A SIMPLY-SUPPORTED BARREL SHELL ROOF

by

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SYNOPSIS

The purpose of this report is to analyse and design a simply-supported barrel shell roof by the method suggested in the ASCE Manual No. 31 published by the American Society of Civil Engineers.

This paper presents the derivation of the internal stress equations by using the membrane theory. The corrective internal stress equations produced by the bending theory are shown. A discussion of the advantages of using shell structures, the development and improvement of the shell theories, and the structural behavior of the barrel shell are included. A design example is given, following the procedures suggested by the above-named manual. The tables and charts in the Manual will be used to facilitate the work of design.

INTRODUCTION

Constructing roofs for large unobstructed areas with a minimum of material has been a goal of engineers for years. The use of deep girders, trusses, arches, etc., makes it possible to attain this goal, but often the appearance of the resulting structures does not satisfy the architects. Thin concrete shells built in the shape of domes and barrel roofs offer the best possibility of achieving the architectural fashion as well as satisfying engineering demands. In recent years, both in the United States and abroad, shells have been utilized in modern architectural design and the competition in architectural showmanship has resulted in complicated shell design. As a result there is a pressing demand for basic information about shells on the part of consulting engineers who face the double task of designing the shells and of explaining them to architects who still retain their own ideas. Therefore, the analysis of thin shells has become an important part of structural analysis.

Shell construction is economical. Even though the cost of form-work and scaffolding increases rapidly with the span, most shell structures are multi-unit, so it is usually possible to re-use the same forms in constructing each unit of the structure, which makes the overall cost lower. That is why concrete shells are the favorite forms of structures adopted by engineers and architects. Shell construction is now expanding rapidly. Different forms of shell roofs can be seen all over the world. In the U. S. alone, over 12 million sq. ft. of shell roofs have already been constructed (3).*

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*Numbers in parentheses refer to items in list of references.

THE HISTORY OF SHELL ROOF CONSTRUCTION AND DESIGN THEORIES

The original work on shell theory was done by G. Lamé and E. Clapeyron of Germany (2), who in 1828 produced the "membrane analogy" in which a shell was considered capable of resisting external loads by direct stresses unaccompanied by any bending.

During the late nineteenth century, G. B. Airy and A. E. H. Love of England (5) made a more accurate analysis than the membrane analogy by taking the shearing forces and moments into consideration. Following this development, Carl Zeiss of Germany (2) produced the mathematical equations based on Love's theory for practical shell design. These formulae were made use of by Dischinger (2), who in 1923 attempted the design of a shell to cover a rectangular floor area in Germany, but the first attempt failed because the difficulties of mathematical computation became too great for solution. In the following year, a simplified version of the design was attempted successfully. In 1930, Finsterwalder of Germany (2) presented an approximate theory which involved displacements and which was proved by experiments. In 1936, H. Schorer (8) of the United States further improved Finsterwalder's equations. Since that time, shell analyses have been studied intensively by many investigators, such as Jakobsen (5), Flüger (6), and Timoshenko (1) and various improved theories have been published. The American Society of Civil Engineers published a manual (2) in 1952 in which were presented a set of practical formulae and numerical tables for cylindrical shell design. This manual is extremely useful and many designers consider it a valuable hand book for shell design.

STRUCTURAL BEHAVIOR OF A SIMPLY-SUPPORTED BARREL SHELL

Structural shells can be divided into three general classes:

