.1) are all different. Perturbatively, this partition function is counting Riemann surfaces of the type:

$$\langle \rho(\lambda_1) \ldots \rho(\lambda_n) \rangle \supset \tag{4.52}$$



where n eigenbrane boundaries are present, but no asymptotic boundary insertions.

[24] Here $d\mu(\kappa_1 \ldots \kappa_L)$ is the measure of (4.19).

Delta spikes and a void

The expectation value of the spectral density $\rho(E) = \sum_{i=1}^{L} \delta(E-\lambda_i)$ in the new ensemble (4.51) is by definition:

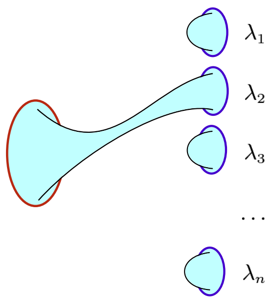
$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} = \frac{1}{\mathcal{Z}_{L,\lambda_1 \dots \lambda_n}} \int d\lambda_{n+1} \dots d\lambda_L \, \rho(E) e^{-LV(\lambda_1 \dots)} \, \Delta(\lambda_1 \dots). \tag{4.53}$$

We immediately obtain:

$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} = \frac{\langle \rho(E)\rho(\lambda_1) \dots \rho(\lambda_n) \rangle_L}{\langle \rho(\lambda_1) \dots \rho(\lambda_n) \rangle_L}. \tag{4.54}$$

This is a conditional probability. As announced, this corresponds to a version of JT gravity where each fixed eigenvalue of the matrix integral translates into the introduction of a fixed-energy boundary on which Riemann surfaces in the path integral are to end. As explained before, in the genus expansion disks and annuli dominate the regime of interest. From (4.54) we read off the type of geometries contributing significantly to the JT gravity path integral:

$$\langle \rho(\lambda_1) \dots \rho(\lambda_n) \rangle \, \langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} \supset$$



(4.55)

There are also multi-annulus configurations where eigenbranes connect to other eigenbranes. Three-holed spheres and handle-body geometries contribute, but not significantly. Using a generalization of formula (4.44) we can rewrite (4.54) as:

$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} = \sum_{i=1}^{n} \delta(E - \lambda_i) + \frac{R(E, \lambda_1 \dots \lambda_n)}{R(\lambda_1 \dots \lambda_n)}. \tag{4.56}$$

At this point our discussion of the previous section comes into play as we can immediately write down the exact answer for a given n using the cluster functions (4.48) etcetera. As a consistency check on the normalization we can take the integral over E of (4.56):[25]

$$\int_{\mathcal{C}} d\lambda \frac{R(\lambda, \lambda_1 \dots \lambda_n)}{R(\lambda_1 \dots \lambda_n)} = L - n. \tag{4.60}$$

[25] From (4.36) one finds:

$$\int_{\mathcal{C}} d\lambda \, \Delta(\lambda, \lambda_1 \dots \lambda_n) \left\langle \psi^2(\lambda)\psi^2(\lambda_1) \dots \psi^2(\lambda_n) \right\rangle_{L-n-1} \mathcal{Z}_{L-n-1}$$
$$= \Delta(\lambda_1 \dots \lambda_n) \left\langle \psi^2(\lambda_1) \dots \psi^2(\lambda_n) \right\rangle_{L-n} \mathcal{Z}_{L-n}. \tag{4.57}$$

We see that the number of eigenvalues in the smooth continuum is exactly down by n as compared to $\rho(E)$, and these eigenvalues are accounted for by the delta functions. Notice that as discussed in section 4.2, the contributions from the annuli connecting the asymptotic boundary to the eigenvalue boundaries is negligible when $|E-\lambda_i| \gg 1/\rho(E)$, and the same holds for all nonperturbative contributions. Therefore, all effects due to the fixed eigenvalues are short-ranged and one has:[26]

$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} \approx \langle \rho(E) \rangle \,, \quad |E-\lambda_i| \gg 1/\rho(E). \tag{4.61}$$

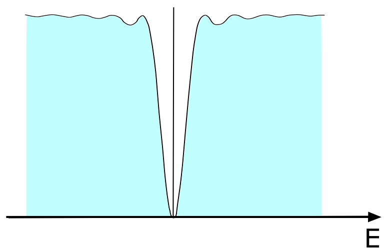
To maximally appreciate the physics in the exact formula (4.56), let's do a small case-by-case study.[27]

- **1 eigenvalue.** We have from (4.56):

$$\langle \rho(E) \rangle_\lambda = \delta(E-\lambda) + \rho(E)(1 - \text{sinc}^2 \pi \rho(\lambda)(E-\lambda)). \tag{4.62}$$

  Close to the fixed eigenvalue this looks like:

$$\langle \rho(E) \rangle_\lambda = \tag{4.63}$$



  This exhibits eigenvalue repulsion: the fixed charge repels the particles of the gas, as modeled here by the Vandermonde factor $(E-\lambda)^2$.

- **2 eigenvalues.** Using the GUE cluster functions (4.48) in (4.56), we find a less

---

Using (4.40), we recognize the definition of the correlators (4.43) on the left and the n-level correlator on the right, completing the proof of (4.60). Since $R(\lambda) = \langle \rho(\lambda) \rangle$, taking $n = 0$ in (4.60) we recover the normalization property:

$$\int_{\mathcal{C}} d\lambda \, \langle \rho(\lambda) \rangle = L. \tag{4.58}$$

We can apply (4.57) recursively to find:

$$\int_{\mathcal{C}} d\lambda_1 \dots d\lambda_n \, \Delta(\lambda_1 \dots \lambda_n) \, \left\langle \psi^2(\lambda_1) \dots \psi^2(\lambda_n) \right\rangle \mathcal{Z}_L = \mathcal{Z}_{L+n}. \tag{4.59}$$

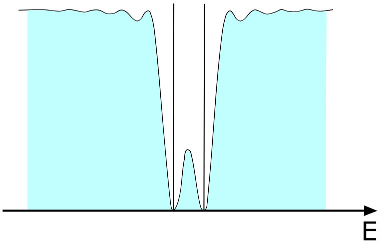
This formula appeared in [153]. It means that we can add eigenvalues to an ensemble by introducing pairs of branes $\psi^2(\lambda)$ and integrating out λ.

[26] This corresponds to the intuition of around (4.29) that far enough from the fixed charges we can't distinguish them from the scenario where the charged gas would fill in this space.

[27] The eigenvalues used to generate these plots are the same as those used in the plot of (4.7). These are exact plots, not cartoons.

elegant answer for the case of two fixed eigenvalues.[28] A plot close to the fixed eigenvalues is much more intuitive:

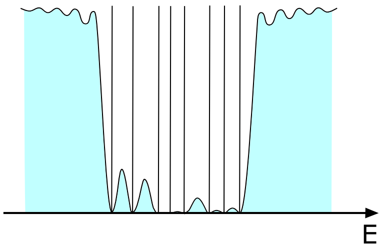
$$\langle \rho(E) \rangle_{\lambda_1,\lambda_2} = \tag{4.65}$$



There is a relatively low probability for another eigenvalue to be found in between λ_1 and λ_2, provided they are close enough.
In general we can think of the initial coarse-grained density as a low-frequency approximation to the series of delta-functions. The maximal frequency here is the typical eigenvalue spacing $1/\rho(E)$. We therefore manifestly see that we are not changing any early-time $t \ll \rho(E)$ physics by fixing eigenvalues.

- **A bin of eigenvalues.** It is not hard to plot (4.56) exactly for an increasing number of consecutive eigenvalues of (4.1) in some region. For example, for $n = 8$ we obtain:

$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_8} = \tag{4.66}$$



We're starting to see the features claimed in formula (4.29). First, fixing a large number of consecutive eigenvalues will create to good approximation a void in the continuum spectral density in the interval $\mathcal{I}$ where the eigenvalues are situated:

$$\langle \rho(E) \rangle_{\lambda_1 \dots \lambda_n} \approx \sum_{i=1}^{n} \delta(E - \lambda_i), \quad E \in \mathcal{I}. \tag{4.67}$$

In particular the total integrated continuum spectral density in the region $\mathcal{I}$ can be found to be between zero and one with almost all of this probability localized near the edges of $\mathcal{I}$. If we take $\mathcal{I}$ large enough such that the number of fixed

[28]
$$\rho(E)_{\lambda_1,\lambda_2} = \delta(E-\lambda_1) + \delta(E-\lambda_2) + \rho(E) - \rho(E)\frac{\text{sinc}^2\pi\rho(E)(E-\lambda_1) + \text{sinc}^2\pi\rho(E)(E-\lambda_2)}{1-\text{sinc}^2\pi\rho(E)(\lambda_1-\lambda_2)}$$
$$- \rho(E)\frac{2\text{sinc}\pi\rho(E)(E-\lambda_1)\text{sinc}\pi\rho(E)(E-\lambda_2)\text{sinc}\pi\rho(E)(\lambda_1-\lambda_2)}{1-\text{sinc}^2\pi\rho(E)(\lambda_1-\lambda_2)}. \tag{4.64}$$

eigenvalues $n \gg 1$ the contributions of the continuum density become negligible and the theory is essentially "locally" discrete. Secondly we see that the effect is not felt far outside of $\mathcal{I}$ and dies out over a range $\sim 1/\rho(E) \ll 1$.

As mentioned before, in the region closer to the spectral edge, where the spectral density $\rho(E)$ changes rapidly, we can no longer trust the sine-kernel type GUE cluster functions (4.48). Fortunately, in that region, we have the exact results of the Airy model available. Using the method of [153] to calculate brane correlators, it is straightforward though slightly tedious to recover the known Airy cluster functions. We do so in the supplementary 4.5.1 for the case $T(E_1, E_2)$ and recover the Airy kernel [119]. We then study the spectral density with one fixed eigenvalue close to the spectral edge. The behavior is very similar to that of (4.63). It would be straightforward to extend this to multiple fixed eigenvalues but we will refrain from doing so. All this points in the direction of the picture (4.29). By including the $1 \ll N \ll \Lambda$ eigenbranes corresponding to the $E < \Lambda$ spectrum (4.1) in the prescription for the JT gravity path integral (4.12), we get a version of JT gravity which is dual to a discrete quantum chaotic system with spectrum (4.1) at $E < \Lambda$. We note that it is highly plausible that the calculations of the brane correlators presented in the supplementary section 4.5 are more subtle when $n \sim e^{S_0}$. In particular, the limit $e^{S_0} \gg 1$ used in [153] to obtain the semiclassical brane correlators is probably more subtle.[29] It would be valuable to understand how the technical calculation is modified and if it would still make sense to paint a smooth geometric picture. There is no reason though to expect any qualitative deviations from the picture (4.29) and our conclusions. In particular we expect no sizeable modification of the correlation function $R(E_1 \dots E_n)$ away from the GUE answer.

Erratic oscillations

To stack up the claim that introducing these eigenbranes in JT gravity allows one to capture the $E < \Lambda$ features of the discrete system with spectrum (4.1), we would like to reproduce the local spectral form factor (4.7) from a purely *bulk* JT gravity calculation. For this we will investigate $\langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1 \dots \lambda_n}$ with emphasis on the terms that contribute to the plateau region $t > 2\pi\rho(E)$. Using the ensemble with n fixed eigenvalues (4.51), one immediately writes down:

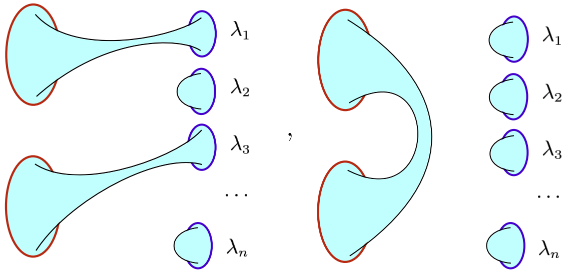
$$\langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1 \dots \lambda_n} = \frac{\langle \rho(E_1)\rho(E_2)\rho(\lambda_1)\dots\rho(\lambda_n)\rangle}{\langle \rho(\lambda_1)\dots\rho(\lambda_n)\rangle}. \tag{4.68}$$

Geometrically we are calculating the correlator of two fixed-energy boundaries in a version of JT gravity that has n eigenbranes hovering in the bulk. The only significant

[29] At least one of the authors of [153] agrees with this.

perturbative contributions are again due to the disk and annuli, for example:

$$\langle \rho(\lambda_1)\dots\rho(\lambda_2)\rangle \langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1\dots\lambda_n} \supset$$



(4.69)

As in (4.55) the eigenvalue boundaries that don't connect to the asymptotic boundaries don't need to be capped off by disks, there can be annuli between them. In particular this contains the annulus connecting the two asymptotic boundaries and the perturbative and nonperturbative corrections on top of it, as well as the two disconnected asymptotic disks plus their genus expansion and nonperturbative corrections. When averaging over the eigenbranes to recover the version of JT gravity of [9], these contributions are the only ones that survive. Other contributions where the asymptotic boundaries connect to the eigenbranes via Euclidean wormholes, such as the first contribution shown in (4.69), account for the gravitational analogue of the off-diagonal contributions mentioned for example in section 7 of [10].[30] As predicted there, including such off-diagonal terms in a gravitational theory renders the theory discrete and in particular it results in a factorizing theory of bulk quantum gravity in the sense that for example $Z(\beta_1,\beta_2) = Z(\beta_1)Z(\beta_2)$. Our construction makes explicit how that happens in JT gravity. This is trivial for a discrete theory. We will discuss this factorization in more detail in the next subsection from a gravitational point of view.
Using the exact formulas for the multi-spectral densities discussed in section 4.2, we obtain from (4.68):

$$\begin{aligned}\langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1\dots\lambda_n} &= \sum_{i=1}^{n}\delta(E_1-\lambda_i)\sum_{j=1}^{n}\delta(E_2-\lambda_j)\\ &+\frac{R(E_1,\lambda_1\dots\lambda_n)}{R(\lambda_1\dots\lambda_n)}\sum_{j=1}^{n}\delta(E_2-\lambda_j)+\frac{R(E_2,\lambda_1\dots\lambda_n)}{R(\lambda_1\dots\lambda_n)}\sum_{i=1}^{n}\delta(E_1-\lambda_i)\\ &+\delta(E_1-E_2)\frac{R(E_1,\lambda_1\dots\lambda_n)}{R(\lambda_1\dots\lambda_n)}+\frac{R(E_1,E_2,\lambda_1\dots\lambda_n)}{R(\lambda_1\dots\lambda_n)}. \qquad (4.70)\end{aligned}$$

Again, using the JT spectral density and the GUE cluster functions, it is easy to calculate and plot this recursively for increasing n, though we won't show any plots here. Numerically investigating the continuous contributions to (4.70) it quickly becomes obvious that if we fix a large number of eigenvalues of (4.1) in some region $\mathcal{I}$, we find to

[30] The paper [4] on which this chapter is based actually appeared essentially on the same day as [10] so this is an a posteriori remark.

good approximation:[31]

$$\langle\rho(E_1)\rho(E_2)\rangle_{\lambda_1\ldots\lambda_n} \approx \sum_{i=1}^{n}\delta(E_1-\lambda_i)\sum_{j=1}^{n}\delta(E_2-\lambda_j), \quad E_1, E_2 \in \mathcal{I}. \tag{4.71}$$

If we take the region $\mathcal{I}$ large enough such that $|\text{bin}(E)| \ll |\mathcal{I}|$ then we trivially recover the discrete version of the local spectral form factor (4.6) in JT gravity, including all the erratic oscillations in (4.7). We would like to understand in a bit more detail the approach of the local spectral form factor to this erratic behavior though. Let us focus on the plateau region $t > 2\pi\rho(E)$ for the remainder of this subsection, and let's consider the case with only a few fixed eigenvalues.[32] In the averaged version of JT gravity, the plateau behavior is only due to the first term in (4.24):

$$\langle\rho(E_1)\rho(E_2)\rangle^{\text{plateau}} = \delta(E_1-E_2)\rho(E_1). \tag{4.72}$$

Via a direct calculation analogous to (4.26) one finds that only the first and penultimate contributions to (4.70) are relevant for the spectral form factor at $t > 2\pi\rho(E)$:[33]

$$\begin{aligned}\langle\rho(E_1)\rho(E_2)\rangle^{\text{plateau}}_{\lambda_1\ldots\lambda_n} &= \delta(E_1-E_2)\frac{R(E_1,\lambda_1\ldots\lambda_n)}{R(\lambda_1\ldots\lambda_n)} + \sum_{i,j}^{n}\delta(E_1-\lambda_i)\delta(E_2-\lambda_j)\\ &= \delta(E_1-E_2)\,\langle\rho(E_1)\rangle_{\lambda_1\ldots\lambda_n} + \sum_{i\neq j}^{n}\delta(E_1-\lambda_i)\delta(E_2-\lambda_j).\end{aligned} \tag{4.73}$$

This formula nicely interpolates between the averaged variant (4.72) and the discretized variant (4.71). The first term contributes a constant plateau of height N.[34] The second term generates ever more erratic oscillations for increasing number of eigenvalues:

$$\langle S_E(t)\rangle_{\lambda_1\ldots\lambda_n} = N + \sum_{i\neq j}^{n}\cos t(\lambda_i-\lambda_j). \tag{4.74}$$

For $n=0$ this is the usual random matrix theory answer. For $n=N$ we recover the discrete answer (4.6). It is straightforward to generalize these calculations to full fledged thermal correlation functions in JT gravity plus eigenbranes, following the discussion of [3, 18]. See also the concluding remarks of this chapter, and the main body of chapter 5.

[31] In particular, much like the depletion of the continuum spectral density in for example (4.66), one observes that well within the bulk of $\mathcal{I}\times\mathcal{I}$, the final term in (4.70) can be made arbitrarily small by increasing n.

[32] An analytic analysis of the plateau region is orders of magnitude simpler than that of the ramp region, perhaps to the surprise of some.

[33] This is checked explicitly in [4].

[34] This is a variant of (4.60) where we take the fixed eigenvalues sufficiently deep in the bin, such that the tails extending outside the bin are negligible. The continuum contributes $N-n$ and the deltas give n.

Factorization

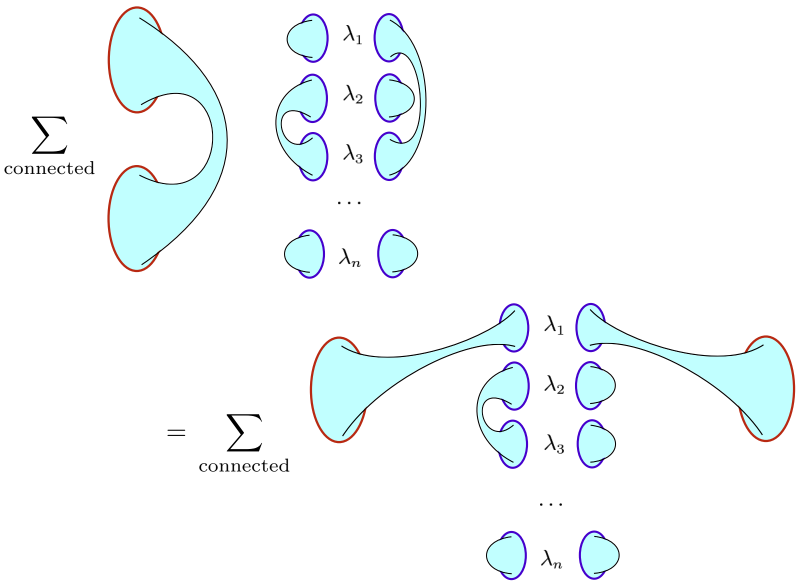
Let us now return to the factorization property of discrete systems discussed below (4.69). The connected part of the two level spectral density is defined as:

$$\langle \rho(E_1)\rho(E_2)\rangle^{\text{conn}}_{\lambda_1\dots\lambda_n} = \langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1\dots\lambda_n} - \langle \rho(E_1)\rangle_{\lambda_1\dots\lambda_n} \langle \rho(E_2)\rangle_{\lambda_1\dots\lambda_n} . \tag{4.75}$$

From (4.71) and (4.67), we get to good approximation:

$$\langle \rho(E_1)\rho(E_2)\rangle^{\text{conn}}_{\lambda_1\dots\lambda_n} \approx 0, \quad E_1, E_2 \in \mathcal{I}. \tag{4.76}$$

This factorization is trivial for a discrete system. However it entails a nontrivial equality in bulk quantum gravity. To appreciate this consider the geometries that contribute to $\langle \rho(\lambda_1)\dots\rho(\lambda_2)\rangle \langle \rho(E_1)\rho(E_2)\rangle_{\lambda_1\dots\lambda_n}$ and compare these geometries to the geometries that contribute to $\langle \rho(\lambda_1)\dots\rho(\lambda_2)\rangle \langle \rho(E_1)\rangle_{\lambda_1\dots\lambda_n} \langle \rho(\lambda_1)\dots\rho(\lambda_2)\rangle \langle \rho(E_2)\rangle_{\lambda_1\dots\lambda_n}$. If we strip off geometries that contribute to both, we end up in the former with connected geometries such as the annulus between the two asymptotic boundaries. This geometry is the epitome of the factorization problem in the ensemble averaged version of JT gravity discussed for example in [8, 9]. In the latter we are left with configurations where the boundaries are indirectly connected via matching pairs of eigenbranes. The sum of what remains in either quantity is non-zero. We can calculate the exact answer for each quantity independently for increasing n using the techniques of section 4.2. It turns out that these quantities match for $E_1, E_2 \in \mathcal{I}$. This proves the following geometric property:



(4.77)

This factorizing property is slightly surprising in the sense that the geometries on the right hand side are never counted in the original perturbative JT gravity path integral prescription for $\langle\rho(E_1)\rho(E_2)\rangle_{\lambda_1\ldots\lambda_n}$. One might have expected that connected contributions to the gravity analogue of a discrete system vanish. We find that instead they are non zero, but their sum can effectively be replaced by a sum over disconnected contributions. This is very much in line with the discussion in section 7 of [10] which states that factorization is not obtained by proving that the annulus contribution between the asymptotic boundaries vanishes in a discretized version of gravity, but rather one should include precisely the appropriate "off-diagonal" contributions. In our story, these "off-diagonal" contributions are accounted for by the asymptotic boundaries connecting to the eigenbranes in (4.69). We note that (4.77) looks somewhat like introducing a "complete set of baby universes" between E_1 and E_2 as hinted towards in [18]. The precise interpretation however turns out to be in some sense orthogonal to that suggestion. What looks like a complete set of states in the Hilbert space of baby universes is actually just the insertion of $|\alpha\rangle\langle\alpha|$ where one such state is an α state of JT gravity [19]. It is labeled by *all* the energy levels $\lambda_1\ldots\lambda_n$ of the dual discrete system. The Hilbert space of baby universes consists of all α states. Here we are studying a discrete system and therefore we should be projecting on just *one* of the α states. In this sense the Hilbert space of JT gravity in Euclidean signature is composed of all the α states. This is nothing like the Hilbert space of the dual quantum mechanical system which has one state $|\lambda\rangle$. for each energy level. Both Hilbert spaces result in the same amplitude though. Holography does not demand equality of Hilbert spaces but only equality of amplitudes. Therefore this is perfectly fine. For more on this see the concluding remarks in chapters 3 and 5.

4.4 Concluding remarks

It would be interesting to understand what these eigenbranes mean for a Lorentzian observer probing the gravitational bulk. Can he somehow obtain information about the branes hovering deep in the bulk? One way to work towards this would be to investigate boundary correlators in the matrix ensemble, see for example [18, 121]. It would be valuable to understand if we can construct bulk observables within JT gravity as a sum over these more complicated geometries, using geodesic localizing, in analogy to the construction of local bulk observables in the disk version of JT gravity [3]. We make comments on this in chapter 4. Work on this is forthcoming [66]. It would also be interesting to extend this discussion to AdS_3 gravity which is likely dual to a product of matrix ensembles, one for each Virasoro descendant level. The perturbative expansion would be closely related to a particular Chern-Simons theory on $\Sigma_{g,n}\times S_1$ and the eigenbranes would be fixed energy boundaries hovering in the bulk with a torus $S_1\times S_1$ topology but with no Virasoro descendants associated with them. Work on this is also in progress [45].

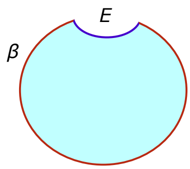
The main conclusion of this chapter seems to be that we might need to include even more exotic effects other than baby universes in our bulk quantum gravity if we want to have a sensible, and hence genuinely discrete, model of quantum gravity. In general it is actually rather natural to imagine that the information about the microstates in terms of energy levels should be imprinted somehow in boundary conditions of our gravitational theory, as discussed for example in the concluding remarks of chapter 3. The very recent work of [19] is strong general evidence in favor of that intuition. The eigenbrane story of this chapter is to be interpreted as a very explicit way of realizing that intuition.[35] Let us put general implications for quantum gravity aside for now though. If anything at least we have provided for a full bulk gravitational answer to Maldacena's information paradox [14] in JT gravity. Let us emphasize though that most of the work had already been done in [7, 8, 9]. Furthermore let us note that we have not technically discussed late time correlators in this chapter but as explained around (3.152) and in [18] this is a rather mild modification of the spectral form factor computation. The point is that we will see the full late time behavior as expected of a discrete theory if we include the possibility for baby universes to be absorbed and emitted by eigenbranes labeled by the energy levels of our discrete model of quantum gravity. This is the bulk explanation of late time correlators in quantum gravity. At least at low energies that is where we expect to find JT gravity rather universally.

We end this chapter with two further remarks.

Boundary mergers

The Dirac deltas that appear in the exact answers for the spectral densities in section 4.3 have an a posteriori interpretation as eigenvalue boundaries merging with the asymptotic boundaries.[36] Considering for example the last equality in (4.41). The first term can be read as counting Riemann surfaces which end on the merger of the two original boundaries of lengths β_1 and β_2, resulting in a boundary of total length $\beta_1+\beta_2$. Let us pretend here to take that interpretation seriously and count Riemann surfaces that end on a merged boundary. It is convenient to introduce the JT gravity disk amplitude between a fixed length state $|\beta\rangle$ and a fixed energy state $|E\rangle$:[37]

$$\langle\beta|E\rangle =$$



$$= e^{-\beta E}. \qquad (4.79)$$

The merger of an asymptotic disk with an eigenbrane results in the genus zero amplitude $\rho_0(\lambda)\,\langle\beta|\lambda\rangle$.[38] This merged boundary can connect to the other eigenvalue boundaries, and develop handles. In taking the sum, the overlap $\langle\beta|\lambda_i\rangle$ is a spectator. We end up with a factor that cancels precisely the denominator in (4.54), and we are left only

[35] As also appreciated by [19].

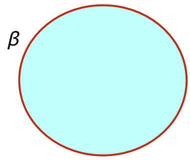
[36] See for example [78, 141].

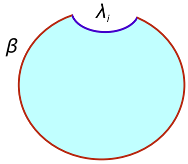
[37] These are the states used in [54, 57, 18], with $|\beta\rangle$ the Hartle-Hawking state of the JT gravity disk. We have:

$$|\beta\rangle = \int_0^\infty dE\, e^{-\beta E}\,\rho_0(E)\,|E\rangle\,, \quad \langle E_1|E_2\rangle = \frac{\delta(E_1-E_2)}{\rho_0(E)}, \quad \langle\beta_1|\beta_2\rangle = Z_{\text{JT}}(\beta_1+\beta_2). \qquad (4.78)$$

[38] This is the inverse Laplace transform of the boundary with length $\beta_1+\beta_2$ with respect to β_2.

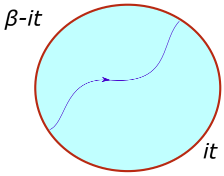
with $\langle \beta | \lambda_i \rangle$. As pointed out in section 4.3, all other contributions to the JT gravity partition function add up to zero. This suggests the net gravitational effect of fixing all eigenvalues (4.1) in JT gravity is the following replacement:

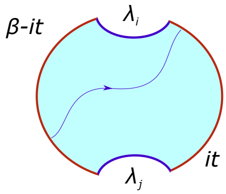
$Z_{\mathrm{JT}}(\beta) =$ 

$\to \sum_{i=1}^{N}$ 

$= \sum_{i=1}^{N} e^{-\beta \lambda_i}$. (4.80)

It is straightforward to extend this to holographic correlation functions.[39] As explained in chapter 3 we have that boundary correlators in JT gravity correspond to Wilson lines traversing the Riemann surfaces [6, 1, 2, 61, 156]. The Wilson line separates the Riemann surface into two disconnected pieces, each connected to a piece of boundary:[40]



$\to \sum_{i=1}^{N} \sum_{j=1}^{N}$ 

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} e^{-(\beta - it)\lambda_i}\, e^{-it\lambda_j} \left| \mathcal{O}_{\ell, \lambda_i \lambda_j} \right|^2. \tag{4.81}$$

Notice that there is precisely one eigenbrane for each eigenvalue. The diagonal contributions on the right hand side, which look as if they could be attributed to two identical eigenbranes merging into the two parts of the asymptotic boundary, are actually due to the two sides of the asymptotic boundary merging together to form an annulus configuration with an inner-boundary-to-outer-boundary Wilson line, and a single eigenvalue boundary merging into that annulus configuration. This is identical to the above picture though because $\delta(E_1 - E_2)\delta(E_1 - \lambda_i) = \delta(E_1 - \lambda_i)\delta(E_2 - \lambda_i)$. Let us highlight the following caveat though. The final expression (4.81) is the two point function of a discrete system for any operator $\mathcal{O}$. It may however be nontrivial given an actual physical system to find a precise operator $\mathcal{O}$ whose matrix elements are (3.145). One might hope to start with such an actual physical system such as one particular realization of the SYK model instead of this more abstract setup, and find a map between operators in that system and the Wilson lines or massive particles traveling through in the JT gravity bulk.

A gravitational hint of ensemble averaging?

[39] See [18, 121] for recent discussions.

[40] A similar such configuration with a vacuum Wilson line does not contribute to the JT gravity partition function, because the eigenvalues are chosen not to be degenerate, and merging two fixed energy boundaries to a fixed length boundaries results in an amplitude proportional to a Dirac delta on those energies.

The statistical ensemble we found from the matrix integral was interpreted gravitationally in terms of multiple boundaries. Here we illustrate that starting with gravity directly, one can get hints of this underlying ensemble, reversing the logic of this work. We start from a property of the n-boundary correlator in JT gravity (4.30):

$$Z(\beta_1 \dots \beta_n) \supset e^{(2-2g-n)S_0} \int_0^\infty db_1\, b_1\, Z(\beta_1, b_1) \cdots \int_0^\infty db_n\, b_n\, Z(\beta_2, b_n)\, V_{g,n}(b_1 \dots b_n). \tag{4.82}$$

Let us take the length of one of the boundaries to zero. The trumpet amplitude $Z(\beta)$ in (3.64) becomes a delta distribution for $\beta \to 0$. Taking $\beta_1 \to 0$ therefore localizes on spacetimes where the neck length b_1 vanishes. Due to the twist factor b_1 and the polynomial behavior of the Weil-Petersson volumes, we see that every perturbative contribution vanishes except for the case when the β_1-boundary is capped off by a disk. We end up with:

$$Z(0, \beta_1 \dots \beta_n) = Z(0)\, Z(\beta_1 \dots \beta_n). \tag{4.83}$$

Doing an n-fold inverse Laplace transform of this equation, we find:

$$\int_0^\infty d\lambda\, \rho(\lambda, \lambda_1 \dots \lambda_n) = \rho(\lambda_1 \dots \lambda_n) \int_0^\infty d\lambda \rho(\lambda) = \rho(\lambda_1 \dots \lambda_n)\, Z(0). \tag{4.84}$$

Recursively one gets from this:

$$\frac{1}{Z(0)^n} \int_0^\infty d\lambda_1 \dots d\lambda_n\, \rho(E, \lambda_1 \dots \lambda_n) = \rho(E), \tag{4.85}$$

$$\frac{1}{Z(0)^n} \int_0^\infty d\lambda_1 \dots d\lambda_n\, \rho(\lambda_1 \dots \lambda_n) = 1. \tag{4.86}$$

This suggests to think of $\rho(\lambda_1 \dots \lambda_n)\, Z(0)^{-n} = w(\lambda_1 \dots \lambda_n)$ as the weight function of a statistical ensemble. This is strengthened by (4.85) and its generalization to multiple E_i: correlators in JT gravity can be calculated as averages in this statistical ensemble. In particular the observable that calculates $\rho(E)$ is extracted from (4.85) as:

$$\int_{\mathcal{C}} d\lambda_1 \dots d\lambda_n\, w(\lambda_1 \dots \lambda_n)\, \rho(E)_{\lambda_1 \dots \lambda_n} = \rho(E), \qquad \rho(E)_{\lambda_1 \dots \lambda_n} = \frac{\rho(E, \lambda_1 \dots \lambda_n)}{\rho(\lambda_1 \dots \lambda_n)}. \tag{4.87}$$

This corresponds to the quantity we considered in the main text.

4.5 Supplementary material

In this supplementary section we calculate brane pair correlators in JT gravity of the type:

$$\left\langle \psi^2(E_1) \dots \psi^2(E_n) \right\rangle. \tag{4.88}$$

A single brane is defined as (4.34). As discussed in the main text around (4.43) this is an efficient way to calculate objects such as $R(E)$ and $R(E_1, E_2)$. We can rewrite the brane operator (4.34) as:

$$\psi(E) = e^{-\frac{LV(E)}{2}} \prod_{i=1}^{L} (E - \lambda_i) = \exp\left(-\frac{LV(E)}{2} + \operatorname{Tr}\log(E - M) \right). \tag{4.89}$$

The operator in the exponential corresponds to the insertion of an unmarked fixed energy boundary in JT gravity [9]:

$$\text{Disk}(E) = -\frac{LV(E)}{2} + \operatorname{Tr}\log(E - M) = -\int_{\mathcal{C}} \frac{d\beta}{\beta}\, e^{\beta E}\, Z(\beta). \tag{4.90}$$

This is the precise analogue of an unmarked FZZT boundary brane in Liouville theory [154, 157]. Equation (4.89) is slightly misleading in combination with (4.90) though. The original brane correlator (4.34) is an analytic function of E, whereas the FZZT brane (4.90) has a discontinuity on the positive real axis. Consequently, to each energy E correspond two different FZZT boundaries in gravity, depending on how we approach the real axis. This is equivalent to specifying the value of $\sqrt{-E}$ for $E > 0$. Let us introduce a new variable $e = i\sqrt{E}$, then $\sqrt{-E} = \pm e$ for $E > 0$. Depending on this sign, exponentiating the FZZT boundary (4.90) gives two distinct gravitational brane correlators $\langle \psi(\pm e) \rangle$. This raises the question which gravitational brane corresponds to inserting the brane operator (4.34) in the matrix integral. The answer was given in [153].[41] The brane correlators have an exact expression for finite e^{S_0} as a Kontsevich matrix integral, or an appropriate JT gravity generalization thereof.[42] For $e^{S_0} \gg 1$ we can use the method of Laplace on this Kontsevich matrix integral. Depending on whether the energy parameters $E_1 \dots E_n$ are positive or negative, different saddles contribute due to Stokes phenomena. Each such saddle and the loop corrections around it correspond to a gravitational brane. It turns out that for all energies $E_1 \dots E_n$ positive, we need to sum over all possible corresponding gravitational branes with equal weight. For each such saddle, if we furthermore take $E \gg e^{-2S_0/3}$ only the exponentiation of disks and annuli contributes significantly.[43] We will be working throughout in the regime $e^{S_0} \gg 1$. In most of this supplementary section we furthermore assume $E \gg e^{-2S_0/3}$. In section 4.5.1 we calculate the correlators close to the spectral edge using the Airy model. The Airy calculations are exact for any e^{S_0} and by construction coincide with the JT gravity answers for $E \ll 1$. For $e^{S_0} \gg 1$ these regions overlap. The result is that we can obtain a precise answer for any value of E.

One brane pair

Consider now the calculation of $R(E)$ corresponding to a single brane pair (4.39). Summing all saddles results in:

$$\left\langle \psi^2(E) \right\rangle = \langle \psi(e)\psi(e) \rangle + \langle \psi(e)\psi(-e) \rangle + \langle \psi(-e)\psi(e) \rangle + \langle \psi(-e)\psi(-e) \rangle\,. \tag{4.91}$$

[41] See also [9].

[42] See [141].

[43] Higher genus surfaces give multiplicative contributions of the type $e^{\chi e^{S_0} \cdots}$. Here $\chi < 0$ and the ... polynomials in $1/E_1$ multiplied with $(E_1 \dots)^{-1/2}$. This follows from a modification of the calculations in section 4.2 to unmarked boundaries.

As explained above and in [153, 9] only disks and annuli are significant in the regime we are focusing on:

$$\langle \psi(e_1)\psi(e_2)\rangle \approx e^{\text{Disk}(e_1)+\text{Disk}(e_2)+\frac{1}{2}\text{Ann}(e_1,e_1)+\frac{1}{2}\text{Ann}(e_2,e_2)+\text{Ann}(e_1,e_2)}. \tag{4.92}$$

One obtains the fixed energy disk and annuli as Laplace transforms of the leading JT gravity fixed length disk (4.13) and annuli amplitude (4.32), as in (4.90). The result is [9]:

$$\text{Disk}(\pm e) = \pm i\pi \int_0^E dM \rho_0(M), \quad \text{Ann}(e_1,e_2) = -\ln(e_1+e_2). \tag{4.93}$$

Note that these indeed both depend explicitly on the sign $\pm e$. We get:

$$\langle \psi(e_1)\psi(e_2)\rangle = \frac{1}{2\sqrt{e_1}\sqrt{e_2}(e_1+e_2)} e^{\text{Disk}(e_1)+\text{Disk}(e_2)}. \tag{4.94}$$

We can use:

$$\begin{aligned}
\langle \psi(e)\psi(e)\rangle + \langle \psi(-e)\psi(-e)\rangle &= -\frac{1}{2E}\cos\left(2\pi\int_0^E dM\rho_0(M)\right) = 2\pi\rho_{\text{nonp}}(E)\\
\lim_{e_1\to e_2} \langle \psi(e_1)\psi(-e_2)\rangle + \langle \psi(-e_1)\psi(e_2)\rangle &= \lim_{e_1\to e_2} \frac{\sinh(\text{Disk}(e_1)-\text{Disk}(e_2)))}{\sqrt{E}(e_1-e_2)}\\
&= \frac{1}{\sqrt{E}}\partial_e \text{Disk}(e).
\end{aligned} \tag{4.95}$$

We then find:

$$\left\langle \psi^2(E)\right\rangle = 2\pi\rho_0(E) - \frac{1}{2E}\cos\left(2\pi\int_0^E dM\rho_0(M)\right) = 2\pi R(E). \tag{4.96}$$

This matches the result of the resolvent-based brane dipole calculation of $R(E)$ in [9].

