THE CONJOINT MEASUREMENT PROBLEM

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A: INTRODUCTION

Conjoint measurement is a type of fundamental measurement in which the empirical ordering of a set of elements, satisfying certain axioms, is used to obtain scales for the elements so ordered, and at the same time to obtain scales for the factors contributing to the ordering.

The need for such a system derives from the fact that many of the variables considered in economics, sociology, and particularly psychology are not amenable to direct measurement. Investigators in these fields are dependent on a subject's externalization of an internal quality which cannot be directly measured. And all too often the measures constructed have only an ordinal relationship. For example, the psychologist investigating the subjective loudness of pure tones has only the subjective statement that tone X is louder than tone Y. He might have his subjects indicate how much louder on an n-point scale, but this is not much more useful than the simple ordering, and certainly cannot be considered to have the desirable property of equal intervals between points on the scale. Many measurement theorists, including Thurstone and Chave (1929), Guilford (1954), and Guttman (1947), have devised means of obtaining scales with equal interval properties and comparable origins in an effort to avoid the drawbacks of working with measures that are essentially only ordinal. Their efforts have produced some useful results but do not entirely avoid the arbitrariness of origin nor the inequality of intervals, and scale points still have little more than ordinal properties.

Conjoint measurement, on the other hand, faces the fact that only an empirical ordering is given, and, instead of trying to get around the limitations of such a system, uses this ordering to derive scales that have interval scale properties, and in many cases even attain to ratio scale status.

Suppose a data structure is given which can be characterized by a data matrix whose entries, d_{ij}, satisfy an empirical ordering and can be expressed as the joint effect of row, a_i, and column, b_j, factor-levels. The basic objective of conjoint measurement is to determine whether or not an additive representation, of the form $f(d_{ij}) = f_1(a_i) + f_2(b_j)$, exists and if it does, to find the triple of functions f, f_1, and f_2 which satisfy the data structure.

Such data structures are very common in many types of research. We will deal mainly with the familiar 2-factor experiment, although results have been extended to the n-factor case. One specialization of the 2-factor experiment to be borne in mind is that there is an empirical ordering defined on the observed variable, d_{ij}. This dependent variable is elicited as the response to the simultaneous application of one level from each of the factors.

The first section of this report compares two research situations, and is intended to point out the type of problem solved by conjoint measurement. The second section presents the general psychological measurement framework in which the conjoint measurement problem is solved. The third gives the historical development of the axiomatization required for solution. The fourth section is devoted to methods of finding the functions stipulated by the axiomatization. And the last section discusses an example of the application of conjoint measurement.

B: HEURISTIC EXAMPLES

1. Campbellian Measurement Situation

Consider an agricultural experiment in which different amounts of fertilizer, in pounds, and of water, in gallons, are applied in a factorial arrangement to subplots in a field of peas. The variable of interest is the length, in inches, of the pea pods produced. The researcher is, no doubt, mainly interested in finding the best combination of fertilizer and water and, assuming that length of pods is his main criterion of the best combination, he can apply an analysis of variance to test which plot produces the best pods.

But suppose he were interested in determining the effect of x gallons of water when combined with y pounds of fertilizer in terms of z inches of pea pods produced. That is, he is concerned with estimating the parameters in the model,

$$E[z_{ij}] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$

(The model is stated in terms of its expected value to avoid complications arising from the experimental error term.) If the analysis of variance shows that the interaction term can be deleted safely from the model, then the parameters α_i and β_j can be estimated.

The statistics of this procedure is not the main concern here. Rather, this example is adduced because of its formal similarity to the conjoint measurement problem and the striking dissimilarity between the numbers available in this example and the measures found in many sociological and psychological experiments.

In the agronomy study all three measures, (pounds, gallons, inches) satisfy all the properties of real numbers and Campbellian measurement.

This means that all the usual operations with real numbers are available and meaningful. Further, all three measures satisfy an empirical concatenation operation; that is, one quantity of fertilizer can be placed in a pan balance with another and the combined weight determined. Similarly quantities of water can be added, and length of 2 pods can be determined by the use of a ruler and the results added. These measures also satisfy the axioms of a full order. For example, if a, b and c are any three measures of the amount of fertilizer (in pounds) then the reflexive, ($a \geq a$), atisymmetric, ($a \geq b$ and $b \geq a$ implies $a = b$), transitive, ($a \geq b$ and $b \geq c$ implies $a \geq c$), and connectedness, ($a \geq b$ or $b \geq a$ or both), properties are satisfied.

2. Conjoint Measurement Situation

Consider the psychologist mentioned in the introduction who was interested in investigating the subjective loudness of pure tones. Now a pure tone consists of the combination of an intensity component (objectively measureable in dynes/cm^2) and a frequency component (objectively measureable in cycles per second). Loudness, as such, cannot be directly measured. It is a quality of sound which can be ordered along its intensity dimension or its frequency dimension, but the joint effect of these two components has meaning only in reference to a being with the faculty of hearing. Since humans can simply state that tone A is louder than tone B, pure tones can be ordered by human subjects.

Suppose that the researcher is considering the hypothesis that loudness is an additive linear function of frequency and intensity. This hypothesis is very similar to that entertained by the agronomist when he entertained the linear additive model, and estimated the marginal effects of fertilizer and water. But unlike the agronomist the psychologist has only an ordering of the subjective loudness of pure tones instead of a measure in inches. And although the frequencies, in cycles per second (cps), and intensities, in dynes/cm^2, are analogous to the pounds of fertilizer and gallons of water, the objective measures of frequency and intensity are not of primary concern. The psychologist wants to find out how these two factors contribute to the loudness of the tones.

Conjoint measurement provides an answer to this problem. Formalizing this example and carrying out some analysis will give a good intuitive idea of what conjoint measurement tries to do.

Suppose the experimenter has a console with 2 dials, F and I, by which he can construct pure tones. Turning the F dial controls frequency, the I dial, intensity. The dials are set at a stop to the left so that turning one to the right increases I or F. Let (F, I) denote the pure tone produced. The subject's task is to order pure tones. The tones are varied systematically by the experimenter in the following way.

An equivalence relation is defined on the set of ordered pairs $\{(F,I)\}$ such that $(F,I) = (F',I')$ iff $(F,I) \leqq (F',I')$ and $(F',I') \leqq (F,I)$, where $\leqq$ means "is judged not louder than". An arbitrary tone (F_0, I_0) is selected as the zero point in the ordering, and another tone (F_1, I_0) is selected such that $(F_0, I_0) \leqq (F_1, I_0)$ but not $(F_0, I_0) = (F_1, I_0)$. Then (F_1, I_0) can be considered as one unit removed from the zero point. An I_1 is then determined by varying the volume while holding the frequency at F_0 until a tone (F_0, I_1) is achieved such that $(F_1, I_0) = (F_0, I_1)$. Thus a shift of one unit on the frequency axis produces the same change in discernable loudness as a shift of one unit along the volume axis. Continuing in this fashion we find an (F_2, I_0) and an (F_0, I_2) such that $(F_2, I_0) = (F_0, I_2)$, and so on along the possible marginal factor levels. Then the condition that the factors be additive is that

$(F_0, I_2) = (F_1, I_1) = (F_2, I_0)$, $(F_0, I_3) = (F_1, I_2) = (F_2, I_1) = (F_3, I_0)$, etc.

That is, the rectangular lattice type structure formed by matching changes produced by varying the levels of one factor with changes produced by varying the levels of the other factor satisfies the condition that the lattice points on the left to right diagonals are judged to be equivalent. If this test indicates additivity then scales of subjective intensity and frequency can be specified uniquely so that:

$f_1 (F_i)$ = the reading on the F dial for frequency F_i

$f_2 (I_j)$ = the reading on the I dial for intensity I_j

$f (F_i, I_j) = f_1 (F_i) + f_2 (I_j)$

Although the loudness of pure tones example illustrates what conjoint measurement tries to do, it has certain drawbacks. Conjoint measurement is not restricted to situations in which a convenient physical analogue is available for assigning a number to subjective qualities. It is not necessary to have some analogue of the frequency and intensity dials. The physical apparatus in this example takes the place of the quite difficult problem, a problem central to conjoint measurement applications, of determing a triple of functions which assign numbers on the basis of the ordering alone.