1. Cylindrical shells; e. g., barrel shells (Fig. 1).

2. Shells of revolution; e. g., domes (Fig. 2).

3. Shells formed by double curves; e. g., hyperbolic paraboloids (Fig. 3).

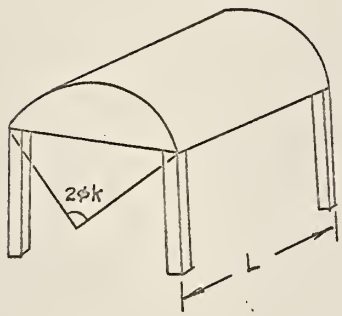


Fig. 1. Barrel Shell

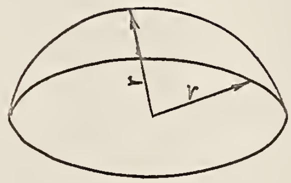


Fig. 2. Dome

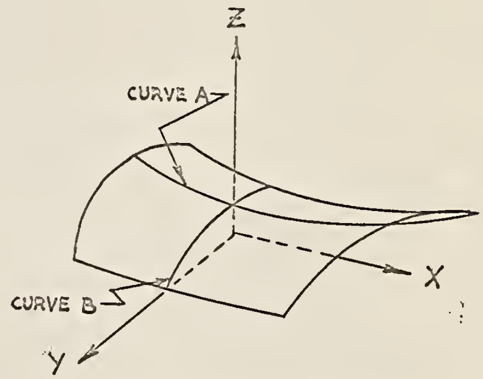
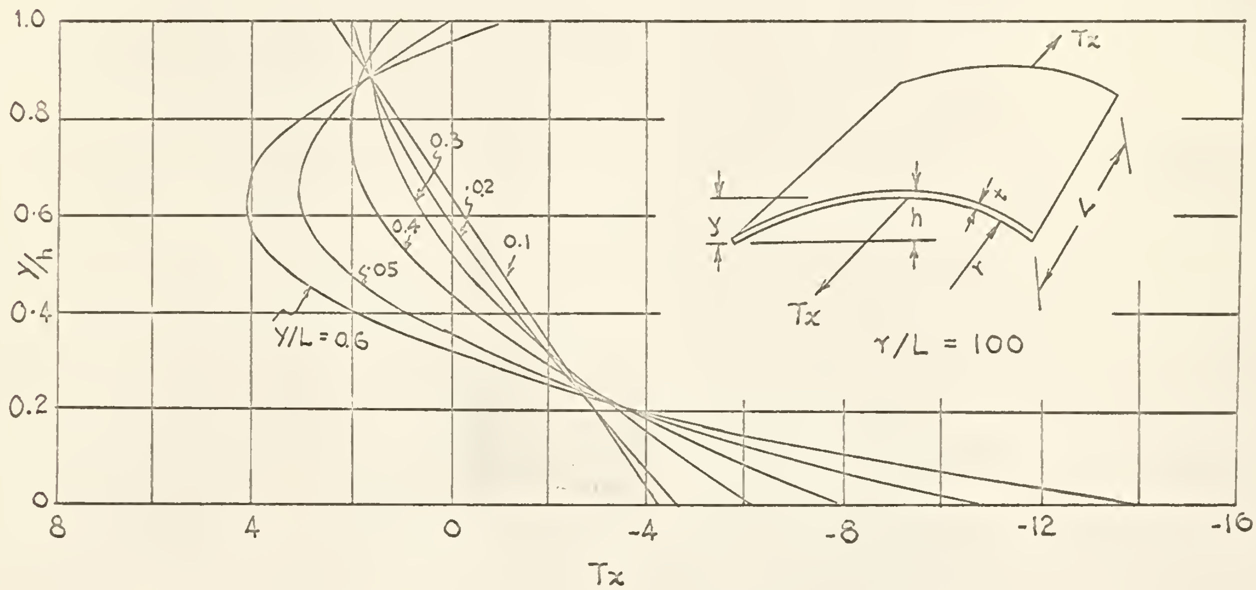


Fig. 3. Hyperbolic Paraboloid Shell

This report deals with a simply-supported shell of the first type. Although the barrel shell is strictly a three-dimensional structure, its structural behavior can be examined as a two dimensional problem by the simple beam theory. Longitudinally, the barrel shell of Fig. 1 can be considered as a beam supported by two transverse supports. This visualized beam (the shell) is different from an ordinary beam because it has a thin curved-slab cross section. When this beam is subjected to load applied along its longitudinal span, it produces bending stresses in the shell. These are called direct stresses, T_X, as if they were flexural stresses, f_c, in a beam. At any cross section in an ordinary beam these bending stresses are always linear, while in the shell, they change as the ratio of the radius to the length increases. The variation of direct stresses and the ratio of the radius to the length can be best illustrated by curves shown in Fig. 4. The



L = span length
t = thickness of slab
y = distance measured from the edge
h = height of the crown
T_x= direct stress

Fig. 4. Longitudinal stress distribution corresponds to the change of radius to length.

direct stress is linear (see the shaded area, which is equivalent to the fiber stress of an ordinary beam) when the span length of the shell exceeds five times the radius, i. e., r/L is less than 0.2. For larger values of r/L, the direct stress at the lower edge is greater than that given by an ordinary beam theory. This increment of stress at the free edge, indicating that additional forces are present at the edge, is necessary for equilibrium.