Two brane pair

Next we calculate the two brane pair correlator $\left\langle \psi^2(E_1)\psi^2(E_2)\right\rangle$. For notational purposes consider $\left\langle \psi^2(E)\psi^2(K)\right\rangle$ with $e = i\sqrt{E}$ and $k = i\sqrt{K}$. Summing the 16 saddles gives:

$$\left\langle \psi^2(E)\psi^2(K)\right\rangle = \sum_{\text{signs}} \langle \psi(\pm e)\psi(\pm e)\psi(\pm k)\psi(\pm k)\rangle\,. \tag{4.97}$$

The generic brane correlator is similar to (4.92):

$$\left\langle \prod_i \psi(e_n)\right\rangle = \prod_i e^{\text{Disk}(e_i)+\frac{1}{2}\text{Ann}(e_i,e_j)} \prod_{j\neq i} e^{\text{Ann}(e_i,e_j)}. \tag{4.98}$$

Using (4.93) we obtain:

$$\langle\psi(e_1)\psi(e_2)\psi(e_3)\psi(e_4)\rangle \approx \frac{1}{4\sqrt{e_1}\sqrt{e_2}\sqrt{e_3}\sqrt{e_4}} \times \frac{e^{\mathrm{Disk}(e_1)+\mathrm{Disk}(e_2)+\mathrm{Disk}(e_3)+\mathrm{Disk}(e_4)}}{(e_1+e_2)(e_1+e_3)(e_1+e_4)(e_2+e_3)(e_2+e_4)(e_3+e_4)}. \tag{4.99}$$

It is now a straightforward but somewhat tedious task to evaluate (4.97). The 16 terms fall in three classes. Firstly there are 4 terms where the signs match within each brane pair:

$$\sum_{s_a,s_b=\pm}\langle\psi(s_a e)\psi(s_a e)\psi(s_b k)\psi(s_b k)\rangle = \frac{2\cosh(2\mathrm{Disk}(e){-}2\mathrm{Disk}(k))}{16EK(e-k)^4}+\frac{2\cosh(2\mathrm{Disk}(e){+}2\mathrm{Disk}(k))}{16EK(e+k)^4}. \tag{4.100}$$

Secondly, there are 8 mixed terms:

$$\begin{aligned}\sum_{s_a,s_b=\pm}&\langle\psi(s_a e)\psi(s_a e)\psi(s_b k)\psi(-s_b k)\rangle+\sum_{s_a,s_b=\pm}\langle\psi(s_a e)\psi(-s_a e)\psi(s_b k)\psi(s_b k)\rangle\\ &=\frac{4\pi^2}{(E-K)^2}\rho_0(K)\,\rho_{\mathrm{nonp}}(E)+\frac{4\pi^2}{(E-K)^2}\rho_0(E)\,\rho_{\mathrm{nonp}}(K)\\ &\quad+\frac{\sinh(2\mathrm{Disk}(k))-\sinh(2\mathrm{Disk}(e))}{i\sqrt{-e^2}\sqrt{-k^2}(e^2-k^2)^3}.\end{aligned} \tag{4.101}$$

The remaining 4 terms have opposite signs within each brane pair. These terms require a double use of the Hopital rule:

$$\sum_{s_a,s_b=\pm}\langle\psi(s_a e)\psi(-s_a e)\psi(s_b k)\psi(-s_b k)\rangle = \frac{4\pi^2}{(E-K)^2}\rho_0(E)\rho_0(K)-\frac{(K+E)}{(K-E)^4}\frac{1}{\sqrt{E}\sqrt{K}}.$$

One recognizes the first term as the product of two perturbative disks and the second term as the perturbative annulus (4.33). Adding these three contributions and multiplying by $(E-K)^2/4\pi^2$ gives the exact pair density correlator for $e^{S_0}\gg 1$ away from the spectral edge. We can distill from the exact answer $T(E,K)$ in JT gravity:

$$\begin{aligned}R(E,K)=&R(E)R(K)-\frac{(K+E)}{4\pi^2(K-E)^2}\frac{1}{\sqrt{E}\sqrt{K}}+\frac{\sinh(2\mathrm{Disk}(k))-\sinh(2\mathrm{Disk}(e))}{4\pi^2 i\sqrt{E}\sqrt{K}(K-E)}\\ &+\frac{2\cosh(2\mathrm{Disk}(e))\cosh(2\mathrm{Disk}(k))}{4\pi^2(K-E)^2}\\ &-\frac{\sinh(2\mathrm{Disk}(e))\sinh(2\mathrm{Disk}(k))(K+E)}{4\pi^2(K-E)^2}\frac{1}{\sqrt{E}\sqrt{K}}.\end{aligned} \tag{4.102}$$

This connected contribution contains the perturbative annulus as first term. The remainder is the nonperturbative contribution to the annulus. To uncover the GUE structure (4.48) we focus on $|E-K| \ll 1$.[44] This simplifies things considerably:

$$\frac{\sinh(2\mathrm{Disk}(k)) - \sinh(2\mathrm{Disk}(e))}{i\sqrt{EK}(K-E)^3} = -\frac{8\pi^2}{(E-K)^2}\rho_0(E_+)\,\rho_{\mathrm{nonp}}(E_+) + \mathcal{O}\left(\frac{1}{E-K}\right). \tag{4.103}$$

Furthermore:

$$\begin{aligned}
&\frac{1}{16EK}\frac{2\cosh(2\mathrm{Disk}(e) - 2\mathrm{Disk}(k))}{(e-k)^4} - \frac{(K+E)}{(K-E)^4}\frac{1}{\sqrt{KE}} \\
&= -\frac{4}{(E-K)^4}\sin^2(\pi\int_K^E dM\,\rho_0(M)) \\
&\quad - \frac{2}{E_+^2(E-K)^2}\sin^2(\pi\int_K^E dM\,\rho_0(M)) + \mathcal{O}\left(\frac{1}{E-K}\right).
\end{aligned} \tag{4.104}$$

Collecting everything, we find for $|E_1 - E_2| \ll 1$:

$$\begin{aligned}
R(E_1,E_2) &= \rho_0(E_1)\rho_0(E_2) + \rho_0(E_1)\rho_{\mathrm{nonp}}(E_2) + \rho_0(E_2)\rho_{\mathrm{nonp}}(E_1) - 2\rho_0(E_+)\rho_{\mathrm{nonp}}(E_+) \\
&\quad - \rho_0(E_1)\rho_0(E_2)\mathrm{sinc}^2\pi\rho_0(E_+)(E_1-E_2) - \frac{1}{2E_+^2}\sin^2\pi\rho_0(E_+)(E_1-E_2) \\
&= \rho_0(E_1)\rho_0(E_2)(1 - \mathrm{sinc}^2\pi\rho_0(E_+)(E_1-E_2)) + R_{\mathrm{wiggle}}(E_1,E_2).
\end{aligned} \tag{4.105}$$

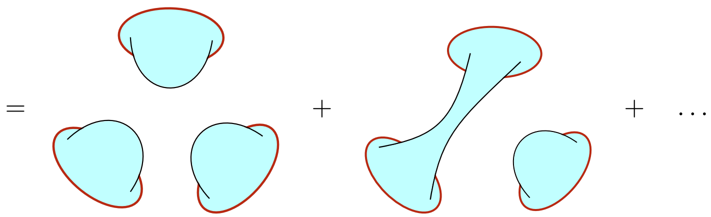
The first term is the well known GUE result. the second term is small and oscillatory. It is the analogue of the wiggles (4.17) in $R(E)$. For the purpose of our story in the main text these wiggles are negligible but for the fact that $R_{\mathrm{wiggle}}(E_1,E_1) = 0$. This implies that $R(E_1,E_1) = 0$ exactly, as demanded by eigenvalue repulsion in the ensemble.

More brane pairs

This procedure readily extends to a generic number of brane pairs. The perturbative contribution is found by picking opposite signs within each brane pair. Only then is there no oscillatory contribution. For example for $n=3$ after a tedious tripple application of Hopital one recognizes the perturbative disks and annuli:

$$\begin{aligned}
&R(E,K,M)_{\mathrm{pert}} \\
&= \frac{(e^2-k^2)^2(k^2-m^2)^2(m^2-e^2)^2}{8\pi^3}\sum_{\mathrm{signs}}\langle\psi(\pm e)\psi(\mp e)\psi(\pm k)\psi(\mp k)\psi(\pm m)\psi(\mp m)\rangle \\
&= \rho_0(E)\rho_0(K)\rho_0(M) + \rho_0(E)\frac{(k^2+m^2)}{(k^2-m^2)^2}\frac{1}{\sqrt{-k^2}\sqrt{-m^2}} \\
&\quad + \rho_0(K)\frac{(m^2+e^2)}{(m^2-e^2)^2}\frac{1}{\sqrt{-m^2}\sqrt{-e^2}} + \rho_0(M)\frac{(e^2+k^2)}{(e^2-k^2)^2}\frac{1}{\sqrt{-e^2}\sqrt{-k^2}}
\end{aligned}$$

[44] We introduce $E_- = (E-K)/2$ and $E_+ = (K+E)/2$.



We didn't draw the other cyclic permutations of the second diagram for comfort. It is reassuring to see these and only these perturbative contributions appear. Notice for example that there is no perturbative three holed sphere contribution, nor are there higher genus geometries. This is consistent, as discussed in the beginning of this section those geometries don't contribute significantly. On the other hand, the full $R(E, K, M)$ does for example contain the nonperturbative corrections associated with the genus expansion seeded by the three-holed sphere, which *are* significant.[45]

4.5.1 Fixing eigenvalues near the spectral edge

Close to the spectral edge $|E| \ll 1$, JT gravity reduces to topological gravity or the Airy model with spectral density:[46]

$$\rho_0(E) = \frac{\sqrt{E}}{\pi}. \tag{4.106}$$

This theory is identical to the $(2,1)$ minimal string. The $(p,1)$ minimal strings are topological, and for these models the multi brane correlators can be calculated exactly for any value of the string coupling.[47] This is the content of formula (1.11) in [153]. In the case of the $(2,1)$ minimal string, we have:

$$\langle \psi(x) \rangle = \mathrm{Ai}(x), \qquad x = -E. \tag{4.107}$$

Multi brane correlators are then calculated using formula (1.11) in [153]:[48]

$$\left\langle \prod_{i=1}^{n} \psi(x_i) \right\rangle = \frac{\Delta^{1/2}(\partial_1 \ldots \partial_n)}{\Delta^{1/2}(x_1 \ldots x_n)} \prod_{i=1}^{n} \langle \psi(x_i) \rangle \,. \tag{4.108}$$

It is again straightforward, but slightly tedious to calculate the multi brane pair correlators that get the Airy cluster functions. We'll show how this goes for $R(E)$ and

[45] We could find more perturbative contributions by including the exponentials of these surfaces in the brane correlators such as (4.92).

[46] We have rescaled the energy, removing the e^{S_0} dependence here.

[47] It is solvable because we can solve the two coupled differential equations that define the single brane correlator. This function is known as a Baker-Akhiezer function of the KP hierarchy. For more on that see for example [153] or the lecture notes [158].

[48] We have $\langle \psi(x,\tau) \rangle = \mathrm{Ai}(x+\tau)$, therefore $\partial_\tau = \partial_x$, which is one of the two differential equations that define the Baker-Akhiezer function in question. The other one is the Airy equation. By rescaling the energy axis we can eliminate the τ dependence.

$R(E_1, E_2)$, and investigate the Airy spectral density with one fixed eigenvalue $\langle\rho(E)\rangle_\lambda$. For the two-brane correlator, we have:

$$\langle\psi(x_1)\psi(x_2)\rangle = \frac{\langle\psi(x_1)\rangle' \langle\psi(x_2)\rangle - \langle\psi(x_1)\rangle \langle\psi(x_2)\rangle'}{x_1 - x_2}. \tag{4.109}$$

Setting $x_1 \to x_2$, one finds:

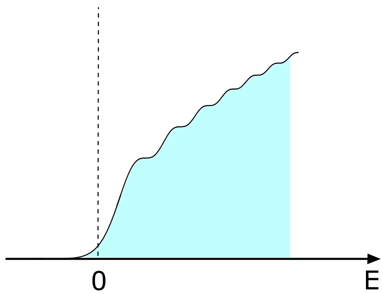
$$\left\langle\psi(x)^2\right\rangle = \langle\psi(x)\rangle'' \langle\psi(x)\rangle - \langle\psi(x)\rangle'^2 . \tag{4.110}$$

Inserting the solution (4.107), and using the Airy equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, this becomes:

$$\left\langle\psi(x)^2\right\rangle = x\mathrm{Ai}(x)^2 - e^{2S_0}\mathrm{Ai}'(x)^2. \tag{4.111}$$

This is proportional to the Airy spectral density:[49]

$$R(E) = -\left\langle\psi^2(-E)\right\rangle = \tag{4.112}$$



To calculate the two-brane-pair correlator, we are led to consider (4.108):

$$\begin{aligned}&\langle\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\rangle\\ &= \frac{(\partial_1-\partial_2)(\partial_1-\partial_3)(\partial_1-\partial_4)(\partial_2-\partial_3)(\partial_2-\partial_4)(\partial_3-\partial_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)}\mathrm{Ai}(x_1)\mathrm{Ai}(x_2)\mathrm{Ai}(x_3)\mathrm{Ai}(x_4).\end{aligned}$$

The partial derivatives generate a total of 64 terms of which some cancel, but 24 remain. For example the first term we would write down is:

$$\langle\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\rangle \supset \frac{\mathrm{Ai}'''(x_1)\mathrm{Ai}''(x_2)\mathrm{Ai}(x_3)\mathrm{Ai}(x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)}. \tag{4.113}$$

Each such term has 6 derivatives to distribute among the Airy functions with a maximum of 3 per Airy. Taking $x_1 \to x_2 = x$ and $x_3 \to x_4 = y$, one ends up with terms such as:

$$\left\langle\psi^2(x)\psi^2(y)\right\rangle \supset \frac{1}{(x-y)^4}\mathrm{Ai}''''(x)\mathrm{Ai}''(x)\mathrm{Ai}''(y)\mathrm{Ai}(y). \tag{4.114}$$

[49]The normalization of the wavefunction (4.107) is chosen different from that in (4.94), hence the different proportionality factor.

Each term has now 8 derivatives to distribute among the Airy functions, with a maximal of 4 per Airy. Repeatedly applying the Airy equation, one finds after what is very much a bookkeeping exercise:

$$\left\langle \psi^2(x)\psi^2(y)\right\rangle = \frac{1}{(x-y)^2}\left\langle \psi^2(x)\right\rangle \left\langle \psi^2(y)\right\rangle - \frac{1}{(x-y)^2}K(x,y)^2, \tag{4.115}$$

with $K(x,y)$ the well-known Airy kernel:

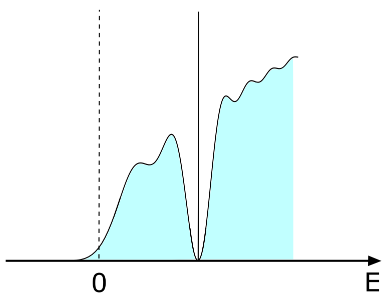
$$K(x,y) = \frac{\mathrm{Ai}'(x)\mathrm{Ai}(y) - \mathrm{Ai}(x)\mathrm{Ai}'(y)}{x-y}. \tag{4.116}$$

This replaces the role of the sine kernel $S(E_i, E_j)$ for GUE away from the spectral edge, also in higher cluster functions [119].
Now that we have the appropriate clusters near the spectral edge, we can redo the analysis of section 4.3 and fix eigenvalues in this region, as formulas (4.56) and (4.70) are completely general. For example:

$$\langle \rho(E)\rangle_\lambda = \delta(E-\lambda)R(E) - \frac{K(E_1,E_2)}{R(\lambda)}, \tag{4.117}$$

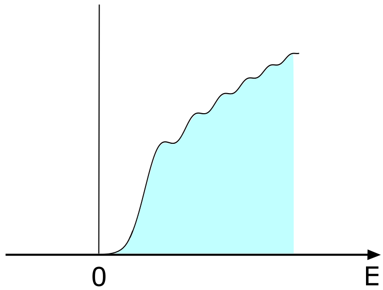
with $R(E)$ from (4.112). For an eigenvalue not too close to the spectral edge, we have:

$\langle \rho(E)\rangle_\lambda =$ (4.118)



One recognizes the same features as in (4.63). It is interesting to see what happens when we insert an eigenvalue very close to the spectral edge or even in the forbidden region $E < 0$. Note that this is a very unlikely situation since the total spectral density in the forbidden region is much less than one. Hence, when inserting an eigenvalue in the forbidden region, we expect a depletion in the continuum of essentially the entire forbidden region and of the region closest to the spectral edge. Armed with our exact Airy formula (4.117) we find this is indeed the case. For example when fixing an eigenvalue at the origin, one finds:

$$\langle \rho(E) \rangle_0 = \qquad (4.119)$$



5 Matter probes close to the horizon

This chapter is based largely on a publication [3] by the author in collaboration with Thomas Mertens and Henri Verschelde. The goal is to understand the physics of matter probes deep in the bulk of JT quantum gravity. We point out that generic properties of late time correlators in finite volume holography [14] translate into equally generic effects on physics close to the black hole horizon. In particular the non decaying behavior of boundary correlators is found to imply a breakdown of the cluster decomposition principle in quantum gravity at large spatial separations. It furthermore implies that quantum field theory in Rindler space is not a good approximation to quantum gravitational physics at Planck scale distances from the semiclassical black hole horizon. This is confirmed via explicit calculations in JT gravity.

5.1 Introduction

Thus far in this work, when discussing matter probes in JT gravity, we've considered inserting matter probes on or very close to the asymptotic boundary. For example, the boundary anchored Wilson lines discussed mostly in chapter 2 can be understood as the quantum gravitational expectation values of boundary to boundary propagators for some bulk scalar particle of mass $m^2 = \ell(\ell-1)$. Alternatively, using the extrapolate holographic dictionary, they can be understood as the primary two point function of a 1d conformal field theory coupled to 1d Schwarzian quantum mechanics. Let us elaborate a bit on this interpretation. Working in AdS_2 with Wick rotated Poincaré coordinates the extrapolate holographic dictionary states:

$$\langle \phi_{m^2}(t_1, z_1)\phi_{m^2}(t_2, z_2)\rangle = z_1^\ell\, z_2^\ell\, \langle \mathcal{O}_\ell(t_1)\mathcal{O}_\ell(t_2)\rangle\,, \quad m^2 = \ell(\ell-1), \quad z_1,\, z_2 \ll 1. \tag{5.1}$$

The right hand side is a boundary 1d conformal field theory correlation function. Normalizing our operators suitably the answer is:

$$\langle \mathcal{O}_\ell(t_1)\mathcal{O}_\ell(t_2)\rangle = \frac{1}{(t_1-t_2)^{2\ell}}. \tag{5.2}$$

Coupling this to the 1d Schwarzian reparameterization mode which we'll here refer to as $f(t)$ this becomes:[1]

$$\langle f \cdot \mathcal{O}_\ell(t_1)\, f \cdot \mathcal{O}_\ell(t_2) \rangle = \frac{f'(t_1)^\ell f'(t_2)^\ell}{(f(t_1) - f(t_2))^{2\ell}}. \tag{5.3}$$

Remember that we're talking about conformal primaries. The associated covariant transformation law gives the nominator in this expression. This function is to be inserted in a Schwarzian path integral in order to couple to quantum gravity. It is indeed the holographic dual to a boundary anchored Wilson line in JT gravity (2.97). This confirms that boundary anchored Wilson lines can be interpreted with suitable holographic renormalization as a boundary to boundary propagator of a massive scalar field $\phi^2_m(t,z)$ in AdS_2.

In this chapter we would like to understand more generally how to compute quantum gravity expectation values of bulk scalar matter probes in AdS_2. In particular, say we denote the matter two point function in some fixed metric g as:

$$\langle g \cdot \phi_{m^2}(t_1, z_1)\, g \cdot \phi_{m^2}(t_2, z_2) \rangle\,. \tag{5.4}$$

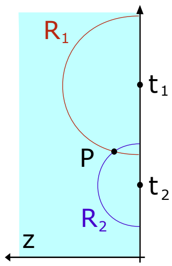
We now imagine that the fields can be anywhere in the bulk. We would like to pinpoint precisely how matter correlation functions like this depend on the different metrics in the JT gravity path integral. Then we would like to do the actual gravitational path integral. Schematically we thus want to calculate for example:

$$\int [\mathcal{D}g]\, \delta(R+2)\, \langle g \cdot \phi_{m^2}(t_1, z_1)\, g \cdot \phi_{m^2}(t_2, z_2) \dots \rangle\, e^{-S[g]}. \tag{5.5}$$

There are several subtleties in trying to do such a calculation. The most pressing one is that we need to specify what is meant by the coordinates such as t_1 and z_1 that enter into this expression. When thinking about the above as just a calculation in quantum field theory that wouldn't be an issue. Here we are not trying to do quantum field theory though. We are doing quantum gravity. Gravity as we all know is a gauge theory of which the associated gauge transformations are diffeomorphisms or coordinate transformations that leave the scalar curvature unaffected. This means that a priori bulk coordinates in a theory of quantum gravity are not physical objects. They are gauge covariant. Therefore just naively calculating a path integral such as (5.5) would give you an answer, but it would have zero physical worth.
What is required, if we want to do reasonable calculations, is a gauge invariant definition of bulk coordinates. In a holographic context this is possible. We need to define the bulk coordinates in a boundary intrinsic way. To understand why this is the case, remember that large gauge transformations (unlike their small cousins) are seldom redundant. In the case of gravity one could say that diffeomorphism invariance is broken on the boundary. Regardless of how one wants to phrase this, boundary coordinates (unlike bulk coordinates) are physical. In JT gravity this is very explicit, much like in AdS_3

[1] In chapter 2 we referred to this as $\gamma_-(t)$.

gravity. The Schwarzian field $f(t)$ was originally understood as a reparameterization mode which assigns a nonzero action $S[f]$ to a boundary reparameterization which takes t to $f(t)$ [21, 22, 23]. The Schwarzian action therefore represents an explicit breaking of the boundary diffeomorphism invariance. Besides boundary coordinates, there are but a hand full of gauge invariant observables we could imagine in gravity. One of those is the length of geodesics. Combining the length along a geodesic with the physicality of boundary coordinates, we are able to define bulk points in an invariant manner. One simply fixes the location of a bulk point by specifying the geodesic distance to this point from several reference points on the boundary. In two dimensional Euclidean space this could look for example like:



. (5.6)

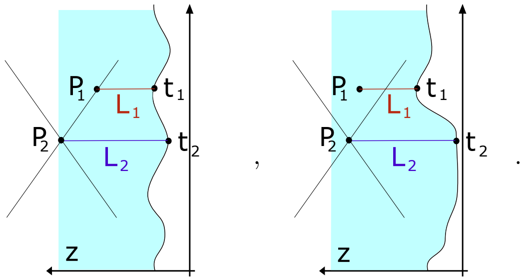
Lines of constant Euclidean geodesic distance to a boundary anchor are semicircles. Two radii and two anchor points uniquely determine a point in the bulk. This is in spirit much like how satellite navigation works, which uses lightlike geodesics in combination with the location of four reference points (the satellites) to determine uniquely one's position. The number of such reference points required to uniquely specify one's location depends on the number of dimensions. In particular in AdS_2 we require only two such points. This way of defining bulk points is known as geodesic localizing, and it is quite popular. See for example [159, 160] in general and [161, 162, 163, 164, 165, 166, 167] for the often studied case of AdS_3.

A further subtlety that arises when trying to define bulk operators in quantum gravity is that of potential non localities. At this point we would like to distinguish two types of non locality. The first is related to UV completeness. The second is related to the diffeomorphism invariant construction of local bulk operators. We associate locality with the question of whether or not there are poles in correlators when two operators are lightlike separated. A related property is the vanishing of commutators for all spacelike separated operators. The UV type of non locality is easily understood by acknowledging that a UV complete theory can by definition not have any divergences. One example is a discrete finite dimensional 1d quantum mechanical system. For example a single copy of the SYK model. Any correlator is a sum of a finite number of finite terms, so the result is manifestly finite. consequently the bulk dual to SYK is nonlocal [27]. A second famous example is string theory. This is manifestly nonlocal as all fundamental observables are built out of nonlocal strings. In summary, UV non-localities are an integer part of any full fledged theory of everything. There are no such non-localities in JT gravity as we will see. This is because the theory is not meant to be UV complete. It is rather meant to be a universal low energy description of a generic theory of everything. For more on this see section 5.4. The second type of non locality has nothing to do with UV completeness. Rather it associated with the inherent non localities in the construction of diffeomorphism invariant bulk operators. Roughly speaking this is because such a construction uses geodesics, which are non local. One expects to see traces of these non-localities in computations involving geodesically localized bulk operators [159, 168, 169, 170, 171, 172, 173, 174]. Giddings and collaborators have argued extensively that these types of non localities are very generic, in the sense that most ways of geodesic localizing will result in non local bulk operators [159, 168, 169, 170, 171, 172, 173, 174]. Most, but not necessarily all. Our point of view is that a sensible definition of geodesic localizing is one where the resulting local bulk operators are actual *local* bulk operators. In other words we take the stance that the Giddings type non-localities are a constraint on bulk reconstruction guiding us towards a more unique definition of bulk operators. This is in line with the general philosophy of this work. We should be looking for physically reasonable principles that guide us towards a sensible definition of what quantum gravity is. Two of these are discreteness and random matrix statistics in the spectrum, which we implemented in chapter 4. Another one would be unitary black hole evaporation, with a related resolution [10, 11, 19]. We will add to that the existence of genuinely *local* bulk operators in the IR effective description. One way to motivate this is as follows. The Giddings type of non localities make life more difficult. They do not however immediately seem to resolve any issues or puzzles. This should be contrasted with the UV type non localities. Surely these also make life more difficult. More importantly though they are consequences of a huge big positive, namely UV completeness. Why not try to avoid an effect that only seems to make life more difficult.

With this in mind we are motivated to consider a "preferred" type of geodesic localizing where the geodesics in question are incoming and outgoing lightrays or null geodesics. We imagine placing a fictitious mirror at the location of some bulk point and define the spatial bulk coordinate of a point to be one half the time it takes a light ray emitted by the boundary observer, to reflect off said mirror, and find its way back to the boundary observer's detector. This implements the old radar definition of constructing a coordinate frame. Via this definition lightlike separation of bulk fields is unambiguously defined. Two bulk points are lightlike separated if and only if they share an "anchor" on the boundary. This may seem like a trivial feature. However when you think about it, it is fairly unique to this lightlike geodesic localizing. Say we would use a spatial geodesic and let us focus on the case of two dimensional gravity. As explained in chapter 2 (and as reemphasized below) we can think about the relevant metrics as defined by some wiggly boundary curve in AdS_2. Different wiggles then effectively move the operators around in the rigid bulk when we specify the location of the operators in the bulk via geodesic localizing with respect to the wiggly boundary.[2] There is then a priori no reason why two operators which are lightlike separated for one wiggle, would remain lightlike separated for another wiggle. Suppose for example we construct a point P in the bulk of Lorentzian AdS_2 by specifying some boundary time coordinate t as well as some regularized distance L along a purely spacelike geodesic from the boundary point t

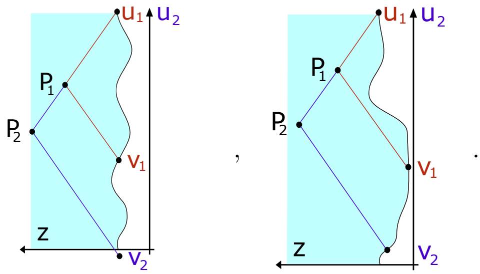
[2] See picture (2.35) for some wiggles.

inwards. The construction is explained in more detail in [3]. Imagine now we construct two such points P_1 and P_2 for different boundary wiggles:



(5.7)

We chose the points such that they were lightlike separated for the first wiggle. Clearly for generic other wiggles the two operators will not be lightlike separated. Therefore there is clearly some kind of non locality a la Giddings in the bulk if we would use this definition. Using lightlike geodesic localizing however bulk locality is automatic:



(5.8)

Independent of the wiggle, bulk lightlike separation follows from the fact that the two operators share a boundary anchor. The wiggles are bijective so sharing a boundary anchor is diff invariant information that is well defined when ensemble averaging over different wiggles. Given that we use lightlike geodesic localizing, bulk locality is then one to one with the existence of a pole in boundary correlators when two boundary operators approach. The latter is ubiquitous in non UV complete theories, see also section 5.4. By consequence we expect to find a sharp causal structure for the corresponding bulk operators in a non UV complete theory of quantum gravity. We will confirm very explicitly that this works as advertised in the case of JT gravity. We note that this light ray geodesic localizing of bulk points was investigated in generic dimensions in [175].

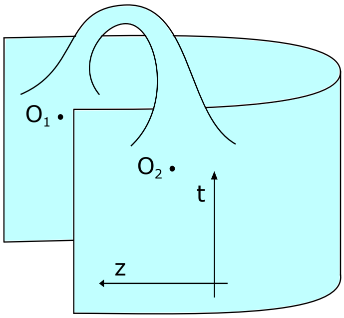
Having specified local bulk operators, we will finally be able to get on with it and just calculate path integrals of the type (5.5). In doing so it will be vital to keep in mind precisely what we mean when we write $\phi_{m^2}(t_1, z_1)$, in particular concerning the meaning of the coordinates. As will become clear, specifics of the anchoring of bulk coordinates to the boundary largely determines how bulk matter couples to the gravitational degrees of freedom. Let us elaborate a bit on this phenomenon. We can think of the metrics contributing to the JT gravity path integral as different patches of rigid AdS_2 space via the $\delta(R+2)$. Let's choose coordinate X for this rigid space. Naively when considering matter-coupled JT gravity one might think of correlators of operators like $\phi(X)$. One might then be led to think that the matter correlator will return some fixed answer independent of the shape of the patch, and factor out of the gravitational path integral. However, a generic invariant definition of bulk coordinates, including our radar definition, implies that the actual location X of the bulk operator insertion for given invariant data, actually depends on the shape of this patch. This is manifest in (5.7) and (5.8). The bulk point becomes fuzzy, which represents the coupling of the matter to quantum gravity. We strongly emphasize therefore that it is incorrect to simply think about quantum fields in rigid AdS_2 with a wiggly boundary. The quantum fields basically "wiggle along" with the boundary.

One main motivation for investigating such bulk matter probes is that they enable us to probe physics in the near horizon region. We have a very strong intuition about physics at late times and close to the boundary via the holographic duality. One of the most striking features of late time physics is known as Maldacena's version of the information paradox [14]. Really it is not so much a paradox but rather a general lesson on how quantum gravity is different from quantum fields in curved spacetime. In particular late time correlators of quantum fields in a classical black hole background decay exponentially fast to zero at late times. On the contrary, quantum gravity is a discrete quantum chaotic system. Its correlators can never actually decay to zero. Rather they oscillate erratically around a generically nonzero average. In the previous chapters we've given a potential bulk explanation of this phenomenon as associated with wildly nonperturbative effects in quantum gravity.[3] This involved the splitting and joining of baby universes which can be furthermore be absorbed in or emitted by eigenbranes. Now basic intuition from Penrose diagrams tells us that late time physics on the boundary should be similar to to near horizon physics at some fixed Cauchy slice. We therefore expect that the non decaying behavior of boundary correlators maps to a non decaying behavior of bulk matter correlators at large spatial separation at some fixed time slice.[4] In other words we expect that Maldacena's information paradox on the boundary maps to a breakdown of the cluster decomposition principle in quantum gravity. The cluster decomposition principle is of course at the hart of local quantum field theory. One might even go as far as to say that you can't have one without having the other [176]. So this seems potentially quite paradoxal. But really it is not. The point is that quantum field theory is defined on a manifold of fixed topology. JT quantum gravity on a fixed topology is thus a quantum field theory. In general though in quantum gravity we want to sum over topologies. The result is not a local quantum field theory. Consequently, adding other topologies immediately results in a breakdown of cluster decomposition. The intuition for this comes from [18]. Two operators which are far apart on some "trivial topology" with no wormholes present, can be brought close together by the possibility of emission of a baby universe close to the first operator, followed by absorption of that same baby

[3] Building strongly on a seminal series of papers by other authors [7, 8, 9].

[4] Obviously they can be at different times, but the physics isn't particularly more or less interesting when we take a finite time separation.

universe close to the second operator:



. (5.9)

The two operators in this configuration are far apart on the "trivial" topology which we have bent here for the purpose of presentation. Via cluster decomposition their correlator goes to zero exponentially fast in a local quantum field theory. This will not happen in quantum gravity. The operators are close by on this baby universe configuration. In summary, baby universes can and will create short cuts so that cluster decomposition from the perspective of quantum fields on the trivial topology does not hold in quantum gravity.
In fact we already expect to see some some traces of this phenomenon in the simple model of JT gravity of chapter 2 where we restrict to the trivial topology. In that case we have that late time correlators decay as power laws in the quantum theory rather than as exponentials. We consequently expect power law decay for correlators of bulk matter at large spatial separations. One could interpret this as a "minor break down" of cluster decomposition in the sense that the correlation decays much slower than expected semiclassically. It is this effect on which we focus most of our attention in this chapter. It is not difficult to include the corrections from the other topologies and even from discretization. We will not elaborate on this too much because such a discussion is yet to appear [66]. The purpose of this work remains to review rather than to innovate. Nevertheless we can't resist making several short comments in section 5.3.

Considering large spatial separations automatically takes at least one of the operators into what semiclassically would be the near horizon region. It is useful for the cluster decomposition argument that there is an infinite amount of space close to the horizon. Otherwise one might mistake the lack of cluster decomposition as arising because space itself is compact. By now it should be obvious that quantum gravity correlators at late time look nothing like their semiclassical counterparts. consequently, bulk matter correlators in quantum gravity close to the "horizon" look nothing like those of bulk matter in a Rindler geometry. It is common folklore that quantum gravity effects cannot play an important role for physics close to the horizon. This is usually motivated by stating that for large black holes the scalar curvature R at the horizon is small. Consequently one does not expect strong coupling effects. The intuition from the Penrose diagram combined with Maldacena's information paradox shows that this piece of folklore is false. We should generically expect quantum effects to proliferate close to the horizon. Yes these are backreaction effects. But backreaction is real. It is not a bad thing to be

avoided at all cost. In fact you can't avoid backreaction in the near horizon region just as you can't avoid backreaction at late time. More importantly there is no reason to try and avoid it. This regime of "backreaction" is where we feel a lot of the interesting physics is.
These results have implications for the more common information paradox as well, the one about unitary black hole evaporation. One of the assumptions of the Hawking calculation is that quantum matter on classical curved space time is a valid approximation to quantum gravity in the near horizon region. We find here via very explicit calculation an example of a matter coupled quantum gravity theory for which this does not hold. Of course we don't learn a whole lot by stating that one of the assumptions of a paradox is false. Few would therefore doubt that information comes out during black hole evaporation in quantum gravity. Therefore it doesn't help to bash on the semiclassical calculation. We would like to actually understand how black holes evaporate unitarily in quantum gravity. Recently we are actually starting to understand how it works. At least in Euclidean signature [10, 11] that is. The partial answer draws heavily on the baby universes discussed in the previous chapters. This should not come as a surprise given their relevance to Maldacena's version of the information problem. We will not discuss in detail those rapidly unfolding developments here. We have had nothing to do with them. Nevertheless it's worth mentioning.

Let us finally emphasize it would be incredibly difficult to find explicit proof of this near horizon behavior of matter probes in any other model of quantum gravity besides JT gravity. To make that point one could even imagine more basic probes. For example one could consider quantum mechanical expectation values of diff invariant geometric observables such as the local metric or more generally geodesic distances.[5] This requires the computation of quantum gravity path integrals of the type:

$$\langle \mathcal{O} \rangle = \int [\mathcal{D}g]\, \mathcal{O}(g) e^{-S[g]}. \tag{5.10}$$

In a generic theory of quantum gravity there are some major obstructions to doing such a calculation. For one, off- hell metrics appearing in the path integral can generically be quite exotic.[6] Consequently there are in general no closed expressions for the geometric observables $\mathcal{O}(g)$. For example there is generically no easy functional relation between g and the geodesic distance between two points. This is not a problem in JT gravity where all metrics are patches of hyperbolic geometry. Secondly there is the technical hurdle of actually being able to calculate the resulting path integral (5.10). Fortunately in JT gravity this turns out to be possible as well.

The remainder of this chapter focuses largely on the disk model of JT gravity discussed in chapter 2 which already largely makes the point. It is organized as follows.
In **section 5.1** we specify in somewhat more detail the radar definition of bulk coordinates, as set up for the path integral calculations that follow.

[5]We will be interested in the metric tensor $ds^2 = g(dx, dx)$. This is a scalar under diffeomorphisms.

[6]Granted, baby universes and eigenbranes may sound quite exotic, but really it just boils down to counting Riemann surfaces in JT gravity.

In **section 5.2** we calculate several path integrals of bulk matter coupled to JT gravity. The focus is primarily on bulk conformal primaries for which calculations simplify. However we also spell out certain calculations for massive bulk scalars.
In **section 5.3** we investigate the effects of large spatial separations on the bulk matter correlators. The focus is again on bulk conformal primaries. These are simple enough so that the calculations are tractable, and complex enough so that we can make a point. We prove our intuition that large distance physics in the bulk is essentially equivalent to late time physics in the 1d dual quantum mechanical system. We comment on the fate of quantum fields in Rindler as an accurate description to near horizon physics.
In **section 5.4** we briefly touch on several important consequences and generalizations. This includes comments on bulk reconstruction, the infalling observer, the firewall debate, locality and a gravitational interpretation of the dramatic observed effects.