Before leaving these examples it is well to note again their similarities and particularly the dissimilarities which make the conjoint measurement approach valuable. The agronomy experiment is representative of a large class of experiments frequently encountered in applied statistics. This class of experiments is characterized by: measurement on extensive quantities, a full order on the set of entities observed, an empirical concatenation operation, i.e., an empirical addition operation, with these entities, and the factors and "response" are independently measurable. On the other hand the psychological experiment represents a class of experiments seldom seen by the novice statistician. This, no doubt, is understandable since the data collected are not amenable to the usual statistical procedures. The characterization of this type of experiment shows that the usual properties of the real continuum cannot be immediately applied. In the situations in which conjoint measurement is applicable the researcher is confronted with: trying to measure non-extensive properties, only a partial ordering of the entities observed, no natural empirical concatenation operation, the property to be measured cannot be measured independently of the contributing factors. Nevertheless, under certain conditions to be elaborated, interval scales can be found, through simultaneous transformations of the factor levels and the ordered data, which result in an additive representation of the data structure. In fact necessary and sufficient conditions for this representation to exist will be given. But first some notation and basic concepts must be introduced.

C: TERMINOLOGY AND DEFINITIONS

Suppes and Zinnes (1963) have formalized the psychological measurement problem in modern algebra terms. The problem is to show that a given empirical relational system that purports to measure a given property of the elements of the domain of the system is isomorphic (homomorphic) to an appropriately chosen numerical relational system, and to determine the scale type of measurements resulting from the numerical assignment. The problem involves stating and proving two theorems: a numerical representation theorem and a uniqueness theorem.

In what follows the loudness of pure tones experiment will be used to exemplify concepts, and the reader is encouraged to make such associations when the connections are not pointed out.

A relational system is a finite (n+1)-tuple of the form $U = (A, R_1, \ldots, R_n)$ where A is a non-empty set of elements, called the domain of U, and $R_1, \ldots, R_n$ are relations defined on A. The loudness example is an example of an empirical relational system in which $U = (T, \lesseqq, =)$ where T is the set of pure tones and $\lesseqq$, = are the binary relations defined on these tones. The adjective empirical is used since the relations are defined operationally on the empirical domain.

A full numerical relational system is a relational system whose domain is the set of all real numbers.

Two relational systems U and V are homomorphic if $U = (A, R_1, \ldots, R_n)$ and $V = (B, S_1, \ldots, S_n)$, and there exists a mapping, f, of A onto B such that for each $i = 1, \ldots, n$ and each subset of A, $(a_1, \ldots, a_{m_i})$, $R_i\ (a_1, \ldots, a_{m_i})$ maps onto $S_i(f(a_1), \ldots, f(a_{m_i}))$ where R_i and S_i are m_i-ary relations.

If the mapping is one-to-one then U is isomorphic to V.

A subsystem of a relational system, U, is a relational system obtained from U by taking a domain that is a subset of U and restricting the relations to this subset.

These definitions enable us to define a scale. A scale is the ordered triple (U,R,f), where U is an empirical relational system, R is a full numerical relational system and f maps U homomorphically onto a subsystem of R.

Then the psychological measurement problem is to show that any given empirical relational system that purports to measure a given property of the elements of the domain is homomorphic to an appropriately chosen numerical relational system. This is the representation problem. The next step is to determine the scale type of the measurements resulting from the procedure, and is called the uniqueness problem.

We here define the four most common types of scales. For this purpose let (U, R, f) be a scale, and g any function such that (U, R, g) is a scale. Also let x, y, a, b be elements of the real line.

1. (U, R, f) is a ratio scale if there exists a similarity transformation, F, such that $g = F \circ f$.

(F is a similarity transformation if F maps the real line onto the real line and there exists a positive constant, a, such that, for all x, $F(x) = ax$.)

2. (U, R, f) is an interval scale if there is a linear transformation, F, such that $F(x) = ax+b$ and $g = F \circ f$.

3. (U, R, f) is an ordinal scale if there is a monotone transformation F, $g = F \circ f$ and $x \gtrless y$ implies $F(x) \gtrless F(y)$.

4. (U, R, f) is a nominal scale if there is a one-to-one transformation, F, such that $g = F \circ f$.

If the function f maps an empirical relational system onto a numerical relational system it is called a fundmental numerical assignment. And this leads to the definition of fundamental measurement. Fundamental measurement of a set A with respect to the empirical relational system U involves the establishment of a fundamental numerical assignment for U.

The reader should have no difficulty in applying these concepts to the loudness of pure tone example. As pointed out before the empirical relational system is $U = (T, \leqq, =)$. And, by means of the two dial console a fundamental numerical assignment was established, namely $f(F, I) = f_1 (F) + f_2 (I)$. The scale obtained is (U, R, f) where U is defined above and R is a subset of the real line with the usual real relations $\leqq$, $=$.

The cps readings taken from the frequency dial also provide an example of a fundamental numerical assignment, as do the dynes/cm^2 readings from the intensity dial. To see that these three scales are in fact interval scales, merely note that calibration of the dials is completely arbitrary and that cps and dynes/cm^2 would not need to be used. We could have done equally well by using degrees of arc through which the dials had to be turned to produce a particular pure tone. Again it is pointed out that in most situations in which conjoint measurement is applicable the marginal numerical assignments are not so easily arrived at.

Although results have been extended beyond the two-factor case, we will be mainly concerned with two way data structures. In this context consider a data matrix, D. Suppose that the cells of the matrix represent the combination of a row contribution, A, and a column contribution, B, (e.g. frequency and intensity) and that numbers have been assigned to the cells by some subjective procedure or by a numerical assignment. Then the data matrix is said to be additive if there exist real valued functions f_1, f_2 and f defined on A, B, and D respectively such that

i) $f(a,b) = f_1 (a) + f_2 (b)$

ii) $f(a,b) \leq f(a', b')$ if and only if (iff)

$D(a,b) \leq D(a', b')$ for all a,a' and b,b' levels of A and B.

An mxn data matrix D is monotone if

i) for each row i, $d_{ij} \gtrless d_{i,i+1}$, $j = 1,\ldots,n-1$ and

ii) for each column j, $d_{ij} \gtrless d_{i+1,j}$, $i = 1,\ldots,m-1$.

It is to be noted that additivity implies monotonicity, but not vice-versa.

Since monotonicity is necessary to additivity, and additivity is what we are after, a quick first check of whether or not the application of conjoint measurement will be useful is to see whether or not the obtained data matrix is in fact monotone. If monotonicity is satisfied then conjoint measurement theory demonstrates the necessary and sufficient conditions for the existence of an additive representation of the data matrix.

D: HISTORICAL DEVELOPMENT

Conjoint measurement has been characterized as a method of determining if a data matrix satisfies a polynomial representation (Tversky, 1965), as a scaling procedure which simultaneously scales the data and the marginal factors so that additivity is satisfied (Luce and Tukey, 1964), or as a mathematical structure which gives insight into the underlying nature of the empirical system (Krantz, 1964).

The type of system which will be considered for the most part is a two-way data matrix in which the marginal factors are unrestricted as to measurement level and there is an empirical relation defined on the responses to the factorial combinations. In psychological measurement terms, it is desired to set up an homomorphism between the given empirical relational system and an appropriate numerical relational system.

Formally, the conjoint measurement problem can be stated as follows: given an empirical relational system in which two (or more) factors, A and P, contribute to the ordering of the domain, find conditions under which there exists a numerical representation of the form

$$f(A,P) = f_1(A) + f_2(P).$$

1. The Luce-Tukey Axiomatization

R. Duncan Luce and John W. Tukey (1964) were the first to give an axiomatization of, and a representation theorem for, the conjoint measurement problem. Earlier Debru (1960) had given an analogous result in a different context and based on topological assumptions.

Their approach is based on a weak ordering of the empirical domain which is considered as a cross product space. The weak ordering gives rise to an equivalence relation. A solution of equations and a concellation axiom are postulated and dual standard sequences defined in order to give meaning to the archemedean axiom for the ordered pairs of the domain. These axioms are sufficient to prove the desired representation and uniqueness theorems. Their formulation is as follows.

Given: A set A, with elements a, b, c,...

A set P, with elements p, q, r,...

Their cross product AxP, with elements ordered pairs.

$\geq$ a binary relation on the pairs of AxP.