The behavior of the shell in the transverse direction exhibits a distinctive action. For a beam, the transverse loads are resisted by the development of bending and shear stresses. A plate is likened to a two-dimensional beam and resists transverse loads by two-dimensional bending and shear (Fig. 5), whereas a shell's resistance to load is through tension and compression as well as small bending moments (Fig. 6). In order to illustrate more clearly how the internal forces, due to the applied loads, operate in the transverse direction, a typical shell strip is shown in Fig. 7. The normal

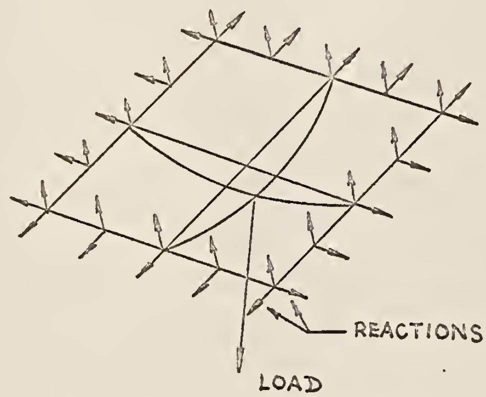


Fig. 5. Plate under applied load.

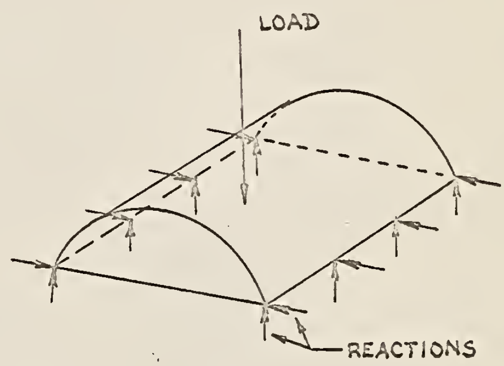


Fig. 6. Shell subjected to applied load.

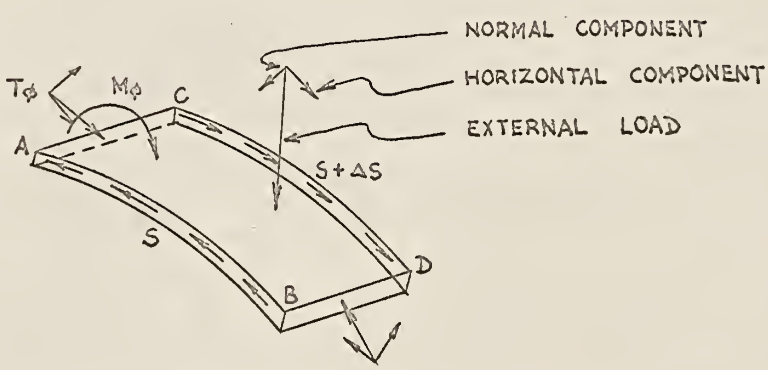


Fig. 7. Transverse action of a unit shell strip.

component of the external load is resisted by the vertical component of the transverse force, T_ϕ , on the radial sections, while the horizontal component of the external load is resisted by the shearing forces and the transverse forces. If face BD is taken at the lower edge of the shell, then the radial moment, M_ϕ , equals the difference between the moment due to external load acting on the unit strip and the moment due to the tangential shears, S. This radial moment at any point is always acting counterclockwise or negative, and in most cases is insignificant.

Besides the longitudinal and transverse action, the boundary displacements and reactions should be given much attention. The edges of the barrel shell in this report are considered as being without restrictions (free). These free edges are the major structural weaknesses in the shell, and thus many problems arise, such as cracking and deflection due to applied loads and the transmission of internal stresses along the edges. These weaknesses can be overcome either by supplying beams along the edges or by thickening the edges. The latter form of reinforcement is used in this paper.

At the ends, the barrel shell is considered to be simply supported by two end diaphragms(Fig. 8). These diaphragms are assumed to be flexible in the longitudinal direction, but rigid in resisting loads in the plane of the diaphragms.