5.1.1 Clocks and rods

As explained around (2.35), all metrics in the topologically trivial version of JT gravity can be thought of as cut-outs of the Poincaré disk or upper half plane.[7] The boundary observer can be thought of as living on a wiggly boundary curve. As explained in great detail in the original papers in this field [21, 22, 23], its trajectory is specified as $Z(t) = \epsilon \dot{f}(t)$ and $T(t) = f(t)$. Here t is meant to be the proper time of the boundary observer. It is this field $f(t)$ which is weighed by the Schwarzian action $S[f]$. From the perspective of the boundary observer, the path integral over $f(t)$ is interpreted as a path integral over inequivalent bulk metrics. To deduce the corresponding bulk metric, the asymptotic observer uses the clocks and rods described in the introduction. By shooting light rays into the bulk and collecting them back he defines a bulk coordinate z. In particular he associates light cone coordinates v and u to every bulk point. Here $v = t_1$ is the time on his clock at which he sends the signal and $u = t_2$ the time at which he receives the signal.[8] Of course we really imagine a mathematical abstraction of this experiment. We are just defining a coordinate frame much like coordinate frames would be defined in general relativity. We can describe this same experiment in the Poincaré frame with identical outcome. A lightray is sent from the boundary at time $T_1 = f(t_1)$. It reflects at the fictitious mirror at the location of the bulk point and impinges back on the boundary at Poincaré time $T_2 = f(t_2)$. Via this experiment we associate coordinates $V = T_1$ and $U = T_2$ to this bulk point. The result is an "experimentally" determined map from the Poincaré frame to our diff-invariant bulk coordinates:

$$U = f(u), \qquad V = f(v). \tag{5.13}$$

[7] To the latter we associate the metric:

$$ds^2 = \frac{1}{Z^2}(dZ^2 - dT^2). \tag{5.11}$$

[8] We have:

$$u = t + z, \qquad v = t - z. \tag{5.12}$$

Via this map we arrive at the following bulk metric:

$$ds^2(f) = \frac{f'(u)f'(v)}{(f(u)-f(v))^2}du\,dv. \tag{5.14}$$

This depends explicitly on boundary diffeomorphisms but that's alright as these are not redundant. Crucially the metric (5.14) is defined not to transform under small diffeomorphisms. Notice the location dependent conformal scaling factor which is uniquely determined by the physical field $f(t)$. An example would be the thermal boundary reparameterization $f(t) = \tanh \pi t/\beta$. This is the solution to the Schwarzian equation of motion. One finds the thermal AdS_2 bulk metric:

$$ds^2(\beta) = \frac{4\pi^2}{\beta^2}\frac{dz^2 - dt^2}{\sinh^2 \frac{2\pi}{\beta} z}. \tag{5.15}$$

This will return in the remainder. Notice that our definition of a bulk frame is similar to that of [159]. We would like to re-emphasize the preferred nature of the boundary observers proper time t. Because diff invariance is broken on the boundary, we can have a "preferred" boundary time. By consequence there can also be a "preferred" bulk frame (t, z).[9] Within this framework we would specify a local bulk operator of a massless scalar field at (u, v) as a collection of fields $\phi(f(u), f(v))$. Notably, as announced in the introduction, the location of this operator in the Poincaré frame depends explicitly on $f(t)$. The path integral over metrics can be thought of as some statistical averaging such that the operator becomes fuzzy and is smeared out in the Poincaré frame. Nevertheless this family of operators still defines a local field theory in the sense that the causal structure is not affected by coupling to quantum gravity. As explained in the above this is very specific to the light cone definition.
This fuzziness is nevertheless conceptually important for what follows. It represents the coupling of the bulk matter degrees of freedom to quantum gravity via explicit dependence on the bulk metrics f of bulk matter correlators. As a result the matter correlator does not factor out of the gravitational path integral:

$$\int [\mathcal{D}f]\phi(f(u), f(v))\ldots e^{-S[f]}. \tag{5.16}$$

This creates a somewhat counterintuitive scenario that is important to understand. For every fixed f the matter correlator is that of a field in AdS_2. However when we average over the degrees of freedom f with a certain weight this structure will generically not be preserved. For example the quantum gravity correlator will not satisfy Klein-Gordon equation for a massless field in AdS_2. This is basic statistics but applied to differential equations with $\mathcal{O}$ some differential operator:

$$\langle \mathcal{O}\phi(x)\rangle \neq \mathcal{O}\langle\phi(x)\rangle. \tag{5.17}$$

Semiclassical gravitational physics is obtained within quantum gravity as a scenario where the ensemble of metrics collapses to its saddle point via localization of the path

[9] This preferred nature of the time coordinate t was already appreciated in [20, 23, 43].

integral as $\hbar \to 0$. When there are field insertions at large time separations or large spatial separations, the semiclassical saddle will no longer be relevant and quantum fluctuations of the metric become important. In fact we will find that the theory is always effectively strongly coupled in the near horizon region. Of course this is back-reaction. But that is precisely what we expect and want. The late time behavior of holographic correlators is a feature and certainly not a bug. The same is true for the near horizon behavior of bulk correlators.

Before we get started with some explicit calculations, one final comment. We've explained why lightlike geodesic localizing is preferred in a sense, as it avoids unwanted non-localities. It seems though as if they are preferred for a stationary observer, but not for an infaller. Indeed, by definition, in these coordinates we can't venture beyond the semiclassical horizon. We would imagine that infaller physics depends only on the trajectory of the boundary wiggles up to some time t_0 at which the infaller jumps of the boundary and into the black hole. Two sensible coordinates might be this t_0 and the affine null distance along the observers' idealized null trajectory starting from the boundary at t_0. There is no a priori reason the resulting bulk operators to be local ones. More importantly there is no a priori reason for expecting the same physics in infaller coordinates as compared to physics in radar bulk coordinates. This is one example of different gauge choices in quantum gravity resulting in different physics. See also [19]. It would be highly interesting to try and construct the physics in such an infaller frame. At the moment of writing we have not yet actively tried to investigate this though.

5.2 Matter probes in the bulk

In this section we exactly calculate several correlators of bulk matter coupled to JT quantum gravity. We imagine correlators of the type:

$$\int [\mathcal{D}g]\, \delta(R+2)\, e^{-S[g]} \int [\mathcal{D}\phi]\, g \cdot \phi_1(u,v) \ldots e^{-S[\phi,g]}. \tag{5.18}$$

The latter factor represents the matter path integral in some fixed background g. Constraining to trivial topologies for the Euclidean metric g, the locus of the gravitational path integral are the metrics (5.14). The gravitational action $S[g]$ becomes the Schwarzian action (2.63) as explained in great detail in chapter 2. We then imagine doing the matter path integral in each fixed background (2.63). This results in a correlator of the type:

$$\int [\mathcal{D}f]\, e^{-S[f]}\, \langle f \cdot \phi_1(u,v) \ldots \rangle . \tag{5.19}$$

Implicitly here the correlator is defined as:

$$\langle f \cdot \phi_1(u,v) \ldots \rangle = \frac{1}{\mathcal{Z}[f]} \int [\mathcal{D}\phi]\, f \cdot \phi_1(u,v) \ldots e^{-S[\phi,f]}. \tag{5.20}$$

Since all frames are local conformal transformations of the Poincaré frame, we know the general expression for this correlator once we know its expression in the Poincaré frame.[10] Either we have an exact answer if the matter is exactly solvable or a perturbative series expansion. Anyway we have some function of which we're supposed to calculate the Schwarzian path integral. At this point our problem has reduced to a technical one. Can we compute such a Schwarzian path integral? Fortunately the Schwarzian theory has a funny way of somehow allowing an exact computation of most observables one would be interested in.[11] Let us note that we have chosen to normalize the matter correlation function *before* evaluating the gravitational path integral.[12] In the remainder of this section we will focus on three example theories of which we can calculate the two point function. This suffices to make the point regarding near horizon physics and cluster decomposition.

5.2.1 Example 1. Massless scalar

As our first example we consider arguably the simplest non-topological theory. A massless scalar with Dirichlet boundary conditions on the asymptotic boundary. The action is:

$$S[\phi, g] = \int dx \sqrt{g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{5.22}$$

We are interested in calculating its two point function in the metrics (5.14). As all metrics are just AdS_2 up to a coordinate transform, and since these are scalars, we can calculate the correlator in plane old AdS_2 and then just do the coordinate transformation (5.13) to find the answer for generic $f(t)$. So we are first calculating:

$$\langle \phi(P)\phi(Q) \rangle = \int [\mathcal{D}\phi]\, \phi(P)\phi(Q)\, e^{-S[\phi]}. \tag{5.23}$$

The propagator of a massless scalar in AdS_2 is common knowledge up to normalization:

$$\langle \phi(P)\phi(Q) \rangle = \ln \left| \frac{(T-T')^2 - (Z-Z')^2}{(T-T')^2 - (Z+Z')^2} \right| = \ln \left| \frac{(U-U')(V-V')}{(U-V')(V-U')} \right|. \tag{5.24}$$

The denominator in the first equality is to enforce the Dirichlet boundary condition [177]. We could think about it as a contribution due to some second image charge, much like we can think about image charges when calculating the field of a point charge in the

[10] An example is a CFT two-point function of a weight h primary field:

$$\langle f \cdot \phi_h(u_1)\, f \cdot \phi_h(u_2) \rangle = f'(u_1)^h f'(u_2)^h \, \langle \phi_h(f(u_1))\phi_h(f(u_2)) \rangle = \frac{f'(u_1)^h f'(u_2)^h}{(f(u_1) - f(u_2))^{2h}}. \tag{5.21}$$

[11] One exception being the "covariant" bilocal, where only one of the two operators is taken to transform. In principle such observables seem relevant to infaller physics.

[12] For conformal bulk matter this is an irrelevant choice. This might seem counterintuitive since the partition function of 2d conformal matter $\mathcal{Z}[f]$ depends explicitly on the metric via the conformal anomaly. However, it was shown in appendix C of [57] that in this particular set up the conformal anomaly generates again a Schwarzian action with a prefactor that is subdominant to the one in the Schwarzian action originating from the JT action.

proximity of a perfectly conducting medium in electromagnetism. Mapping to a generic metric using (5.13) we find:

$$\langle f \cdot \phi(p)\, f \cdot \phi(q) \rangle = \ln \left| \frac{(f(u) - f(u'))(f(v) - f(v'))}{(f(v) - f(u'))(f(u) - f(v'))} \right|. \tag{5.25}$$

We want to compute the Schwarzian path integral of this function. This is the log of a product of four elementary bilocals of the type (5.3). When computing path integrals of functions of operators there is always a hidden question of operator ordering. Indeed. As a classical correlation function (5.25) is just a function. But as an object in the Schwarzian path integral it is a function of operators. Therefore there are possible ordering ambiguities. We will imagine the usual time ordering unless otherwise specified.[13] In this example there really isn't much of an issue because we can write the answer as a "sum" of simple operators with no ordering ambiguities as we'll see. However care is needed in more generic situations. A pragmatic way to proceed in general is as follows. Just choose the ordering which results in the expected physics in the semiclassical limit with no extra uncalled-for shockwave interactions. An observation which greatly helps our computation of the Schwarzian path integral is now the following:

$$\int_v^u dt \int_{v'}^{u'} dt' \frac{f'(t) f'(t')}{(f(t) - f(t'))^2} = \ln \left| \frac{(f(u) - f(u'))(f(v) - f(v'))}{(f(v) - f(u'))(f(u) - f(v'))} \right|. \tag{5.26}$$

The integrand on the left hand side is precisely the Schwarzian bilocal (5.3) and corresponds to a boundary anchored Wilson line in JT gravity. We can think of this as bulk reconstruction in each background (5.14) with a bulk to boundary propagator which doesn't depend on $f(t)$ explicitly. It relates via the holographic dictionary $m^2 = \ell(\ell - 1)$ a boundary 1d conformal primary with $\ell = 1$ to a massless field in the bulk. For more on this see section 5.4. It is sometimes instructive to transform to to the angular coordinate $\theta(t)$ as $f(t) = \tanh \pi\theta(t)/\beta$. The monodromy constraint on the field configurations is now just periodicity in Euclidean time $\theta(t + i\beta) = \theta(t) + i\beta$. This makes it easier to read off the semiclassical answer for which $\theta(t) = t$.

$$\int_v^u dt \int_{v'}^{u'} dt' \frac{\theta'(t)\theta'(t')}{\frac{\beta}{\pi} \sinh \frac{\pi}{\beta}(\theta(t) - \theta(t'))^2} = \ln \left| \frac{\sinh \frac{\pi}{\beta}(\theta(u) - \theta(u')) \sinh \frac{\pi}{\beta}(\theta(v) - \theta(v'))}{\sinh \frac{\pi}{\beta}(\theta(v) - \theta(u')) \sinh \frac{\pi}{\beta}(\theta(u) - \theta(v'))} \right|. \tag{5.27}$$

The classical solution corresponds to $\theta(t) = t$. Anyway, we can factor the integrals in (5.26) out of the Schwarzian path integral:

$$\langle \phi(t, z)\phi(t, z') \rangle = \int_v^u dt \int_{v'}^{u'} dt' \int [\mathcal{D}f]\, \frac{f'(t) f'(t')}{(f(t) - f(t'))^2}\, e^{-S[f]}. \tag{5.28}$$

[13] This is also implicitly done in the complexity computation of [57] and the entanglement computation of [178].

The latter path integral was done in chapter (2) via an equivalent computation of a boundary anchored Wilson line in JT gravity. From (2.259) we find:

$$\int [\mathcal{D}f]\,\frac{f'(t)f'(t')}{(f(t)-f(t'))^2}\,e^{-S[f]} \tag{5.29}$$
$$= \int_0^\infty dE_1\,\sinh 2\pi\sqrt{E_1}\,e^{-\beta E_1}\int_0^\infty dE_2\,\sinh 2\pi\sqrt{E_2}\,e^{i(t-t')E_2}\,\Gamma(1\pm ik_1\pm ik_2).$$

It is straightforward to do the double integral in (5.28). We obtain the bulk propagator of a massless scalar in topologically trivial JT quantum gravity:

$$\langle\phi(t,z)\phi(t',z')\rangle$$
$$= \int_0^\infty dE_1\,\sinh 2\pi\sqrt{E_1}\,e^{-\beta E_1}\int_0^\infty dE_2\,\sinh 2\pi\sqrt{E_2}\,e^{i(t-t')(E_2-E_1)}$$
$$\Gamma(1\pm i\sqrt{E_1}\pm i\sqrt{E_2})\,z\,\mathrm{sinc}\,z(E_2-E_1)\,z'\,\mathrm{sinc}\,z'(E_2-E_1). \tag{5.30}$$

The function $z\,\mathrm{sinc}\,z\omega$ is the bulk to boundary propagator in the Fourier domain. Indeed,it is the Fourier transform of the Heaviside bulk to boundary propagator $\theta(t-|z|)$. The function is smooth everywhere. In particular there are no new IR divergences as compared to the boundary two point function. The full correlation function (5.30) can be plotted numerically for Euclidean times. This is done in [3]. Rather than doing so, let us focus on different regimes in which we have analytic control. Notice immediately that the analytic structure of this formula is that of a two-point function of *local* bulk operators. In particular the integral is finite and smooth except at the lightcone singularities at $t\pm z\pm z'=0$.[14] Let us for future reference introduce the parameter $\min|t-t'\pm z\pm z'|$. This is a measure for how far from the lightcone we are. Furthermore for a more detailed analysis let us focus on the microcanonical ensemble instead of the canonical one. This is obtained effectively by replacing the thermal density matrix $e^{-\beta E_1}$ by a microcanonical one $\delta(E_1-E)$.[15] This corresponds to imposing fixed energy rather than fixed length boundary conditions on one side of the boundary anchored Wilson line in (2.258) and fixing the length of the second boundary to $i(t-t')$. It is not obvious how to think of fixed energy boundaries in the path integral language in the sense that there is no obvious analogue of (2.33). But we don't need to understand this intermediate step because we already understand the answer for the correlator.[16] To discuss different regimes it is useful to introduce $\omega=E_2-E_1$. Different parametric regimes are determined by the scaling of $\min|t-t'\pm z\pm z'|$ as compared to $1/E$. For future reference let us define a macroscopic black hole as a system for which the integral (5.30) is dominated by its semiclassical saddle $E_1=(\pi/\beta)^{1/2}$. We distinguish the following regimes.

[14] Notice that the exact answer (5.30), unlike the semiclassical approximation (5.33) further on has no singularities [57, 179] at $\tau=\pm in\pi/\sqrt{M}$.

[15] A more appropriate definition would be to consider a narrow Heaviside on some energy bin of size δE around E.

[16] One could optionally just define the relevant path integral as the inverse Laplace transform of the usual prescription.

- Suppose $\min(|t-t'\pm z\pm z'|)\ll 1/E$. The integral over E_2 in the microcanonical version of (5.30) is then dominated by $\omega \gg E$. By consequence we can Taylor expand the integrand in E and keep only the lowest order term. The resulting correlator corresponds to the zero temperature $\beta=\infty$ thermal ensemble, for every microcanonical ensemble. So probing close to the light cones we find the classical Poincaré answer (5.24). Indeed in the Poincaré frame the insertion of the bilocal is effectively a UV effect $\omega \gg E$, explaining why we find Poincaré physics when $\omega \gg E$ dominates. As we are close to the lightcone the answer (5.24) reduces essentially to the logarithmic leading lightcone divergence:

$$\langle \phi(t,z)\phi(t',z')\rangle \approx \ln\min(|t-t'\pm z\pm z'|). \tag{5.31}$$

- Suppose $1/E \ll |t-t'\pm z\pm z'| \ll 1$.[17] The integral (5.30) can be written as a sum of four terms by expanding the sines. Each one is associated with a single light cone coordinate. If all the light cone coordinates scale as considered here the integral over ω in each of these terms is dominated by $\omega \ll E$.[18] For a pure state the integral in (5.30) can then be approximated by a suitable modification of the results of [100]:

$$\sqrt{E}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\, z\,\mathrm{sinc}\, z\omega\, z\,\mathrm{sinc}\, z'\omega\, e^{\frac{\pi\omega}{\sqrt{E}}}\,\Gamma(1\pm i\omega/\sqrt{E}). \tag{5.32}$$

Doing the Fourier transform we recover the semiclassical answer for the two point function in a mass E black hole:

$$\langle \phi(t,z)\phi(t',z')\rangle \approx \ln\left|\frac{\sinh\sqrt{E}(t+z+z')\sinh\sqrt{E}(t-z-z')}{\sinh\sqrt{E}(t+z-z')\sinh\sqrt{E}(t-z+z')}\right|. \tag{5.33}$$

See (5.27). It makes sense that we recover the semiclassical answer when the relevant integral is dominated by $\omega \ll E$. This is the probe approximation where the Wilson line doesn't significantly backreact on the semiclassical background, which in this case is the metric corresponding to a black hole of energy E.

- There is a transient regime when $|t-t'\pm z\pm z'|\sim 1/E$ where backreaction does become important. The energy injection ω is now of the same order of magnitude as the black hole mass E. This transient region is clearly visible in numerical plots but only exists for small black holes when $E\ll 1$.

The most interesting regimes as far as we are concerned are those of very late time $t\gg 1$ and close to the horizon $z\gg 1$. They are discussed separately in section 5.3 once we've discussed some further examples of matter in JT gravity. We note that the above discussion extends trivially to the thermal ensemble.

[17] More precisely we require that the absolute values of all the light cone coordinates satisfy these constraints.

[18] We are invoking the Lorentzian variant of the saddle point method known as the Riemann-Lebesgue theorem or the stationary phase approximation.

5.2.2 Example 2. Conformal primary

The massless scalar propagator in two dimensions is kind of peculiar in terms of its IR properties. Generically in quantum field theory, connected correlators decay to zero when the spatial separation of either two of its operators is taken to be large. This is the cluster decomposition principle. Schematically:

$$\langle \phi(0)\phi(z)\dots\rangle_{\text{conn}} = 0, \qquad z \to \infty. \tag{5.34}$$

The massless scalar two point function on the half plane in two dimensions is an "annoying" exception to this because it goes to a constant at large z. Therefore it is arguably the worst possible example to make our point about large distance and near horizon physics in quantum gravity. Let us therefore discuss two "better" examples which do have the expected semiclassical decay at large separations.

As a second example of matter coupled to JT quantum gravity we will consider the two point function of generic 2d conformal primary fields in AdS_2. Semiclassically the corresponding correlators do decay at large separations as exponentials if we pick a natural coordinate system. The 2d conformal field theory propagator of weight $(h, \bar{h})$ primaries on the half plane is well known:

$$\left\langle \phi_{h,\bar{h}}(u,v)\phi_{h,\bar{h}}(u',v')\right\rangle = \frac{1}{(u-u')^{2h}}\frac{1}{(v-v')^{2\bar{h}}} - (u' \leftrightarrow v'). \tag{5.35}$$

The second term is a mirror term that imposes Dirichlet boundary conditions at the asymptotic boundary just like we had for the scalar field in (5.24). We would now like to do the relevant Schwarzian path integral. It is well known how correlators of conformal primaries transform when going to the metric (5.14), see (5.21)

$$\left\langle f\cdot\phi_{h,\bar{h}}(u,v)\, f\cdot\phi_{h,\bar{h}}(u',v')\right\rangle = \frac{f'(u)^h f'(u')^h}{(f(u)-f(u'))^{2h}}\frac{f'(v)^{\bar{h}} f'(v')^{\bar{h}}}{(f(v)-f(v'))^{2\bar{h}}} - (u' \leftrightarrow v'). \tag{5.36}$$

Notice that this decays exponentially at large spatial separations in the semiclassical limit. To see this we can just put t and t' equal to zero. Furthermore one needs to keep in mind that the semiclassical answer in this case is $f(t) = \tanh \pi t/\beta$. Taking the large distance limit, and ignoring prefactors, this becomes:

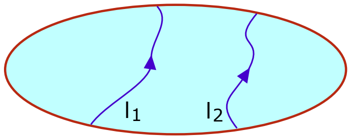
$$\left\langle \phi_{h,\bar{h}}(0,z)\phi_{h,\bar{h}}(0,z')\right\rangle \approx e^{-\frac{2\pi}{\beta}(h+\bar{h})(z-z')} - e^{-\frac{2\pi}{\beta}(h+\bar{h})(z+z')}, \quad z > z'. \tag{5.37}$$

This is the decay for increasing z which we claim cannot go on forever in any sensible theory of quantum gravity, regardless of any UV details of the model. To prove this we are initially interested in computing the Schwarzian path integral of (5.36). This is incredibly straightforward as the above is just the sum of two products of two of our usual Schwarzian bilocals. So the task at hand is just to compute four point functions in Schwarzian quantum mechanics. In a canonical language we would write the above as:

$$\mathcal{O}_h(u,u')\,\mathcal{O}_{\bar{h}}(v,v') - \mathcal{O}_h(u,v')\,\mathcal{O}_{\bar{h}}(v,u'), \quad \mathcal{O}_h(u,v) = \frac{f'(u)^h f'(u')^h}{(f(u)-f(u'))^{2h}}. \tag{5.38}$$

It is important to note the operator ordering here. Regardless of the ordering of the four times u, u', v, v' the bilocal operators are inserted as a whole one after the other. This determines the integration contour for a path integral calculation as one where Wilson lines in the Euclidean disk do not cross. See for example [1]. This means in particular that no $6j$-symbols appear in the calculation, and correspondingly there are no "unwanted" semiclassical shockwaves [100]. The relevant Schwarzian calculations thus corresponds to a bulk JT gravity calculation of the type:

$$Z(\ell_1, \ell_2, \beta_1, \beta_2, \beta_3, \beta_4) = \quad . \tag{5.39}$$



In this chapter we will suppress in the graphics the labels that denote the boundary conditions on each boundary segment such as β_1. This amplitude is easily calculated using the techniques discussed in chapter 2. A precise answer can be found for example in [43, 1]. Let us just give the answer in the microcanonical ensemble for a black hole of energy E:

$$\begin{aligned}
&\left\langle \phi_{h,\bar{h}}(u,v)\phi_{h,\bar{h}}(u',v')\right\rangle \\
&\qquad = \int dE_1 \sinh 2\pi\sqrt{E_1}\,\frac{\Gamma(h \pm i\sqrt{E} \pm i\sqrt{E_1})}{\Gamma(2h)} e^{i(u-u')(E_1-E)} \\
&\qquad\qquad \int dE_2 \sinh 2\pi\sqrt{E_2}\,\frac{\Gamma(\bar{h} \pm i\sqrt{E} \pm i\sqrt{E_2})}{\Gamma(2\bar{h})} e^{i(v-v')(E_2-E)} \\
&\qquad\qquad - (u' \leftrightarrow v').
\end{aligned} \tag{5.40}$$

One checks that this vanishes identically on the asymptotic boundary. This would not be the case had we adopted another definition for the time contour in the path integral. Clearly this strengthens our belief that the naive contour is indeed the most sensible one. The properties of (5.40) are fairly unsurprising and mainly identical to those discussed around equation (5.30). One noticeable feature is that the left movers and right movers remain uncoupled in a pure state, whereas in a thermal ensemble quantum gravity effects couple them. The main structural difference with the scalar propagator is in its behavior at large spatial separations. We will focus on this in section 5.3. For future reference let us introduce some notation. We define:

$$|\mathcal{O}_{\ell,E_1E_2}|^2 = \frac{\Gamma(\ell \pm i\sqrt{E_1} \pm i\sqrt{E_2})}{\Gamma(2\ell)}. \tag{5.41}$$

Furthermore we reintroduce some notation from chapter 4:

$$\rho_0(E) = \sinh 2\pi\sqrt{E}. \tag{5.42}$$

We can then write the bulk to bulk propagator in the microcanonical ensemble more efficiently as:

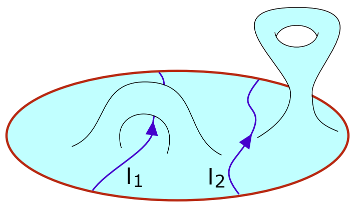
$$\langle \phi_{\ell,\ell'}(u,v)\phi_{\ell,\ell'}(u',v')\rangle = \int_0^\infty dE_1\, e^{i(u-u')(E_1-E)} \int_0^\infty dE_2\, e^{i(v-v')(E_2-E)} \rho_0(E)\rho_0(E_1)\rho_0(E_2)\left|\mathcal{O}_{\ell,E_1E}\right|^2 \left|\mathcal{O}_{\ell',E_2E}\right|^2 - (u' \leftrightarrow v'). \quad (5.43)$$

Writing the answer in this way has the obvious advantage that we can immediately write down the exact answer in the two other versions of JT gravity discussed in chapter (4). In case of the ensemble averaged description of [9] whose gravitational description involves all types of higher genus Riemann surfaces ending on the three disk shaped regions in the JT gravity amplitude (5.39) we replace the integration kernel by:

$$\langle \rho(E)\rho(E_1)\rho(E_2)\rangle\ \left|\mathcal{O}_{\ell,E_1E}\right|^2 \left|\mathcal{O}_{\ell',E_2E}\right|^2. \quad (5.44)$$

A typical contribution to a JT gravity path integral which ends up computing this kernel includes Riemann surfaces of generic topology ending on three geodesic necks. For a detailed explanation see chapter 3. There is one such "neck" for each topological disk region in (5.39). For example one contribution is:

$$\langle \rho(E_1)\rho(E_2)\rho(E_3)\rangle\ \left|\mathcal{O}_{\ell_1,E_1E_2}\right|^2 \left|\mathcal{O}_{\ell_2,E_2E_3}\right|^2 \supset$$



. (5.45)

In case of a discretized system whose gravity dual includes eigenbranes for the Riemann surfaces to end on, we replace the integration kernel by:

$$\langle \rho(E)\rho(E_1)\rho(E_2)\rangle_{\lambda_1\ldots\lambda_N}\ \left|\mathcal{O}_{\ell,E_1E}\right|^2 \left|\mathcal{O}_{\ell',E_2E}\right|^2. \quad (5.46)$$

See chapter 4. The first factor in this expression can be essentially replaced by the delta spikes (4.1) describing a discrete quantum chaotic dual. Furthermore in both these examples we need to replace the energy contour by $\mathcal{C}$. We can summarize all versions by one single notation where we introduce $\rho(E, E_1, E_2)$ to be interpreted differently in each version of JT gravity.

5.2.3 Example 3. Massive scalar

As a final example and a proof of principle we would like to compute the two point function of a massive scalar field coupled to JT quantum gravity. The matter action is:

$$S[\phi, g] = \int d^2x\sqrt{g}\, g^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi + m^2\phi\right). \quad (5.47)$$

The two point function of a massive scalar in AdS_2 with Dirichlet boundary conditions is well known. Up to normalization we have [177]:

$$\langle \phi_{m^2}(u,v)\phi_{m^2}(u',v')\rangle = \delta^{-2\ell}\,{}_2F_1(\ell,\ell,2\ell,-1/\delta^2), \qquad m^2 = \ell(\ell-1). \tag{5.48}$$

This formula features one of the isometric invariants of AdS_2 [180, 182]:

$$\sigma = \frac{(z^2+z'^2)-(t-t')^2}{2zz'} = 2\delta^2+1, \tag{5.49}$$

This is furthermore related to the invariant d introduced in [177] as $\sigma = 1+d$. The geodesic distance D itself can be written as:

$$D = 2\operatorname{arcsinh}\delta = \operatorname{arccosh}\sigma. \tag{5.50}$$

By definition of what we mean by a scalar field in general relativity, the correlator transforms by a simple coordinate transformation when going to the metric (5.14). As a result we have an identical answer:

$$\langle f\cdot\phi_{m^2}(u,v)\, f\cdot\phi_{m^2}(u',v')\rangle = \delta^{-2\ell}\,{}_2F_1(\ell,\ell,2\ell,-1/\delta^2). \tag{5.51}$$

The difference is what we mean by δ in this formula:

$$\delta^2 = -\frac{(f(u)-f(u'))(f(v)-f(v'))}{(f(u)-f(v))(f(u')-f(v'))}. \tag{5.52}$$

At first sight one might think it's ridiculous to hope one could calculate the Schwarzian path integral of this global conformal block. Nevertheless as it turns out this is very much feasible. As was proven very explicitly in [3] we can rewrite this function as:

$$\begin{aligned}&\langle f\cdot\phi_{m^2}(u,v)\, f\cdot\phi_{m^2}(u',v')\rangle\\ &= \int_{v_1}^{u_1} d\tau \int_{v_2}^{u_2} d\tau'\, \frac{f'(\tau)^\ell f'(\tau')^\ell}{(f(\tau)-f(\tau'))^{2\ell}}\, \frac{(f(u)-f(\tau))^{\ell-1}(f(v)-f(\tau))^{\ell-1}}{f'(\tau)^{\ell-1}(f(u)-f(v))^{\ell-1}}\\ &\qquad\qquad \frac{(f(u')-f(\tau'))^{\ell-1}(f(v')-f(\tau'))^{\ell-1}}{f'(\tau')^{\ell-1}(f(u')-f(v'))^{\ell-1}}.\end{aligned} \tag{5.53}$$

Before proceeding with an in principle calculation let us note the structure of this formula and explain why it holds true. Essentially what we've done here is reverse engineered bulk reconstruction a la HKLL [180, 181, 183, 184] in a fixed background (5.14). The relevant bulk to boundary propagator in this case is:

$$f\cdot K_{m^2}(\tau,t,z) = \frac{(f(u)-f(\tau))^{\ell-1}(f(v)-f(\tau))^{\ell-1}}{f'(\tau)^{\ell-1}(f(u)-f(v))^{\ell-1}}\theta(z-|t-\tau|). \tag{5.54}$$

This is recognized as the second term on the first line of (5.53) with a similar term on the second line. The other factor is the usual boundary to boundary propagator (5.3).

This should be contrasted to the bulk to boundary propagator for the massless scalar bulk field in (5.26):

$$f \cdot K_0(\tau, t, z) = \theta(z - |t - \tau|). \tag{5.55}$$

Such a reverse engineered bulk reconstruction perspective on bulk correlators could be expected to be useful in this context because the Schwarzian is the holographic dual to JT gravity. It then makes sense to rewrite correlators of massive bulk fields as integrals over correlators of some dual boundary 1d conformal field theory in the hope that the corresponding Schwarzian path integrals are natural. In all examples that we've studied we indeed find a "natural" boundary correlator to be inserted in the Schwarzian path integral. Sticking to the notation of (5.38) we can for example write (5.53) as an integral over products of seven Schwarzian bilocal operators:

$$\begin{aligned} &\langle f \cdot \phi_{m^2}(u,v)\, f \cdot \phi_{m^2}(u',v') \rangle \\ &= \int_u^{u'} d\tau \int_v^{v'} d\tau' \mathcal{O}_\ell(\tau,\tau')\, \mathcal{O}_{\ell-1}(u,v)\, \mathcal{O}_{1-\ell}(u,\tau)\, \mathcal{O}_{1-\ell}(v,\tau) \\ &\qquad\qquad \mathcal{O}_{\ell-1}(u',v')\, \mathcal{O}_{1-\ell}(u',\tau')\, \mathcal{O}_{1-\ell}(v',\tau') \end{aligned} \tag{5.56}$$

Though seemingly a daunting task, computing the associated Schwarzian path integral is on a technical level in principle no more difficult that the one with two bilocals in (5.21). The difficulty is certainly not in calculating a Wilson line correlator in JT gravity. We can immediately write down the answer for any such correlator. Rather the subtlety here is to identify a suitable operator ordering. It is clear that we want to avoid Wilson line crossings unless we would be interested in computing some bulk out of time ordered correlators.[19] But which Wilson line configuration computes the HKLL kernel? Do we just insert the three bilocals one after the other? Intuition suggests we might want a picture where one Wilson line envelops around the other two, because the relevant boundary times are cyclic permutations of for example u, v and τ. On the other hand, one might argue in favor of simply inserting the seven bilocals one after another, resulting in a sort of wheel like diagram. A reason why this is intuitive, is that upon considering a microcanonical ensemble as before, there would be no funny coupling of the different parts of the diagram. To appreciate this, remember formula (5.40). We would just have a single energy integral for each Wilson line. Therefore the ordering of the Wilson lines in this particular picture would not be relevant, which makes it "preferred" in some sense. More importantly we expect that this picture won't give any surprises in the semiclassical limit. For what this work is concerned though we will not try to fully resolve this issue. The point we are trying to make about long distance physics can already be strongly motivated based on the exact answers for the conformal primaries, combined with our intuitive expectations, and does not necessarily demand we solve this operator ambiguity. In the future it would be interesting to get a firmer grip on the operator ordering principles for bulk matter correlators in JT gravity. For now it is more important to keep in mind that on a technical level the problem can be solved.

[19] This is in fact demanded by the extrapolate holographic dictionary [3].

5.3 No late time decay nor cluster decomposition

We would like to prove the point made in the introduction that the late time behavior of boundary correlators a la Maldacena [14] is one to one with a breakdown of cluster decomposition in the bulk. Put in a more careful way the statement is that quantum gravitational corrections become important at large time like separations, but also generically at large spatial separations of operators. By large, we mean parametrically larger as compared to the scale set by the gravitational coupling. There are generically four types of effects which we'll see, analogous to those discussed for the spectral form factor in chapter 4. Firstly we have corrections due to the Schwarzian wiggles which come with a transition from exponential decay to power law decay. Secondly there are the corrections due to annuli which results in the ramp. Correlators now increase with separation. Thirdly there is the plateau type behavior due to nonperturbative corrections in the sum over all Riemann surfaces connecting to the relevant boundaries. The latte two effects are inherent to quantum chaotic systems and accurately described by the matrix ensemble non-perturbative definition of JT gravity [9]. Finally in both the ramp and plateau regions we have erratic oscillations. These are at the heart of Maldacena's information paradox [14] and are entirely due to the inherent discreteness of any sensible theory of quantum gravity. This results in erratic oscillations for a systems whose eigenvalue spacing satisfies a certain universality class of random matrix statistics. This should be contrasted to the usual somewhat rhythmic picture of Poincaré recurrences. Quantum chaotic systems, such as black holes, will come with erratic oscillations. In fact this difference in late time behavior of correlators is arguably one of the sharpest ways to experimentally distinguish a quantum chaotic system from a non chaotic quantum system [17].

To prove our point we would like to compare the large distance behavior of the 2d conformal primary bulk two point function in JT gravity to that of the boundary two point function discussed in [3, 18]. The boundary two point function can be written following the notation introduced below (5.40) as:

$$\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(t')\rangle = \int_{\mathcal{C}} dE\, e^{-\beta E} \int_{\mathcal{C}} dE_1\, e^{-\beta_1 E_1}\, e^{i(t-t')(E_1-E)}\, \rho(E,E_1)\, |\mathcal{O}_{\ell,E_1 E}|^2. \tag{5.57}$$

On the other hand we will consider the bulk two point function. We are interested in large spatial separations. We imagine choosing t and t' such that they are separated by some finite Euclidean time β_1. This doesn't change any of the physics but the integrals will look better behaved. We've applied a similar logic in the previous equation. The thermal bulk two point functions for a conformal primary (5.40) becomes:

$$\begin{aligned}&\langle \phi_{\ell,\ell'}(u,v)\phi_{\ell,\ell'}(u',v')\rangle \\ &= \int_{\mathcal{C}} dE\, e^{-\beta E} \int_{\mathcal{C}} dE_1\, e^{-\beta_1 E_1} \int_{\mathcal{C}} dE_2\, e^{-\beta_1 E_2} \\ &\quad e^{i(z-z')(E_1-E_2)}\, \rho(E,E_1,E_2)\, |\mathcal{O}_{\ell,E_1 E}|^2\, |\mathcal{O}_{\ell',E_2 E}|^2 - (z' \leftrightarrow -z').\end{aligned} \tag{5.58}$$

Comparing (5.57) and (5.58) is suggestive. It should be intuitively clear that the behavior of (5.57) for large timelike separations in principle closely resembles the behavior of (5.58) for large spatial separation. Let us nevertheless give some more intuition and present a few more detailed calculations.
The Schwarzian or disk corrections are almost exclusively related to IR features of the spectrum associated with the fact that $\rho_0(E)$ has a rather sharp spectral edge of the type $\sqrt{E}\theta(E)$. This should be contrasted to the smooth Cardy rise at high energies. It might be surprising to the reader that we are discussing IR corrections due to quantum gravity. Intuitively one might have thought that quantum gravity is well described by Einstein gravity at low energies which in this context would be semiclassical JT gravity. One would only imagine corrections due to quantum gravity in the UV. One of the main points we are trying to make here is that such an intuition is false. Any sensible theory of quantum gravity is a unitary discrete quantum mechanical system. As compared to semiclassical gravity there will be several types of corrections in such a realistic model of quantum gravity. On the one hand are UV effects associated with the fact that the dimension of the Hilbert space is finite. Put differently and in a more general context there are UV effects associated with the fact that the Cardy rise changes at some high energy scale. We can't and shouldn't try to probe such effects in JT gravity. It is not UV complete. Rather one could try to probe for this physics if one for example understood the bulk dual to a single representative of SYK. It might also be possible in the $T\bar{T}$ deformed version of JT gravity [185]. On the other hand there are two other types of profound effects. One is intrinsically IR and is associated with the departure from Cardy rise close to the spectral edge. The other is very local on the energy axis and is associated with the fact that when we look very closely, we see that the spectrum is not smooth, but rather a sum of delta spikes. Moreover we would find that these delta spikes are never "very close" together. This is the hallmark of a quantum chaotic system.[20] To see such effects we do not need to be probing at high energies. We can remain well and comfortably in our low energy effective theory of quantum gravity. In this theory, as explained in the previous chapter, we can still very much see the random matrix statistics, the spectral edge and the delta spikes. The key is that we can see them when probing at exponentially long time scales or at exponentially long distances. For an intuitive explanation consider the Fourier transform:

$$f(t) = \int_{-\infty}^{+\infty} d\omega\, e^{i\omega t}\, f(\omega). \tag{5.59}$$

Now imagine that $f(w)$ is a sum of delta spikes with some specific weight:

$$f(\omega) = \sum_{\lambda} f(\lambda)\delta(\omega - \lambda). \tag{5.60}$$

Imagine now we reconstruct a smooth signal $\tilde{f}(\omega)$ from this a la Nyquist sampling theorem. In other words imagine smearing out the delta's over a region larger than the average separation ϵ between deltas. The question is whether or not we can distinguish

[20] There are more precise ways to formulate this, but this is the most elementary one.