Definition 1. A weak ordering satisfies the reflexive, transitive and connectedness properties. i.e., for all a, b, c, in A and p, q, r in P,

i) $(a,p) \geq (a,p)$

ii) $(a,p) \geq (b,q)$ and $(b,q) \geq (c,r)$ implies $(a,p) \geq (c,r)$

iii) either $(a,p) \geq (b,q)$ or $(b,q) \geq (a,p)$ or both.

Definition 2. (a,p) is equivalent to (b,q) iff $(a,p) \geq (b,q)$ and $(b,q) \geq (a,p)$, and we write $(a,p) = (b,q)$.

Definition 3. A dual standard sequence is a doubly infinite sequence of pairs $\{a_i,p_i\}$, $i = 0, \pm 1, \pm 2,...$, where a_i is in A and p_i is in P, and for each i,

i) $(a_i, p_{i+1}) = (a_{i+1}, p_i)$,

ii) $(a_{i+1}, p_{i-1}) = (a_i, p_i)$.

Axiom 1. $\leq$ is a weak ordering.

Axiom 2. Solution of equations. For each a in A and p, q in P there exists an f in A such that $(f,p) = (a,q)$, and for each a, b in A and p in P there exists an x such that $(a,x) = (b,p)$.

Axiom 3. Cancellation. For all a, f, b in A and p, x, q in P, (a, x) $\geq$ (f, q) and (f, p) $\geq$ (b, x) implies (a, p) $\geq$ (b, q).

Axiom 4. Archimedean. If $\{a_i, p_i\}$ is a non-trivial dual standard sequence, b is in A and q is in P, there exist integers m, n such that $(a_m, p_m) \geq (b, q) \geq (a_n, p_n)$

Using these axioms the following representation and uniqueness theorems result.

Representation Theorem. If axioms 1 to 4 hold, there exist real valued functions defined on A and P respectively, such that:

i) $f_1(a) + f_2(p) \geqq f_1(b) + f_2(q)$ iff $(a, p) \geq (b, q)$,

ii) $f_1(a) \geqq f_1(b)$ iff $a \geq b$,

iii) $f_2(p) \geqq f_2(q)$ iff $p \geq q$.

Uniqueness Theorem. If f_1, f_2 and f_1', f_2' are two pairs of functions satisfying i, ii, iii of the representation theorem, there exists real constants s > 0, k and c such that

$f_1'(a) = s\ f_2(a) + k$, and $f_2'(p) = s\ f_2(p) + c$.

Thus f_1 and f_2 are unique up to a linear transformation and determine interval scales with a common unit.

The connectedness property of the ordering can be particularly bothersome in survey experiments. An example in which connectedness does not hold is given by the situation in which n voters rank m candidates. There is no way of determining if voter i prefers candidate x more than voter j preferes candidate y. And often the transitivity assumption is quite optimistic, particulary in economic research where particular combinations in commodity bundles have a sometimes peculiar way of inducing preferences.

The solution of equations axiom assures that each of the two factors be rich enough in levels so that given any cell and any row, a column can be found which contains a cell equivalent in effect to the given cell. Figure 1 exemplifies the implications. Given the cell (c,p) and the row a, factor P must be fine enough to assure that there is an r which satisfies (c,p) $\geq$ (a,r) and (a,r) $\geq$ (c,p). When difference thresholds are small this could lead to an unmanageably large number of factor levels, and in fact leads to an unbounded system. This can be seen by solving the equation (c,r)=(a,x) which necessitates a solution for (c,x)=(a,y) which in turn requires a z such that (c,y)=(a,z), etc. And of course the same expansion is required for the A factor. Obviously, such a system is quite unmanageable empirically.

The cancellation axiom does not present such problems and it does not seem unreasonable to expect its fullfillment in empirical situations. It is analogous to the transitivity property of the order and states that if a change from a to f on the A factor overbalances a change from x to q on the P factor and a change from f to b overbalances a change from p to x, then the overall change from a to b will overbalance the change from p to q. Figure 2 illustrates the concept. The central arrow indicating the conclusion, and the cell at the tail preferred to the cell at the head. The cancellation axiom is basic to conjoint measurement theory. The other axioms have been relaxed but the cancellation axiom which assures a type of commutativity of the ordered pairs, remains intact. The archimedean axiom assures that any element of the domain can be bracketed by some multiple of the "units" found by the conjoint measurement procedure. It also implies that the unit is small enough so that any change in either factor will be detectable. It was pointed out that the solution to equation axiom requires an unbounded system.

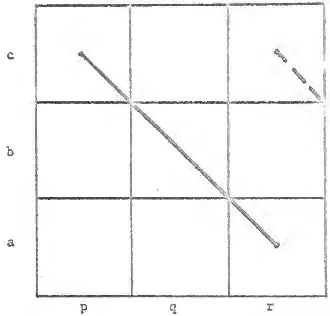

Fig. 1. An illustration of the solution of equations axiom.

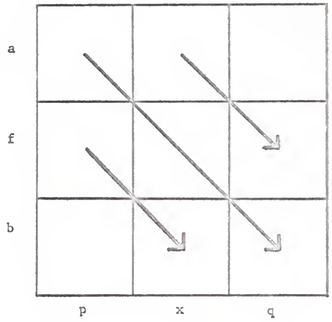

Fig. 2. An illustration of the cancellation axiom.

The archimedion axiom leads, in the limit, to a continuum of factors levels.

These axioms assure the existance of the desired additive representation and the existence of interval scales for each of the factors and for their joint effects.

The proof of the existance of an additive representation in a system satisfying the Luce-Tukey axioms is given very rigorously in their paper. As will be seen subsequently finding the functions which satisfy this representation is quite a problem in its own right. Also the Luce-Tukey axioms are sufficient to prove their representation theorem, but it has been shown that their axioms are not all necessary. The major drawback of their system is that it requires an empirical relational system which is unbounded and continuous.

2. Krantz's Reformulation

David H. Krantz (1964) reformulated and extended the Luce-Tukey axiomatization. Luce and Tukey used dual standard sequences to overcome the lack of an empirical concatenation operation. Krantz shows that their results, in terms of two factors and their formal cross-product along with an empirical weak order, can be obtained by assuming an equivalence relation on the empirical system and specializing the solution of equations and cancellation axioms for the equivalence relation. He then defines a formal concatenation operation on the equivalence classes of the empirical domain, introduces a full ordering on the domain and proves the representation and uniqueness theorems using the structure of an archimedean fully ordered group. Although the formulation is somewhat repetitive we give it in full for completeness.

Suppose we are interested in an empirical system which can be characterized as the cross-product of two non-empty sets A= (a, b, c,...); P= (p, q, r,...) .

The system is also assumed to have an empirically defined equivalence relation. Two axioms are postulated:

<u>Axiom 1</u>. Solvability. For any a, p, a', and p' there exists a'' and p'' such that (a', p'') = (a, p) = (a'', p').

<u>Axiom 2</u>. Cancellation. (a, p') = (a', p'')

and (a', p) = (a'', p')

implies (a, p) = (a'', p'').

These axioms partition the A x P space into equivalence classes, denoted by $\overline{(a, p)}$. Next a binary operation is defined on A x P/ =, the set of equivalence classes, as follows. Let a_o, p_o be fixed, (a, p) and (a', p') are given. Choose b, q by axiom 1 such that (a, p) = (b, p_o) and (a', p') = (a_o, q). Then $\overline{(a, p)} \circ \overline{(a', p')} = \overline{(b, q)}$ is a well defined binary operation on the class of equivalence classes analogous to a concatenation operation. It can be shown that $\circ$ satisfies the closure, associative, and commutative laws and that for each element $\overline{(a, p)}$ of A x P/ = there exists a unique $\overline{(a', p')}$ in A x P/ = such that $\overline{(a, p)} \circ \overline{(a', p')} = \overline{(a_o, p_o)}$, so that (A x P/ = , o) is a commutative group with identity element $\overline{(a_o, p_o)}$. Next an order relation is added to the group structure. First some definitions and a theorem are needed.

(G, $\circ$, R) is an ordered group if (G, $\circ$) is a group, R is a partial ordering of G, and if x R y then for any z, $x \circ z$ R $y \circ z$ and $z \circ x$ R $z \circ y$, for all x, y, z in G.