$f(t)$ from $\tilde{f}(t)$. The answer is that they are identical up to late times of order $1/\epsilon$ but very different afterwards. The smooth function will drop to zero in a sufficiently fast manner, whilst the function $f(t)$ shows its true colors at late times as a sum over pure phases:

$$f(t) = \sum_\lambda e^{i\lambda t}. \tag{5.61}$$

This oscillates erratically, but not at times shorter than the inverse average level spacing $1/\epsilon$. More generally there is the intuition from Fourier transforms that we need large times to resolve sharp features in the energy domain. One such sharp feature is the spectral edge $\sqrt{E}\theta(E)$ which is still orders of magnitude smoother than the delta's. This is why as a plot in function of time we first see the effects from this spectral edge, resulting in power law behavior, and only later on see effects from the random matrix statistics. The key is that both the late time propagator (5.57) and the large distance propagator (5.58) contain such a "late time" Fourier transform.
The conclusion of this story is that one should only expect Einstein gravity to be an accurate coarse grained description of quantum gravity at low energies. Notably focusing on low energies does not automatically imply coarse graining. Coarse graining is an additional constraint. By consequence we should in general expect quantum gravity modifications of Einstein gravity both for short distance or high energy phenomena, as well as for very late time or very large distance phenomena. This corresponds to the statement that quantum mechanics reduces to classical mechanics at large quantum numbers with all the genuine quantum behavior taking place at low energies. It is the latter that we are probing throughout this work.

5.3.1 Schwarzian quantum corrections

Let us now be very explicit and do some calculations using (5.57) and (5.58). To see the Schwarzian corrections it is advised to tune all the energies over which we're integrating to be small. So we imagine $\beta \gg 1$ and $\beta_1 \gg 1$. Furthermore to see the Schwarzian corrections we should imagine the simple disk model where for example:

$$\rho(E_1, E_2) \approx \rho_0(E_1)\rho_0(E_2). \tag{5.62}$$

If we take furthermore $t - t' \gg 1$ in (5.57) and $z - z' \gg 1$ in (5.58) then all integrals become dominated by small energies. The vertices behave smoothly in this region and the integrals are well approximated by using the lowest order Taylor approximation:

$$|\mathcal{O}_{\ell,E_1 E_2}|^2 \approx \frac{\Gamma(\ell)}{\Gamma(2\ell)}, \qquad E_1, E_2 \ll 1. \tag{5.63}$$

We will drop these constant factors. Furthermore the integrals are well approximated by Taylor expanding the spectrum near the spectral edge:

$$\rho_0(E) \approx \sqrt{E}\theta(E), \qquad E \ll 1. \tag{5.64}$$

In both cases (5.57) and (5.58) this is supported by a numerical analysis. In conclusion, in the regime of interest we find for the boundary two point function:

$$\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(t')\rangle \approx \int_0^\infty dE\, e^{-(\beta+it-it')E}\,\sqrt{E}\int_0^\infty dE_1\, e^{-(\beta_1-it+it')E_1}\,\sqrt{E_1}. \quad (5.65)$$

Doing these integrals, taking $\beta_1 \ll |t-t'|$ and dropping overall prefactors we find [48, 49, 43]:

$$\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(t')\rangle \approx |t-t'|^{-3}, \qquad |t-t'| \gg \beta \gg 1. \quad (5.66)$$

This should be contrasted to the exponential semiclassical decay in (5.3). The analysis for the bulk two point function is very similar:

$$\langle \phi_{\ell,\ell'}(u,v)\phi_{\ell,\ell'}(u',v')\rangle \approx \int_0^\infty dE\, e^{-\beta E}\int_0^\infty dE_1\, e^{-(\beta_1-iz+iz')E_1}$$
$$\int_0^\infty dE_2\, e^{-(\beta_1+iz-iz')E_2}\,\sqrt{E}\sqrt{E_1}\sqrt{E_2} - (z' \leftrightarrow -z'). \quad (5.67)$$

Doing these integrals and taking $\beta_1 \ll |z-z'|$ we find very similar behavior:

$$\langle \phi_{\ell,\ell'}(0,z)\phi_{\ell,\ell'}(0,z')\rangle \approx \beta^{-3/2}|z-z'|^{-3} - \beta^{-3/2}|z+z'|^{-3}. \quad (5.68)$$

This still decays but much slower than the semiclassical exponential decay (5.37). This can be considered a first indication that cluster decomposition is not looking good in quantum gravity.

5.3.2 Black holes and not so random matrices

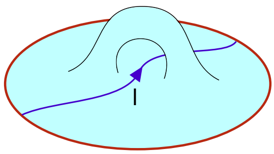
Secondly we would like to understand the effects of eigenvalue repulsion and random matrix statistics.[21] In particular we would like to understand the analogue of the ramp for the bulk two point function at large spatial separation. One reason to focus on the ramp is that the plateau calculation corresponding to (5.58) in the ensemble averaged theory of [9] will end up giving a vanishing result. The bulk correlator analogue to the plateau corresponds to a contribution of the type:

$$\rho(E,E_1,E_2) \supset \rho(E,E_1)\delta(E_1-E_2). \quad (5.69)$$

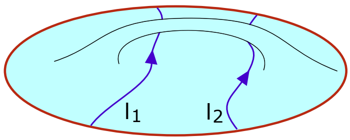
Because of the delta here, all dependence on z and z' drops out in (5.58). By consequence the second term enforcing the Dirichlet boundary conditions has an identical plateau as the first term, with the net result hence vanishing. Do not mistake this as a sign that all is well for cluster decomposition after all, or that the correlator does eventually decay to zero. This symmetry under exchanging z' and $-z'$ does not uphold for a genuinely discrete system. We will see erratic oscillations in (5.58) at large spatial separation for a genuinely discrete system. They just oscillate around a vanishing average. The situation

[21]The matrices are “not so random” in the sense that we can imagine a single matrix and find the relevant GUE statistics in the spacing statistics of its eigenvalues with no ensemble averaging required.

is analogous to that of the partition function $Z(\beta+it)$ for a discrete system. The partition function certainly doesn't decay to zero at late times and it is a great intuitive example of the implications of Maldacena's information paradox. It is just that upon ensemble averaging, it is not a good example. This is actually one of the reasons why it became popular to investigate ensemble averages of the spectral form factor $Z(\beta+it)Z(\beta-it)$ in stead of ensemble averages of the partition function. At late time the spectral form factor measures the averaged norm of the erratic oscillations of the partition function [7]. By consequence the more sensible Plateau calculation for the bulk two point function in the ensemble averaged version of JT gravity would probe for the variance of the erratic oscillations and would hence involve two copies of for example (5.45). We would have two circular boundaries with two Wilson lines stretching between each boundary circle, like two very poor tennis rackets.[22] We are then supposed to sum over all geometries ending on this configuration and the calculation will involve $\rho(E_1,E_2,E_3,E_4,E_5,E_6)$. Notably there are geometries connecting the two copies. These end up breaking the symmetry discussed above.
It is not difficult to do so but for the purpose of this work it would take us too far. A more detailed investigation of these effects is forthcoming [66]. We will settle for the ramp here then. The ramp or its analogue is due to an annulus connecting two distinct regions [8, 3, 9, 18]. In the case of the boundary two point function the relevant JT gravity path integral is the following:

$$\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(t')\rangle \supset$$  . (5.70)

In the case of the bulk two point function the relevant contribution to the JT gravity path integral that results in the analogue of the ramp is:

$$\langle \phi_{\ell_1,\ell_2}(0,z)\phi_{\ell_1,\ell_2}(0,z')\rangle \supset$$  . (5.71)

This corresponds to effectively using the following two answers for the spectral densities in (5.57) and (5.58), up to prefactors:

$$\begin{aligned}\rho(E,E_1) &\approx -\frac{E+E_1}{\sqrt{E}\sqrt{E_1}(E-E_1)^2} \approx -\frac{1}{(E-E_1)^2}\\ \rho(E,E_1,E_2) &\approx -\rho_0(E)\frac{E_1+E_2}{\sqrt{E_1}\sqrt{E_2}(E_1-E_2)^2} \approx -\rho_0(E)\frac{1}{(E_1-E_2)^2}.\end{aligned} \tag{5.72}$$

To get a feeling for the precise answer let us neglect the vertex functions in (5.57) and (5.58). They will not fundamentally change the physicals. It is just that these factors

[22] We might not want to allow the Wilson lines to connect the two circles, otherwise we will not get the analogue of the spectral form factor in the large β limit.

are somewhat “annoying” because they hamper us in trying to get an analytic answer for the integrals. One way to motivate this is to imagine taking again $\beta \gg 1$ and $\beta_1 \gg 1$ such that we effectively are in the Airy region where we can use (5.63) and ship in the relevant Airy formulas. Another way to motivate this, is that the physics in the spectral form factor for the boundary two point function is very similar to the physics of the actual boundary two point function. Of course one sensible option is to just abandon analytics and plot the result which does display an analogue to the ramp. We feel it is more insightful to present this analytic (though rather schematic) calculation. In the case of the boundary two point function one is led to the following calculation:

$$\begin{aligned}\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(t')\rangle &\approx -\int dE\, e^{-(\beta+it-it')E}\int dE_1\, e^{-(\beta_1-it+it')E_1}\,\frac{E+E_1}{\sqrt{E}\sqrt{E_1}(E-E_1)^2}\\ &\approx \frac{2\pi}{\beta}|t-t'|.\end{aligned} \tag{5.73}$$

The $\approx$ is to be interpreted with a huge grain of salt. In the second equality we took $\beta_1 \ll |t-t'|$. We can do a similar calculation for the bulk two point function. We find:

$$\begin{aligned}\langle \phi_{\ell,\ell'}(\beta_1,z)\phi_{\ell,\ell'}(0,z')\rangle \approx -\int_0^\infty dE\, e^{-\beta E}\,\rho_0(E)\int dE_1\, e^{-(\beta_1+iz-iz')E_1}&\\ \int dE_2\, e^{-(\beta_1-iz+iz')E_2}\,\frac{E_1+E_2}{\sqrt{E_1}\sqrt{E_2}(E_1-E_2)^2}&\\ \approx Z(\beta)\frac{2\pi}{\beta_1}|z-z'| - Z(\beta)\frac{2\pi}{\beta_1}|z+z'|&\\ \approx Z(\beta)\frac{z'}{\beta_1},\quad z\gg z'.&\end{aligned} \tag{5.74}$$

This is the analogue to the ramp in the spectral form factor for the bulk two point function. We see that the answer goes to a constant value at large distance separations. This should be contrasted to the semiclassical exponential decay in (5.37) and to the Schwarzian type power law decay in (5.68). This is as clear a breakdown of cluster decomposition as one could imagine. This should suffice to make our point. It is intuitive that it comes from the annulus contribution which corresponds to the emission and re-absorption of a baby universe. As pictured schematically in (5.9) and explained in detail in [18], the baby universe emission-absorption process essentially creates a shortcut. Two operators which are far apart in terms of spatial coordinate distance z can be effectively close together if there is a Euclidean wormhole connecting nearby regions. We note that cluster decomposition does hold on each fixed topology, so there is a possible semantics discussion just around the corner. We define cluster decomposition as the question whether or not connected correlators in quantum gravity of the type (5.5) decay to zero at large spatial separations of the operator insertions, using our lightlike geodesic localization definition of a bulk coordinate. With those definitions, the answer is clearly that we do not have cluster decomposition.

5.3.3 No quantum fields in Rindler

One potentially puzzling aspect of this story is the following. When probing large spatial separations we are basically pushing one or both of our operators into the would-be Rindler region of the semiclassical large black hole. It is common lore that quantum gravity effects cannot be important close to the horizon of large black holes because the local curvature R is tiny. However here we find very explicitly that quantum gravity effects proliferate close to the would-be semiclassical horizon. The confusion is the same as the one that pushes us to make statements like "quantum gravity is only relevant at high energies". The effects we are discussion are not associated with high energy phenomena, but rather with low energy phenomena. At large distance scales, such as close to the horizon where there in an infinite amount of space in infalling coordinates, we are probing very tiny energies or energy differences. This can be thought of as a manifestation of the generic fact that the horizon is a probe for ultra-low energy physics, and hence for quantum effects.[23] Therefore our basic Fourier intuition suggests that on the contrary one should expect inherently *quantum* effects close to the horizon. In fact as it turns out, these quantum effects are so dramatic that they change the effective spacetime as perceived by an observer. The effective geometry close to the would-be horizon is nothing like Rindler and in fact does crazy things. Let us note though that the modifications are only visible at a Planck proper distance to the horizon. Observations that are not probing this Planckian regime would be good to go semiclassically.

We would like to prove this statement about near horizon geometry from first principles. Let us therefore just compute the bulk metric as measured by an asymptotic observer. As we are doing quantum physics, this is the expectation value of some metric operator g in some density matrix ρ:

$$\langle g \rangle = \mathrm{Tr}(\rho g). \tag{5.75}$$

This translates into a JT gravity path integral:

$$\langle g \rangle = \int [\mathcal{D}g]\, \delta(R+2)\, g\, e^{-S[g]}. \tag{5.76}$$

This boils down to a Schwarzian path integral when we're restricting to trivial topologies:

$$\langle g \rangle = \int [\mathcal{D}f]\, \frac{f'(u)f'(v)}{(f(u)-f(v))^2} du\, dv\, e^{-S[f]}. \tag{5.77}$$

In [3] it is motivated in more detail why this is the appropriate operator to consider, based on measurements of geodesic distances by the boundary observer. We choose not to spell out that discussion here as we fear it might distract from the point we are trying to make. Notice though that this is by construction a Lorentz scalar. As before in equation (5.36) we should specify an operator ordering or equivalently a contour for the path integral, whenever we promote a classical object such as the metric (5.14) to

[23] For related comments in gauge theories, see [5, 6, 2].

a canonical operator. We want the metric as a quantum mechanical operator to be Hermitian. A suitable Hermitian metric operator is obtained as the average of both time orderings of the two point function. Therefore we define the metric to be the real part of (5.57). We end up with the following metric expectation value:

$$\langle g\rangle = \int_{\mathcal{C}} dE_1\, e^{-\beta E_1} \int_{\mathcal{C}} dE_2\, \cos 2z(E_1-E_2)\,\rho(E_1,E_2)\,\lvert\mathcal{O}_{1,E_1E_2}\rvert^2. \tag{5.78}$$

This is rigorous in case we restrict to the disk topologies but more of a prescription in other versions of JT gravity. The difficulty in finding a more rigorous proof of this formula is that it is not obvious how to translate the baby universes and eigenbranes into a Lorentzian story and so it is not a priori obvious how to define a "quantum metric" in general. The above seems the most sensible guess. It is important to keep in mind though that what we are going to say is rigorous when restricting to disks. The disk model is actually sufficient to point out that dramatic things are going on close to the horizon. There are several interesting regimes through which we're moving with ever increasing z.

- For $z \ll 1$ we can ignore higher genus corrections. Let us furthermore assume $\beta \ll 1$ such that we are dealing with macroscopic black holes. The double integral with $\omega = E_1 - E_2$ is dominated by $\omega \ll 1/\beta^2$ as around (5.33). We recover the classical metric (5.15) of a black hole with inverse temperature β. It has an event horizon at $z=\infty$. Closer to the horizon the classical metric reduces to the Rindler metric:[24]

  $$ds^2 \approx e^{-\frac{2\pi}{\beta}z}\, du\, dv. \tag{5.79}$$

  This corresponds to the exponential decay at "late times" of the boundary two point function.

- For $z \gg 1$ we become sensitive to the Schwarzian corrections. Let us imagine $z \ll e^{S_0}$ or powers of it such that we can still neglect higher genus. It might not be obvious a priori, but in the units we are working the Planck length is essentially of order one. The proper distance from z of order one to the semiclassical horizon is then of order the Planck length.[25] This means we are probing closer than a Plank's length to the horizon. The transition to power law decay in the boundary correlator directly implies a transition to power law behavior of the conformal scaling factor:

  $$ds^2 \approx z^{-3}\, du\, dv. \tag{5.80}$$

  It is an amusing thought experiment to calculate the curvature tensor in this "geometry". One finds that it blows up linearly with z. One should think of this curvature tensor as constructed operationally by the boundary observer. The asymptotic observer constructs a manifold using our radar definition of bulk points and endows this effective classical manifold with the near horizon metric (5.80).

[24] This is Rindler space in tortoise coordinates $\rho = e^{-\frac{\sqrt{M}}{C}z}$, $\tau = 2\sqrt{M}t$, with metric $ds^2 = \rho^2 d\rho^2 - d\tau^2$.
[25] See [3] for more details.

He then measures the effective curvature tensor by parallel transporting a vector around a small loop on the effective manifold. The result will be different than the mathematical curvature $R + 2 = 0$ because the points along the loop are now defined using our definition of bulk points and not by some fixed Poincaré coordinates. The result (5.80) is quite fundamental in the sense that it makes manifest that Rindler space breaks down as effective geometry close to the horizon in any sensible theory of quantum gravity. It is not that one should now imagine quantum fields on (5.80) as an appropriate effective description of near horizon physics. The take away is rather that one should not try to use quantum fields on *any* spacetime if one is interested in probing closer than a Planck's length to the would-be horizon. You cannot neglect quantum effects at that distance and you'll have to use a genuine quantum mechanical description of gravity if you want to get meaningful results. Notice again that we emphasize the word quantum, rather than emphasizing the potential UV features of quantum gravity. It is funny in that regard to probe the strength of quantum fluctuations in the near horizon metric. In our context the metric is nothing more than a variable in a statistical ensemble and we could for example calculate the covariance of the metric:[26]

$$\text{Cov}(g(z_1), g(z_2)) = \langle g(z_1)g(z_2)\rangle_\beta - \langle g(z_1)\rangle_\beta \langle g(z_2)\rangle_\beta . \tag{5.81}$$

This probes fluctuations away from the saddle $g_0(z)$. In the parametric regions where semiclassical physics holds the covariance vanishes. When probing close to the horizon though, this changes. For example if we take $z_1,\ z_2 \gg 1$ and take furthermore $z_1 \gg z_2$, the covariance blows up:[27]

$$\text{Cov}(g(z_1), g(z_2)) \simeq z_2^{3/2}. \tag{5.82}$$

In this sense the closer we get to the would-be horizon, the more prominent quantum fluctuations become.

- For $z \sim e^{S_0}$ we become sensitive the the underlying random matrix statistics, and higher genus contributions become important. Working in the ensemble averaged version of JT gravity of [9] we would first encounter the analogue of a ramp region, dominated by the annulus:

$$ds^2 \approx \frac{2\pi}{\beta} z\, du\, dv. \tag{5.83}$$

Finally very deep in the bulk we would find ourselves in genuine flat space associated with the plateau region in the spectral form factor:

$$ds^2 \approx du\, dv. \tag{5.84}$$

We propose not to read too much into this for the following reason. The plateau is an averaged approximation to the physics of a sensible quantum gravity, which is always one single discrete quantum system. It is the averaging which results in

[26] We thank Zhuo-Yu Xian for suggesting this.
[27] One should normalize appropriately.

> this apparent “flat space”. In any given realization of a discrete system though, we would have erratic oscillations both in the ramp region and in the plateau region. The corresponding metrics, though positive definite, do not seem to make any sense, they fluctuate as erratically with z as does the late time two point function. What might be interesting though, is that the erratic oscillations carry information about the specific “microstate” of quantum gravity. In other words it contains the information about all the energy levels in the spectrum of the discrete system. Therefore in principle we can retrieve the information about the microstate of the black hole, by probing very close to the would-be semiclassical horizon. We don’t need to go into the semiclassical interior.

Let us end this section with a comment. Formula (5.78) raises a subtle question. What is a metric in quantum gravity? Our perspective stems from the quantum mechanical point of view. The metric is just another operator in the algebra of which we can calculate expectation values. More in particular, the boundary observer can measure its expectation value. The fact that the near-horizon metric expectation value is not the semiclassical answer, means that in a sense this metric operator creates backreaction. When one tries to think of this from a general relativity perspective this is puzzling. In quantum mechanics though, when we calculate the expectation value of an operator, we really do not care whether this operator significantly effects the state it works on or not. So in our context this backreaction should not be considered surprising. In neither context should if be considered “bad”. This does highlight that if we want to think about metric fluctuations in quantum gravity, we should leave some if not most of our general relativity intuition behind.

5.3.4 Averting an information paradox

The following is but a comment in which we briefly explain why Maldacena’s information paradox averts an actual information paradox.
In semi-classical gravity information is lost. This is in contradiction with a quantum gravity, which is unitary. The result is the information paradox of [186], which was reincarnated as the infaller or firewall paradox [187]. A sharp way to state a paradox is as a set of hypotheses that are all assumed to be true but which are in logical contradiction. The paradox is resolved if one can *prove* one of the hypotheses wrong [188, 189, 190, 191, 192]. For the information paradox these hypothesis are basically unitary quantum mechanics and the assumptions that go into the Hawking calculation [186]. If we decide to go with quantum mechanics then logic dictates that one of the assumptions in the Hawking calculation *needs* to be invalid. In particular one of these assumptions is that quantum gravity effects are suppressed at microscopic distances from the horizon [192]. This is necessary to motivate the use of quantum field theory in curved space and in particular in the Rindler geometry, which directly implies the Unruh effect. The above calculations show that such an assumption is plainly false in quantum gravity. What is funny depending on one’s perspective is, that this essentially follows directly from Maldacena’s version of the information paradox combined with some intuition about Penrose diagrams. The fact that late time correlators don’t decay to zero exponentially

in quantum gravity as they would in a semiclassical black hole, implies almost directly that near horizon correlators of matter in quantum gravity can not have the expected behavior of correlators in Rindler space.
Let us emphasize though that this observation that there is no sharp paradox doesn't solve anything. In writing this we are not on the verge of understanding infaller physics of unitary black hole evaporation from the bulk perspective. Fortunately, some people are [10, 11]. The apparent key towards understanding unitary black hole evaporation is to include appropriate corrections from baby universes to *any* Euclidean calculation that involves gravity.

5.4 Concluding Remarks

We end this chapter with several concluding remarks.

Relation to the bulk reconstruction program

One way to read the calculations of the bulk matter correlators in section 5.2 is as providing a type of HKLL prescription [180, 181, 183, 184] for bulk reconstruction in JT quantum gravity. The prescription is the following. In any fixed metric g the HKLL prescription [180, 181, 183, 184] associates a bulk operator with a linear combination of boundary operators via a convolution integral:

$$\phi_{m^2}(t,z) = \int_{-\infty}^{+\infty} d\tau\, K_{m^2}(\tau,t,z)\,\mathcal{O}_\ell(\tau). \tag{5.85}$$

The kernel represents the bulk to boundary propagator in the metric g. The right hand side represents an operator in the 1d boundary conformal field theory. The parameters are related as per usual via the holographic dictionary $m^2 = \ell(\ell-1)$. This formula ignores matter interactions in the bulk. Furthermore it ignores gravitational fluctuations. The calculations of section 5.2 show that these are correctly accounted for by doing bulk reconstruction in each off shell metric g before path integrating over the metrics with some action $S[g]$. In the case of topologically trivial JT gravity this reduces to a Schwarzian path integral. Let us limit ourselves to that case here. Purely from the 1d boundary point of view we would be led to calculate bulk correlators as:

$$\int [\mathcal{D}f] \int_{-\infty}^{+\infty} d\tau_1\, f\cdot K_{m^2}(\tau_1,t_1,z_1)\ldots \langle f\cdot \mathcal{O}_\ell(\tau_1)\ldots\rangle\, e^{-S[f]}. \tag{5.86}$$

The average denotes the matter path integral in some fixed background (5.14). We would now like to point out that this matches exactly with our calculation of bulk matter correlators. In particular let us focus on the massless scalar bulk two point function (5.28) and on the massive scalar two point function (5.53). Before proceeding let us note that it shouldn't be a surprise this works out. Otherwise the HKLL prescription itself wouldn't make sense. Nevertheless we feel it is interesting to explicitly see how it works in JT quantum gravity. For AdS_2 in Poincaré coordinates and a primary field with

$\ell = 1$ the appropriate Kernel is just a Heaviside function [182, 160]. This transforms as a scalar between frames:

$$\phi_0(T,Z) = \int_V^U dT\, \mathcal{O}_1(T) = \int_v^u dt\, f'(t)\, \mathcal{O}_1(f(t)) = \int_v^u dt\, f\cdot \mathcal{O}_1(t) = f\cdot\phi_0(t,z). \quad (5.87)$$

The final equality holds because the bulk field is a scalar and hence does not transform. From this, one then finds the HKLL kernel in a generic frame:

$$f\cdot K_0(\tau,t,z) = \theta(z-|t-\tau|). \quad (5.88)$$

With this knowledge we can now indeed immediately read the calculation of the massless scalar bulk (5.28) as an implementation of the HKLL prescription (5.86). For a massive bulk field the steps are identical. One determines the bulk to boundary propagator by solving the Klein Gordon equation in AdS_2 and finds its reparameterized version by demanding that $\phi_{m^2}(T,Z)$ is a scalar under (5.13). Furthermore one uses the fact that $\mathcal{O}_\ell(\tau)$ is a conformal primary of weight ℓ. One finds:

$$f\cdot K_{m^2}(\tau,t,z) = \frac{(f(u)-f(\tau))^{\ell-1}(f(v)-f(\tau))^{\ell-1}}{f'(\tau)^{\ell-1}(f(u)-f(v))^{\ell-1}}\theta(z-|t-\tau|). \quad (5.89)$$

The calculation of the massive bulk two point function (5.53) is now indeed manifestly of the form (5.86)

Beware of semiclassical intuition

Let us emphasize again an important point. Step zero in computing correlators of bulk matter in quantum gravity is to define a bulk frame in a diffeomorphism invariant manner. To define such a bulk frame, as explained in the introduction, one needs to anchor bulk points to the asymptotic boundary via some type of geodesic construction [159]. Skipping this step, one would end up calculating covariant observables in a diffeomorphism invariant theory. The results would thus be non physical.[28] Upon carefully defining a bulk point via geodesic anchoring, one immediately stumbles onto potentially counterintuitive facts. For example, all metrics in the JT gravity path integral are patches of AdS_2. However, correlators of bulk matter coupled to JT quantum gravity do not need to and generically will not have the precise structure of matter correlators in a fixed AdS_2 background. This can be understood in layman terms as due to the properties of expectation values in statistical ensembles. For example, with an ensemble average over x we have:

$$\langle f(x)\rangle \neq f(\langle x\rangle). \quad (5.90)$$

We could imagine $f(x)$ is some property like "satisfies the AdS_2 Klein Gordon equation". More than anything this highlights we should be very careful when shipping in semiclassical intuition into inherently quantum problems.

[28] This would be like calculating the expectation value of $A(x)\cdot A(x)$ in pure electromagnetism. You will get an answer, but it means nothing.

Caveats regarding quantum fields in Rindler space

One could draw several conclusions from the results of section 5.3. All of them would be based on the fact that we can essentially take any generic property of late time holographic correlators and map this to a similar property of large distance bulk matter correlators. One of the more important conclusions is associated with the fact that large distance implies that at least one of the operators is taken closer than a Planck's length to the semiclassical horizon. One finds that correlators in such a regime are nothing like correlators in a the semiclassical Rindler geometry, much like late time correlators in the dual boundary theory deviate strongly from quasi normal mode decay [14]. The conclusion is that it would be completely wrong to use quantum fields in Rindler to describe near horizon physics in quantum gravity, if one wants to probe very close to the horizon.

The fact that the local curvature R is tiny at the semiclassical horizon of large black holes is only a motivation to neglect UV effects in quantum gravity. It is not a motivation to neglect IR quantum effects such as the discreteness of quantum gravity and the GUE statistics of nearby energy levels. Such IR effects are universal in any model of quantum gravity. Furthermore they are "local" properties of a theory in terms of energy levels. Therefore they do not depend on for example the UV details. These IR effects were found to dictate very near horizon physics. This is because close to the horizon there is an "infinite amount" of space. In this sense the semiclassical horizon is anything but a "harmless" smooth surface in the quantum theory.

We note that in general one should expect the UV details to be important for other questions such as the fate of semiclassical curvature singularities like the one inside a black hole or like the one at the beginning of time.

Gravitational explanation for backreaction

Notice that there is no way to "avoid" the type of backreaction discussed in the previous point by tuning some parameters. No matter how light one chooses a certain operator, or how tiny one makes the Newton constant, there will always be a region close to the horizon where quantum gravitational fluctuations will become important. Similarly one would not try to "avoid" the slope, ramp and plateau in holographic correlators. They are there. We have to accept this and understand their origin from a gravitational point of view. The same goes for similar behavior in bulk matter correlators close to the horizon. The gravitational explanation is identical in both cases. It involves baby universes and eigenbrane boundaries or equivalently α state boundaries [8, 9, 4, 19]. Let us elaborate a bit on the equivalence of the latter two concepts. Applied to JT gravity, the so called α states of [19] are labeled by the energy levels of a single discrete quantum chaotic system. So we have for example the states $|\lambda_1 \dots \lambda_N\rangle$. They should be thought of roughly as a product of single brane correlators, with appropriate normalization:

$$\langle \mathrm{HH}|\lambda_1 \dots \lambda_N\rangle \approx \frac{1}{(2\pi)^{n/2}} \Delta(\lambda_1 \dots \lambda_N)^{1/2} \langle \psi(\lambda_1) \dots \psi(\lambda_N)\rangle \,. \tag{5.91}$$

The Hartle Hawking state $|\mathrm{HH}\rangle$ represents as always the state with no boundaries. The branes (as explained in chapter 4) represent "coherent states" of fixed energy boundaries. This makes sense. One expects in general that the α states are "coherent states" in terms of the number of boundaries. Expectation values of operators in an α state now map to the eigenbrane calculations of chapter 4. For example:

$$\langle\rho(E)\rangle_{\lambda_1\dots\lambda_N} \approx \frac{\langle\lambda_1\dots\lambda_N|\,\rho(E)\,|\lambda_1\dots\lambda_N\rangle}{\langle\lambda_1\dots\lambda_N|\lambda_1\dots\lambda_N\rangle} \approx \frac{\langle\rho(E)\rho(\lambda_1)\dots\rho(\lambda_N)\rangle}{\langle\rho(\lambda_1)\dots\rho(\lambda_N)\rangle}. \tag{5.92}$$

The α state perspective and the eigenbrane perspectives are essentially identical. They are just phrased in a slightly different language.

It is amusing that a single state $|\lambda_1\dots\lambda_N\rangle$ in the nonperturbative Euclidean JT gravity Hilbert space corresponds to essentially the whole of the discrete dual system. This is not at all at odds with the holographic dictionary. We do not need Hilbert spaces to map to Hilbert spaces. Rather only the correlation functions need to match. In fact when we think about it, associating $|\lambda_1\dots\lambda_N\rangle$ to the Hilbert space of Euclidean JT gravity is very natural from the point of view of holography. It is the analogue to for example the statement (2.31). Boundary moduli generically label states in the Hilbert space of a topological field theory when the Cauchy surface in question is the boundary surface. See also the discussion of chapter 3.

The horizon could be smooth and also have a firewall

It is interesting to contemplate a bit the potentially far stretching implications of (5.90) when it comes to observations in quantum gravity. In the introduction we have argued in favor of using lightlike geodesic anchoring to define local bulk observables. As explained below (5.17) though, they are naturally associated with a stationary observer. More importantly they are certainly inadequate to describe physics in an infalling frame. Say we would find a suitable definition of infaller frames. Then a priori on behalf of (5.90) there is no reason for physics in those frames to be equivalent to physics in the bulk frames we've defined using lightlike geodesic anchoring.[29] Indeed we would just have found two physically sensible but inequivalent ways to partially gauge fix diff invariance in the bulk. Different observers indeed correspond to different ways of dressing bulk operators with gravitational Wilson lines. Specifying such a dressing is a gauge choice which partially gauge fixed diff invariance in the bulk. We have not tried to define such a frame. However it is a funny exercise to think about what could be. It is not beyond the realms of possibility that the infaller would not associate special properties to the horizon. In fact this is expected because the horizon is not in any sense a special surface to him. For example he will cross it after a finite proper time, so we might not a priori

[29] This is not contradictory to the equivalence principle from general relativity. Compare for example the Poincaré physics to the physics in our family of frames. For each off shell metric, there is a relation $(u,v) \leftrightarrow (f(u),f(v))$ between different frames. This relation however, does not exist after the path integral over bulk metrics f. As a consequence one should not a priori expect the existence of some generalized equivalence principle to hold in full quantum gravity. This is for example the reason that the exact matter correlators discussed in this work do not have the precise structure of correlators of quantum matter in a fixed AdS_2 background.

expect to see late time of large distance physics at work. This should be contrasted to the stationary observer for whom the horizon is clearly special. In our "stationary" frames, physics near the horizon goes bonkers, at least when we compare it with our expectations of plain old quantum fields in Rindler. This isn't precisely like a real firewall, but it might just qualify as the type of wicked behavior the authors of [187, 193] had in mind. Anyway. It is important to realize the following point. Because generically in quantum gravity physics in one "family" of frames is not expected to be equivalent to physics in another "family" of frames. It is very much within the realms of possibility that there are frames with firewalls and frames with smooth horizons, within the same theory, with no contradiction. This being said, the purpose in life of firewalls is unclear given the recent progress in resolving the Page curve "paradox" [10, 11]. Then again, the purpose in life of strings was initially not to be a model of quantum gravity, and yet here we are. In this sense it is most likely too soon to stop thinking about infalling observers and to no longer take firewalls into serious consideration.

Comments on the purpose of probing locality

The operators we've constructed are local bulk operators in a theory of quantum gravity. Let us prove this potentially slightly confusing fact. JT gravity is time-reversal invariant. We have:

$$G^+(t_1, z_1; t_2, z_2) = \langle \phi(t_2, z_2)\phi(t_1, z_1)\rangle\,. \tag{5.93}$$

Time reversal invariance implies:

$$\langle \phi(t_1, z_1)\phi(t_2, z_2)\rangle = \langle \phi(t_2, z_2)\phi(t_1, z_1)\rangle^*\,. \tag{5.94}$$

We find because of reality that $\langle [\phi(t_1, z_1), \phi(t_2, z_2)]\rangle$ vanishes at $t = 0$.[30] Furthermore we see that the bulk propagators have logarithmic divergences on the lightcone. Combining these two observations results in bulk locality. This was expected. We used lightlike localizing precisely to construct operators which are "as local as possible". The fact that we can obtain genuinely local operators is however a bug of JT gravity rather than a feature. Indeed. This locality can be understood as arising because the theory is not UV complete in the sense that the spectrum $\rho(E)$ goes on forever. We do not expect this kind of locality in a UV complete quantum gravity. For example, a finite dimensional quantum system cannot correspond to a local bulk simply because every calculation gives a finite answer. We can not have divergences when two operators "approach". We expect that bulk locality can be achieved as long as there still exists a smooth geometric dual. For example we could imagine local bulk operators in the double scaled matrix integral of [9] but we can't imagine local bulk operators in the SYK model. Let us explain this in a bit more detail. Schematically for JT gravity the situation is as follows, let us focus on the boundary two point function (5.57) and let's take $\beta \gg 1$ such that one of the energies is effectively small. The vertex behaves up to constants as:

$$|\mathcal{O}_{EE_1}|^2 \approx E_1 e^{-2\pi\sqrt{E_1}}. \tag{5.95}$$

[30] In the case of (5.40) for example, this is invariance under the exchange of E_1 and E_2.

This results in a UV divergence in the topologically trivial version of JT gravity of the type:

$$\int_0^\infty dE_1\,\rho_0(E_1)\,|\mathcal{O}_{EE_1}|^2 \approx \int_0^\infty dE_1\,E_1. \tag{5.96}$$

In writing this, we leave out the contribution from the integral over E which just gives a finite factor $Z(\beta)$. We will also neglect this in the following two examples.[31] In double scaled matrix integral of [9] we would be lead to a factor of the type:

$$\int_{\mathcal{C}} dE_1\,\langle\rho(E_1)\rangle\,|\mathcal{O}_{EE_1}|^2. \tag{5.97}$$

In one representative of this ensemble averaged system or in some specific α state of JT gravity we would have

$$\sum_{i=1}^{\infty}|\mathcal{O}_{M\lambda_i}|^2. \tag{5.98}$$

Both of these expressions have the same type of UV divergence as the simple disk boundary two point function. This makes sense. Both theories correspond to smooth geometric bulk duals. There is then for these models no a priori reason to expect one could not define local bulk operators. For a finite dimensional discrete boundary system though, we would have a finite sum:

$$\sum_{i=1}^{L}|\mathcal{O}_{M\lambda_i}|^2. \tag{5.99}$$

An example would be a single realization of the SYK model or a single representative of a single cut matrix integral that has not been double scaled to zoom in on a spectral edge.[32] There can then be no genuine light cone divergences for “bulk operators”. By consequence there can never be genuinely local bulk physics in a candidate dual quantum gravity to such a finite dimensional system. This is well known in the SYK model [27]. The conclusion in terms of probing bulk locality is that we would be asking JT gravity to answer questions it is simply not meant to answer. In general we should avoid asking UV questions in JT gravity. Rather we should be asking IR questions such those related to late time or near horizon physics. Alternatively we should be asking questions local on the energy axis such as those related to spectral fluctuations. For such questions there is hope to find a universal answer in quantum gravity regardless of the UV completion.

[31] Corrections due to eigenvalue repulsion such as those associated with annuli connecting the two regions are irrelevant in this point because the integral over E_1 is dominated by large energies whereas the integral over E is dominated by small energies. The region where they are close together is thus effectively suppressed.

[32] The spectral edges should coarse grain to the JT gravity density of states.

6 Concluding remarks

There is no point in repeating each of the conclusions of the individual chapters. We will take this chapter rather as an opportunity to introduce some interesting open questions. We are already actively working on some of these. Others could be considered as exercises for the reader. Before doing so, let us quickly repeat only the most important conclusions of this work.

Baby universes

There is good reason to believe that a lot of the magic of the holographic dictionary lies in understanding highly exotic configurations in the bulk gravitational path integral. These include baby universes, wormholes and spacetime D-branes. Only when we account for all of these appropriately are we able to understand fundamental properties of the boundary dual such as discreteness and unitarity, from a bulk gravitational point of view.

Cluster decomposition in quantum gravity

Including these exotic configurations we can understand for example the late time behavior of boundary correlators from a bulk point of view. Finding a gravitational explanation for this behavior is Maldacena's information paradox. We can phrase a bulk version of this information paradox. Bulk matter correlators in a sensible theory of quantum gravity with a discrete dual cannot decay to zero at large spatial separations. By consequence there can be no cluster decomposition in quantum gravity. That is fine. Because we are summing over topologies, quantum gravity is not a local quantum field theory. Therefore there should not a priori be cluster decomposition. The gravitational explanation is that we can have Euclidean wormholes connecting distant regions of spacetime. This results in finite correlators between any two regions.

Quantum effects near the horizon

One way to obtain large distances in the bulk is to probe very close to the semiclassical black hole horizon. By consequence quantum effects proliferate close to the horizon. This counters the intuition that quantum effects would only be important when the gravitational curvature is large such as close to a singularity. The more appropriate statement is that UV modifications of gravity will only be important when the gravitational curvature is large such as close to a singularity. Quantum effects though are rather naturally associated with very small energies, or very small energy differences. The latter will be important at late times and large distances. The UV corrections on the other hand will be important at early times and short distances. This proves that quantum fields in Rindler is a very poor approximation to physics very close to the semiclassical black hole horizon. Quantum effects should not be neglected very close to a black hole.

To close off this work let us present some intriguing open questions and speculations. Others can be found in the concluding remarks of individual chapters. The questions which we have chosen to present here are special in the sense that one could almost immediately get started with doing calculations. By consequence they are very realistic research projects waiting to be done.

More quantum chaos in quantum gravity

As explained for example in [7] we expect generically that suitable averages of observables in a theory with quantum black holes are described by random matrix theory. In JT gravity this is very explicit. One version of JT gravity *is* a random matrix theory. One might wonder whether this holds true in other models of quantum gravity. The only one for which we know the calculations might be analytically tractable is AdS_3 gravity [101, 83]. Can we somehow sum over topologies in three dimensional gravity and obtain a description of AdS_3 as a matrix integral? We have multiple convincing pieces of evidence that this is indeed the case [45].

Cluster decomposition in cosmology

A particularly natural setting to relate the lack of cluster decomposition to Maldacena's information paradox is cosmology. It would be interesting to investigate this using the JT gravity description of dS_2 quantum gravity [62, 63]. One would be inclined to study S matrix elements. This boils down to studying Wilson lines on two copies of global dS_2. One interesting effect in this case is that the copies could connect when we sum over topologies. By consequence in quantum gravity the cross section is not just the S matrix squared. There are connected contributions where one copy of the S matrix connects to the other via Euclidean wormholes. Is the S matrix unitary in different versions of JT gravity?

Are Euclidean wormholes real?

Can we understand unitary evaporation in a discrete system of quantum gravity such as the version of JT gravity with eigenbranes? What comes of the replica wormholes of [10, 11]? It might be that they factorize, but only effectively as in (4.77). A fundamental question is which of the two pictures in (4.77) is "true". Do we really have Euclidean wormholes connecting the black hole interior to the Hawking radiation or is this emergent after summing over all energies on the right hand side of (4.77)? Similar questions have been raised in the concluding remarks of [8, 9, 18, 10]. It might be that the two different pictures in (4.77) just represent different gauge choices [19].

The infalling observer

Is there a natural set of frames in quantum gravity to describe the physics of an infalling observer? Using affine null coordinates might be the way to go. Is it possible to do the resulting Schwarzian path integrals exactly? At least we would do them perturbatively. Perhaps there is already some insight at the perturbative level.

Bulk reconstruction on the annulus

How does bulk reconstruction work on the annulus? How do we anchor a point on the annulus to the boundary? To which boundary do we anchor? Can we do the Schwarzian path integrals? Questions of this type are currently under investigation [194] and are relevant to studying quantum chaos in dS_2.

Comments on a potential holy grail

There is some recent interesting work on so called $T\bar{T}$ deformations of Schwarzian quantum mechanics [185]. Allegedly this pushes the boundary of JT gravity deeper into the bulk. The resulting formulas might no longer look, at first sight, as if they are consistent with the fact that JT gravity in the bulk is an $SL(2,\mathbb{R})$ BF theory. For example the spectrum of the $T\bar{T}$ deformed theory looks like that of a UV complete theory. One might wonder (as the authors of [185] did) if this theory could be related to a non double scaled matrix integral when summing over topologies. A priori that seems very unlikely because the recursion relation of the matrix integral is a purely bulk feature of JT gravity. It depends only on the topological properties of the theory such as the Weil-Petersson volumes and not at all on the Schwarzian boundary conditions. In this sense it would be surprising if moving the boundary deep into the bulk would end up changing the topological recursion relation very much.
We believe the truth it that the topological recursion does not change one bit. In particular we believe that the theory is still a double scaled matrix integral but that by choosing very specific boundary conditions we are just asking "weird" questions about this double scaled matrix integrals. Because we are asking "weird" questions the answers might look at first sight as if they arise from a finite cut matrix integral. In particular we believe that the sum over topologies would end up computing for example schematically:

$$\langle \rho(f(E_1))\rho(f(E_2))\rangle \,. \tag{6.1}$$

There is a specific formula for $f(E)$ in [185]. In these coordinates E_1 and E_2 there would be a finite support for the perturbative answers. But the point is that we are just acting with a coordinate transformation on the answers for the double scaled matrix integral only after doing the actual matrix integral calculation. This is not the same as studying a finite cut matrix integral. We would consequently not call this a UV complete theory of quantum gravity. Rather we have introduced some UV cutoff in a theory which is inherently not UV complete. This is fundamentally different from an actual UV complete model which roughly speaking "knows more" than the low energy UV incomplete description. This deformed version of JT gravity on the other hand "knows less". We need to add stuff like matter fields to JT gravity in order to find a UV complete quantum gravity like we would do for example in string theory. In this context we should not be removing stuff like we would do for example in lattice approximations to quantum fields. In general such lattice approximations are useful among other things because they enable controlled calculations on a computer, but of course in such a context you would not probe for high energy behavior. In JT gravity the calculations are already controllable analytically. We do not need to create a controlled environment by "throwing away" information. In particular one way to realize that this deformed version of JT gravity is different from some finite L matrix integral is to fix eigenvalues in the ensemble by considering eigenbranes or some α state in the bulk. We can distinguish the deformed version of JT gravity from some finite L matrix integral basically because the Dirac deltas transform covariantly under the coordinate transform on the energy axis. The spectral density in an α state will hence not end up looking like a sum of delta spikes with unit weight. In particular they will have weights $1/f'(E)$. This spoils a potential interpretation as a discrete quantum mechanical system. We remind that reader that for a finite L matrix integral, considering an α state in the bulk would result in a spectrum that is a bunch of delta spikes with precisely unit weight. Our conclusion it that this $T\bar{T}$ deformation is not the holy grail in two dimensional quantum gravity.[1] The holy grail would remain to find a precise bulk dual to the full SYK model.

More geometry

It would be great to have an intuitive geometric interpretation of the group theoretic $6j$ symbols that show up when two gravitational Wilson lines cross. Maybe the extra fields γ in the group integrals besides the ϕ which have an interpretation as geodesic lengths, could be interpreted as integrating over offsets in the geodesic distances due to shockwave effects in the quantum theory. That is just random speculation though.

Deriving the matrix integral from gravity

Could we obtain the matrix integral formulation of JT gravity directly from a gravitational calculation? This question is closely related to the recent work of [19]. To attack this question it may be natural to do an infinite number of integral transforms on the matrix integral itself such that we would end up integrating over geodesic lengths $b_1 \dots b_L$ instead of energies $E_1 \dots E_L$. This should correspond to JT gravity ending on any number of geodesic boundaries. This might then be related to topological recursion

[1] Of course it is interesting for other reasons.

for the Weil-Petersson volumes. Potentially the action in the matrix integral could be associated with the nontrivial integration measures which we have for the Weil-Petersson volumes due to the modding by the mapping class group.

A Electromagnetic edge states

In this supplementary chapter we review a work by the author in collaboration with Thomas Mertens, Henri Verschelde and Valya Zakharov [5] on electromagnetic edge states. The logic of this chapter is important to understand the comments on the factorization debate in JT gravity in chapter 3.

A.1 Introduction and summary

One of the open problems in black hole physics is to provide for an understanding of black hole entropy in terms of microscopic degrees of freedom. Essentially this demands we pick a certain theory of quantum gravity, such as string theory, quantize it exactly and take a trace in the resulting Hilbert space.

The entropy of a black hole is the entanglement entropy of said theory of quantum gravity across a codimension two surface, which is some appropriate generalization of the classical event horizon. This resulting entanglement entropy is the von Neumann entropy of the theory on a codimension one Cauchy slice bordered by this generalized horizon and potential asymptotic boundaries. To every subregion one associates a modular Hamiltonian K with which we further associate a reduced density matrix ρ as $K = -\ln \rho$. The entropy of this region is then $S = -\operatorname{Tr} \rho \ln \rho$.[1] One is therefore led to construct and diagonalize the modular Hamiltonian.

For a half-space, as we will be considering, the modular Hamiltonian is the generator of Lorenz boosts orthogonal to the entangling surface. Close to the entangling surface this becomes the Rindler Hamiltonian. The main hurdle in quantizing any theory in a half-space is to find the appropriate boundary conditions to be implemented at the entangling surface. The boundary conditions are to be chosen in such a way that they are “non invasive”. This means as much as demanding that the resulting Hilbert space has precisely the “correct amount” of degrees of freedom. Let us make this more precise by discussing Hilbert space factorization.

- Any physical state in the Hilbert space of the theory on the parent space should exist as a particular state in the tensor product of the respective Hilbert spaces on the subregions of that parent space.

[1] We will always take the state on the parent Cauchy slice to be the vacuum.

- Entanglement entropies can be calculated indirectly via Euclidean path integrals. The canonical example is when the entangling surface is a horizon. The Euclidean path integral then computes the thermal entropy of fields around a black hole. Though often technically feasible, such a calculation does not reveal the underlying Hilbert space structure.[2] In other words, we do not directly infer from such a calculation the microscopic degrees of freedom of black holes. Nevertheless, it provides an important consistency check: only when we impose the appropriate non invasive boundary conditions on the entangling surface, will the corresponding von Neumann entropy of the subregion match this thermodynamic entropy. For example if the boundary conditions are not restrictive enough, there will be "too many" states in the Hilbert space of the subregion, and the von Neumann entropy overshoots the canonical or Renyi entropy.

In summary, we can use the Euclidean calculation as a guide to find the appropriate boundary conditions at the entangling surface. We would like to apply this rationale to a full fledged theory of quantum gravity such as string theory. In this chapter however, we will have a more modest goal: to understand the Hilbert space structure of a subregion and the appropriate boundary conditions for gauge theories.[3]
To make matters as simple as possible, we will focus on electromagnetism, or Maxwell theory. Factorization and boundary conditions in gauge theories were actively investigated in recent years.[4] The canonical entropy of Maxwell theory was calculated in [126, 127, 128]. Donnelly and Wall [220, 221] have pinpointed a statistical interpretation that fully accounts for the thermal answer. The key is to allow a nonzero electric flux in the boundary conditions. They interpreted the resulting horizon flux configurations or "edge modes" as classical backgrounds. In this chapter we promote these edge modes to bona fide states and operators featuring in the factorized Hilbert space of the subregion. We will refer to the result as "edge states". The classic backgrounds are as such replaced by electric flux eigenstates. They are created by acting with large gauge transformations on the vacuum, represented by Wilson lines puncturing the entangling surface. Wilson lines in electromagnetism in general create an electric flux line, and in combination with electric flux measurement operators they make up the gauge invariant algebra of Maxwell theory. Their non locality is at the foundation of the factorization issues associated with gauge theories [222]. The work presented in this and the subsequent chapter shows how to split Wilson lines at a boundary.

A.1.1 Constraint equations

Consider a generic classical field theory in the Hamiltonian formulation, characterized by a set of evolution equations which contain second-order time derivatives, and a set of constraint equations which constrain the phase space. Consider now a Cauchy surface Σ with a dividing surface, separating it into Σ_1 and Σ_2. Ultra local constraint equations

[2]Examples of such calculations in different models include [195, 196, 197, 198, 199, 200, 201].
[3]See chapter 3 for an application to JT gravity which is also a sensible theory of quantum gravity.
[4]See for example [202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219].

such as $\phi = 0$ can be imposed independently on Σ_1 and Σ_2, and the Hilbert space neatly factorizes. Indeed, the constraints imposed on $\mathcal{H}$ are the same as those on $\mathcal{H}_1$ and $\mathcal{H}_2$. However if the constraints come with spatial derivatives such as $\partial_\mu \phi = 0$, then these imply a matching constraint on the entangling surface such as $\phi^1|_\partial = \phi^2|_\partial$. We are now faced with a choice of boundary conditions for the theory in for example Σ_1. The fact that the boundary conditions are non invasive means amongst other things that there should exist an appropriate embedding $\mathcal{H} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$. In the theory leading to $\mathcal{H}$ the entangling surface is not a special surface, consequently $\phi|_\partial$ can take any value. For an embedding $\mathcal{H} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ to exist, we should then clearly allow $\phi^1|_\partial$ and $\phi^2|_\partial$ to take arbitrary values at the entangling surface. We should interpret this as "summing over boundary conditions" for the subspace, a fact that will be made more precise from a path integral construction in chapter B.
Suppose we construct a basis of states of $\mathcal{H}_1$ that diagonalizes $\phi^1|_\partial$. If we denote the other quantum numbers specifying a state in $\mathcal{H}_1$ by ψ_1 this would be $\phi^1|_\partial |\lambda_1, \psi_1\rangle = \lambda_1 |\lambda_1, \psi_1\rangle$. Doing the same for $\mathcal{H}_1$ we end up with a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$ that consists of $|\lambda_1, \psi_1\rangle \otimes |\lambda_2, \psi_2\rangle$. The constraint embeds $\mathcal{H}$ in this direct product as the diagonal sector:

$$|\psi_1, \psi_2, f\rangle = \sum_\lambda f(\lambda)\ |\lambda, \psi_1\rangle \otimes |\lambda, \psi_2\rangle \,. \tag{A.1}$$

Here f encodes the value of $\phi|_\partial$ which can take any value depending on the state in $\mathcal{H}$. From the point of view of an observer constrained to Σ_1 whose evolution is governed by the modular Hamiltonian K_1, the boundary is an infinite redshift surface. This is so because close to the entangling surface, the modular Hamiltonian is the boost operator, so there is a Rindler horizon. This infinite redshift means in particular that an observer constrained to Σ_1 cannot measure or affect the value of $\phi^1|_\partial$. This implies that the algebra of that observer commutes with the operators that change and measure λ_1, and can only affect ψ_1. We will refer to the degrees of freedom ψ_1 as associated with a "bulk" Hilbert space, and to the λ_1 as edge states associated with an "edge" Hilbert space. These edge states contribute significantly to the von Neumann entropy of Σ_1 and explain the answers for the canonical entropies of numerous theories from a Hilbert space perspective. The lesson of this subsection is that non-ultralocal constraints generically result in these so called edge states. In the remainder of this chapter we will make precise how this goes for Maxwell theory.

A.1.2 Horizon charges

Consider Maxwell theory in Lorenz gauge:

$$S = \int_{\mathcal{M}_1} dx \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \nabla_\mu A^\mu \nabla_\nu A^\nu - \partial^\mu \bar{c} \partial_\mu c \right). \tag{A.2}$$

The path integral is over field configuration that satisfy Lorentz gauge:

$$\nabla^\mu A_\mu = 0. \tag{A.3}$$

This is a first class constraint of the type discussed in the previous subsection. We'll ignore the ghosts for now. We have:

$$\delta S = \int_{\mathcal{M}_1} dx\sqrt{-g}\,(\nabla_\mu \nabla^\mu A^\nu)\delta A_\nu + \int_{\partial\mathcal{M}_1} dx\sqrt{-h}\, n_\mu \left(F^{\mu\nu} + g^{\mu\nu}\nabla^\sigma A_\sigma\right)\delta A_\nu. \quad \text{(A.4)}$$

There are massless fields in the bulk:

$$\nabla^\mu \nabla_\mu A_\nu = 0. \quad \text{(A.5)}$$

Furthermore we have the boundary conditions:

$$n_\mu \left(F^{\mu\nu} + g^{\mu\nu}\nabla^\sigma A_\sigma\right)\delta A_\nu|_\partial = 0. \quad \text{(A.6)}$$

Suppose now that $\mathcal{M}_1$ is a submanifold of some larger space $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. When we vary the action of Maxwell theory on $\mathcal{M}$ we find a matching constraint on the would-be dividing surface:

$$n_\mu \left(F^{\mu\nu} + g^{\mu\nu}\nabla^\sigma A_\sigma\right)|_{\partial\mathcal{M}_1} = n_\mu \left(F^{\mu\nu} + g^{\mu\nu}\nabla^\sigma A_\sigma\right)|_{\partial\mathcal{M}_2}. \quad \text{(A.7)}$$

Following the discussion of the previous paragraph, it is clear that we should relax the boundary conditions on the theory on $\mathcal{M}_1$ to allow for boundary currents:

$$n_\mu F^{\mu\nu}|_{\partial\mathcal{M}_1} = \mathcal{J}_\nu. \quad \text{(A.8)}$$

We are then led to "sum over" all such boundary current configurations in an appropriate manner.[5] Due to the linearity of Maxwell or $U(1)$ Yang-Mills theory, we can decompose the quantum theory into the quantization of a "bulk" theory with boundary conditions (A.6) and a decoupled "edge" theory associated with the quantization of the boundary currents (A.8). The structure is just that of the solution space of a sourced linear differential equation, which is the sum of a particular solution to the sourced equation and a general solution to the homogeneous equation. In this case, the homogeneous equations come with perfect magnetic conductor boundary conditions:

$$n_\mu F^{\mu\nu}|_{\partial\mathcal{M}_1} = 0, \qquad n_\mu A^\mu|_{\partial\mathcal{M}_1} = 0. \quad \text{(A.9)}$$

The discussion of the previous paragraph suggests to associate edge states with $n \cdot F|_{\partial\mathcal{M}_1} |\mathcal{J}\rangle = \mathcal{J}\,|\mathcal{J}\rangle$. However as it turns out the transverse magnetic field is not actually significantly constrained by the boundary conditions (A.9) when we are interested in an entanglement calculation. A redshift argument in section A.3.1 shows that the transverse magnetic fields are in fact only constrained to be regular at the entangling surface, a perfectly sensible constraint. This is confirmed in the more rigorous path integral construction of the subsequent chapter, where the path integral localizes due to the redshift on $\mathcal{J}_i = 0$. This leaves only boundary charges $n^\mu \cdot F_{\mu t}|_{\partial\mathcal{M}_1} |\mathcal{Q}\rangle = \mathcal{Q}\,|\mathcal{Q}\rangle$ as edge states. A pragmatic way to prove this, is to check that the von Neumann entropy we end up with matches the canonical Maxwell entropy, which, as we'll see, it does. Adding

[5]This is made more rigorous in the subsequent chapter.

more degrees of freedom to $\mathcal{H}_1$ would result in a mismatch, and would then by definition not correspond to the correct edge theory.

Notice that the Lorenz gauge constraint (A.3) is equivalent to Gauss' law on solutions to the equations of motion:

$$\nabla_\mu F^{\mu t} = 0. \tag{A.10}$$

The edge states allow for factorization of the Gauss' law constraint or equivalently of the Lorentz gauge constraint near the entangling surface, so we could alternatively think of the edge states as associated with a nonzero radial gauge component at the entangling horizon $n^\mu A_\mu|_{\partial\mathcal{M}_1}|\mathcal{A}\rangle = \mathcal{A}\,|\mathcal{A}\rangle$. The bases $|\mathcal{A}\rangle$ and $|\mathcal{Q}\rangle$ are conjugates as it turns out.

This rationale carries through to other gauge theories such as non-Abelian Yang-Mills discussed in the next chapter [6], higher spin gauge theories [5], massive spin fields such as Proca theory [5], weakly coupled Einstein gravity in arbitrary dimensions and open string field theory, where the role of the Lorentz gauge constraint is played by the Virasoro constraints.[6] In each of these models with the exception of Proca theory, we have large gauge degrees of freedom associated with the constraints or equivalently boundary charges associated with some generalization of Gauss law, representing a flux of some field strength. Quantizing these in a manner very similar to the procedure discussed below for Maxwell theory, results in a von Neumann entropy that matches the canonical entropy in question.

Notice that due to the infinite redshift the edge degrees of freedom are static from the perspective of the Rindler observer, consequently we will be led to quantize the zero mode solutions to (A.5).[7] Furthermore, we will see that the redshift ensures a vanishing spatial extend of the corresponding electric field. This corresponds to the intuition that an exterior observer has no access to the horizon degrees of freedom.

A.1.3 Wilson lines

As explained in the previous subsection, we need to include edge modes to allow for states in $\mathcal{H}$ with a nonzero electric flux through the entangling surface, which is arbitrary from the perspective of Σ. In similar spirit, we can directly infer the quantization rules of these edge degrees of freedom from the Maxwell equal time algebra. Canonical quantization of (A.2) corresponds to:[8]

$$[A_\mu(\rho, \mathbf{x}), \Pi^\nu(\rho', \mathbf{y})] = i\delta^\nu_\mu \delta(\rho - \rho')\delta(\mathbf{x} - \mathbf{y}). \tag{A.11}$$

Here,

$$\Pi^\nu = \frac{\partial \mathcal{L}}{\partial \partial_t A_\nu} = \sqrt{-g}\left(F^{\mu t} - g^{t\mu}\nabla_\sigma A^\sigma\right). \tag{A.12}$$

Its spatial components represent electric flux. The only invariant or physical information in (A.11) is Wilson loops creating electric flux:

$$[\Phi_\Omega, \mathcal{W}_\mathcal{C}] = \theta(\Omega \cap \mathcal{C})\,\mathcal{W}_\mathcal{C}. \tag{A.13}$$

[6] We have unpublished work on this.

[7] In Euclidean coordinates this static feature is trivial, as the Rindler horizon reduces to a point.

[8] Notice that here one also quantizes the longitudinal and temporal polarization. These are later projected out by imposing (A.3) and by modding out by null states.

Here Φ_Ω is the flux through a co-dimension two surface $\Omega \subset \Sigma$ and $\mathcal{W}_\mathcal{C}$ is the oriented $U(1)$ Wilson loop on the curve $\mathcal{C}$:[9]

$$\Phi_\Omega = n^\Omega_\nu \Pi^\nu, \quad \mathcal{W}_\mathcal{C} = e^{i\int_\mathcal{C} A}. \tag{A.14}$$

The boundary conditions (A.9) on the "bulk theory" don't allow for electric flux through the horizon. Instead, horizon flux is accounted for by quantization of the edge modes. They are configurations of local horizon electric flux or equivalently of boundary charge, created by Wilson lines piercing the horizon. The entire algebra acting on $\mathcal{H}_1$ consists then of bulk Wilson loops, horizon-anchored Wilson lines, and electric flux throughout the bulk as well as "into" the horizon.
From (A.14) one notices that the horizon-anchored Wilson lines correspond to the radial components $n^\mu A_\mu|_{\partial\mathcal{M}_1}$ discussed in the previous subsection. We will see that the contributions to this quantity are associated with the zero modes of the null (or pure gauge) polarization of the gauge field. These zero modes represent large gauge degrees of freedom. Unlike their bulk counterparts these are seldom redundant and often acquire dynamics. Famous examples of this include the duality between 3d Chern-Simons theory and 2d Wess-Zumino-Witten models. Another example is the duality between JT gravity and Schwarzian quantum mechanics discussed in chapter 2.

The remainder of this chapter is organized as follows.
In **section A.2** we discuss 2d Maxwell theory in Rindler space. The bulk Hilbert space of the model is found to be trivial. Indeed 2d Yang-Mills is topological with correspondingly no propagating degrees of freedom. The edge states in 2d are $U(1)$ charges on the entangling surface created by space-filling Wilson line operators. In the asymptotically flat context on which we focus here, these states have a volume divergence in their energy, which localizes on the state with no edge modes. The theory is then truly thermodynamically trivial.
In **section A.3** we include transverse dimensions. We quantize the propagating sector as well as the topological edge sector and calculate the corresponding partition as trace over the modular Hamiltonian K. In particular we point out how to implement the boundary conditions on the Rindler horizon. The resulting von Neumann entropy matches the Euclidean calculation of [126, 127, 128]. This confirms we have included precisely "enough freedom" on the entangling surface.

A.2 Maxwell theory in two dimensions

Let us first introduce some useful coordinate systems for Minkowski space, with t the proper time of a Rindler observer accelerating through flat space:

$$ds^2 = -dT^2 + dX^2 = e^{2r}\left(-dt^2 + dr^2\right). \tag{A.15}$$

We furthermore define light cone coordinates $U = T - X$, $V = T + X$ and $u = t - r$, $v = t + r$. They are related by a conformal transformation $u = -\ln -U$ and $v = \ln V$

[9]We have specified to the $q = 1$ representation, a generalization is straightforward.

in the right Rindler wedge or R wedge $U < 0$ and $V > 0$. This is readily extended to the L wedge $U > 0$ and $V < 0$. The Rindler horizon is located at $V = 0$. The proper distance to this entangling surface ρ follows as $r = \ln \rho$. Let's now write down the solution space of the massless wave equation (A.5), where one notes the coupling in the Rindler coordinate system:

$$(\partial_r^2 - \partial_t^2)A_r + 2(\partial_t A_t - \partial_r A_r) = 0, \quad (\partial_r^2 - \partial_t^2)A_t + 2(\partial_t A_r - \partial_r A_t) = 0. \tag{A.16}$$

We are looking for positive frequency Rindler modes $\partial_t A_\mu = -i\omega A_\mu$ and intend to construct a basis for the solution space of (A.16). This requires a choice of inner product, which we take to be the generalized Klein-Gordon inner product:[10]

$$(A, B) = i \int d\Sigma_\mu \mathcal{J}^\mu, \quad \mathcal{J}^\mu = \frac{1}{\sqrt{-g}} \left(B_\nu \Pi^{\mu\nu}_{A^*} - A_\nu{}^* \Pi^{\mu\nu}_B \right). \tag{A.18}$$

Here Σ is a Cauchy surface and $\mathcal{J}$ is checked to be a conserved current. At a fixed time slice the electric fluxes (A.12) enter. To write down the basis for the solution space of (A.16) in a compact manner, it is convenient to introduce some notation:

$$\phi_k^R = \frac{1}{\sqrt{16\pi k}} e^{-ik(t-r)}, \quad \phi_k^L = \frac{1}{\sqrt{16\pi k}} e^{-ik(t+r)}, \quad \epsilon = \omega + i. \tag{A.19}$$

We find that the basis consists of the 4 sets of orthogonal modes:

$$\begin{aligned} A^{(1)}_{\mu,k} &= \frac{\epsilon}{|\epsilon|} \left(e^{2r} + \right) \phi_k^R, \quad A^{(2)}_{\mu,k} = \frac{1}{|\epsilon|} \left(\epsilon e^{2r} - \bar{\epsilon} \right) \phi_k^R, \\ A^{(3)}_{\mu,k} &= -\frac{\bar{\epsilon}}{|\epsilon|} \left(+ e^{2r} \right) \phi_k^L, \quad A^{(4)}_{\mu,k} = \frac{1}{|\epsilon|} \left(\epsilon - \bar{\epsilon} e^{2r} \right) \phi_k^L. \end{aligned} \tag{A.20}$$

The modes $A^{(2)}$ and $A^{(4)}$ have positive norm and the modes $A^{(1)}$ and $A^{(3)}$ have negative norm. The quantum field A decomposes into these modes as:

$$A_\mu = \int_0^{+\infty} dk \left(a_k^{(1)} A^{(1)}_{\mu,k} + a_k^{(2)} A^{(2)}_{\mu,k} + a_k^{(3)} A^{(3)}_{\mu,k} + a_k^{(4)} A^{(4)}_{\mu,k} + \text{h.c.} \right). \tag{A.21}$$

Canonical quantization (A.11) boils down to:[11]

$$\begin{aligned} \left[a_k^{(1)}, a_{k'}^{(1)\dagger} \right] &= -\delta(k - k'), \quad \left[a_k^{(2)}, a_{k'}^{(2)\dagger} \right] = \delta(k - k'), \\ \left[a_k^{(3)}, a_{k'}^{(3)\dagger} \right] &= -\delta(k - k'), \quad \left[a_k^{(4)}, a_{k'}^{(4)\dagger} \right] = \delta(k - k'). \end{aligned} \tag{A.23}$$

[10] We define:

$$\Pi^{\mu\nu}_A = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu}. \tag{A.17}$$

[11] This is a consistency check on our choice of inner product. The oscillator commutation relations ought to imply the canonical equal time commutation relations and vice versa. Indeed:

$$\begin{aligned} [A_\mu(t, r), \Pi^\nu(t, r')] &= \int_0^{+\infty} dk \left(A^{(2)}_{\mu,k} \Pi_k^{(2)\nu *} - A^{(1)}_{\mu,k} \Pi_k^{(1)\nu *} + ((1,2) \leftrightarrow (3,4)) - c.c. \right) \\ &= i\delta^\mu_\nu \delta(r - r'). \end{aligned} \tag{A.22}$$

At this point the Hilbert space is the usual Fock space with the above creation and annihilation operators. However, we still need to impose Lorenz gauge and mod out null states. Lorenz gauge is imposed as $\nabla^\mu A^{(+)}_\mu |\psi\rangle = 0$.[12] This translates into the constraints $\langle\phi| n^{(2)}_k - n^{(1)}_k |\psi\rangle = 0$ and $\langle\phi| n^{(4)}_k - n^{(3)}_k |\psi\rangle = 0$ with occupation number defined in the usual way. All states in the original Fock space that meet this constraint are of the form:

$$|\psi\rangle = \prod_k \left(a^{(2)^\dagger}_k + a^{(1)^\dagger}_k\right)^{n_k} \left(a^{(4)^\dagger}_k + a^{(3)^\dagger}_k\right)^{m_k} |0\rangle . \tag{A.24}$$

All of them are null hence the physical field configurations are pure gauge $A_\mu |\psi\rangle = \partial_\mu \phi |\psi\rangle$ with $\Box\phi = 0$. As expected,there are no propagating degrees of freedom left in two dimensional electromagnetism after modding out the null states.

A.2.1 Edge states

We did not explicitly impose the boundary conditions (A.9) on the above propagating modes. We will show how to do so in the next section in general dimensions. We can understand that the perfect magnetic boundary conditions don't allow for electric flux through the boundary in the "bulk" Hilbert space discussed in the previous section. To account for this in $\mathcal{H}_1$ we are led to quantize the zero mode sector of the theory. We will find that in fact the 2d edge states represent a constant flux profile throughout the entire subregion, say Σ_1. In other words the edge theory of 2d Maxwell captures *all* of the degrees of freedom of the theory. The zero mode solutions to (A.3) include:

$$A_t(\rho, t) = -\frac{q}{2}\rho^2, \quad A_\rho(\rho, t) = -a\frac{1}{\rho \ln \epsilon}. \tag{A.25}$$

We chose the expansion coefficients such that q represents the electric flux $F^{\rho t}(\rho, t) = q/\rho$. From (A.14) and (A.12) we then find $\Phi(\rho, t) = q$. This is also the electric field in Minkowski coordinates. This makes manifest that the zero mode sector contains constant electric field solutions. The second contribution is pure gauge $A_\mu = \partial_\mu \phi$ with $\phi(\rho, t) = -a \ln \rho / \ln \epsilon$. By taking ϵ to zero the function $\ln \rho / \ln \epsilon$ becomes a type of one-sided Heaviside distribution with unit value at the entangling surface $\rho = 0$ and which vanishes for all $\rho > 0$. We are left at $t = 0$ with:

$$\Phi(\rho) = q, \quad \mathcal{W}_\mathcal{C} = e^{ia}. \tag{A.26}$$

Here the Wilson line contour $\mathcal{C}$ is space threading from $\rho = 0$ to $\rho = \infty$. Quantization of this (a, q) phase space is achieved by imposing the Maxwell algebra (A.13) which results in:

$$[a, q] = i. \tag{A.27}$$

As advertised around (A.10) the edge state bases $|a\rangle$ and $|q\rangle$ are conjugates. The first basis represents the value of a would-be gauge field at the horizon $\phi|_{\partial\mathcal{M}_1} |a\rangle = a |a\rangle$

[12] The $(+)$ denotes we single out the positive frequency part of the field.

whilst the latter represents horizon flux eigenstates $\Phi|_{\partial\mathcal{M}_1} |q\rangle = q\,|q\rangle$. We can create an electric flux eigenstate $|q = \mathcal{E}\rangle$ by working with a space treading Wilson line in the representation $\mathcal{E}$ of $U(1)$. We can denote this as $\mathcal{W}_{\mathcal{C}}^{\mathcal{E}} = e^{i\mathcal{E}a}$. Here a is an operator and $\mathcal{E}$ is crucially just a number. We have:

$$|\mathcal{E}\rangle = \mathcal{W}_{\mathcal{C}}^{\mathcal{E}}\,|0\rangle\,. \tag{A.28}$$

The energy of these states suffers from a volume divergence:[13]

$$H_1\,|q\rangle = \int_0^\infty d\rho\sqrt{-g}\left(-\frac{1}{2}F_{\rho t}F^{\rho t}\right)|q\rangle = \frac{Vq^2}{2}\,|q\rangle \tag{A.30}$$

As explained in the introduction, Gauss' law (A.10) or equivalently the Lorenz gauge constraint (A.3) embeds the Hilbert space $\mathcal{H}$ for the Minkowski evolution in $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the diagonal sector:[14]

$$|f\rangle = \sum_q f(q)\,|q\rangle \otimes |q\rangle = \sum_a g(a)\,|a\rangle \otimes |a\rangle\,. \tag{A.31}$$

The Minkowski vacuum has a thermal density matrix in either $\mathcal{H}_1$ or $\mathcal{H}_2$, so we have the thermofield double:

$$|\mathrm{HH}\rangle = \sum_q e^{-Vq^2/2}\,|q\rangle \otimes |q\rangle\,. \tag{A.32}$$

Here we are considering an asymptotically flat setup, so we must take V to ∞ which localizes on the vacuum in the edge sector as well. The full theory of two dimensional electromagnetism is thus confirmed to be thermodynamically trivial.

A.3 Maxwell theory in more dimensions

In more than two dimensions we do have propagating degrees of freedom. We will see that implementing the boundary conditions (A.9) requires us to introduce a regulator, which we need to keep track of carefully in order to make sense of the thermodynamics. Indeed, it is well known that quantum field theory in Rindler suffers from short distance divergences. If one wants to say anything meaningful, a regulator has to be introduced to pinpoint "how divergent" certain quantities are. For the edge degrees of freedom we won't have trouble with the volume divergence of the two dimensional example no more, but again we'll have to carefully keep track of the same short distance regulator to make sense of the thermodynamics.

[13] To find the correct sign, one has do define the Hamiltonian as a component of the stress tensor with

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{A.29}$$

Furthermore, we have in this coordinate system $V = \int_0^\infty d\rho\,\rho$.

[14] The function $g(a)$ is the Fourier transform of $f(q)$.