R is a full (total, simple) order if R is a binary relation such that R is reflexive, antisymmetric, transitive and connected.

R is a partial order if the connectedness requirement is dropped.

R is a weak order if the antisymmetric requirement is dropped.

If $(G, \circ, R)$ is a fully ordered group with identity e, then G is archimedean if for all x, y in G, $x \neq e$, there exists an integer $n \neq 0$ such that x^n R y, where x^n means $x \circ x$ n times.

<u>Theorem</u>: Suppose $(A \times P/=, \circ, \leq)$ is a fully ordered group. Define order relations on A and P, denoted by $\leq$, by

$a \leq a'$ iff $\overline{(a, p)} \leq \overline{(a', p)}$, for all p in P,

$p \leq p'$ iff $\overline{(a, p)} \leq \overline{(a, p')}$, for all a in A.

Then these orders are weak orders on A and P respectively.

Making use of the theorem (Birkhoff, 1948, p. 226) "An archemedean simply ordered group is isomorphic to a subgroup of the (ordered) additive group of real numbers. Moreover, the isomorphism is unique up to multiplication by a positive real number", the measurement theorems follow.

<u>Krantz's Representation and Uniqueness Theorem</u>

Suppose that $(A \times P/=, \circ, \leq)$ is an archimedean fully ordered group, with $\circ$ as defined above and identity $\overline{(a_o, p_o)}$. Then

A. There exists a real valued function, f, on A x P/= such that

i) $\overline{(a, p)} \leq \overline{(a', p')}$ iff $f\overline{(a, p)} \leq f\overline{(a', p')}$

ii) If (a_1, p_1) is arbitrary and * is the group operation on A x P/= with $\overline{(a_1, p_1)}$ as identity then $f(\overline{(a, p)} * \overline{(a', p')}) = f\overline{(a,p)} + f\overline{(a',p')} - f\overline{(a_1,p_1)}$

It follows that $(A \times P/=, *, \leq)$ is also an archimedean group.

B. If g is another real-valued function on A x P/= satisfying A i) and ii), then there exists real numbers t and u, $t > 0$, such that $g = tf + u$.

C. There exist real valued functions f_1 and f_2 on A and P respectively such that

i) $a \leq a'$ iff $f_1(a) \leq f_1(a')$

$p \leq p'$ iff $f_2(p) \leq f_2(p')$

ii) $f(a, p) = f_1(a) + f_2(p)$

D. If g_1 and g_2 are functions satisfying C i) and ii) then there exists a real number v such that $g_1 = f_1 + v$, $g_2 = f_2 - v$

A i) demonstrates the existence of a real valued, order preserving numerical assignment from the empirical space into the reals. A ii) shows that this assignment is independent of choice of identity. B indicates that the assignment is unique up to a positive linear transformation, i.e., interval scale is attained. C i) demonstrates the existence of order preserving numerical assignments for the marginal scales. C ii) determines the desired additive representation, and D demonstrates the uniquness, up to a translation, of the marginal scales.

Krantz then goes on to make a further generalization by dropping the formal cross product structure of the empirical system. He considers as given merely a set of elements which have three equivalence relations defined on them. This is intuitively reasonable. For to return to the loudness of pure tones example, there are empirical orderings of the intensity and frequency factors as well as of the tones themselves. So the tones can be ordered on their intensity and frequency components as well as on their combined effect. This is a particularly interesting approach in view of the fact that it dispenses with the necessity of singling out one variable, from several interrelated variables, as being the dependent variable.

Dropping the cross product structure of the empirical system leads to a generalization to the case in which more than three variables are conjointly measured.

Still Krantz's reformulation is just an extension of the Luce-Tukey system. The axiomatization is still in terms of an unbounded data structure, gives sufficient but not necessary conditions for the additive representation, applies only to additive measurement models, and requires a full order. (Luce-Tukey required only a weak order.)

3. Tversky's General Theory

Amos Tversky (1965) added an elegant extension of the theory. His formulation has the advantages that it: is formed in terms of partially ordered data (connectedness is not required), applies to finite as well as infinite data structures, provides necessary as well as sufficient conditions for measurement, and these conditions apply to any polynomial measurement model. Thus a complete characterization of all data that are measureable by such models is provided.

The notation in this formulation gets rather complicated and the proof of the representation theorem is based on constructing a polynomial ring with integral coefficients over the factor set. In what follows a data structure, D, is an extension of the empirical relational system defined previously. A polynomial measurement model is the generalization of the additive model considered in the Luce-Tukey axiomatization. Relations defined on different domains are subscripted to differentiate them.

a. Definitions

1. A data structure, $D=(C, \leq_o)$, is any partially ordered set of data such that each data element, $x=(a,b,\ldots,k)$, can be regarded as an element of a subset of the cross-product set $A \times B \times \ldots \times K$.

2. A polynomial measurement model, M, is a combination rule which represents each data element as a polynomial function of its components.

3. A data structure, D, satisfies a polynomial measurement model, M, whenever there exist real valued functions f_A, f_B, ..., f_K defined on A, B, ..., K respectively such that for any x=(a,b,...,k),

i) $f(x) = M(f_A(a), f_B(b), \ldots, f_K(k))$

ii) for all x, x'

$x >_o x'$ implies $f(x) > f(x')$

$x =_o x'$ implies $f(x) = f(x')$

4. A ring R is the triple (R(C),@,*) where R(C) is a set and @, * are distinct binary operations on R(C), such that

i) (C,@) is a commutative group

ii) (C,*) is a semigroup (the operation satisfies only the associative property)

iii) The distributive law holds between @ and *

5. A partially ordered ring is an ordered pair (R, $\geq$) which satisfies

i) R is a ring

ii) $\geq$ is an antisymmetric partial order on R

iii) the order satisfies the ring operation, that is,

a) $x \geq y$ implies $x@z \geq y@z$

$x \geq y$ implies $x*z \geq y*z$.

6. A fully ordered ring satisfies i) and iii) of definition 5 but $\geq$ is a full order on R.

7. A fully ordered ring is archimedean if, for any x, y in R, there exists an integer n such that $ny \geq x$.

8. A polynomial is any combination of formal sums, differences and products of the elements of a set.

9. A binary relation, $\geqq_i$, is said to be an extension of another, $\geqq_j$ whenever $x \geqq_j y$ implies $x \geqq_i y$ for all x,y in $\geqq_j$.

10. If there exist x and y, elements of R, neither of which is zero, and $x*y=0$, then R contains zero divisions.

b. Construction

A theorem of Hion states: "A fully ordered Archimedian ring without zero divisors is order-isomorphic to a unique subring of the real number field taken with its usual ordering." The proof of Tversky's representation theorem follows directly from this theorem after a construction showing that any data structure which satisfies a polynomial measurement model can be imbedded in a fully ordered Archimedean ring.

Consider the polynomial ring, R, with integral coefficients, generated by the component set A x B x ... x K. Then R(C) is the set of polynomials formed by adding, subtracting and multiplying elements of the form $x=(a, b, \ldots, k)$. The measurement model M assigns to each data point $x=(a, b, \ldots, k)$ in C a unique M(x) in R(C). Thus M maps the data structure $D=(C,\geqq_o)$ into R and induces an order, $\geqq_1$, on the images of C such that $M(x) \geqq_1 M(y)$ iff $x \geqq_o y$. It is to be noted that different measurement models define different order relations on R(C). Although this mapping does define a partial order on the polynomial ring R, it does not necessarily make R a partially ordered polynomial ring. To assure that R is partially ordered the relation $\geqq_1$ is extended by defining a relation $\geqq_2$ so that iii) of definition 5 is satisfied.

Let p_i be an element of R(C). The relation $\geqq_2$ is defined separately for its symmetric, $=_2$, and antisymmetric, $>_2$, parts. When elements p_1 and p_2 of R(C) are identical except for order of operations we write $p_1 \equiv p_2$.

Definition 11.

$p_1 =_2 p_2$ whenever one of the following holds:

i) $p_1 =_1 p_2$ or $p_1 \equiv p_2$

ii) $p_1 \equiv p_3 @ p_4$, $p_2 \equiv p_5 @ p_6$ and

$p_3 =_2 p_5$, $p_4 =_2 p_6$

iii) $p_1 \equiv p_3 * p_4$, $p_2 \equiv p_5 * p_6$ and

$p_3 =_2 p_5$, $p_4 =_2 p_6$

iv) $p_1 * p_3 =_2 p_2 * p_3$, $p_3 \neq_2 0$, where 0 is the zero polynomial.