A.3.1 Propagating bulk modes

We start by constructing and quantizing the phase space of propagating degrees of freedom subject to the boundary conditions (A.9). Some details are banished to the supplementary section A.5.1. It is convenient to introduce the scalar modes:[15]

$$\phi_{\omega,\mathbf{k}} = \sqrt{\sinh(\pi\omega)} K_{i\omega}(k\rho) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}. \tag{A.34}$$

These solve $\Box\phi = 0$. We define furthermore a basis of unit vectors:

$$e_\mu^{(0)} = (\rho, 0, \mathbf{0}), \quad e_\mu^{(1)} = (0, 1, \mathbf{0}), \quad e_\mu^{(k)} = \left(0, 0, \frac{\mathbf{k}}{k}\right), \quad e_\mu^{(a)} = (0, 0, \mathbf{n}^a). \tag{A.35}$$

The first has norm minus one the others plus one. The solution space of $\nabla^\mu \nabla_\mu A_\nu = 0$ is spanned by the basis [223]:

$$\begin{aligned} A^{(1)}_{\mu,\omega\mathbf{k}} &= \frac{1}{k}\left(\rho\partial_\rho, \frac{1}{\rho}\partial_t, \mathbf{0}\right)\phi_{\omega,\mathbf{k}} = \frac{1}{k}\left(e_\mu^{(0)}\partial_\rho\phi_{\omega,\mathbf{k}} + e_\mu^{(1)}\frac{1}{\rho}\partial_t\phi_{\omega,\mathbf{k}}\right), \\ A^{(0)}_{\mu,\omega\mathbf{k}} &= \frac{1}{k}\left(\partial_t, \partial_\rho, \mathbf{0}\right)\phi_{\omega,\mathbf{k}} = A^{(G)}_{\mu,\omega\mathbf{k}} - A^{(k)}_{\mu,\omega\mathbf{k}}, \\ A^{(k)}_{\mu,\omega\mathbf{k}} &= i e_\mu^{(k)}\phi_{\omega,\mathbf{k}}, \\ A^{(a)}_{\mu,\omega\mathbf{k}} &= i e_\mu^{(a)}\phi_{\omega,\mathbf{k}}. \end{aligned} \tag{A.36}$$

The phase space is then spanned by the expansion coefficients of A:

$$A_\mu = \sum_{\omega,\mathbf{k}} \alpha^{(1)}_{\omega\mathbf{k}} A^{(1)}_{\mu,\omega\mathbf{k}} + \alpha^{(0)}_{\omega\mathbf{k}} A^{(0)}_{\mu,\omega\mathbf{k}} + \alpha^{(k)}_{\omega\mathbf{k}} A^{(k)}_{\mu,\omega\mathbf{k}} + \alpha^{(a)}_{\omega\mathbf{k}} A^{(a)}_{\mu,\omega\mathbf{k}} + (hc). \tag{A.37}$$

The modes are normalized using the Klein-Gordon norm and imposing the canonical commutation relations (A.11) results in a Fock space:

$$\begin{aligned} &\left[\alpha^{(0)}_{\omega\mathbf{k}}, \alpha^{(0)\dagger}_{\omega'\mathbf{k}'}\right] = -\delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'), \quad &&\left[\alpha^{(1)}_{\omega\mathbf{k}}, \alpha^{(1)\dagger}_{\omega'\mathbf{k}'}\right] = \delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}') \\ &\left[\alpha^{(k)}_{\omega\mathbf{k}}, \alpha^{(k)\dagger}_{\omega'\mathbf{k}'}\right] = \delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'), \quad &&\left[\alpha^{(a)}_{\omega\mathbf{k}}, \alpha^{(b)\dagger}_{\omega'\mathbf{k}'}\right] = \delta_{ab}\delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'). \end{aligned} \tag{A.38}$$

The modes $A^{(k)}$ don't respect Lorenz gauge.[16] Imposing the constraint $\nabla^\mu A_\mu^{(+)} |\psi\rangle = 0$ then takes the operators $\alpha^{(k)\dagger}$ out of the equation. The modes $A^{(G)}$ are pure gauge and consequently the operators $\alpha^{(G)\dagger}$ create null states which are redundant and to be modded out. The resulting Hilbert space is the Fock space associated with $\alpha^{(1)\dagger}$ and

[15] These are normalized as:

$$\int_0^{+\infty} \frac{d\rho}{\rho} dx\, \phi_{\omega,\mathbf{k}}(\rho,\mathbf{x})\, \phi_{\omega',\mathbf{k}'}(\rho,\mathbf{x}) = \frac{1}{\omega}\delta(\omega-\omega')\delta_{\mathbf{k},\mathbf{k}'}. \tag{A.33}$$

[16] In particular:

$$\nabla^\mu A_\mu^{(k)} = -k\phi \neq 0. \tag{A.39}$$

$\alpha^{(a)\dagger}$. A caveat is that we are yet to impose the boundary conditions (A.9) which will further restrict the phase space. We would like to emphasize that the Klein-Gordon inner product is meaningless without boundary conditions at $\partial\Sigma_1$. It does not make sense to discuss the classical solution space of a differential equation without specifying boundary conditions. In Rindler coordinates the boundary is an infinite redshift surface, making it somewhat subtle how to impose the boundary conditions in a meaningful way. This is related to the short distance divergence in QFT discussed above. To deal with this, we will regularize the boundary surface from $\rho = 0$ to $\rho = \epsilon$ and only take ϵ to zero at the very end. This results in a discretized phase space and finite thermodynamic quantities which diverge upon $\epsilon \to 0$. Any interpretation though is only sensible at $\epsilon = 0$. The boundary conditions (A.9) translate to:

$$\Pi^\rho|_{\rho=\epsilon} = 0, \quad \rho F_{\rho i}|_{\rho=\epsilon} = 0, \quad A^\rho|_{\rho=\epsilon} = 0.$$

These look more familiar in 4d:

$$\mathbf{E}_\perp|_{\rho=\epsilon} = 0, \quad \rho\mathbf{B}_\parallel|_{\rho=\epsilon} = 0, \quad A^\rho|_{\rho=\epsilon} = 0. \tag{A.40}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are as perceived by a stationary Rindler observer. Notice the metric factor in the magnetic boundary condition. For ϵ to zero this merely implies regularity of the transverse magnetic field, a very sensible constraint. This is the reason for the absence of edge degrees of freedom associated with the transverse magnetic field. In the supplementary section A.5.1 we write out the field strength in function of the modes (A.36). One infers from this that the boundary conditions (A.9) enforce Dirichlet boundary conditions on the scalar wavefunctions associated with $A^{(1)}$:

$$\phi_{\omega,\mathbf{k}}|_{\rho=\epsilon} = 0. \tag{A.41}$$

The wavefunctions associated with all other modes are restricted to satisfy Neumann boundary conditions:

$$\rho\partial_\rho\phi_{\omega,\mathbf{k}}|_{\rho=\epsilon} = 0. \tag{A.42}$$

These discretize the range of ω in (A.37) for a given value of $k\epsilon$. We are left with:[17]

$$A_\mu = \sum_{\mathbf{k}} \left(\sum_{\omega\in\sigma_D} \alpha^{(1)}_{\omega\mathbf{k}} A^{(1)}_{\mu,\omega\mathbf{k}} + \sum_{\omega\in\sigma_N} \sum_a \alpha^{(a)}_{\omega\mathbf{k}} A^{(a)}_{\mu,\omega\mathbf{k}} \right) + (\text{h.c.}). \tag{A.46}$$

[17]The Dirichlet and Neumann spectra are:

$$\sigma_D = \{\omega | K_{i\omega}(k\epsilon) = 0\}, \quad \text{and} \quad \sigma_N = \{\omega | \rho\partial_\rho K_{i\omega}(k\epsilon) = 0\}. \tag{A.43}$$

The modified Bessel function behaves asymptotically as:

$$K_{i\omega}(x) \approx \frac{1}{2}\Gamma(i\omega)e^{-i\omega\ln\frac{x}{2}} + \text{c.c.}, \quad x \ll 1, \qquad K_{i\omega}(x) \approx \frac{\sqrt{\pi}}{2}\frac{e^{-x}}{\sqrt{x}}, \quad x \gg 1. \tag{A.44}$$

It oscillates erratically near $\rho \approx 0$. For $k\epsilon \ll 1$ the spectra (A.43) become approximately equidistant:

$$\sigma_D := \{\omega_n, |\omega_n = \frac{\pi n}{\ln\frac{2}{k\epsilon}}\}, \quad \text{and} \quad \sigma_N := \{\omega_n|\omega_n = \frac{\pi(n-1/2)}{\ln\frac{2}{k\epsilon}}\}, \quad n > 0. \tag{A.45}$$

Upon taking $\epsilon \to 0$ we recover a continuum for $\omega > 0$.

We note that the zero mode $\omega = 0$ never enters in these discrete spectra because $K_0(x)$ behaves fundamentally different near the origin than $K_{i\omega}(x)$.[18] In summary, the bulk photon with perfect magnetic conductor boundary conditions has one polarization with Dirichlet boundary conditions and $d-2$ polarizations with Neumann boundary conditions.[19] Thermodynamically this is significant. A Dirichlet polarization contributes differently than does a Neumann polarization. For example in the supplementary section A.5.2 we find that the ratio of the partition functions is nontrivial:[20]

$$\frac{Z_D(\beta)}{Z_N(\beta)} = \prod_k \left(\frac{\beta}{2\pi \ln \frac{2}{k\epsilon}} \right)^{1/2} . \tag{A.47}$$

The partition function of the propagating modes is then $Z_N(\beta)^{d-1}$ times this ratio.

A.3.2 Edge states

The purpose in life of the edge degrees of freedom is to account for nonvanishing horizon electric flux in $\mathcal{H}_1$ or equivalently to allow for a nonvanishing radial component of the gauge field at the entangling surface. Such nonzero values are manifestly not included in the propagating sector on account of the boundary conditions (A.40). As explained below (A.14) we should look for the edge modes in the zero mode sector of the solution space of (A.5). As discussed below (A.46) this is related to the particular behavior of $K_0(x)$ as compared to its finite frequency nephews. Consider the following subsector of the zero mode solution space of (A.5), which should be compared to (A.25):

$$A = -\sum_{\mathbf{k}} \left(\frac{1}{k} q_{\mathbf{k}} A^{(1)}_{\mathbf{k}} + k a_{\mathbf{k}} A^{(G)}_{\mathbf{k}} \right) . \tag{A.48}$$

Here we defined the normalization of the zero mode scalar such that it takes unit value on the entangling surface:[21]

$$\phi_{\mathbf{k}}(\rho, \mathbf{x}) = \frac{K_0(k\rho)}{K_0(k\epsilon)} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \phi_{\mathbf{k}}(\epsilon, \mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{A.49}$$

Taking ϵ to zero this function reduces to a plane wave multiplied with the same Heaviside distribution as we had for the zero mode sector in (A.25) with unit value at the entangling surface $\rho = 0$ and which vanishes for all $\rho > 0$. This delta localizes the pure gauge component of (A.48) on the entangling surface resulting in a finite contribution to a

[18] We have $K_0(x) \approx \ln \frac{2}{x}$ for $x \ll 1$.

[19] Here d is dimensionality of Σ.

[20] This is only precise for $k\epsilon \ll 1$. The partition function gets contributions from arbitrarily high k. The same is true for the edge partition function. The cancellation of the explicit ϵ dependence in the total partition function (A.63) however, does hold for any k.

[21] This is to be compared to (A.34).

Wilson line going into the entangling surface at position $\mathbf{x}$:[22]

$$\mathcal{W}_{\mathcal{C}} = e^{ia_{\mathbf{x}}}. \tag{A.51}$$

The radial electric flux associated with the solution space (A.48) has a finite value on the entangling surface but vanishes for all $\rho > 0$ due to the Heaviside in (A.49). Defining the Fourier transform of the operator $q_{\mathbf{k}}$ as $q_{\mathbf{x}}$ we have the radial flux profile:

$$\Phi(0, \mathbf{x}) = q_{\mathbf{x}}, \qquad \Phi(\rho, \mathbf{x}) = 0, \quad \rho > 0. \tag{A.52}$$

The total flux through the entangling surface Φ_Ω featuring in (A.13) is the integral of $\Phi(0, \mathbf{x})$. The edge phase space parameterized by $a_{\mathbf{k}}$ and $q_{\mathbf{k}}$ is quantized by imposing the Maxwell algebra (A.13) stating that a Wilson line piercing through the entangling surface creates electric flux through that surface. This implies:[23]

$$[a_{\mathbf{k}}, q_{-\mathbf{k}'}] = i\delta_{\mathbf{k},\mathbf{k}'}. \tag{A.53}$$

In writing this we imagine there are some boundary conditions in the transverse directions which discretize $\mathbf{k}$. A state with a plane wave horizon electric flux $\Phi(\mathbf{k})$ is obtained as:

$$|\Phi(\mathbf{k})\rangle = e^{i\Phi(\mathbf{k})a_{-\mathbf{k}}} |0\rangle . \tag{A.54}$$

Indeed:

$$q_{\mathbf{k}} |\Phi(\mathbf{k})\rangle = \Phi(\mathbf{k}) |\Phi(\mathbf{k})\rangle . \tag{A.55}$$

Working with products of such operators we can manufacture a generic horizon flux profile eigenstate. Let us make some remarks.

- Via a different set of techniques the authors of [224] reached a very similar conclusion. A set of canonical conjugate degrees of freedom lives on the entangling surface for Maxwell and Yang-Mills theory. They understood these degrees of freedom as "new" degrees of freedom introduced in order to restore gauge invariance $A_\mu \sim A_\mu + \partial_\mu \phi$ in the entire subregion including on the boundary. Our point of view is that the edge degrees of freedom include the gauge field on the entangling surface which is then *not redundant.* There is indeed no a priori reason to demand invariance under large gauge transformations in general. The difference between these point of vies is essentially semantics though. The theory itself doesn't actually care whether or not we introduce extra degrees of freedom and mod the same amount of freedom out, or do nothing at all.

[22] We defined the Fourier transform of the operator:

$$a_{\mathbf{x}} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}. \tag{A.50}$$

[23] The components of the gauge field and of the conjugate field should be hermitian operators, therefore $q_{-\mathbf{k}} = q_{\mathbf{k}}^\dagger$.

- The total radial electric field in the Rindler wedge measured by either a Rindler or a Minkowski observer is:

$$\mathcal{E}(t,\rho,\mathbf{x}) = \sum_{\mathbf{k}} q_{\mathbf{k}} \frac{K_0(k\rho)}{K_0(k\epsilon)} e^{i\mathbf{k}\cdot\mathbf{x}} - \sum_{\omega\in\sigma_D,\mathbf{k}} \alpha^{(1)}_{\omega,\mathbf{k}} k\sqrt{\sinh(\pi\omega)} K_{i\omega}(k\rho) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}.$$

  For ϵ to zero this is complete in the space of all square integrable functions on the Rindler wedge. The same is true for the radial component of the gauge field.

- The action of a configuration of Wilson lines on the edge sector is only determined by its set of boundary punctures with an orientation. The bulk profile of $\mathcal{C}$ has no effect because the horizon flux only has a nonvanishing commutator with the radial component of the gauge field at the entangling surface per (A.11). Notice that such Wilson line endpoints also create electric flux tangential to the horizon from the perspective of a single Rindler wedge.[24] This is just the Coulomb field of a point charge on the horizon. There is no such tangential flux due to the edge sector in the Minkowski Hilbert space $\mathcal{H}$ though. There is an equal but opposite charge on the other side of the horizon which cancels the contribution of the charge in the right wedge.[25] In the end we just have one field line flowing through the entangling surface with no charges.

A.3.3 A thermodynamic consistency check

Although we believe that the idea of making the entangling surface transparent by introducing a "minimal amount" of edge degrees of freedom is the way to go, it seems hard to prove this in general for any arbitrary theory.[26,27] Therefore we are forced to do a consistency check. In particular we must calculate the entanglement entropy and match with the replica trick calculations of [126, 127, 128]. We will calculate the partition function of the right wedge here, which is equivalent information. For the contributions of the propagating degrees of freedom one uses (A.47). Let's now calculate the regularized energy of the edge states (A.55) and the corresponding edge partition function. Consider the Maxwell Hamiltonian:

$$H_{\text{edge}} = \int dx d\rho\rho \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - F^{t\mu}F_{t\mu}\right). \tag{A.56}$$

[24] The zero mode configuration (A.48) results in $E_i = \sum_{\mathbf{k}} i\frac{k_i}{k^2} q_{\mathbf{k}} \partial_\rho \phi_{\omega,\mathbf{k}}$.

[25] The charges are not taken to infinity as they would for a dipole.

[26] As discussed in the introduction though, this upholds very well in a case by case study.

[27] Certainly from a Hilbert space point of view there is no a priori restricting on the amount of edge degrees of freedom one introduces. For example we could just add a quantum number n to the states in $\mathcal{H}_1$ with no real physical meaning and with n running from 0 to N. We can then manipulate the entanglement entropy at will by playing with N.

Inserting the edge mode expansion (A.48) this simplifies to:[28]

$$H = \sum_{\mathbf{k}} |q_{\mathbf{k}}|^2 \frac{1}{2k^2 \ln \frac{2}{k\epsilon}}. \tag{A.59}$$

As a consistency check note that the Hamiltonian generates the correct classical equations of motion $\dot{a_{\mathbf{k}}} = 0$ and $\dot{q_{\mathbf{k}}} = 0$ for the static solutions when we take ϵ to zero. We see that the edge theory is essentially a collection of free non-interacting particles in 2d. There is one particle for each transverse momentum with a mass that depends on the transverse momentum and diverges for ϵ to zero. Sticking to the notation of (A.1) one immediately identifies the Minkowski ground states in $\mathcal{H}$ as the thermofield double:

$$|\mathrm{HH}\rangle = \sum_{\Phi} e^{-\pi H(\Phi)} \sum_{\psi} e^{-\pi H(\psi)} |\Phi, \psi\rangle \otimes |\Phi, \psi\rangle . \tag{A.60}$$

Here the edge states are denoted by Φ and the propagating Hilbert space is spanned by ψ. Tracing over $\mathcal{H}_2$ results in a thermal density matrix for $\mathcal{H}_1$ the von Neumann entropy of which is the entanglement entropy of the Minkowski ground state (or Hartle-Hawking state or thermofield double state) across the entangling surface. We can calculate the contribution of the edge degrees of freedom to the right wedge partition function as a trace over the continuous edge Hilbert space (A.55) which is just a Gaussian integral:[29]

$$\int [\mathcal{D}\Phi]\, e^{-\beta E(\Phi)} = \prod_{\mathbf{k}} k \left(\frac{\beta}{2\pi \ln \frac{2}{k\epsilon}} \right)^{-1/2} \tag{A.62}$$

Combining this with the contribution of the propagating modes discussed around (A.47) we end up with:

$$Z(\beta) = Z_N(\beta)^{d-1} \prod_{\mathbf{k}} k. \tag{A.63}$$

The second factor is precisely the contact term found in the replica trick calculation. This completes the proof that we have introduced precisely the "correct amount" of edge degrees of freedom in our quantization procedure.

[28] The only nonzero terms are:

$$H = \frac{1}{2} \left(\int dx \int \frac{d\rho}{\rho} (F_{t\rho})^2 + \sum_i \int dx \int \frac{d\rho}{\rho} (F_{ti})^2 \right). \tag{A.57}$$

Plugging in the expansion (A.48) this becomes:

$$H = \frac{1}{2} \sum_{\mathbf{k}} q_{\mathbf{k}} q_{-\mathbf{k}} \frac{1}{k^2 K_0(k\epsilon)^2} \left(k^2 \int d\rho \rho K_0^2(k\rho) + \int d\rho \rho \left(\partial_\rho K_0(k\rho)\right)^2 \right). \tag{A.58}$$

One then uses partial integration and the asymptotics of the Bessel function.

[29] With the density $p(\Phi) = e^{-2\pi E(\Phi)}/Z$ the contribution of the edge modes to the von Neumann entropy becomes:

$$S = - \int [\mathcal{D}\Phi(\mathbf{x})]\, p(\Phi) \ln p(\Phi). \tag{A.61}$$

A.4 Concluding remarks

We end this chapter with two closing remarks.

No microstates without quantum gravity

The ϵ regularization is important to make sense of thermodynamics in Rindler but as alluded to earlier any physical interpretation demands taking ϵ to zero. Doing so reveals that the edge states represent a degeneracy on top of the propagating Hilbert space $H(\Phi) = 0$. This is somewhat trivial given that they are zero mode solutions to (A.5). Remember furthermore that they are inaccessible to an outside observer, whose algebra is restricted to work in the Hilbert space of propagating degrees of freedom. In combination this suggests to think of the edge states as providing electromagnetic hair to the black hole, or electromagnetic microstates if you will. Of course there can be no hope of finding the microstates of a black hole in the quantization of quantum fields on a fixed background. Obvious species problems aside, it makes no sense whatsoever to talk about black hole microstates unless you have just quantized a full fledged theory of quantum gravity such as string theory or a lower dimensional version of quantum gravity such as pure JT gravity or pure AdS_3 gravity. In other words, we can understand corrections to $S_0 = A/4G$ by investigating quantum fields on a classical background if $G \ll 1$. But there is no hope of understanding S_0 itself within that context. This is a gravitational object demanding a counting of states in a quantum theory of gravity.

Soft photons are electromagnetic edge modes

Let us denote the conjugate edge state basis to $|\Phi\rangle$ as $|\mathcal{A}\rangle$. These are eigenstates of large gauge field components $a_{\mathbf{k}}$ at the boundary or equivalently of the boundary anchored Wilson line. For example $a_{\mathbf{k}}\,|\mathcal{A}(\mathbf{k})\rangle = \mathcal{A}(\mathbf{k})\,|\mathcal{A}(\mathbf{k})\rangle$. Consider now the weighted charge operator:

$$q(\mathcal{A}) = \int_\Omega d\mathbf{x}\, q_{\mathbf{x}}\, \mathcal{A}(\mathbf{x}). \tag{A.64}$$

Exponentiating this operator we find an operator that changes the large gauge field at the boundary. This is the analogue of the Wilson line operator which changes the fluxprofile when working on an electric flux eigenstate $|\Phi\rangle$. One difference is that the Wilson line acts locally. Here we are considering the analogue of an operator that creates an entire electric field instead of a single fluxline. It creates, a coherent combination of Wilson lines:

$$e^{iq(\mathcal{A}_2)}\,|\mathcal{A}_1\rangle = |\mathcal{A}_1 + \mathcal{A}_2\rangle\,. \tag{A.65}$$

For an infinitesimal deformation $\mathcal{A}_2 = \delta\mathcal{A}$ this becomes:

$$i[\mathcal{A}, q(\delta\mathcal{A})] = \delta\mathcal{A}. \tag{A.66}$$

This is yet another way of stating that the charges can be thought of as static photons, generating a large $U(1)$ gauge transformation. This picture of edge states is hence manifestly identical to the “soft photon” discussion of [225, 226] up to a choice of Cauchy surface. In [225, 226] these soft photons were argued to provide electromagnetic hair. In conclusion electromagnetic edge modes and soft photons are essentially identical phenomena.

A.5 Supplementary material

We gather some of the more technical material associated with the discussion on electromagnetic edge states presented in the main body of this chapter.

A.5.1 Some details on propagating bulk modes

With the scalar mode ϕ defined in (A.34), a set of orthonormal modes (for $\omega \neq 0$) is (A.36). We have $\nabla^\mu A_\mu^{(k)} = -k\phi = -\nabla^\mu A_\mu^{(0)}$. Each component of the gauge field expands as:

$$\begin{aligned} A_i &= \sum_{\omega,\mathbf{k}} i\frac{k_i}{k}\phi\alpha_{\omega\mathbf{k}}^{(k)} + in_i^a\phi\alpha_{\omega\mathbf{k}}^{(a)} + (hc) \\ A_t &= \sum_{\omega,\mathbf{k}} -\frac{i\omega}{k}\phi\alpha_{\omega\mathbf{k}}^{(0)} + \frac{1}{k}\rho\partial_\rho\phi\alpha_{\omega\mathbf{k}}^{(1)} + (hc) \\ A_\rho &= \sum_{\omega,\mathbf{k}} \frac{1}{k}\partial_\rho\phi\alpha_{\omega\mathbf{k}}^{(0)} - \frac{i\omega}{k\rho}\phi\alpha_{\omega\mathbf{k}}^{(1)} + (hc). \end{aligned} \tag{A.67}$$

The conjugate momenta are via (A.12):

$$\begin{aligned} \Pi^i &= \sum_{\omega,\mathbf{k}} -\omega\frac{k_i}{k\rho}\phi\alpha_{\omega\mathbf{k}}^{(0)} - i\frac{k_i}{k}\partial_\rho\phi\alpha_{\omega\mathbf{k}}^{(1)} + \frac{k_i}{k\rho}\omega\phi\alpha_{\omega\mathbf{k}}^{(k)} + \frac{n_i^a}{\rho}\omega\phi\alpha_{\omega\mathbf{k}}^{(a)} + (hc), \\ \Pi^t &= \sum_{\omega,\mathbf{k}} \frac{1}{\rho}k\phi(\alpha_{\omega\mathbf{k}}^{(0)} - \alpha_{\omega\mathbf{k}}^{(k)}) + (hc), \\ \Pi^\rho &= \sum_{\omega,\mathbf{k}} -k\phi\alpha_{\omega\mathbf{k}}^{(1)} + (hc). \end{aligned}$$

One checks that these satisfy the algebra (A.11) if one imposes:

$$\begin{aligned} &\left[\alpha_{\omega\mathbf{k}}^{(0)}, \alpha_{\omega'\mathbf{k}'}^{(0)\dagger}\right] = -\delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'), \quad \left[\alpha_{\omega\mathbf{k}}^{(1)}, \alpha_{\omega'\mathbf{k}'}^{(1)\dagger}\right] = \delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}') \\ &\left[\alpha_{\omega\mathbf{k}}^{(k)}, \alpha_{\omega'\mathbf{k}'}^{(k)\dagger}\right] = \delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'), \qquad \left[\alpha_{\omega\mathbf{k}}^{(a)}, \alpha_{\omega'\mathbf{k}'}^{(b)\dagger}\right] = \delta_{ab}\delta_{\omega\omega'}\delta(\mathbf{k}-\mathbf{k}'). \end{aligned} \tag{A.68}$$

The magnetic field components are:

$$F_{\rho i} = \sum_{\omega,\mathbf{k}} i\frac{k_i}{k}\partial_\rho\phi(\alpha_{\omega\mathbf{k}}^{(k)} - \alpha_{\omega\mathbf{k}}^{(0)}) + in_i^a\partial_\rho\phi\alpha_{\omega\mathbf{k}}^{(a)} + \frac{\omega k_i}{k\rho}\phi\alpha_{\omega\mathbf{k}}^{(1)}, \tag{A.69}$$

$$F_{ij} = \sum_{\omega,\mathbf{k}} (n_i^a k_j - n_j^a k_i)\phi\alpha_{\omega\mathbf{k}}^{(a)}. \tag{A.70}$$

The boundary conditions (A.9) boil down to Dirichlet boundary conditions $\phi|_{\partial\mathcal{M}} = 0$ on the modes $A^{(1)}_{\mu,\omega\mathbf{k}}$ and Neumann boundary conditions $\partial_\rho\phi|_{\partial\mathcal{M}} = 0$ on the other sets.

A.5.2 Thermodynamics of Dirichlet and Neumann scalar fields

Here we present the proof of (A.47). One is led to compute the determinant associated with the operator:

$$\left(-\partial_\tau^2 - \partial_r^2 + (k^2+m^2)e^{2r}\right)\psi(t,r) = \lambda\psi(t,r). \tag{A.71}$$

We set $m=0$ for what follows but it is easy to restore its dependence. Consider now the ratio of functional determinants defined on the interval $r \in [\ln\epsilon, \ln R]$ with either Neumann or Dirichlet boundary conditions on the end points. Let us first take $\partial_\tau = 0$. A generalization of the Gelfand-Yaglom theorem gives [227]:[30]

$$\frac{\det\left(-\partial_r^2 + k^2e^{2r}\right)^{ND}}{\det(-\partial_r^2 + k^2e^{2r})^{DD}} = \frac{\psi_{(1)}(\ln R)}{\psi_{(2)}(\ln R)}. \tag{A.72}$$

Here $\psi_{(1)}$ and $\psi_{(2)}$ are two solutions to the initial value problem $(-\partial_r^2 + k^2e^{2r})\psi(r) = 0$ with boundary conditions:[31]

$$\begin{aligned}&\psi_{(1)}(\ln\epsilon) = 1, \quad \psi'_{(1)}(\ln\epsilon) = 0,\\ &\psi_{(2)}(\ln\epsilon) = 0, \quad \psi'_{(2)}(\ln\epsilon) = 1.\end{aligned} \tag{A.75}$$

Solving this gives:

$$\psi_{(1)}(\ln R) = -k\epsilon\left(K_0'(k\epsilon)I_0(kR) - I_0'(k\epsilon)K_0(kR)\right), \tag{A.76}$$
$$\psi_{(2)}(\ln R) = K_0(k\epsilon)I_0(kR) - I_0(k\epsilon)K_0(kR). \tag{A.77}$$

In the limit $k\epsilon \to 0$ with R fixed,we can then use the Bessel asymptotics to find:

$$\frac{\det\left(-\partial_r^2 + k^2e^{2r}\right)^{ND}}{\det(-\partial_r^2 + k^2e^{2r})^{DD}} = \ln 2/k\epsilon. \tag{A.78}$$

[30] The notation $\det(O)^{AB}$ refers to the functional determinant of the operator O with A boundary conditions at $r = \ln\epsilon$ and B boundary conditions at $r = \ln R$.

[31] This formula can be proven by writing

$$\frac{\det\left(-\partial_x^2 + V(x)\right)^{ND}}{\det(-\partial_x^2 + V(x))^{DD}} = \frac{\det\left(-\partial_x^2 + V(x)\right)^{ND}}{\det(-\partial_x^2)^{ND}}\frac{\det\left(-\partial_x^2\right)^{DD}}{\det(-\partial_x^2 + V(x))^{DD}}\frac{\det\left(-\partial_x^2\right)^{ND}}{\det(-\partial_x^2)^{DD}}. \tag{A.73}$$

The first two factors are directly evaluated using the standard Gelfand-Yaglom theorem. The last factor is evaluated explicitly as:

$$\frac{\det\left(-\partial_x^2\right)^{ND}}{\det(-\partial_x^2)^{DD}} = \frac{\pi}{2\ln R/\epsilon}\prod_{n\geq 1}\left(1 - \frac{1}{(2n)^2}\right) = \frac{1}{\ln R/\epsilon} = \frac{\psi^{V=0}_{(1)}(\ln R)}{\psi^{V=0}_{(2)}(\ln R)}. \tag{A.74}$$

Generalizing to include discrete momentum n along the thermal circle, we find similarly:

$$\frac{\det\left(-\partial_r^2 + k^2 e^{2r} + \left(\frac{2\pi n}{\beta}\right)^2\right)^{ND}}{\det\left(-\partial_r^2 + k^2 e^{2r} + \left(\frac{2\pi n}{\beta}\right)^2\right)^{DD}} = \frac{\delta_{n,0}}{\ln 2/k\epsilon} + \frac{2\pi|n|}{\beta}. \tag{A.79}$$

Using zeta regularization for the infinite product $\prod_{n\in\mathbb{Z}_0} \frac{2\pi|n|}{\beta} = \beta$ one recovers:

$$\frac{Z_D}{Z_N} = \prod_k \left(\frac{\beta}{\ln 2/k\epsilon}\right)^{\frac{1}{2}}. \tag{A.80}$$

This completes the derivation of (A.47).

B Edge dynamics as a path integral

In this supplementary chapter we summarize a work by the author in collaboration with Thomas Mertens and Henri Verschelde on dynamics of boundary degrees of freedom in Yang-Mills theory via a path integral prescription [6]. The logic of this chapter is instrumental to understanding the factorization debate in chapter 3 as well as more generally the description of 2d BF theory and JT gravity on generic topologies.

B.1 Introduction

In the previous chapter we learned from a canonical quantization point of view how to assign edge degrees of freedom to an entangling surface. In particular we learned that it is natural in the single-sided theory to sum over boundary conditions, which boils down to summing over charge profiles on the entangling surface. In this chapter we aim to place that discussion in a more general path integral framework and to generalize the discussion to Yang-Mills theory for non-Abelian groups. More precisely we would like to answer the following questions. What is the path integral for the single sided theory? How is the gluing constraint enforced in the path integral?
By providing with a path integral perspective on edge modes in gauge theories we aim to place edge modes in non-topological gauge theories such as Maxwell theory and Yang-Mills on the same footing as boundary degrees of freedom in topological field theories such as 3d Chern-Simons theory and 2d BF theory. In those cases the entire topological field theory is dual to a boundary theory. Depending on the boundary conditions this can be for example a 2d Wess-Zumino-Witten model in the case of 3d Chern-Simons theory or for example 1d quantum mechanics on a group in the case of 2d BF theory.[1] We can think about the boundary dynamics which we're going to write down as describing the "topological subsector" of the non-topological gauge theories under study. This topological sector is associated with large gauge degrees of freedom much like for example a Wess-Zumino-Witten model is essentially a path integral over boundary values of "pure gauge" variables $g|_\partial$ where the Chern-Simons path integral over A localizes on "pure gauge" or flat connections $A = g^{-1}dg$ for which $F = 0$. The ruling principle

[1]Some might say duality is a slight misnomer here. A more neutral statement is that path integrals of the topological field theory calculate a certain correlator in the conformal field theory respectively quantum mechanical model. However this is basically how we expect holography to work when placed in a rigorous framework so we'll stick to using the word duality.

here is that boundary values of "pure gauge" variables are actually part of the physical phase space, unlike their redundant small cousins. Non-topological gauge theories have a "topological subsector" in the sense that there are also gauge fields g whose bulk values are redundant but whose boundary values are genuine phase space variables. The action which we'll obtain should in this sense be compared to the action of the Wess-Zumino-Witten model or to that of free quantum mechanics on a group depending on the number of dimensions. The reader who is interested in this relation is invited to compare the discussion on edge dynamics in the current chapter with the discussion on edge dynamics and gluing in 2d BF theory and JT gravity in chapter 3.
In the case of non-Abelian Yang-Mills, which is an interacting theory, different sectors do not decouple so technically it doesn't make sense to consider a "topological subsector". As this is essentially the first time large gauge degrees of freedom are considered as bona fide physical fields in the path integral we will take the stance that it is a sensible first step to neglect interactions between the "topological subsector" and the propagating bulk degrees of freedom. This enables us to study the edge dynamics as they are. For 2d Yang-Mills and for Maxwell theory in generic dimensions which are both one loop exact there is no such caveat.
With the path integral for the boundary degrees of freedom comes obviously some canonical structure, which in the case of Maxwell was discussed in the previous chapter. It is important to note that the canonical structure we obtain for Yang-Mills matches that derived in [224]. The rationale used in [224] is not identical but rather isomorphic to ours. They introduce "new" fields on the boundary to save gauge invariance, whilst our point of view is that in the single-sided theory there is no a priori reason why gauge transformation on the boundary should be redundant. The role of their "new" degrees of freedom is played in our story by the non-redundant large gauge degrees of freedom.[2]

This chapter is organized as follows.
In section **B.2** we spell out our logic and write down the relevant path integrals for the single sided theories. In some sense we are postulating the answer but bearing in mind the extended Hilbert space discussion of the previous section it is kind of obvious that we are giving the correct prescription.
In section **B.3** we stack this prescription up with several consistency checks. Further application are discussed in [6] but will be left out here.

B.2 Edge dynamics

Let's start by discussing factorization and the extended Hilbert space from the path integral point of view. Later on we will focus on the single-sided theory and zoom in on the emergent boundary dynamics on cutting surfaces.

[2] The difference between these point of view is semantics.

B.2.1 Cutting and gluing

Consider Maxwell theory on a Lorentzian manifold $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ with no boundaries:

$$S[A] = -\frac{1}{4e^2} \int_{\mathcal{M}} dx \sqrt{-g}\, F^{\mu\nu} F_{\mu\nu}. \tag{B.1}$$

Let's take $e = 1$ in what follows. Defining separate fields A_1 and A_2 restricted to respectively $\mathcal{M}_1$ and $\mathcal{M}_2$ we can rewrite the full path integral as:

$$\int [\mathcal{D}A_1]\, e^{iS[A_1]} \int [\mathcal{D}A_2]\, e^{iS[A_2]}\, \delta(A_1|_\partial - A_2|_\partial). \tag{B.2}$$

This is somewhat imprecise. There is a lot of gauge redundancy in the path integral so a more appropriate way to write this is as:

$$\int \frac{[\mathcal{D}A_1]}{\mathrm{Vol}(G_1)} e^{iS[A_1]} \int \frac{[\mathcal{D}A_2]}{\mathrm{Vol}(G_2)} e^{iS[A_2]}\, \frac{\delta(A_1|_\partial - A_2|_\partial)}{\mathrm{Vol}(G_\partial)}. \tag{B.3}$$

The notation should be quite obvious. Let us now decompose both A_1 and A_2 into a completely gauge-fixed part and a pure gauge part as $A_1 + d\phi_1$ and $A_2 + d\phi_2$. This should be read with the understanding that A_1 and A_2 are now gauge fixed. We define:

$$\int \frac{[\mathcal{D}\phi_1]}{\mathrm{Vol}(G_1)} = \int [\mathcal{D}\phi_1|_\partial]. \tag{B.4}$$

This makes sense because $\mathrm{Vol}(G_1)$ denotes small gauge transformations in $\mathcal{M}_1$. There is obviously a ghost path integral from the determinant of the transformation, but it does not play a role in this story. We are left with:

$$\begin{aligned} &\mathrm{Vol}(G_\partial)^{-1} \int [\mathcal{D}A_1][\mathcal{D}\phi_1|_\partial] e^{iS[A_1]} \\ &\qquad \int [\mathcal{D}A_2][\mathcal{D}\phi_2|_\partial] e^{iS[A_2]}\, \delta(A_1|_\partial + d\phi_1|_\partial - A_2|_\partial - d\phi_2|_\partial). \end{aligned} \tag{B.5}$$

Let us now rewrite the functional integral by introducing a Lagrange multiplier one form field which we'll refer to as the current:

$$\begin{aligned} &\delta(A_1|_\partial + d\phi_1|_\partial - A_2|_\partial - d\phi_2|_\partial) \\ &\qquad = \int [\mathcal{D}\mathcal{J}] \exp\left(i \int_\partial dx\, \mathcal{J} \cdot (A_1 + d\phi_1 - A_2 - d\phi_2) \right). \end{aligned} \tag{B.6}$$

We can further rewrite this as:

$$\begin{aligned} &\int [\mathcal{D}\mathcal{J}] \exp\left(i \int_\partial dx\, \mathcal{J} \cdot (A_1|_\partial + d\phi_1|_\partial - A_2|_\partial - d\phi_2|_\partial) \right) \\ &\qquad = \int [\mathcal{D}\mathcal{J}_1] \exp\left(i \int_\partial \mathcal{J}_1 \cdot (A_1 + d\phi_1) \right) \\ &\qquad\qquad \int [\mathcal{D}\mathcal{J}_2] \exp\left(i \int_\partial \mathcal{J}_2 \cdot (A_2 + d\phi_2) \right) \delta(\mathcal{J}_1 + \mathcal{J}_2). \end{aligned} \tag{B.7}$$

One could argue that nothing has happened so far. Nevertheless the way we've rewritten the path integral is very suggestive keeping in mind the extended Hilbert space discussion of the previous chapter. Clearly the Lagrange multiplier currents are nothing but the currents which act as sources on the boundaries for the electromagnetic field. We learned in the previous chapter that the extended Hilbert is obtained by summing independently over the currents $\mathcal{J}_1$ and $\mathcal{J}_2$ whilst the Hilbert space of the glued theory is obtained by constraining $\mathcal{J}_1 + \mathcal{J}_2 = 0$ or in other words there is no net charge on the cutting surface. Schematically we have the following association of the extended Hilbert space with a path integral:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \sim \int [\mathcal{D}\mathcal{J}_1] \cdots \int [\mathcal{D}\mathcal{J}_2] \ldots . \tag{B.8}$$

The ... represent the exponential of the sourced term in the action but also the usual path integral of Maxwell theory on $\mathcal{M}_1$. This association is to be contrasted with the schematic association of the physical Hilbert space with a path integral:

$$\mathcal{H} \sim \int [\mathcal{D}\mathcal{J}_1] \cdots \int [\mathcal{D}\mathcal{J}_2] \ldots \frac{\delta(\mathcal{J}_1 + \mathcal{J}_2)}{\text{Vol}(G_\partial)}. \tag{B.9}$$

This schematical translation of the extended Hilbert space construction in a path integral prescription is the central idea of this chapter. We are often interested in a single-sided description where the edge degrees of freedom manifest themselves. The previous observation suggests to consider as single sided theory the following path integral:[3]

$$\int [\mathcal{D}\mathcal{J}] Z[\mathcal{J}] = \int [\mathcal{D}A][\mathcal{D}\mathcal{J}][\mathcal{D}\phi] \exp\left(i \int_\partial \mathcal{J} \cdot (A + d\phi) + iS[A] \right). \tag{B.10}$$

Here it is understood that A is completely gauge-fixed. Notice that in this description the large degrees of freedom are manifestly not redundant. They are weighed with a nontrivial action and furthermore there is no additional denominator signalling any modding. The gluing rule is then the following:

$$Z = \int [\mathcal{D}\mathcal{J}_1] Z_1[\mathcal{J}_1] \int [\mathcal{D}\mathcal{J}_2] Z_2[\mathcal{J}_2] \frac{\delta(\mathcal{J}_1 + \mathcal{J}_2)}{\text{Vol}(G_\partial)}. \tag{B.11}$$

The modding by the boundary gauge volume might seem somewhat unnatural from this point of view. It's much more intuitive when we rewrite this path integral a little bit though. For this notice that the boundary action in the glued theory (B.6) only depends on the difference between gauge transformations on either side of the gluing surface $\phi_- = \phi_1 - \phi_2$. Transforming from ϕ_1 and ϕ_2 to ϕ_+ and ϕ_- this means that the diagonal large gauge transformations *are* redundant in the gluing integral. This is already manifest in the initial formula (B.2) which is invariant under such a smooth gauge transformation. In other words:

$$\int \frac{[\mathcal{D}\phi_+|_\partial]}{\text{Vol}(G_\partial)} = 1 \tag{B.12}$$

[3]We are dropping some labels henceforth.

Therefore we are left with:[4]

$$Z = \int [\mathcal{D}A_1] e^{iS[A_1]} \int [\mathcal{D}A_2] e^{iS[A_2]} \int [\mathcal{D}\mathcal{J}][\mathcal{D}\phi_-] \exp\left(i \int_\partial \mathcal{J} \cdot (A_1 - A_2 + d\phi_-) \right). \tag{B.13}$$

It is now very natural to do the path integral over ϕ_-. This simply localizes on conserved boundary currents:

$$Z = \int [\mathcal{D}A_1] e^{iS[A_1]} \int [\mathcal{D}A_2] e^{iS[A_2]} \int [\mathcal{D}\mathcal{J}]\, \delta(\nabla \cdot \mathcal{J}) \exp\left(i \int_\partial \mathcal{J} \cdot (A_1 - A_2) \right). \tag{B.14}$$

Formula (B.13) will be very useful when we discuss gluing in 2d Yang-Mills as an application in the next section. Some remarks are first in place.

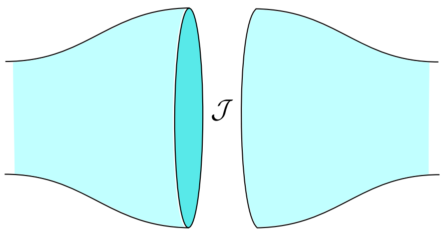
- Because it is so crucial let us emphasize again that the values of the pure gauge fields on the cutting surface are not redundant.

- Formula (B.11) represents in a very precise manner factorization of the path integral. This is more manifest when we switch to Euclidean signature where we are free to choose the Cauchy slices. In this picture one is led to associate a state $|\mathcal{J}\rangle$ with every choice of boundary conditions $n \cdot F|_\partial = \mathcal{J}$ in the path integral on the Cauchy slice. We should then think about $Z[\mathcal{J}]$ more precisely as some inner product $Z[\mathcal{J}] = \langle \psi | \mathcal{J} \rangle$ where the state $\langle \psi |$ is obtained by evolving from some (in general) multi boundary configuration up to the Cauchy slice. Schematically we have $|\psi\rangle = \mathcal{U}\, |\mathcal{J}_1 \dots \mathcal{J}_n\rangle$ when there are n boundaries in the "past" with boundary conditions $\mathcal{J}_1 \dots \mathcal{J}_n$. The gluing formula should then be read as simply inserting a complete set of states in the $|\mathcal{J}\rangle$ basis. Suppose that $Z = \langle \psi_1 | \psi_2 \rangle$ and that $Z_1[\mathcal{J}] = \langle \psi_1 | \mathcal{J} \rangle$ and $Z_2[\mathcal{J}] = \langle \psi_2 | \mathcal{J} \rangle$. We then have:

$$Z = \int [\mathcal{D}\mathcal{J}]\, \langle \psi_1 | \mathcal{J} \rangle \, \langle \mathcal{J} | \psi_2 \rangle = \int [\mathcal{D}\mathcal{J}] Z_1[\mathcal{J}] Z_2[-\mathcal{J}]. \tag{B.15}$$

  This reproduces formula (B.11). Graphically the path integral is glued on some cutting surface with boundary conditions $\mathcal{J}$ as:

$Z = \int [\mathcal{D}\mathcal{J}] \quad \langle \psi_1 | \dots$



$\dots | \psi_2 \rangle \quad .$ (B.16)

  Here we leave implicit whatever set of boundaries or potentially complicated operator insertions correspond to the states $|\psi_1\rangle$ and $|\psi_2\rangle$.

[4]We have dropped the subscript ∂ in the integration measure over the gauge field ϕ_- and will often do so for any gauge field in what follows. Small gauge configurations always neatly cancel so they don't play any role throughout.

- We note that there are subtle aspects of the above construction related to zero modes of the fields ϕ_1 and ϕ_2 as well as possible winding modes if the gluing surface contains non contractable circles. None of these will be important in the examples we discuss in this chapter. Both however are absolutely crucial when it comes to gluing in 2d BF theory and in JT gravity. In particular the winding modes correspond to conjugacy class elements λ and the zero modes correspond to twists τ.[5] For the moment let us simply state that in general we consider $\phi \to \phi+\tau$ as redundant. Winding is introduced as $\int A \to \int A + \lambda$. These holonomies are actual physical information and therefore not redundant. Here A denotes the gauge connection prior to gauge fixing. Whether one considers the holonomies λ to be part of the phase space associated with the gauge fixed A or with the pure gauge field ϕ is a semantics discussion. What is relevant is that it is not included in $\text{Vol}(G)$. In other words it is not redundant. We will choose to consider it part of the pure gauge field in which case (B.12) becomes:

$$\int \frac{[\mathcal{D}\phi_+]}{\text{Vol}(G_\partial)} = \int d\lambda_+. \tag{B.17}$$

  Of course a cutting surface can have a wildly complex topology. Here the right hand side should be read as including holonomies around all non-contractable and non-intersecting circles in the boundary.

- Gluing in the sense of (B.13) is along the lines of the construction of a gauge/fiber bundle where $\mathcal{M}_1$ corresponds to one of the patches and where the boundary is the common region. The compatibility of two connections only requires the fields on the common region to be related by a gauge transformation. Doing the path integral over $\mathcal{J}$ in (B.13) we recover precisely this statement:

$$Z = \int [\mathcal{D}A_1] e^{iS[A_1]} \int [\mathcal{D}A_2] e^{iS[A_2]} \int [\mathcal{D}\phi_-] \delta(A_1 - A_2 + d\phi_-). \tag{B.18}$$

  We are thus gluing by summing over all “compatible” connections. Again we emphasize that for such formulas to make sense the fields A_1 and A_2 should be understood as being completely gauge-fixed.

B.2.2 Boundary theory

We would like to proceed and actually evaluate the single sided path integral. Schematically this means we would like to calculate the following Euclidean path integral:

$$\text{Tr}_{\mathcal{H}_1}(e^{-\beta H}) \sim \int [\mathcal{D}A][\mathcal{D}\mathcal{J}][\mathcal{D}\phi] \exp\Big(- \int_\partial \mathcal{J} \cdot (A + d\phi) - S[A] \Big). \tag{B.19}$$

We can check the validity of this formula in cases where the left-hand side is well understood. One example studied in the previous chapter, is that of electromagnetism

[5] To make matters worse for the reader, sometimes we will refer to λ as twists. It is important not to confuse these two types of “twists”.

in Rindler space. We will do this consistency check explicitly in the ensuing section. Because Maxwell theory is a free theory we can immediately do the path integral over A. Indeed, the path integral is quadratic in A so the result is just the exponential of the on shell action multiplied with a determinant of quadratic fluctuations. The latter contribution represents the thermodynamic contribution of the propagating degrees of freedom which in this case is a photon subject to $n \cdot F|_\partial = 0$ boundary conditions. Let us denote its contribution by $\det \mathcal{O}^{-1/2}$. We end up with:

$$\mathrm{Tr}_{\mathcal{H}_1}(e^{-\beta H}) \sim \det \mathcal{O}^{-1/2} \int [\mathcal{D}\mathcal{J}][\mathcal{D}\phi] \exp\Big(- \int_\partial \frac{1}{2} \mathcal{J} \cdot A[\mathcal{J}] + \mathcal{J} \cdot d\phi \Big). \tag{B.20}$$

Here the connection A is to be understood as being evaluated on shell as a particular solution to the bulk Maxwell equations $\nabla \cdot F = 0$ and to the boundary conditions $n \cdot F|_\partial = \mathcal{J}$. Because the connection is considered completely gauge fixed and because we are essentially modding out by bulk photons - these are solutions to the sourceless equations and hence do not affect the value of $\mathcal{J} \cdot A$ on the boundary - this actually defines an isomorphism $A_\partial[\mathcal{J}]$. We can evolve this solution into the bulk using $\nabla \cdot F = 0$ which then uniquely determines $A[\mathcal{J}]$. We'll see very explicitly how this isomorphism works for electromagnetism in Rindler. Other examples can be found in [6]. In some sense this procedure is in spirit similar to holography a la Witten [228]. See also chapter 2. The factor 1/2 in the above action comes from on shell evaluation of the bulk action $S[A]$. Indeed using integration by parts and the boundary condition $n \cdot F|_\partial = \mathcal{J}$ we find:

$$-\frac{1}{2} \int dx \sqrt{-g}\, F^{\mu\nu} \partial_\mu A_\nu = -\frac{1}{2} \int dx\, \partial_\mu \left(\sqrt{-g} F^{\mu\nu} A_\nu \right) = -\frac{1}{2} \int_\partial \mathcal{J} \cdot A[\mathcal{J}]. \tag{B.21}$$

Notice that on shell $n \cdot \mathcal{J} = 0$ because F is anti symmetric and $\mathcal{J} = n \cdot F$. In other words $F(n,n) = 0$. Obviously this contribution only depends on the completely gauge-fixed part of A as the bulk action depends but on F.
Stripping of the determinant of bulk fluctuations we isolate the "topological subsector" of the theory:

$$\int [\mathcal{D}\mathcal{J}][\mathcal{D}\phi] \exp\Big(- \int_\partial \frac{1}{2} \mathcal{J} \cdot A[\mathcal{J}] + \mathcal{J} \cdot d\phi \Big). \tag{B.22}$$

This action is in general higher derivative so this model can be quite unwieldy.[6] We will restrict in what follows to examples where the resulting dynamics isn't so complicated. Let us make some remarks first.

- From this path integral one reads off the canonically conjugate field of ϕ as $\pi_\phi = \mathcal{J}^t = \mathcal{Q}$. This implies $[\phi, \mathcal{Q}] = i$ or $[\mathcal{Q}, g] = -ig$ with $g = e^\phi$. In this sense we can read the above as a phase space path integral.

- We can do the path integral over ϕ which localizes on conserved currents. Alternatively and more in line with the treatment of edge dynamics in topological field

[6] A canonical analysis was initiated in [6].

theories such as 3d Chern-Simons theory and 2d BF theory we can path integrate out the currents $\mathcal{J}$ because $A[\mathcal{J}]$ is linear in $\mathcal{J}$. We end up, schematically, with:

$$\int [\mathcal{D}\phi] \exp\left(-\frac{1}{2}\int_\partial A^{-1}[d\phi]\cdot d\phi\right). \tag{B.23}$$

This looks eerily similar to the respective action of a 2d $U(1)$ Wess-Zumino-Witten model or the action of a 1d particle on $U(1)$ which would have $A[\mathcal{J}] \sim \mathcal{J}$ in this action. The similarity only goes so far though. As discussed in a bit more detail in chapter 3 for example the boundary dynamics of2d BF theory shouldn't quite be confused with the emergent dynamics on entangling surfaces. For such topological theories essentially by definition the on shell evaluation vanishes $A[\mathcal{J}] = 0$ and the connections are flat. The $\mathcal{J}\cdot d\phi$ term in the action remains though. One then obtains nontrivial boundary energy by *by hand* adding a boundary term to the action of the schematic form $\sim \mathcal{J}\cdot\mathcal{J}$. When we read (B.22) as a phase space path integral this means we are introducing a nonzero Hamiltonian. Obviously when cutting on a random surface in the bulk BF there is no such Hamiltonian, as the theory was topological to begin with. Indeed how would you choose for example a particular nonzero complex structure on the cutting surface? There is no sensible unique way of doing so and therefore the only sensible answer is that there is none. This is consistent with the fact that gluing in 2d BF theory comes with the $\mathcal{J}\cdot d\phi$ action only.

B.2.3 Generalization to other groups

It is fairly straightforward to generalize this construction to Yang-Mills theory for generic Lie groups G such as say $SU(2)$, $SU(3)$ or $\mathrm{SL}(2,\mathbb{R})$. In fact the cutting and gluing works exactly the same way. The fields A and $\mathcal{J}$ are now algebra valued for example $A = A_a J^a$ where J^a are the generators of G.[7] Let's choose these generators such that $\mathrm{Tr}\big(J^a J^b\big) = \delta_{ab}$. We can take the dimension of the "matrices" J^a to be whatever we want. We can denote the commutator of these matrices as $\big[J^a, J^b\big] = f^{ab}{}_c J^c$. Yang-Mills on some closed manifold is then defined by:

$$S[A] = -\frac{1}{4e^2}\int_\mathcal{M} dx\sqrt{-g}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}). \tag{B.24}$$

The field strength is also algebra valued with components defined as:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc}A^b_\mu A^c_\nu. \tag{B.25}$$

Notice that this is nonlinear in A. Therefore Yang-Mills, unlike electromagnetism, is an interacting field theory. We can now go through the same steps as we did to understand cutting and gluing in electromagnetism. In particular we introduce an algebra valued current $\mathcal{J} = \mathcal{J}_a J^a$ on the cutting surface. We can write the space of connections

[7] More in particular they generate the algebra which is an infinitesimal piece of G close to the identity group element.

as $g^{-1}Ag + g^{-1}dg$ with A now understood to be completely gauge fixed and g a group valued field parameterizing the "gauge" field. We end up with the following path integral for the single sided theory:

$$\int [\mathcal{D}A][\mathcal{D}\mathcal{J}][\mathcal{D}g] \exp\Big(- \int_\partial \mathrm{Tr}\big(\mathcal{J} \cdot g^{-1}Ag + \mathcal{J} \cdot g^{-1}dg\big) - S[A]\Big). \tag{B.26}$$

Gluing is completely analogous to (B.7).

This is about as far as we can go exactly for Yang-Mills theory in any number of dimensions greater than two. We'll focus on 2d a bit more in the next section. The reason we can't proceed without making approximation is that it isn't quite funny to "do the path integral over gluons". The theory is interacting so loops of higher order than one will be important if we want to get an exact result. We could consider the theory at weak coupling but that is cheating. Rather here we will only do the path integral over A up to one loop and keep in mind that actually we should be doing a loop expansion around each classical solution or in other words around each distinct boundary source $g\mathcal{J}g^{-1}$ if we want to get an actual precise answer for this path integral. Given such an approximation we can trivially repeat the steps of the previous subsection and obtain the path integral:

$$\det \mathcal{O}^{-1/2} \int [\mathcal{D}\mathcal{J}][\mathcal{D}g] \exp\Big(- \int_\partial \frac{1}{2}\mathrm{Tr}\big(g\mathcal{J}g^{-1}A[\mathcal{J}]\big) + \mathrm{Tr}\big(\mathcal{J} \cdot g^{-1}dg\big)\Big). \tag{B.27}$$

The on shell connection $A[\mathcal{J}]$ is here uniquely determined by the bulk Yang-Mills equations of motion as well as the boundary conditions $n \cdot F|_\partial = g\mathcal{J}g^{-1}$. In general the relation $A[\mathcal{J}]$ might be highly nonlinear though. Consequently finding a solution may be difficult. Fortunately and perhaps somewhat miraculously we *can* actually always obtain a boundary action quadratic in $\mathcal{J}$ by adopting radial gauge $n \cdot A = 0$. Indeed we see from (B.25) that under those circumstances there is no quadratic term in A in $n \cdot F$. Therefore we have a linear relation $g^{-1}Ag[\mathcal{J}]$. The result is a quadratic action. Doing the path integral over $\mathcal{J}$ we end up schematically with:

$$\det \mathcal{O}^{-1/2} \int [\mathcal{D}g] \exp\Big(-\frac{1}{2} \int_\partial \mathrm{Tr}\big(A^{-1}[g^{-1}dg] \cdot g^{-1}dg\big)\Big). \tag{B.28}$$

Again this is in some sense in the same class of models as branes propagating freely on a group manifold which pop up for topological field theories. Examples of this include the 2d Wess-Zumino-Witten model which are 1d strings on a group manifold. Unfortunately though the only "nice" theory that seems to pop up is when we are considering 2d Yang-Mills. We will focus on this example further on.

One thing which we can do rigorously is infer the canonical structure on the boundary. This obviously doesn't mind about loop corrections. More importantly we can read it off directly from the exact formula (B.26). The conjugate of the group element g is $\pi_g = g^{-1}\mathcal{J}^t$. More particularly element per element one defines:

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ji}}, \tag{B.29}$$

The appropriate charges are defined using this conjugate momentum as $\mathcal{Q} = g\pi_g = \mathcal{J}^t$. Its components are:

$$\mathcal{Q}^a = (g\pi_g)^a = \mathrm{Tr}(g\pi_g\tau^a) = (g\pi_g)_{ij}(\tau^a)_{ji}. \tag{B.30}$$

From the canonical algebra $[g_{ij}, \pi_{kl}] = i\delta_{il}\delta_{jk}$ we can derive the algebra of the charges:[8]

$$[\mathcal{Q}^a, g] = -i\tau^a g \quad , \qquad \left[\mathcal{Q}^a, \mathcal{Q}^b\right] = f^{ab}{}_c\,\mathcal{Q}^c. \tag{B.33}$$

In other words the current components satisfy the same algebra as the generators in which we expanded the currents. The latter are classical matrices though whilst the current components are quantum operators acting in some Hilbert space. In other words the above algebra would become trivial for $\hbar \to 0$ but the algebra of generators is always the same.

The charges $\mathcal{Q}$ generate large gauge transformations which are physical fields on the cutting surface. As announced in the introduction this is precisely the same boundary phase space obtained in [224]. Ours and their boundary fields are identical in spirit and separated only by a semantics discussion on whether or not we like to think of large gauge degrees of freedom as redundant or not redundant. Our methods and the body of literature on topological gauge theories suggest they are not redundant. Furthermore our methods naturally provide the phase space with a Hamiltonian. The resulting action would also be the correct one to use in the context of [224].

B.3 Some consistency checks

In this section we provide evidence that "single sided" path integrals are indeed of the type (B.10). Furthermore we perform several nontrivial checks on the gluing formula (B.13).

The first example is electromagnetism in Rindler space for which we computed the left hand side of (B.20) in great detail in the previous chapter. We will reproduce the answer by doing the Rindler path integral on the right hand side. As a bonus the boundary action provides a convincing argument for the absence of static tangential magnetic fields or spatial boundary currents $\mathcal{J}_i$ as horizon degrees of freedom.

---

[8] Some intermediate steps:

$$[\mathcal{Q}^a, g_{kl}] = [(g\pi_g)_{ij}(\tau^a)_{ji}, g_{kl}] = (\tau^a)_{ji} g_{im}[\pi_{mj}, g_{kl}] = -i(\tau^a)_{ki} g_{il} = -i(\tau^a g)_{kl}. \tag{B.31}$$

Furthermore:

$$\begin{aligned}\left[\mathcal{Q}^a, \mathcal{Q}^b\right] &= \left[(g\pi_g)_{ij}(\tau^a)_{ji}, (g\pi_g)_{kl}(\tau^b)_{lk}\right] \\ &= (\tau^a)_{ji}(\tau^b)_{lk}[g_{im}\pi_{mj}, g_{ks}\pi_{sl}] \\ &= g_{ks}\pi_{sj}\left((\tau^a)_{ji}(\tau^b)_{ik} - (\tau^a)_{ik}(\tau^b)_{ji}\right) \\ &= f^{ab}{}_c(g\pi_g)_{kj}(\tau^c)_{jk} = f^{ab}{}_c\,\mathcal{Q}^c.\end{aligned} \tag{B.32}$$

The second set of examples includes 2d Maxwell and 2d Yang-Mills which are quasi-topological gauge theories and more importantly which are one loop exact. We rewrite the 2d disk path integral with appropriate boundary conditions as the path integral of 1d quantum mechanics on a group living on the boundary of the disk. The latter is the result of our boundary action applied to this two dimensional case. These theories provide with strong checks on our claims.
The task in both cases essentially boils down to finding an explicit solution $A[\mathcal{J}]$. This is merely the equivalent of solving a Laplace equation with boundary conditions.

B.3.1 Electromagnetic edge states revisited

We would like to explicitly do the path integral on the right hand side of (B.20) in the case where ∂ is the Rindler horizon. Wick rotating to Euclidean signature this becomes just a point on the unit disk.[9] Because of this horizon currents are necessarily "static". Otherwise they would be singular. We will be working in Lorenz gauge $\nabla \cdot A = 0$. To compare with the results of the previous chapter we will initially regularize the boundary to be at the Euclidean circle $\rho = \epsilon$. In the end we need to take ϵ to zero though.
Because the boundary currents are assumed to be static, the field $A[\mathcal{J}]$ is static as well. The boundary conditions $n \cdot F|_\partial = \mathcal{J}$ read in Rindler coordinates:

$$-g^{\alpha\alpha}\rho\left(\partial_\rho A_\alpha - \partial_\alpha A_\rho\right)\big|_\partial = \mathcal{J}^\alpha. \tag{B.34}$$

It is then pretty straightforward to determine a bijection between such static currents $\mathcal{J}$ and static solutions A to the Maxwell equations in Rindler. Using the notation of (A.36) in the previous chapter we find the following zero mode sectors:

$$\mathcal{J} = \sum_{\mathbf{k}} \left(\mathcal{J}^{(a)}_{\mathbf{k}} e^{(a)} + \mathcal{Q}_{\mathbf{k}} e^{(0)}\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad A = \sum_{\mathbf{k}} \left(c^{(a)}_{\mathbf{k}} A^{(a)}_{\mathbf{k}} + c^{(1)}_{\mathbf{k}} A^{(1)}_{\mathbf{k}}\right). \tag{B.35}$$

The normalization of the scalar wavefunctions is as in (A.49) in the previous chapter. Let us furthermore define $s = -\ln\epsilon$. It is now straightforward to use the asymptotic behavior of the scalar wavefunctions to obtain from (B.34) a relation for the expansion coefficients c in function of those of the boundary current. Inserting these into A returns $A[\mathcal{J}]$. From the solutions for $c^{(1)}$ we find a contribution:

$$\mathcal{J}^t A[\mathcal{J}]_t\big|_{\text{bdy}} = \mathcal{Q}\frac{1}{s\Delta}\mathcal{Q}. \tag{B.36}$$

From the solutions for the $c^{(a)}$ one finds:

$$\mathcal{J}^i A[\mathcal{J}]_i\big|_{\text{bdy}} = s\mathcal{J}^i\mathcal{J}^i. \tag{B.37}$$

Inserting these into the path integral (B.20) one finally obtains:

$$\int [\mathcal{D}\mathcal{J}][\mathcal{D}\phi] \exp\left(-\int_\partial dx \left(\mathcal{J}\cdot\partial\phi + \mathcal{Q}\frac{1}{2s\Delta}\mathcal{Q} + \frac{s}{2}\mathcal{J}^i\mathcal{J}^i\right)\right). \tag{B.38}$$

[9] In Rindler coordinates $ds^2 = \rho^2 d\tau^2 + d\rho^2 + \ldots$ the horizon is indeed a point.

For $s \to \infty$ the path integral over $\mathcal{J}^i$ localizes on the saddle point $\mathcal{J}^i = 0$. Indeed, the last contribution to the action is of the type to which we can apply the saddle point method. This translates to the fact that there are no electromagnetic edge states associated with magnetic field configurations on the horizon because of the infinite redshift. We are left with a phase space path integral over ϕ and $\mathcal{Q}$:

$$\int [\mathcal{D}\mathcal{Q}][\mathcal{D}\phi] \exp\left(- \int_\partial dx\, \mathcal{Q}\partial_t \phi + \mathcal{Q}\frac{1}{2s\Delta}\mathcal{Q} \right). \tag{B.39}$$

This is precisely the path integral we would write down naively to go with the Hilbert space of edge degrees of freedom discussed in the previous chapter and with the Hamiltonian (A.59). The first term in the action localizes on static configurations when we integrate either of the fields out. The resulting path integral over $\mathcal{Q}$ is manifestly the contribution of the edge states to thermodynamics. This proves that our cutting and gluing prescriptions (B.8) and (B.9) at least work in this specific example.

B.3.2 Quasi topological field theories in two dimensions

As a second example we will consider gauge theories in two dimensions. Given the main topic of this work it is obvious why we are interested in this example. In particular it is clarifying to compare the discussion on 2d Yang-Mills in this section with the discussion on cutting and gluing in 2d BF and JT gravity in chapter 3.

Two dimensional electromagnetism

Let's consider two dimensional electromagnetism on a sphere. We imagine cutting this sphere on some circular surface. We are left with two dimensional electromagnetism on a Euclidean disk. Doing the path integral over the gauge field A is extremely straightforward. The one loop correction actually vanishes. Indeed. The one loop correction is associated with propagating modes which solve the bulk Maxwell equations and also satisfy $n \cdot F|_\partial = 0$. The only such solution up to redundancy is the trivial one $A = 0$.
We are now just faced with finding a particular solution with boundary charges $\mathcal{Q}$. The boundary conditions contain no time derivatives of the gauge field hence they can be evaluated at a fixed angle in polar coordinates. The solution is $A[\mathcal{Q}] = -\rho^2 \mathcal{Q} d\tau/2$. Let's say that the location of the boundary $\rho_\partial(\tau)$ is such that the proper distance from the origin at some τ is $a(\tau)$. We then have $\mathcal{J} \cdot A[\mathcal{J}]|_\partial = -a^2 \mathcal{Q}^2 d\tau/2$. The single sided path integral is then:

$$\int [\mathcal{D}\mathcal{Q}][\mathcal{D}\phi] \exp\left(- \int_\partial d\tau \frac{a}{2}\mathcal{Q}^2 + \mathcal{Q} d\phi \right). \tag{B.40}$$

Doing the path integral over ϕ localizes on static configurations. The boundary integral of $a d\tau$ gives the total area A of the disk regardless of the precise shape of the surface. We end up with:

$$\int_0^\infty dq\, e^{-\frac{A}{2}q^2}. \tag{B.41}$$

It is funny to instead to the integral over $\mathcal{Q}$. We end up with the partition function of a particle propagating on a straight line, though with a coupling that depends on the Euclidean time coordinate:

$$\int [\mathcal{D}\phi] \exp\left(-\int_\partial d\tau \frac{(\partial_\tau \phi)^2}{2a}\right). \tag{B.42}$$

We can absorb this time dependent coupling in a redefinition of the time coordinate. The coordinate on the gluing surface doesn't have an a priori meaning. It makes sense from a physical point of view to define it in such a way that the effective coupling of the boundary quantum mechanics is a constant:

$$\int [\mathcal{D}\phi] \exp\left(-\frac{\pi}{A}\int_\partial d\tau (\partial_\tau \phi)^2\right). \tag{B.43}$$

In writing this we have chosen the new τ coordinate to range over 2π. The fact that (B.41) only depends on the total area is a reflection of a fundamental feature of 2d Maxwell but also 2d Yang-Mills. The theory is "quasi topological" which means that the only physical information in the metric is the area and the topology of certain patches of Euclidean spacetime. This should be contrasted with a completely topological theory whereby the dependence on the metric is via only the topology of certain patches of Euclidean spacetime. Topological theories are obviously also quasi topological.
In this example the twists discussed around equation (B.17) don't play an important role. This is because the topology of the surface under consideration is a disk. Therefore when considering the path integral over all connections A_1 in (B.2) prior to gauge fixing, the holonomy of A_1 around any circle is fixed to zero. This is the case because on a disk all circles are contractible therefore there can be no nontrivial holonomy. More in general however we are interested in understanding cutting and gluing as a local property on arbitrary surfaces though. With that in mind we might write our single sided path integral as follows:

$$\int_{-\infty}^{+\infty} d\lambda\, \delta(\lambda) \int [\mathcal{D}\mathcal{Q}][\mathcal{D}\phi] \exp\left(-\int_\partial d\tau \frac{a}{2}\mathcal{Q}^2 + \mathcal{Q} d\phi + i\mathcal{Q}\lambda\right). \tag{B.44}$$

We have extracted the integral over boundary holonomies from the integral over the gauge field ϕ which is now understood to have trivial winding. The $\delta(\lambda)$ can be thought of as associated with the central point of the disk. This can be further evaluated as:

$$\int_{-\infty}^{+\infty} d\lambda\, \delta(\lambda) \int_{-\infty}^{+\infty} dq\, e^{-\frac{A}{2}q^2 + 2\pi i \lambda q}. \tag{B.45}$$

We will not return to discussing the holonomies for 2d Yang-Mills on the disk but they played a pivotal role when discussing gluing in BF and JT gravity on generic surfaces in chapter 3. Anyway. The amplitude which we end up with is that of a disk in 2d electromagnetism with boundary holonomy $\lambda = 0$. What we find here is that this has a dual description as 1d quantum mechanics on the real line.
Before proceeding to generic groups let us return to gluing along the lines of formula

(B.13) in two dimensional electromagnetism. Following the previous discussion it is straightforward to do the integral over both A_1 and A_2 given that both regions are topological disks. We are left with the path integral:

$$Z = \int [\mathcal{D}\mathcal{Q}]\,[\mathcal{D}\phi_-]\exp\left(-\int_0^\beta d\tau \frac{a_1+a_2}{2}\mathcal{Q}^2 + \mathcal{Q}d\phi_-\right). \tag{B.46}$$

Integrating out the gauge variable we end up with the known answer for two dimensional electromagnetism on a sphere:

$$\int_0^\infty dq\, e^{-\frac{A_1+A_2}{2}q^2}. \tag{B.47}$$

Notably this only depends on the total area $A_1 + A_2 = A$ of the sphere again in agreement with the fact that the theory was quasi topological to begin with.

Generalization to other groups

An interesting generalization of this discussion is to consider cutting two dimensional Yang-Mills on a sphere into disks. Unlike its higher dimensional nephews we can actually isolate the boundary path integral in 2d. The reason is that its path integral is one loop exact [124]. Moreover much like for electromagnetism there are no propagating bulk degrees of freedom. Therefore we don't have a determinant of quadratic fluctuations. Indeed the only solutions to the Yang-Mills equations of motion which satisfy the boundary conditions $n \cdot F|_\partial = 0$ are flat connections. Modding out redundancy only the identity remains. One check on this is the fact that we will be able to recover the full answer for 2d Yang-Mills on a sphere from a path integral on the cutting surface after evaluating the bulk piece on shell. This proves that we are not "missing" degrees of freedom.
We are tasked with finding a solution to the 2d Yang-Mills equation of motion which also satisfies the boundary conditions $n \cdot F|_\partial = g\mathcal{Q}g^{-1}$. As discussed in the previous section, in general F depends non linearly on A and as such a linear relation between $A(\mathcal{Q})$ and $g\mathcal{Q}g^{-1}$ is not guaranteed by the boundary condition. Fortunately we can obtain a particular solution that does ensure such a linear relationship. The required solution is the precise equivalent to the Maxwell solution $A[\mathcal{Q}] = -\rho^2 g\mathcal{Q}g^{-1}d\tau/2$. One checks that this solution satisfies Lorenz gauge as well as the bulk Yang-Mills equations. More details can be found in [6]. Completely analogously we find $\mathcal{J} \cdot A[\mathcal{J}]|_\partial = -a^2\mathcal{Q}^2 d\tau/2$. The single sided path integral is:

$$\int [\mathcal{D}\mathcal{Q}][\mathcal{D}g]\exp\left(-\int d\tau\,\mathrm{Tr}\left(\frac{a}{2}\mathcal{Q}^2 + \mathcal{Q}\partial_t g g^{-1}\right)\right). \tag{B.48}$$

This phase space path integral is actually a type of coadjoint orbit path integral and it can be evaluated brute force [75, 76, 77]. We can solve it alternatively by first doing the integral over $\mathcal{Q}$. We are left with the path integral of free 1d quantum mechanics on a group. Just like in the Maxwell case initially we seem to find a quantum mechanics with a time dependent coupling. We can deal with this be choosing a new "preferred" time coordinate where the coupling is constant. We are left with simply quantum mechanics on the group:

$$\int [\mathcal{D}g] \exp\left(-\frac{1}{2a}\int_{\partial} d\tau \, \mathrm{Tr}\big(g^{-1}\partial_\tau g\big)^2\right). \tag{B.49}$$

This is a textbook quantum mechanics system which we can solve without much further ado using your usual canonical quantization. Alternatively we can just head on calculate the propagator on the group manifold. A basis of orthonormal square integrable wavefunctions on the group manifold is provided by the representation matrices:

$$\psi^R_{ab}(g) = \dim R^{1/2} R_{ab}(g). \tag{B.50}$$

These are eigenfunctions of the Hamiltonian of this quantum mechanical system (which is the Casimir) and can be thought of as the generalization of the Laplace operator to generic groups. The thermal propagator is then immediately written down:

$$\sum_{R,a,b} \psi^R_{ab}(g)\psi^R_{ab}(g)^* \, e^{-\frac{A}{2}\mathcal{C}(R)} = \sum_R \dim R^2 \, e^{-\frac{A}{2}\mathcal{C}(R)}. \tag{B.51}$$

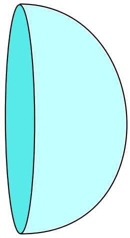
This indeed matches the result of [75, 76, 77]. More details on this are provided in chapter 2 where quantum mechanics on homogeneous spaces is the star of the show. We see again that the "temperature" of the boundary theory is determined by the area of the bulk disk. Much like in the electromagnetism story this answer for the single sided theory represents a 2d Yang-Mills disk with boundary holonomy $\lambda = 0$.
As a check on this discussion we would like to glue two such disks together using the generalization of (B.13) to generic groups. Assuming that the theory is one loop exact we can do the path integrals over A_1 and A_2. We are left with the path integral:

$$\int [\mathcal{D}\mathcal{Q}][\mathcal{D}g_-] \exp\left(-\int_{\partial} d\tau \frac{a_1 + a_2}{2} \, \mathrm{Tr}\big(\mathcal{Q}^2\big) + \mathrm{Tr}\big(\mathcal{Q}g^{-1}dg\big)\right). \tag{B.52}$$

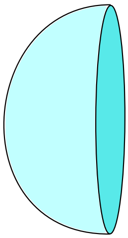
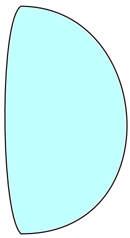
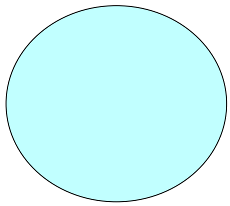
Doing the path integral over $\mathcal{Q}$ and rescaling the time coordinate in a different way we can obtain quantum mechanics on the group with coupling $(A_1 + A_2)/2\pi$. One immediately writes down the answer:

$$\sum_R \dim R^2 \, e^{-\frac{A_1+A_2}{2}\mathcal{C}(R)}. \tag{B.53}$$

Notably this again only depends on the total area of the glued surface. More importantly this is exactly equal to the known sphere partition function of 2d Yang-Mills. Let us end this discussion with a pictorial representation of this cutting and gluing. Graphically the single sided path integral can be represented as:

$$\int [\mathcal{D}\mathcal{Q}] \quad \mathcal{Q}$$  $$. \qquad \text{(B.54)}$$

The amplitude represents the path integral of 2d Yang-Mills on a disk with boundary condition specified by $\mathcal{Q}$. This is very much in spirit of the discussion of around (2.29). This ends up calculating a Yang-Mills disk amplitude with vanishing boundary holonomy. The gluing of the two sides is represented as:

$$\int [\mathcal{D}\mathcal{Q}]$$  $$\mathcal{Q}$$  $$=$$  $$\text{(B.55)}$$

This reproduces the Yang-Mills sphere amplitude.

B.4 Concluding remarks

To end this chapter and to strengthen the relation to the main chapters let us come back briefly to the similarities and differences with edge dynamics in topological gauge theories.

Relation to BF theory

The relation to BF theory was discussed below equation (B.22). Let us just remember that the dynamics on a cutting or gluing surface of BF theory can be considered as the topological limit of the dynamics for 2d Yang-Mills. Basically this sets $A_1 = A_2 = 0$. The dynamics on "physical" boundaries in BF theories though is structurally different. In general we choose to add a nontrivial boundary term to the action for these "physical" boundaries which grants them dynamics. In fact it is convenient to choose the Hamiltonian dynamics such that the boundary dynamics is quantum mechanics on the group, or a coset of the group [44, 52, 1, 2, 229, 230]. This includes the cases of AdS_2, dS_2 and flatspace JT gravity. At cutting and gluing surfaces though we have no such choice. The dynamics on such an interface is what it is. This distinction is important if one is to understand the discussion on JT gravity on higher genus Riemann surfaces in chapter 3.

Relation to Chern-Simons theory

We would like to compare with the appearance of Wess-Zumino-Witten dynamics at the boundaries of Chern-Simons theory [79, 231]. This discussion is actually very similar to the discussion in BF theory. In fact it is true that Chern-Simons is in a precise sense just a q-deformation of BF theory. Similarly the Wess-Zumino-Witten models are in a way quantizations of the particle on a group. Anyway, Wess-Zumino-Witten dynamics arises by choosing as boundary Hamiltonian the Casimir roughly speaking. One then evaluates the Chern-Simons action on shell. This on shell evaluation stems from a path integral over the bulk degrees of freedom. In this topological example the determinant of quadratic fluctuations is again the identity.
One difference with BF theory is that the Chern-Simons action is gauge variant when it comes to large gauge transformations. So evaluating the bulk action on flat connections already gives a nontrivial contribution to the boundary action.