$p_1 >_2 p_2$ whenever one of the following holds:

i) $p_1 >_1 p_2$

ii) $p_1 \equiv p_3 @ p_4$, $p_2 \equiv p_5 @ p_6$ and

$p_3 >_2 p_5$, $p_4 \geqq_2 p_6$

iii) $p_1 \equiv p_3 * p_4$, $p_2 \equiv p_5 * p_6$ and either

$p_3 >_2 p_5 \geqq_2 0$, $p_4 \geqq_2 p_6 >_2 0$ or

$0 >_2 p_5 >_2 p_3$, $0 >_2 p_6 \geqq_2 p_4$

iv) $p_1 \equiv p_3^2 \neq_2 0$, $p_2 =_2 0$

Parts i) of the definition assure that the relations are extensions of $\geqq_1$, by definition 9. Parts ii) and iii) guarantee that iii) of definition 5 will be satisfied. Parts iv) establish multiplicative cancellation, and the "positivity" of non-zero squares.

Now the ordered pair $(R(C), \geqq_2)$ does not necessarily constitute a partially ordered ring. The fact that $\geqq_1$ is a partial order does not guarantee that its extension $\geqq_2$ is a partial order as required by ii) of definition 5. Further the antisymmetric requirement can be violated since a polynomial can be expressed as different combinations of the data elements. That is, it is possible for both $p_1 \geqq_2 p_2$ and $p_2 >_2 p_1$. To overcome this difficulty an axiom is introduced.

Assymmetry axiom: There are no p_1, p_2 in $R(C)$ such that $p_1 \geqq_2 p_2$ and $p_2 >_2 p_1$.

To illustrate some of the concepts developed so far, consider the following example.

Let $D=(A \times B, >_o)$. A has components (levels) a_1 and a_2, B has components b_1 and b_2. Suppose the following rank-order of their combined effects is obtained.

	b_1	b_2
a_1	1	3
2_2	4	2

Then $(a_2, b_1) >_o (a_1, b_2) >_o (a_2, b_2) >_o (a_1, b_1)$. Suppose the measurement model is an additive model, $M(f_A(a), f_B(b))=a+b$. Then the chain of inequalities becomes, in terms of $R(C)$ and $>_1$,

$$a_2 + b_1 >_1 a_1 + b_2 >_1 a_2 + b_2 >_1 a_1 + b_1.$$

Let $p_1 \equiv a_2 + b_1$, $p_2 \equiv a_1 + b_2$, $p_3 \equiv a_2 + b_2$, $p_4 \equiv a_1 + b_1$, $p \equiv p_1 + p_2$, $q \equiv p_3 + p_4$.

Then $p \overset{=}{=} a_2 + b_1 + a_1 + b_2$ and $q \overset{=}{=} a_2 + b_2 + a_1 + b_1$.

So $p \overset{=}{=} q$ implies $p =_2 q$ by i) of definition 11. On the other hand we have

$$(p_1 >_1 p_2), (p_3 >_1 p_4)$$

implies $(p_1 >_2 p_2), (p_3 >_2 p_4)$, by i) of definition 11

but $p_1 >_1 p_3$ and $p_2 >_1 p_4$

implies $(p_1 + p_2) >_2 (p_3 + p_4)$, by ii) of definition 11

implies $p >_2 q$ by i) of definition 11.

Thus the asymmetry axiom is not satisfied.

It is interesting to note that a data structure with this rank ordering does not satisfy the Luce-Tukey cancellation axiom. For we have

$(a_2, a_b) >_o (a_1, b_1)$ and $(a_1, b_2) >_o (a_2, b_2)$ implies $(a_2, b_2) >_o (a_2, b_1)$.

But $(a_2, b_1) >_o (a_2, b_2)$ in the given data structure. So this structure would not satisfy an additive measurement model in the Luce-Tukey formulation. Also note that the data matrix is not monotone.

To see that the additive representation does not hold, assume that it does. Then

$1=a_1+b_1$ implies $b_1=1-a_1$

$2=a_2+b_2$ implies $a_2=2-b_2$

$3=a_1+b_2$ implies $a_1=3-b_2$

$4=a_2+b_1$ implies $b_1=4-a_2$

From which

$b_1=1-3+b_2$ implies $b_1-b_2= -2$

$b_1=4-2+b_2$ implies $b_1=b_2 = 2$.

So, the assumption that the additive model holds leads to a contradiction. This is an example demonstrating that the asymmetry axiom is necessary if the data structure is to satisfy the given measurement model.

If the same data structure is considered, but a multiplicative model $M(f_A(a), f_B(b)) = f_A(a)\ f_B(b)$ is used then the following scale values satisfy the multiplicative model.

$f_A(a_1)=2, f_A(a_2)=-3, f_B(b_1)=-2, f_B(b_2)=1$

The obtained order under the multiplicative model is:

$a_2b_1 >_1 a_1b_2 >_1 a_2b_2 >_1 a_1b_1$ and, using the specified numerical assignment it is seen that indeed $6>2>-3>-4$.

It can be shown that the asymmetry axiom is sufficient to guarantee that $\geqq_2$ is a partial order. But $\geqq_2$ is not necessarily antisymmetric as required by ii) of definition 5. That is, $p \geqq_2 q$ and $q \geqq_2 p$ does not imply that $p =_2 q$. But it is immediate from the definitions of $\geqq_o$, $\geqq_1$, and $\geqq_2$ that $=_2$ is an equivalence relation on $R(C)$. So this defect can be remedied by using $=_2$ to partition $R(C)$ into equivalence classes. This set of equivalence classes is denoted by $R(C)/=_2$, and the element of this set containing the polynomial p is denoted by $\bar{p}$. Addition and multiplication in $R(C)/=_2$ are defined by

$$\bar{p}+\bar{q}=\overline{p+q}$$

$$\bar{p}\ \bar{q}=\overline{pq}$$

It can be shown that these operations are well defined and independent of the representative chosen to represent the equivalence class. Then $R(C)/=_2$ is a ring without zero-divisors. Now define $\geqq_3$ on $R(C)/=_2$ by $\bar{p} \geqq_3 \bar{q}$ iff $p \geqq_2 q$, and show that $\geqq_3$ is: a) independent of choice of p and q in $\bar{p}$ and $\bar{q}$, b) an antisymmetric partial order on $R(C)/=_2$, and c) compatible with the ring operations. Then $(R(C)/=_2, \geqq_3)$ is a partially ordered ring without zero-divisors. And "a partial order of a ring without zero-divisors is extendable to a full order if and only if $\sum_i p_i q_i^2 \neq 0$ where p_i is any product of positive ring elements and q_i is any product of non-zero elements." It can be shown that $\geqq_3$ on $R(C)/=_2$ satisfies this theorem (Fuchs, as given in Tversky 1965 a). But this extension is not unique in general. It should be remembered from definition 9 that if $\geqq_i$ is an extension of $\geqq_j$ then $\geqq_j$ implies $\geqq_i$. Let E be the set of all full orders of $R(C)/=_2$ which include $\geqq_3$. The object of this construction is to assure that the given data structure can be represented as an archimedian fully ordered ring. But unless $\geqq_3$ is itself archimedian (definition 7) E will contain non-archimedian extensions. Consequently a second axiom is introduced.

<u>Archimedean axiom</u>: There exists an element $\geqq_e$ in E such that, for any $\bar{p}$, $\bar{q}$ in $R(C)/=_2$, $n\bar{q} \geqq_e \bar{p}$ for some integer n.

This axiom gives the desired result: $(R(C)/=_2, \geqq_e)$ is a fully ordered archimedian ring without zero-divisions. Then Hion's Theorem (Fuchs, 1963, p. 126) assures the existence of a unique order preserving isomorphism from $(R(C)/=_2, \geqq_e)$ into the real numbers.

<u>Tversky's Representation Theorem</u>:

"For a data structure D to satisfy a polynomial measurement model M it is necessary and sufficient that it satisfies the following conditions:

i) Asymmetry: There are no p, q in R(C) such that $p \geqq_2 q$ and $q >_2 q$.

ii) Archimedian: There exists a $\geqq_e$ in E such that for any $\bar{p}$, $\bar{q}$ in $R(C)/=_2$ $n\,\bar{q} \geqq_e \bar{p}$ for some integer n.