Bibliography

[1] A. Blommaert, T. G. Mertens and H. Verschelde, "The Schwarzian theory - A Wilson line perspective," JHEP **1812**, 022 (2018) arXiv:1806.07765 [hep-th].

[2] A. Blommaert, T. G. Mertens and H. Verschelde, "Fine structure of Jackiw-Teitelboim quantum gravity," JHEP **1909**, 066 (2019) arXiv:1812.00918 [hep-th].

[3] A. Blommaert, T. G. Mertens and H. Verschelde, "Clocks and rods in Jackiw-Teitelboim quantum gravity," JHEP **1909**, 060 (2019) arXiv:1902.11194 [hep-th].

[4] A. Blommaert, T. G. Mertens and H. Verschelde, "Eigenbranes in Jackiw-Teitelboim gravity," arXiv:1911.11603 [hep-th].

[5] A. Blommaert, T. G. Mertens, H. Verschelde and V. I. Zakharov, "Edge State Quantization: Vector Fields in Rindler," JHEP **1808**, 196 (2018) arXiv:1801.09910 [hep-th].

[6] A. Blommaert, T. G. Mertens and H. Verschelde, "Edge dynamics from the path integral - Maxwell and Yang-Mills," JHEP **1811**, 080 (2018) arXiv:1804.07585 [hep-th].

[7] J. S. Cotler *et al.*, "Black Holes and Random Matrices," JHEP **1705** (2017) 118 Erratum: [JHEP **1809** (2018) 002] `arXiv:1611.04650 [hep-th]`.

[8] P. Saad, S. H. Shenker and D. Stanford, "A semiclassical ramp in SYK and in gravity," `arXiv:1806.06840 [hep-th]`

[9] P. Saad, S. H. Shenker and D. Stanford, "JT gravity as a matrix integral," `arXiv:1903.11115 [hep-th]`.

[10] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, "Replica wormholes and the black hole interior," arXiv:1911.11977 [hep-th].

[11] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, "Replica Wormholes and the Entropy of Hawking Radiation," arXiv:1911.12333 [hep-th].

[12] G. 't Hooft, "Dimensional reduction in quantum gravity," Conf. Proc. C **930308**, 284-296 (1993) arXiv:gr-qc/9310026 [gr-qc].

[13] L. Susskind, "The World as a hologram," J. Math. Phys. **36**, 6377-6396 (1995) arXiv:hep-th/9409089 [hep-th].

[14] J. M. Maldacena, "Eternal black holes in anti-de Sitter," JHEP **0304** (2003) 021 `arXiv:0106112 [hep-th]`

[15] M. Srednicki, "Chaos and quantum thermalization," Physical Review E **50** no. 2, (1994) 888.

[16] J. Deutsch, "Quantum statistical mechanics in a closed system," Physical Review A **43** no. 4, (1991) 2046.

[17] F. Haacke, "Quantum Signatures of Chaos."

[18] P. Saad, "Late Time Correlation Functions, Baby Universes, and ETH in JT Gravity," `arXiv:1910.10311 [hep-th]`

[19] D. Marolf and H. Maxfield, "Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information," arXiv:2002.08950 [hep-th].

[20] A. Almheiri and J. Polchinski, "Models of AdS_2 backreaction and holography," JHEP **1511**, 014 (2015) `arXiv:1402.6334 [hep-th]`.

[21] K. Jensen, "Chaos in AdS_2 Holography," Phys. Rev. Lett. **117** (2016) no.11, 111601 `arXiv:1605.06098 [hep-th]`.

[22] J. Maldacena, D. Stanford and Z. Yang, "Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space," PTEP **2016**, no. 12, 12C104 (2016) `arXiv:1606.01857 [hep-th]`.

[23] J. Engelsoy, T. G. Mertens and H. Verlinde, "An investigation of AdS_2 backreaction and holography," JHEP **1607**, 139 (2016) `arXiv:1606.03438 [hep-th]`.

[24] R. Jackiw, "Lower Dimensional Gravity," Nucl. Phys. B **252**, 343 (1985)

[25] C. Teitelboim, "Gravitation and Hamiltonian Structure in Two Space-Time Dimensions," Phys. Lett. **126B** (1983) 41.

[26] A. Kitaev, Talk given at the Fundamental Physics Prize Symposium, Nov. 10, 2014; A. Kitaev, KITP seminar, Feb. 12, 2015; "A simple model of quantum holography," talks at KITP, April 7, 2015 and May 27, 2015.

[27] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," Phys. Rev. D **94**, no. 10, 106002 (2016) arXiv:1604.07818 [hep-th].

[28] S. Sachdev and J.-w. Ye, "Gapless spin fluid ground state in a random, quantum Heisenberg magnet," Phys. Rev. Lett. **70** (1993) 3339, `arXiv:cond-mat/9212030 [cond-mat]`.

[29] J. Polchinski and V. Rosenhaus, "The Spectrum in the Sachdev-Ye-Kitaev Model," JHEP **1604**, 001 (2016) arXiv:1601.06768 [hep-th].

[30] A. Jevicki, K. Suzuki and J. Yoon, "Bi-Local Holography in the SYK Model," JHEP **1607**, 007 (2016) arXiv:1603.06246 [hep-th].

[31] A. Jevicki and K. Suzuki, "Bi-Local Holography in the SYK Model: Perturbations," JHEP **1611**, 046 (2016) arXiv:1608.07567 [hep-th].

[32] G. Turiaci and H. Verlinde, "Towards a 2d QFT Analog of the SYK Model," JHEP **1710** (2017) 167 arXiv:1701.00528 [hep-th].

[33] D. J. Gross and V. Rosenhaus, "The Bulk Dual of SYK: Cubic Couplings," JHEP **1705** (2017) 092 arXiv:1702.08016 [hep-th].

[34] D. J. Gross and V. Rosenhaus, "All point correlation functions in SYK," JHEP **1712** (2017) 148 arXiv:1710.08113 [hep-th].

[35] S. R. Das, A. Jevicki and K. Suzuki, "Three Dimensional View of the SYK/AdS Duality," JHEP **1709** (2017) 017 arXiv:1704.07208 [hep-th].

[36] S. R. Das, A. Ghosh, A. Jevicki and K. Suzuki, "Space-Time in the SYK Model," arXiv:1712.02725 [hep-th].

[37] M. Cvetic and I. Papadimitriou, "AdS_2 holographic dictionary," JHEP **1612**, 008 (2016) Erratum: [JHEP **1701**, 120 (2017)] arXiv:1608.07018 [hep-th].

[38] G. Mandal, P. Nayak and S. R. Wadia, "Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models," JHEP **1711** (2017) 046 arXiv:1702.04266 [hep-th].

[39] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal, "On the Dynamics of Near-Extremal Black Holes," JHEP **1809** (2018) 048 arXiv:1802.09547[hep-th].

[40] A. Strominger, "Les Houches lectures on black holes," hep-th/9501071.

[41] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal, "On the Dynamics of Near-Extremal Black Holes," JHEP **1809**, 048 (2018) arXiv:1802.09547 [hep-th].

[42] U. Moitra, S. P. Trivedi and V. Vishal, "Extremal and near-extremal black holes and near-CFT_1," JHEP **1907**, 055 (2019) arXiv:1808.08239 [hep-th].

[43] T. G. Mertens, G. J. Turiaci and H. L. Verlinde, "Solving the Schwarzian via the Conformal Bootstrap," JHEP **1708** (2017) 136 arXiv:1705.08408 [hep-th].

[44] T. G. Mertens, "The Schwarzian Theory - Origins," JHEP **1805** (2018) 036 arXiv:1801.09605 [hep-th].

[45] A. Blommaert and T. G. Mertens, "Work in progress on quantum chaos in three dimensional gravity"

[46] A. Blommaert, T. G. Mertens and G. J. Turiaci, "Work in progress on quantum Liouville theory with fixed length boundaries"

[47] A. Kitaev and S. J. Suh, "The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual," JHEP **1805**, 183 (2018) arXiv:1711.08467 [hep-th].

[48] D. Bagrets, A. Altland and A. Kamenev, "Sachdev-Ye-Kitaev model as Liouville quantum mechanics," Nucl. Phys. B **911** (2016) 191 arXiv:1607.00694 [cond-mat.str-el].

[49] D. Bagrets, A. Altland and A. Kamenev, "Power-law out of time order correlation functions in the SYK model," Nucl. Phys. B **921** (2017) 727 arXiv:1702.08902 [cond-mat.str-el].

[50] D. Stanford and E. Witten, "Fermionic Localization of the Schwarzian Theory," JHEP **1710** (2017) 008 arXiv:1703.04612 [hep-th].

[51] V. V. Belokurov and E. T. Shavgulidze, "Exact solution of the Schwarzian theory," Phys. Rev. D **96**, no. 10, 101701 (2017) arXiv:1705.02405 [hep-th].

[52] H. A. Gonzalez, D. Grumiller and J. Salzer, "Towards a bulk description of higher spin SYK," arXiv:1802.01562 [hep-th].

[53] J. Maldacena and X. L. Qi, "Eternal traversable wormhole," arXiv:1804.00491 [hep-th].

[54] D. Harlow and D. Jafferis, "The Factorization Problem in Jackiw-Teitelboim Gravity," arXiv:1804.01081 [hep-th]

[55] J. Lin, "Entanglement entropy in Jackiw-Teitelboim Gravity," arXiv:1807.06575 [hep-th].

[56] A. Kitaev and S. J. Suh, "Statistical mechanics of a two-dimensional black hole," arXiv:1808.07032 [hep-th].

[57] Z. Yang, "The Quantum Gravity Dynamics of Near Extremal Black Holes," arXiv:1809.08647 [hep-th].

[58] A. R. Brown, H. Gharibyan, H. W. Lin, L. Susskind, L. Thorlacius and Y. Zhao, "Complexity of Jackiw-Teitelboim gravity," Phys. Rev. D **99**, no. 4, 046016 (2019) arXiv:1810.08741 [hep-th].

[59] J. Maldacena, G. J. Turiaci and Z. Yang, "Two dimensional Nearly de Sitter gravity," arXiv:1904.01911 [hep-th]

[60] H. W. Lin, J. Maldacena and Y. Zhao, "Symmetries Near the Horizon," JHEP **1908**, 049 (2019) arXiv:1904.12820 [hep-th].

[61] L. V. Iliesiu, S. S. Pufu, H. Verlinde and Y. Wang, "An exact quantization of Jackiw-Teitelboim gravity," arXiv:1905.02726 [hep-th]

[62] J. Cotler, K. Jensen and A. Maloney, "Low-dimensional de Sitter quantum gravity," arXiv:1905.03780 [hep-th]

[63] J. Cotler and K. Jensen, "Emergent unitarity in de Sitter from matrix integrals," arXiv:1911.12358 [hep-th].

[64] T. G. Mertens and G. J. Turiaci, "Defects in Jackiw-Teitelboim Quantum Gravity," JHEP **1908** (2019) 127 arXiv:1904.05228 [hep-th]

[65] D. Stanford and E. Witten, "JT Gravity and the Ensembles of Random Matrix Theory," arXiv:1907.03363 [hep-th]

[66] A. Blommaert, "Cluster decomposition and factorization in JT gravity"

[67] A. Blommaert, T. G. Mertens and H. Verschelde, "Unruh detectors and quantum chaos in JT gravity," arXiv:2005.13058 [hep-th].

[68] T. Fukuyama and K. Kamimura, "Gauge Theory of Two-dimensional Gravity," Phys. Lett. **160B**, 259 (1985).

[69] K. Isler and C. A. Trugenberger, "A Gauge Theory of Two-dimensional Quantum Gravity," Phys. Rev. Lett. **63**, 834 (1989).

[70] A. H. Chamseddine and D. Wyler, "Gauge Theory of Topological Gravity in (1+1)-Dimensions," Phys. Lett. B **228**, 75 (1989).

[71] R. Jackiw, "Gauge theories for gravity on a line," Theor. Math. Phys. **92** (1992) 979 [Teor. Mat. Fiz. **92** (1992) 404] arXiv:hep-th/9206093 [hep-th].

[72] E. D'Hoker and D. H. Phong, "Multiloop Amplitudes for the Bosonic Polyakov String," Nucl. Phys. B **269**, 205 (1986).

[73] E. D'Hoker and D. H. Phong, "On Determinants of Laplacians on Riemann Surfaces," Commun. Math. Phys. **104**, 537 (1986).

[74] E. Witten, "On quantum gauge theories in two-dimensions," Commun. Math. Phys. **141** (1991) 153.

[75] A. Alekseev and S. L. Shatashvili, "From geometric quantization to conformal field theory," Commun. Math. Phys. **128** (1990) 197.

[76] A. Alekseev, L. D. Faddeev and S. L. Shatashvili, "Quantization of symplectic orbits of compact Lie groups by means of the functional integral," J. Geom. Phys. **5** (1988) 391.

[77] A. Y. Alekseev and I. T. Todorov, "Quadratic brackets from symplectic forms," Nucl. Phys. B **421** (1994) 413 hep-th/9307026].

[78] P. H. Ginsparg and G. W. Moore, "Lectures on 2-D gravity and 2-D string theory," arXiv:9304011 [hep-th]

[79] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, "Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory," Nucl. Phys. B **326** (1989) 108.

[80] G. W. Moore and N. Seiberg, "Taming the Conformal Zoo," Phys. Lett. B **220** (1989) 422.

[81] R. F. Picken, "The Propagator for Quantum Mechanics on a Group Manifold From an Infinite Dimensional Analog of the Duistermaat-heckman Integration Formula," J. Phys. A **22**, 2285 (1989).

[82] G. W. Moore and N. Seiberg, "Lectures On Rcft," RU-89-32, YCTP-P13-89.

[83] J. Cotler and K. Jensen, "A theory of reparameterizations for AdS_3 gravity," JHEP **1902**, 079 (2019) arXiv:1808.03263 [hep-th].

[84] L. Donnay, "Asymptotic dynamics of three-dimensional gravity," PoS Modave **2015**, 001 (2016) arXiv:1602.09021 [hep-th].

[85] A. L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, "Exact Virasoro Blocks from Wilson Lines and Background-Independent Operators," JHEP **1707** (2017) 092 `arXiv:1612.063858 [hep-th]`.

[86] D. J. Gross and W. Taylor, "Twists and Wilson loops in the string theory of two-dimensional QCD," Nucl. Phys. B **403** (1993) 395 `arXiv:9303046 [hep-th]`.

[87] W. Donnelly and G. Wong, "Entanglement branes in a two-dimensional string theory," JHEP **1709** (2017) 097 `arXiv:1610.01719 [hep-th]`.

[88] S. Cordes, G. W. Moore and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories," Nucl. Phys. Proc. Suppl. **41** (1995) 184 `arXiv:9411210 [hep-th]`.

[89] G. W. Moore and N. Seiberg, "Classical and Quantum Conformal Field Theory," Commun. Math. Phys. **123**, 177 (1989).

[90] R. Camporesi, "Harmonic analysis and propagators on homogeneous spaces," Phys. Rept. **196**, 1 (1990).

[91] J. R. David, M. R. Gaberdiel and R. Gopakumar, "The Heat Kernel on AdS(3) and its Applications," JHEP **1004**, 125 (2010) arXiv:0911.5085 [hep-th].

[92] B. Ponsot and J. Teschner, "Liouville bootstrap via harmonic analysis on a non-compact quantum group," `arXiv:hep-th/9911110`.

[93] B. Ponsot and J. Teschner, "Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of U(q)(sl(2,R))," Commun. Math. Phys. **224** (2001) 613 `arXiv:math/0007097 [math-qa]`.

[94] I. C-H. Ip, "Representation of the quantum plane, its quantum double and harmonic analysis on $GL_q^+(2, R)$," Selecta Mathematica New Series, Vol 19 (4), 987-1082, (2013).

[95] N. Y. Vilenkin, "Special Functions and the Theory of Group Representations," American Mathematical Society, 1968.

[96] N. Y. Vilenkin and A. U. Klimyk, "Representation of Lie Groups and Special Functions: Volume 1," Kluwer Academic Publishers, 1991.

[97] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, "String propagation in a black hole geometry," Nucl. Phys. B **371** (1992) 269.

[98] C. P. Constantinidis, O. Piguet and A. Perez, "Quantization of the Jackiw-Teitelboim model," Phys. Rev. D **79**, 084007 (2009) arXiv:0812.0577 [gr-qc].

[99] I. S. Gradstein and I. M. Ryzhik, "Tables of integrals, sums, series and derivatives," Izd. Fizmatgiz (1963).

[100] H. T. Lam, T. G. Mertens, G. J. Turiaci and H. Verlinde, "Shockwave S-matrix from Schwarzian Quantum Mechanics," `arXiv:1804.09834 [hep-th]`.

[101] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," Nucl. Phys. B **311**, 46 (1988).

[102] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[103] H. Jacquet, "Fonctions de Whittaker associees aux groupes de Chevalley," Bull.Soc.Math. France, (1967), 95, 243-309.

[104] G. Schiffmann, "Integrales d'entrelacement et fonctions de Whittaker," Bull.Soc.Math. France, (1971), 99, 3-72.

[105] M. Hashizume, "Whittaker models for real reductive groups," J.Math.Soc. Japan, (1979), 5, 394-401.

[106] M.Hashizume, "Whittaker functions on semisimple Lie groups," Hiroshima Math.J., (1982), 12, 259-293.

[107] S. Kharchev, D. Lebedev and M. Semenov-Tian-Shansky, "Unitary representations of U(q) (sl(2, R)), the modular double, and the multiparticle q deformed Toda chains," Commun. Math. Phys. **225** (2002) 573

[108] S. Jackson, L. McGough and H. Verlinde, "Conformal Bootstrap, Universality and Gravitational Scattering," Nucl. Phys. B **901**, 382 (2015) arXiv:1412.5205 [hep-th].

[109] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121** (1989) 351.

[110] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory."

[111] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string."

[112] E. P. Verlinde, "Fusion Rules and Modular Transformations in 2D Conformal Field Theory," Nucl. Phys. B **300**, 360 (1988).

[113] J. Teschner, "The Minisuperspace limit of the sl(2,C) / SU(2) WZNW model," Nucl. Phys. B **546** (1999) 369 arXiv:9712258 [hep-th].

[114] J. Teschner, "Operator product expansion and factorization in the H+(3) WZNW model," Nucl. Phys. B **571** (2000) 555 arXiv:9906215 [hep-th].

[115] J. M. Maldacena and H. Ooguri, "Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum," J. Math. Phys. **42** (2001) 2929 arXiv:0001053 [hep-th].

[116] M. Bershadsky and H. Ooguri, "Hidden SL(n) Symmetry in Conformal Field Theories," Commun. Math. Phys. **126** (1989) 49.

[117] S. Carlip, "Dynamics of asymptotic diffeomorphisms in (2+1)-dimensional gravity," Class. Quant. Grav. **22** (2005) 3055 arXiv:0501033 [gr-qc].

[118] G. Compere, L. Donnay, P. H. Lambert and W. Schulgin, "Liouville theory beyond the cosmological horizon," JHEP **1503** (2015) 158 arXiv:1411.7873 [hep-th].

[119] M. Mehta, "Random Matrices," Vol. 142. Elsevier, 2004.

[120] R. Dijkgraaf and E. Witten, "Developments in Topological Gravity," Int. J. Mod. Phys. A **33**, no. 30, 1830029 (2018) arXiv:1804.03275 [hep-th]

[121] J. Cotler and N. Hunter-Jones, "Spectral decoupling in many-body quantum chaos," arXiv:1911.02026 [hep-th].

[122] M. Mirzakhani, "Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered riemann surfaces," Inventiones mathematicae **167** no. 1, (2007) 179–222.

[123] M. Mirzakhani, "Weil-Petersson volumes and intersection theory on the moduli space of curves," J. Am. Math. Soc. **20**, no. 01, 1 (2007).

[124] E. Witten, ''Two-dimensional gauge theories revisited," J. Geom. Phys. **9** (1992) 303 hep-th/9204083].

[125] D. L. Jafferis and D. K. Kolchmeyer, "Entanglement Entropy in Jackiw-Teitelboim Gravity," arXiv:1911.10663 [hep-th].

[126] D. N. Kabat, "Black hole entropy and entropy of entanglement," Nucl. Phys. B **453** (1995) 281 [hep-th/9503016].

[127] D. N. Kabat, S. H. Shenker and M. J. Strassler, "Black hole entropy in the O(N) model," Phys. Rev. D **52** (1995) 7027 hep-th/9506182].

[128] D. Kabat and D. Sarkar, "Cosmic string interactions induced by gauge and scalar fields," Phys. Rev. D **86** (2012) 084021 arXiv:1206.5642 [hep-th].

[129] W. Donnelly and G. Wong, "Entanglement branes, modular flow, and extended topological quantum field theory," JHEP **1910**, 016 (2019) arXiv:1811.10785 [hep-th].

[130] W. Donnelly, S. Timmerman and N. Valdés-Meller, "Entanglement entropy and the large N expansion of two-dimensional Yang-Mills theory," arXiv:1911.09302 [hep-th].

[131] Y. I. Manin and P. Zograf, "Invertible Cohomological Field Theories and Weil-Petersson volumes," arXiv:9902051 [math].

[132] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A Conjecture," Phys. Rev. D **55**, 5112 (1997) hep-th/9610043].

[133] B. Eynard, T. Kimura and S. Ribault, "Random matrices," arXiv:1510.04430 [math-ph].

[134] B. Eynard and N. Orantin, "Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models," arXiv:0705.3600 [math-ph].

[135] B. Eynard, "A short overview of the "Topological recursion"," arXiv:1412.3286 [math-ph].

[136] E. Witten, "Quantum gravity in de Sitter space," hep-th/0106109.

[137] G. Penington, "Entanglement Wedge Reconstruction and the Information Paradox," `arXiv:1905.08255 [hep-th]`

[138] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, "The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole," `arXiv:1905.08762 [hep-th]`

[139] A. Almheiri, R. Mahajan, J. Maldacena and Y. Zhao, "The Page curve of Hawking radiation from semiclassical geometry," `arXiv:1908.10996 [hep-th]`

[140] C. Akers, N. Engelhardt and D. Harlow, "Simple holographic models of black hole evaporation," `arXiv:1910.00972 [hep-th]`

[141] K. Okuyama and K. Sakai, "JT gravity, KdV equations and macroscopic loop operators," `arXiv:1911.01659 [hep-th]`

[142] J. Maldacena, S. H. Shenker and D. Stanford, "A bound on chaos," JHEP **1608**, 106 (2016) `arXiv:1503.01409 [hep-th]`.

[143] S. H. Shenker and D. Stanford, "Black holes and the butterfly effect," JHEP **1403** (2014) 067 `arXiv:1306.0622 [hep-th]`.

[144] S. H. Shenker and D. Stanford, "Stringy effects in scrambling," JHEP **1505** (2015) 132 `arXiv:1412.6087 [hep-th]`.

[145] A. M. García-García and J. J. M. Verbaarschot, "Spectral and thermodynamic properties of the Sachdev-Ye-Kitaev model," Phys. Rev. D **94**, no. 12, 126010 (2016) arXiv:1610.03816 [hep-th].

[146] B. Eynard, "Topological expansion for the 1-Hermitian matrix model correlation functions," JHEP **0411**, 031 (2004) arXiv:0407261 [hep-th]

[147] F. J. Dyson, "Statistical theory of the energy levels of complex systems. I," Journal of Mathematical Physics, 1962, 3.1: 140-156.

[148] F. J. Dyson, "Statistical theory of the energy levels of complex systems. II," Journal of Mathematical Physics, 1962, 3.1: 157-165.

[149] F. J. Dyson, "Statistical theory of the energy levels of complex systems. III," Journal of Mathematical Physics, 1962, 3.1: 166-175.

[150] M. Mehta and F. J. Dyson, "Statistical theory of the energy levels of complex systems. IV," Journal of Mathematical Physics, 1963, 4.5: 701-712.

[151] M. Mehta and F. J. Dyson, "Statistical theory of the energy levels of complex systems. V," Journal of Mathematical Physics, 1963, 4.5: 713-719.

[152] M. Mehta and F. J. Dyson, "A Brownian-motion model for the eigenvalues of a random matrix," Journal of Mathematical Physics, 1962, 3.6: 1191-1198.

[153] J. M. Maldacena, G. W. Moore, N. Seiberg and D. Shih, "Exact vs. semiclassical target space of the minimal string," JHEP **0410**, 020 (2004) arXiv:0408039 [hep-th]

[154] V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, "Boundary Liouville field theory. 1. Boundary state and boundary two point function," arXiv:0001012 [hep-th]

[155] H. Gharibyan, M. Hanada, S. H. Shenker and M. Tezuka, "Onset of Random Matrix Behavior in Scrambling Systems," JHEP **1807** (2018) 124 arXiv:1803.08050 [hep-th]

[156] L. V. Iliesiu, "On 2D gauge theories in Jackiw-Teitelboim gravity," arXiv:1909.05253 [hep-th]

[157] B. Ponsot and J. Teschner, "Boundary Liouville field theory: Boundary three point function," Nucl. Phys. B **622**, 309 (2002) arXiv:0110244 [hep-th]

[158] R. Dijkgraaf, "Intersection theory, integrable hierarchies and topological field theory," arXiv:9201003 [hep-th]

[159] W. Donnelly and S. B. Giddings, "Diffeomorphism-invariant observables and their nonlocal algebra," Phys. Rev. D **93**, no. 2, 024030 (2016) Erratum: [Phys. Rev. D **94**, no. 2, 029903 (2016)] arXiv:1507.07921 [hep-th].

[160] A. Almheiri, T. Anous and A. Lewkowycz, "Inside out: meet the operators inside the horizon. On bulk reconstruction behind causal horizons," JHEP **1801**, 028 (2018) arXiv:1707.06622 [hep-th].

[161] H. Verlinde, "Poking Holes in AdS/CFT: Bulk Fields from Boundary States," arXiv:1505.05069 [hep-th].

[162] A. Lewkowycz, G. J. Turiaci and H. Verlinde, "A CFT Perspective on Gravitational Dressing and Bulk Locality," JHEP **1701**, 004 (2017) arXiv:1608.08977 [hep-th].

[163] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, "Continuous Multiscale Entanglement Renormalization Ansatz as Holographic Surface-State Correspondence," Phys. Rev. Lett. **115**, no. 17, 171602 (2015) arXiv:1506.01353 [hep-th].

[164] Y. Nakayama and H. Ooguri, "Bulk Locality and Boundary Creating Operators," JHEP **1510**, 114 (2015) arXiv:1507.04130 [hep-th].

[165] K. Goto, M. Miyaji and T. Takayanagi, "Causal Evolutions of Bulk Local Excitations from CFT," JHEP **1609**, 130 (2016) arXiv:1605.02835 [hep-th].

[166] H. Chen, A. L. Fitzpatrick, J. Kaplan and D. Li, "The AdS_3 propagator and the fate of locality," JHEP **1804**, 075 (2018) arXiv:1712.02351 [hep-th].

[167] H. Chen, A. L. Fitzpatrick, J. Kaplan and D. Li, "The Bulk-to-Boundary Propagator in Black Hole Microstate Backgrounds," JHEP **1906**, 107 (2019) arXiv:1810.02436 [hep-th].

[168] W. Donnelly and S. B. Giddings, "Observables, gravitational dressing, and obstructions to locality and subsystems," Phys. Rev. D **94**, no. 10, 104038 (2016) arXiv:1607.01025 [hep-th].

[169] W. Donnelly and S. B. Giddings, "How is quantum information localized in gravity?," Phys. Rev. D **96**, no. 8, 086013 (2017) arXiv:1706.03104 [hep-th].

[170] S. B. Giddings and A. Kinsella, "Gauge-invariant observables, gravitational dressings, and holography in AdS," JHEP **1811**, 074 (2018) arXiv:1802.01602 [hep-th].

[171] S. B. Giddings, "Quantum gravity: a quantum-first approach," LHEP **1**, no. 3, 1 (2018) arXiv:1805.06900 [hep-th].

[172] W. Donnelly and S. B. Giddings, "Gravitational splitting at first order: Quantum information localization in gravity," Phys. Rev. D **98**, no. 8, 086006 (2018) arXiv:1805.11095 [hep-th].

[173] S. B. Giddings, "Gravitational dressing, soft charges, and perturbative gravitational splitting," Phys. Rev. D **100**, no. 12, 126001 (2019) arXiv:1903.06160 [hep-th].

[174] S. Giddings and S. Weinberg, "Gauge-invariant observables in gravity and electromagnetism: black hole backgrounds and null dressings," arXiv:1911.09115 [hep-th].

[175] N. Engelhardt and G. T. Horowitz, "Towards a Reconstruction of General Bulk Metrics," Class. Quant. Grav. **34**, no. 1, 015004 (2017) arXiv:1605.01070 [hep-th].

[176] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations."

[177] M. Spradlin and A. Strominger, "Vacuum states for AdS(2) black holes," JHEP **9911**, 021 (1999) hep-th/9904143].

[178] T. G. Mertens, "Towards Black Hole Evaporation in Jackiw-Teitelboim Gravity," JHEP **1907**, 097 (2019) arXiv:1903.10485 [hep-th].

[179] A. L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, "On information loss in AdS_3/CFT_2," JHEP **1605**, 109 (2016) arXiv:1603.08925 [hep-th].

[180] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, "Local bulk operators in AdS/CFT: A Boundary view of horizons and locality," Phys. Rev. D **73**, 086003 (2006) hep-th/0506118].

[181] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, "Holographic representation of local bulk operators," Phys. Rev. D **74**, 066009 (2006) hep-th/0606141].

[182] D. A. Lowe and S. Roy, "Holographic description of asymptotically AdS(2) collapse geometries," Phys. Rev. D **78**, 124017 (2008) arXiv:0810.1750 [hep-th].

[183] D. Kabat, G. Lifschytz and D. A. Lowe, "Constructing local bulk observables in interacting AdS/CFT," Phys. Rev. D **83**, 106009 (2011) arXiv:1102.2910 [hep-th].

[184] D. Kabat and G. Lifschytz, "Local bulk physics from intersecting modular Hamiltonians," JHEP **1706**, 120 (2017) arXiv:1703.06523 [hep-th].

[185] D. J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian, "$T\overline{T}$ in AdS_2 and Quantum Mechanics," Phys. Rev. D **101**, no. 2, 026011 (2020) arXiv:1907.04873 [hep-th].

[186] S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," Phys. Rev. D **14**, 2460 (1976).

[187] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, "Black Holes: Complementarity or Firewalls?," JHEP **1302**, 062 (2013) arXiv:1207.3123 [hep-th].

[188] D. Marolf, "The Black Hole information problem: past, present, and future," Rept. Prog. Phys. **80**, no. 9, 092001 (2017) arXiv:1703.02143 [gr-qc].

[189] J. Polchinski, "The Black Hole Information Problem," arXiv:1609.04036 [hep-th].

[190] D. Harlow, "Jerusalem Lectures on Black Holes and Quantum Information," Rev. Mod. Phys. **88**, 015002 (2016) arXiv:1409.1231 [hep-th].

[191] S. D. Mathur, "The Information paradox: A Pedagogical introduction," Class. Quant. Grav. **26**, 224001 (2009) arXiv:0909.1038 [hep-th].

[192] G. Compère, "Are quantum corrections on horizon scale physically motivated?," Int. J. Mod. Phys. D **28**, no. 14, 1930019 (2019) arXiv:1902.04504 [gr-qc].

[193] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford and J. Sully, "An Apologia for Firewalls," JHEP **1309**, 018 (2013) arXiv:1304.6483 [hep-th].

[194] A. Blommaert and J. S. Cotler, "Work in progress on bulk probes of quantum chaos in de Sitter"

[195] T. G. Mertens, H. Verschelde and V. I. Zakharov, "Near-Hagedorn Thermodynamics and Random Walks: a General Formalism in Curved Backgrounds," JHEP **1402**, 127 (2014) arXiv:1305.7443 [hep-th].

[196] T. G. Mertens, H. Verschelde and V. I. Zakharov, "Random Walks in Rindler Spacetime and String Theory at the Tip of the Cigar," JHEP **1403**, 086 (2014) arXiv:1307.3491 [hep-th].

[197] T. G. Mertens, H. Verschelde and V. I. Zakharov, "Perturbative String Thermodynamics near Black Hole Horizons," JHEP **1506**, 167 (2015) arXiv:1410.8009 [hep-th].

[198] S. N. Solodukhin, "Entanglement entropy of black holes," Living Rev. Rel. **14** (2011) 8 arXiv:1104.3712 [hep-th].

[199] S. N. Solodukhin, "Newton constant, contact terms and entropy," Phys. Rev. D **91** (2015) no.8, 084028 arXiv:1502.03758 [hep-th].

[200] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, "Notes on Entanglement Entropy in String Theory," JHEP **1505** (2015) 106 arXiv:1412.5606 [hep-th].

[201] T. G. Mertens, H. Verschelde and V. I. Zakharov, "String Theory in Polar Coordinates and the Vanishing of the One-Loop Rindler Entropy," JHEP **1608** (2016) 113 arXiv:1606.06632 [hep-th].

[202] P. V. Buividovich and M. I. Polikarpov, "Entanglement entropy in gauge theories and the holographic principle for electric strings," Phys. Lett. B **670** (2008) 141 arXiv:0806.3376 [hep-th].

[203] H. Casini, M. Huerta and J. A. Rosabal, "Remarks on entanglement entropy for gauge fields," Phys. Rev. D **89** (2014) 8, 085012 arXiv:1312.1183 [hep-th].

[204] H. Casini and M. Huerta, "Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice," Phys. Rev. D **90** (2014) 10, 105013 arXiv:1406.2991 [hep-th].

[205] D. Radicevic, "Notes on Entanglement in Abelian Gauge Theories," arXiv:1404.1391 [hep-th].

[206] D. Radicevic, "Entanglement in Weakly Coupled Lattice Gauge Theories," JHEP **1604** (2016) 163 arXiv:1509.08478 [hep-th].

[207] D. Radicevic, "Entanglement Entropy and Duality," JHEP **1611** (2016) 130 arXiv:1605.09396 [hep-th].

[208] W. Donnelly, "Entanglement entropy and nonabelian gauge symmetry," Class. Quant. Grav. **31** (2014) 21, 214003 arXiv:1406.7304 [hep-th].

[209] W. Donnelly, "Decomposition of entanglement entropy in lattice gauge theory," Phys. Rev. D **85** (2012) 085004 arXiv:1109.0036 [hep-th].

[210] W. Donnelly and A. C. Wall, "Do gauge fields really contribute negatively to black hole entropy?," Phys. Rev. D **86** (2012) 064042

[211] K. W. Huang, "Central Charge and Entangled Gauge Fields," Phys. Rev. D **92** (2015) 2, 025010 arXiv:1412.2730 [hep-th].

[212] S. Ghosh, R. M. Soni and S. P. Trivedi, "On The Entanglement Entropy For Gauge Theories," JHEP **1509** (2015) 069 arXiv:1501.02593 [hep-th].

[213] L. Y. Hung and Y. Wan, "Revisiting Entanglement Entropy of Lattice Gauge Theories," JHEP **1504** (2015) 122 arXiv:1501.04389 [hep-th].

[214] S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba and H. Tasaki, "On the definition of entanglement entropy in lattice gauge theories," JHEP **1506** (2015) 187 arXiv:1502.04267 [hep-th].

[215] R. M. Soni and S. P. Trivedi, "Aspects of Entanglement Entropy for Gauge Theories," JHEP **1601** (2016) 136 arXiv:1510.07455 [hep-th].

[216] K. Van Acoleyen, N. Bultinck, J. Haegeman, M. Marien, V. B. Scholz and F. Verstraete, "The entanglement of distillation for gauge theories," Phys. Rev. Lett. **117** (2016) no.13, 131602 arXiv:1511.04369 [quant-ph].

[217] H. Casini and M. Huerta, "Entanglement entropy of a Maxwell field on the sphere," Phys. Rev. D **93** (2016) no.10, 105031 arXiv:1512.06182 [hep-th].

[218] R. M. Soni and S. P. Trivedi, "Entanglement entropy in $(3 + 1)$-d free U(1) gauge theory," JHEP **1702** (2017) 101 arXiv:1608.00353 [hep-th].

[219] B. Michel, "Remarks on Rindler Quantization," arXiv:1612.03158 [hep-th].

[220] W. Donnelly and A. C. Wall, "Entanglement entropy of electromagnetic edge modes," Phys. Rev. Lett. **114** (2015) 11, 111603 arXiv:1412.1895 [hep-th].

[221] W. Donnelly and A. C. Wall, "Geometric entropy and edge modes of the electromagnetic field," arXiv:1506.05792 [hep-th].

[222] D. Harlow, "Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture," JHEP **1601** (2016) 122 arXiv:1510.07911 [hep-th].

[223] A. Higuchi, G. E. A. Matsas and D. Sudarsky, "Bremsstrahlung and Fulling-Davies-Unruh thermal bath," Phys. Rev. D **46** (1992) 3450.

[224] W. Donnelly and L. Freidel, "Local subsystems in gauge theory and gravity," JHEP **1609** (2016) 102 arXiv:1601.04744 [hep-th].

[225] S. W. Hawking, M. J. Perry and A. Strominger, "Soft Hair on Black Holes," Phys. Rev. Lett. **116** (2016) no.23, 231301 arXiv:1601.00921 [hep-th].

[226] S. W. Hawking, M. J. Perry and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," JHEP **1705** (2017) 161 arXiv:1611.09175 [hep-th].

[227] G. V. Dunne, "Functional determinants in quantum field theory," J. Phys. A **41** (2008) 304006 arXiv:0711.1178 [hep-th].

[228] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) hep-th/9802150.

[229] H. L. Verlinde and E. P. Verlinde, "Scattering at Planckian energies," Nucl. Phys. B **371** (1992) 246 hep-th/9110017.

[230] S. Dubovsky, V. Gorbenko and M. Mirbabayi, "Asymptotic fragility, near AdS_2 holography and $T\overline{T}$," JHEP **1709** (2017) 136 arXiv:1706.06604 [hep-th].

[231] S. Carlip, "Inducing Liouville theory from topologically massive gravity," Nucl. Phys. B **362** (1991) 111.