Furthermore, the resultant numerical assignment is unique."

The sufficiency follows directly from the construction and the definition of an extension of an order relation. The necessity follows from definition II and contradiction arguments.

c. Comments

It was noted that the Luce-Tukey (1964) formulation introduced a solution of equations axiom which led to their system being applicable only to unbounded, and at least dense if not continuous, data structure. This drawback is avoided in Tversky's formulation and is the main improvement over previous results.

The second important advance is that this theory is applicable to any polynomial measurement model. For example, Hull's and Spence's performance models provide models amenable to conjoint measurement analysis. These models postulate that some performance measure, P, is a polynomial function of habit strength (learning), H, drive, D, and incentive, K. Hull's performance model, P=H x D x K, could be reduced to an additive model by a log transformation and either Luce-Tukey's or Tversky's formulation applied. On the other hand Spence's performance model, P=H(D+K), can only be attacked through Tversky's formulation. These are only two of many sociological and psychological models in which a conjoint measurement approach could be highly informative.

Another advantage is that Tversky's approach does not limit the investigator to factorial type data structures. The data structure can be any subset of the cross-product space A x B x . . . x K. This advantage also makes conjoint measurement methods applicable to data usually analyzed by nonmetric scaling methods.

It is to be noted that the particular measurement model used determines the nature of the relations $\geqq_2$ and $\geqq_e$. This means that the data must satisfy the asymmetry and archimedean axioms in terms of the particular model employed. Once the model is specified the axioms will be determined so that the data may be tested to see if the model is satisfied. One immediate result that can be obtained is that if an additive two factor model is considered, then the Luce-Tukey cancellation axiom follows from Tversky's asymmetry axiom.

The main drawback to this elegant theoretical formulation is that it does not provide a general solution to the problem of finding a numerical solution when a data structure and measurement model are given. Luce-Tukey also showed that solutions exist but did not show how to obtain them. Some approaches to this problem are given in the next section.

E: METHODS OF FINDING SOLUTIONS

At the outset, it must be admitted that no general approach to finding functions satisfying a specified measurement model has been found. All three formulations in the previous section proved that, under certain conditions, a representation of the form

$$f(x)=M(f_A(a),\dots,f_K(k))$$

exists. Luce-Tukey's formulation was restricted to the additive two-factor linear model. Krantz extended it to the additive n-factor case, and Tversky to the general polynomial model. But none of the three gives a method for finding the required numerical assignments.

There are four essentially different types of data structures to which conjoint measurement procedures can be applied. In the first type the marginal factors are purely nominal and their joint effects can be ordered but there is no numerical assignment associated with them. In the second type the marginal factors are measurable on at least interval scales and their joint effects are ordered independently of the marginal scales, again without providing a numerical assignment. In the third case the marginal factors are only nominal but there is a real-valued function defined on the joint effects which induces the ordering. The last case is characterized by marginal factors measurable on interval scales and an ordering of their joint effects given by a real valued function.

The research problem also determines the way in which conjoint measurement is to be applied. Some studies may want to account only for the obtained ordering and so test the validity of the model. Others may want to recover the numerical values of the joint effects and find marginal scales.

To formalize these ideas Tversky (1965a) gives two definitions:

i) A numerical data structure, Dg, is a data structure whose ordering is given by a real-valued function, g, defined for all its elements.

ii) A strict measurement model attempts not only to account for the order but also to recover the numerical values of the data.

A strict measurement model is applicable only when the data structure is numerical. Generally conjoint measurement tries to find functions of the marginal factors which account for the given order. But for a data structure to satisfy a strict measurement model the data structure must be numerical, the model account for the observed order, and $f(x)=M(f_A(a),\ldots,f_K(k))=g(x)$ for all x in Dg.

Hull's performance model P = H x D x K gives an example of this type of structure. Suppose that in a linear maze task the measure of performance, P, is the time it takes a rat to reach the food box after it is placed in the entrance box. Let the habit strength, H, be measured by the number of reinforced trials, the drive, D, by hours of food deprivation, and incentive, K, by amount of food given on the previous trial. Then for any data point $x=(h,d,k), g(x)=P$ and $f(x)=f_H(h) \times f_D(d) \times f_K(k)$. It is desired to find functions f_H, f_D, f_K and f that not only satisfy

$x \geq_o y$ iff $f(x) \geq f(y)$

but also

$f(x)=g(x)$.

The loudness of pure tones example does not stipulate any real-valued function, g, on the joint effects of the marginals, and so is not an example of a numerical data structure. It falls into the second case described above.

Some approaches to applying conjoint measurement theory to each of the four cases described above can be found in the literature.

In the first case, when the marginal factors are purely nominal and their joint effects are orderable, Luce and Tukey (1964, p.6) suggest that the hypothesis of additivity could be tested by collecting a "sufficiently voluminous body of ordinal data" and testing if their cancellation axiom is satisfied. Luce and Tukey maintain that the cancellation axiom is the most essential axiom in their formulation and that it is a necessary condition for an additive representation. It obviously is the most important axiom substantively, the solution to equations and archimedean axioms pertaining more to the underlying formation of the data structure. If the cancellation axiom is satisfied, an additive model could be accepted in view of Tversky's (1965a) subsequent formulation. Tversky points out that in the case of finite data structures the archimedian axiom is not necessary. And it has been pointed out, by an example in discussing Tversky's asymmetry axiom, that the asymmetry axiom is equivalent to Luce and Tukey's cancellation axiom in the finite case.

The loudness of pure tones example, although falling into the second case, was treated as though the marginals were purely nominal in testing for additivity. When the experimental situation lends itself to the approach of constructing equal interval scales on the basis of just noticeable differences, the hypothesis that the marginal factors are additively related to their joint effect can be tested by testing for the judged equality of the obtained effects on the left to right diagonals.

In the second case, where the marginal factors are measured on at least ordinal scales and the joint effects are ordered independently of the factors, the observed ordering can be used to define a chain of linear inequalities. Each cell (a,p) of the data matrix is represented as the sum of two unknowns,

X_a+Y_p. If there are m rows and n columns then a chain of mn-1 inequalities in m+n unknowns of the form,

$$X_{i_1} + Y_{j_1} <\ldots< X_a + Y_p <\ldots< X_{i_m} + Y_{i_n} ,$$

is defined, the order being determined by the observed ordering. Then the test that an additive representation exists, or that an additive model is satisfied, is to determine whether or not this chain of inequalities has a solution. If the hypothesis of additivity is supported and it is desired to obtain a numerical assignment for the joint effects, there is some problem since the solution to the chain of inequalities is not unique (Tversky,1965b,p.11).

Of course if the marginal scale values employed are substantively meaningful then they can be used directly in the postulated model to give a numerical assignment for their effects. This situation is analogous to testing the hypothesis that the subjective loudness of pure tones can be expressed as the sum of the objective frequency and objective intensity. But adding frequencies and intensities would, most likely, not be substantively meaningful.

To date there is no method of solving for the functions f, f_1, f_2 which theoretically exist and satisfy f(a,p)=f (a)+f (p) in this case.

Where the data are ordered through a given numerical assignment but the marginal factor levels are only nominal, various approaches to obtain the functions f, f_1, f_2 have been tried. This situation is closely related to the two factor analysis of variance situation in which it is desired to estimate the marginal effects. A recent article by Box and Cox (1964) gives one approach.

Their article is devoted to finding transformations in the analysis of variance situation which not only lead to a satisfaction of the usual analysis of variance assumptions but induce a meaningful metric on the data. Thus they try to go a step beyond Tukey's (1949) test for additivity. In their general approach to transformations a particular method of determining a monotonic function which satisfies the additive model is introduced.

It is obvious that an order preserving transformation must be monotonic But the entire family of monotone transformations is quite difficult to deal with analytically. J. B. Kruskall, in an unpublished paper, tried to find the optimum transform among all monotonic transformations. He minimized the sum of squared residuals of monotone transformations of the data values from the values postulated by the model. But his programming procedure is subject to relative minima, and the "reductio ad absurdum" that the sum of squares can be made as small as one likes by taking the range of the transformation to be as large or as small as one chooses.

Box and Cox choose a family of monotonic transformations that is fairly broad, but does not guarantee a solution in the conjoint measurement situation. Still in some cases it will provide a solution, and is indicative of a general procedure that could be used with other families of transformations. Their approach is as follows.

Given a vector of observations, $Y=(y_1,\ldots,y_n)$, and the known design matrix A, the m dimensional vector of parameters, θ, is to be estimated for the model

$$E(y_{ij})=\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}$$

under the constraint that the $(\alpha\beta)_{ij}$ elements of θ are 0.

The families of transformations used are, for $y_i > 0$,

$$y_i^{(\lambda)} = \begin{cases} \dfrac{y_i^{(\lambda)}-1}{\lambda} & ,\lambda \neq 0 \\ \log y_i & ,\lambda = 0 \end{cases}$$

$$y_i^{(\lambda)} = \begin{cases} \dfrac{(y_i+\lambda_2)^{\lambda_1}-1}{\lambda_1} & ,\ \lambda \neq 0 \\ \log y_i+\lambda_2 & ,\ \lambda_1 = 0 \end{cases}$$

Let $Y^{(\lambda)}$ be the transformed observation vector. Then the model can be stated:

$$E(Y^{(\lambda)})=A\theta.$$

It is assumed for some parameter (or vector of parameters)λ, to be determined, that the $y_i^{(\lambda)}$ are normally and independently distributed. Then the likelihood function of Y is

$$L(Y)=L(Y^{(\lambda)})J(\lambda;Y)$$

where $J(\lambda;Y)= \prod_{i=1}^{n} \dfrac{dy_i^{(\lambda)}}{dy_i}$

Then the parameters, θ, can be estimated, when A is of full rank, as:

$$\hat{\theta} = (A'A)^{-1} A'Y^{(\lambda)}.$$

And the estimate of the variance is

$$s^2(\lambda;Y) = \frac{1}{n}(Y^{(\lambda)} - A\hat{\theta})'(Y^{(\lambda)} - A\hat{\theta})$$

$$= \frac{1}{n}(Y^{(\lambda)'} A_r Y^{(\lambda)}) = \frac{S(\lambda;Y)}{n}$$

where $A_r = I - A(A'A)^{-1}A'$.

It follows that the log maximum likelihood is given by

$$L_{max}(\lambda;Y) = -\frac{n}{2}\log s^2(\lambda;Y) + \log J(\lambda;Y)$$

Under the transformation

$$Z^{(\lambda)} = Y^{(\lambda)}/J^{1/n}(\lambda;Y)$$

$L_{max}(\lambda;Y)$ simplifies to

$$L_{max}(\lambda;Z) = -\frac{n}{2}\log s^2(\lambda;Z)$$

$$= \log(S(\lambda;Z))^{-n/2} + (\log n)^{-n/2}$$

so that $L_{max}(\lambda;Z)$ is proportional to

$$(S(\lambda;Z))^{-n/2}.$$

Then the maximum likelihood estimate of λ can be obtained by minimizing $S(\lambda;Z)$. It is pointed out that $S(\lambda;Z)$ equals the interaction sum of squares plus the within cells sum of squares in the usual two way analysis of variance with replications within cells.

A graphical procedure is used to find the maximizing value $\hat{\lambda}$. Values of $L_{max}(\lambda;Z)$ are plotted for various values of λ, a smooth curve drawn and the maximizing value determined. Further, since $-2\log L_{max}$ is approximately distributed as χ^2, a confidence interval on the values of λ, in which additivity holds, can be determined.

The application to the conjoint measurement problem is as follows. It is desired to determine functions f, f_1, f_2 such that

$$f(a,p)=f_1(a)+f_2(p)$$

The Box and Cox procedure provides the f_1 (a) and f_2 (p) by giving a solution for the vector of parameters θ. And the parameter λ is determined by a certain confidence interval within which the additive model holds. Thus a transformation of the numerical assignment defined on the pairs (a,p) is determined so that the additive model is satisfied.

In this third case the approach of solving a system of linear inequalities can also be used to test for additivity and a least squares approach used for finding the required numerical assignments. This method is used in the example to be given in the next section so it will not be elaborated here.

Luce and Tukey (1965, p.4) point out that conjoint measurement can be used even in situations where ordinary measurement procedures suffice, and outline a procedure for conjointly measuring momentum as the joint effect of mass and gravitational potential. This is an example of the fourth case described above where the researcher is blessed with measurable marginal factors and data ordered by a numerical assignment.

The fact that physical entities are conjointly measureable is satisfying from a theoretical point of view, but of little practical importance. What is important is that psychological and sociological investigators are not basically interested in the physical scales of their variables but in the hypothetical subjective scales in their subjects. A further complication arises since the joint effect often cannot be measured independently of the marginal factors. This empirical situation is the basis of the following example.

F: AN APPLICATION OF CONJOINT MEASUREMENT

Tversky (1965b) provides an example of the application of conjoint measurement to an investigation of various models in utility theory. The models considered are for decisions under risk, applied to a gambling situation. Tversky considers four models for investigation.

The subjective expected utility, SEU, model hypothesizes that people act to maximize their subjective expected utility, which is the sum of the subjective utilities of the possible outcomes weighted by their subjective probability of occurence. The expected value, EV, model, on the other hand, hypothesizes that objective probabilities and monetary values can be substituted in the SEU model for subjective probabilities and subjective utility. The other two models, expected utility, EU, and subjective expected value, SEV, state that monetary values or objective probabilities respectively can be substituted in the SEU model. For our purposes, it will suffice to consider only the SEU and the EV models.

Formally the models state: if G is a gamble with possible outcomes $o_1, \ldots, o_n$ contingent upon events $e_1, \ldots, e_n$, and G' is another gamble defined by o_i' and e_i', then there exist real-valued functions, u and s, defined on outcomes and events respectively, such that G' is not preferred to G iff

$$\sum_{i=1}^{n} u(o_i)s(e_i) \geqq \sum_{i=1}^{n} u(o_i')s(e_i').$$

In the SEU model $u(o_i)$ is the subjective utility, or subjective value, of the outcome o_i and $s(e_i)$ is the subjective probability of the occurrence of the event e_i. The EV model defines these quantities as the monetary values of the outcomes and objective probabilities that the events occur, respectively.

Tversky's experiment was concerned with simple two-outcome gambles of the form (a, p) in which the gambler won a positive amount a if the event p occurs, and nothing if p did not occur. A factorial combination of four amounts with four probabilities defines a data matrix D. The gambles in D were ordered by asking subjects to give the minimal price for which they would sell the right to play the gamble.

The empirical structure under the EV model is an example of the situation described earlier in which the marginal scales are measurable and the data are ordered by a numerical assignment. Under the SEU model the marginal factor levels are only nominal.

Using the definition of an additive data matrix given previously, it can be shown that "for gambles of the form (a, p) the SEU model is satisfied if and only if D is additive." (Tversky, 1965b, p. 5) Further, the EV model is satisfied iff D is additive under the constraints that u and s correspond to monetary values and objective probabilities respectively. Thus we have two situations in which conjoint measurement can be used to determine whether or not an additive representation exists, and, for the SEU model, to find numerical scale values for the marginal factors.

Eleven convicts were used as subjects and cigarettes were used as payoffs since they had exchange value among the prisoners. The gambles used consisted of all 16 combinations of 1, 2, 3, 4 packs of cigarettes at probabilities .2, .4, .6, .8. They were presented on slides which, when projected, presented a wheel of fortune arrangement representing the gamble. Fig. 3 gives an example of one such gamble. The subjects were asked to give the minimal selling price for which they would sell the gamble. They were informed that the experimenter would take advantage of them by electing to

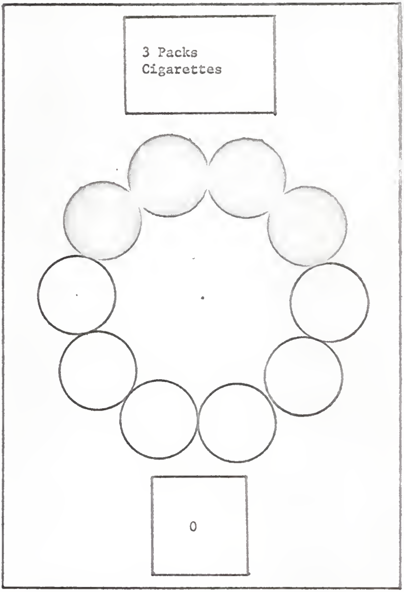

Fig. 3. A typical gamble

buy gambles which they underpriced, or allowing them to pay and play those they overpriced, when they were allowed to play 3 gambles at the end of the session. Thus the subjects could do no better than write down their minimal selling price.

The treatment of the data is similar under both models so the SEU model will be considered for the most part, pointing out the difference in treatment when the EV model is being considered. The definitions and notation used in Tversky's (1965a) formulation will be followed.

The data structure is D = (C, $\geqq$), where C is the cross product set A x P, A takes on the levels 1, 2, 3, 4 packs of cigarettes, and P has the probability levels .2, .4, .6, .8. The relation, $\geqq$, meaning "selling price not less than" is determined by the real valued function g(a, p) = minimal selling price offered for the gamble (a, p). So the analysis will be concerned with a "numerical data structure," Dg.

The polynomial measurement model is

$$M(f_A(a),\ f_B(b)) = \sum_{i=1}^{2} u(a_{ij})s(p_{ik}),\ \text{for } j,\ k=1,\ \ldots,\ 4.$$

Since the gamblers lose nothing if the specified outcome does not occur, the utility of zero gain is assumed to be zero. Under this assumption the model reduces to

$$M(f_A(a),\ f_B(b)) = u(a_j)s(p_k),\ \text{for } j,\ k=1,\ldots,\ 4.$$

Then, taking logarithms, M is equivalent to

$$\log u(a_j) + \log s(p_k).$$

Let $f_A(a) = \log u(a)$ and $f_P(p) = \log s(p)$.

It is pointed out that under the EV model $f_A(a) = \log a$ and $f_P(p) = \log p$. Whereas, under the SEU model, functions u and s must be determined if the polynomial measurement model, which under the logarithmic transformation is equivalent to an additive model, is satisfied.

The conditions that Dg satisfy M are:

i) $f_A(a, p) = f(a) + f_P(p)$,

ii) for all $D = (a, p)$, $D' = (a', p')$

$g(D) \geq g(D')$ iff $f(D) \geq f(D')$.

i.e., D is additive.

Additivity was tested by using the chain of inequalities method on the data provided by each subject under each model. Each cell, (a, p), was represented as the sum of two unknowns, $X_a + Y_p$, the chain of inequalities being determined from the bidding matrix, D. A computer was programmed to solve the inequalities. In some cases the inequalities proved to have no solution, implying non-additivity. In such cases the maximal solvable subset was solved, and a solution matrix computed. Once the solution matrices were obtained, their cell entries were compared with the cell entries in the bidding matrices and the proportion, p, of inversions determined. Then Kendall's rank correlation coefficient, t, is the difference between the proportion of solution matrix entries ordered like their bidding matrix and the proportion of inversions. That is,

$t = (1-p)-p$.

Then t can be used as a measure of additivity. Table 1 gives the obtained results.

Table 1

Rank Correlations between the Solution Matrix and

the Bidding Matrix for each Subject

Subj	1	2	3	4	5	6	7	8	9	10	11	AVE
SEU	1	1	1	1	.950	1	.950	1	.966	1	1	.988
EV	.812	.966	.932	.854	.949	.966	..932	.966	.949	.966	.854	.922

Using the relation, $p = \frac{1}{2}(1-t)$, and the average t for each model, the proportion of inversions for the SEU model is found to be 0.006 and for the EV model it is 0.044. The additive model is not rejected in either case.

It remains to calculate the subjective utility and subjective probability scales for the SEU model. They are already determined by the objective values for the EV model. The least squares condition, that

$$\sum_a \sum_p (f(a,p)-D(a,p))^2$$

be a minimum, was employed to find the functions $f_A(a)$ and $f_P(p)$. The solution was obtained using a computer program which gives a linear approximation. The obtained scales are unique up to a positive additive constant, which was determined using substantive considerations depending upon whether or not the subjective probabilities of complementary events are constrained to sum to one or not. Under the assumption that subjective probabilities sum to one, we have

$$M(a, p) = f_A(a) + f_P(p) = u(a)s(p)$$

so that

$$\sum_p M(a, p) = u(a)(s(.2) + s(.4) + s(.6) + s(.8))$$

$$= u(a)(1+1) = 2u(a).$$

Then $u(a) = \frac{1}{2} \sum_p M(a, p)$

so that once the utility scale is constructed, the subjective probability scale can be obtained by solving for s(p) in the equation $M(a, p) = u(a)s(p)$.

Table 2

Some Examples of Numerical Values Obtained for Certain Subjects

	Utility				Probability			
Obj. Values	30	60	90	120	.2	.4	.6	.8
Subj. 1	32	83	135	195	.35	.50	.55	.60
Subj. 2	32	68	105	137	.25	.39	.62	.72
Subj. 6	30	60	83	112	.21	.42	.58	.80
Subj. 10	30	68	90	120	.20	.39	.60	.80

The interpretation of the results is interesting but not germane to the purpose of the present report. It is only desired to point out that numerical scale values were obtained.

The procedure does raise one particularly important point from a statistical point of view. There is no provision made for, nor assumptions about, an experimental error theory. The models are investigated on a purely subject by subject basis. The only part of the study in which a statistical decision is used is in testing the additive model where the over-all proportion of inversions for each model is determined, and found to be less than 0.05 for both models. The figure, 0.05, is a statistically impressive one, but without an error theory little can be said for its significance. One is still faced with the question, under the hypothesis of additivity how many chance inversions can one expect in an obtained data matrix? If the data do satisfy the additive model perfectly there is no problem. But the vagaries of experimentation preclude that happy occurrence in most cases.

Those engaged in research involving variables which cannot be directly measured should welcome the approach to testing models and finding subjective scales provided by conjoint measurement. Theoretically all that is required is a set of empirical entities that arise from the joint application of levels from two or more variables and which can be ordered by a subject, along with a model which relates the variables under consideration. Given this empirical situation, conjoint measurement provides a method of testing whether or not the model is satisfied by the empirical data structure and of finding ordinal scales not only for the variable ordered by the subject but also for the variables contributing to the ordering. These scales, when used in the model, show how the variables contribute to the ordering given by the subject. Practically, the problem of finding these scales can be quite difficult.

Still the problem which conjoint measurement attempts to solve is real and important. Hopefully a more workable approach to the problem will be forthcoming along with an error theory which will allow known statistical procedures to be applied.

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THE CONJOINT MEASUREMENT PROBLEM

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Abstract

The simplest case of the conjoint measurement problem can be expressed: given, a set of data elements which are the combined effects of two factors, A and B, and which can be empirically ordered, find a triple of real valued functions, f_A, f_B and f defined on the marginal factors and their joint effects such that the joint effects can be expressed as the sum of the obtained factor contributions. If the factors and their joint effects can be measured in the usual Campbellian sense the problem is readily solved by analysis of variance procedures. Conjoint measurement is concerned with empirical situations in which the factors represent variables which may or may not be directly measureable and their joint effects may or may not have a numerical assignment associated with them, but at least have an empirical ordering defined on them.

The theory underlying conjoint measurement is still in its formative stages, the first axiomatization of the problem appearing in 1964(Luce and Tukey). David Krantz(1964) reformulated and extended the theory to the n-factor case. Amos Tversky(1965) extended the theory from the additive model case to general polynomial models and gave necessary and sufficient conditions for non-factorial, as well as factorial, empirical data structures to satisfy a polynomial measurement model. These developements are reviewed extensively in this report.

The practical problem of testing whether a given data structure satisfies a particular measurement model presents no great problem under the present theory. Finding the scale values for the contributing variables is a difficult problem since the theory stipulates only that these functions be monotonic. To date no family of monotonic functions rich enough to provide the required solutions has been isolated.

Another problem arises from the difficulty of applying a theory of experimental error to the conjoint measurement situation. Obtained results are on a per subject basis and the data must satisfy the model exactly. There is no provision for even a single deviation of the data from the model-prescribed order.

Heuristic examples are given to introduce the problems and some terminology introduced to help make the exposition of the theory meaningful. Approaches to finding the scale values are discussed and a numerical example from the literature presented.