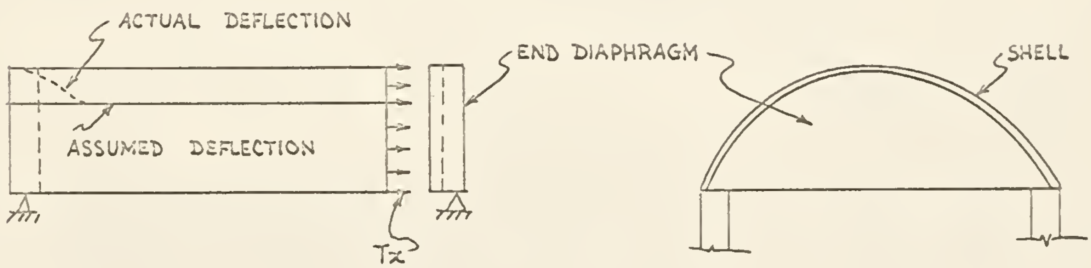


Fig. 8. Flexural deflection at supports.

METHODS OF ANALYSIS

There are three general methods that can be used in the analysis of barrel shells:

1. "Exact" solutions of the differential equations for given boundary conditions.

2. Application of the strength of materials beam theory.

3. The method outlined in ASCE Manual No. 31.

In the first method, eleven equations due to the applied load in terms of direct forces, shear forces, bending moments and displacements are first derived. By successive differential elimination and applying suitable boundary conditions, an eighth-order compatibility equation in terms of one variable is then obtained. The general solution of this compatibility equation involves eight unknown constants. Determining these eight constants requires a lot of tedious computation and consumes innumerable hours. The equations involved in this method are so complicated that it is not practical for shell design purposes.

In the second method, the shell is assumed to act as a beam of circular cross section spanning the two end diaphragms. Thus, the longitudinal moment is determined as for a simply-supported beam, and the longitudinal stresses are determined from the moment of inertia of the cross-section. Unfortunately, this method is unable to predict the transverse bending moment, M_ϕ, and is limited to the analysis of long shells.

The two above-mentioned methods have vital weaknesses. They are either complicated or uncertain. Thus, a systematic and complete analysis is desired. The ASCE Manual No. 31 presents a method which is considered sound and easy for barrel shell design. It is the purpose of this paper to present the ASCE method.

The ASCE Manual states, "Contrary to a popular misconception, the analysis of a concrete shell entails no more computation and time than the analysis by elastic weights of an ordinary indeterminate structure." In this method, the procedures of analysis of a thin barrel shell are the same as the analysis of an ordinary indeterminate structure. The preliminary step (called "membrane analysis") is that the surface load is assumed to be transmitted to the supports by membrane stresses (direct stresses) only. The membrane stresses cause displacements and reactions to occur along the longitudinal edges of the shell, which do not comply with the boundary conditions. This is equivalent to the first step in the analysis of an indeterminate structure, that is removing some redundant forms to make the structure determinate, thereby allowing rotation and displacement to occur at the supports. To satisfy the boundary displacements and reactions, the following step is to apply line loads along the longitudinal edges of the shell by using the bending theory. Due to the line load not only direct stresses and shearing stresses appear, but bending moments will also be produced. The second step is similar to applying reactions to the ordinary indeterminate structure so as to bring the boundary back to its original position. Finally, the stresses induced by the line loads are added to the previously calculated membrane stresses to obtain the final stresses. This is the same as the last step in the analysis of the indeterminate structure, that is summing the stresses found in the determinate state and those produced by the redundants.

Thus, the procedures of analysis of a simply-supported barrel shell with free edges can be summarized as follows:

1. Find the membrane stresses (membrane theory).

2. Correct the membrane stresses for effects produced by the line load acting along the free edges (bending theory).

3. Add the corrections and the membrane stresses to obtain the final stresses.

INTERNAL STRESSES DUE TO SURFACE LOAD SOLVED BY MEMBRANE THEORY

In the membrane theory, all forces are assumed to lie in the shell surface and no bending moments are deemed to exist. Thus, only three forces: T_X, T_ϕ and S are assumed to act on a shell element as shown in Fig. 9. The direct force component T_X and T_ϕ, measured in pounds per unit length, are positive when they create tension. The shearing force, S, also measured in pounds per unit length, is positive when it produces tension in the diagonal direction of increasing values of X and ϕ. The surface load R, in pounds per unit area acting radially, is considered positive when it acts outwardly. The surface load Φ, likewise measured in pounds per unit area, is positive when it acts clockwise in tangential direction.

The equilibrium equations of the element in three directions, radial, tangential and transverse, are:

In the radial direction (Fig. 9b)

$$(T_\phi + \frac{\partial T_\phi}{\partial \phi} d\phi)\, dX \sin\frac{d\phi}{2} + T_\phi \, dX \sin\frac{d\phi}{2} - Rrd\phi \, dX$$

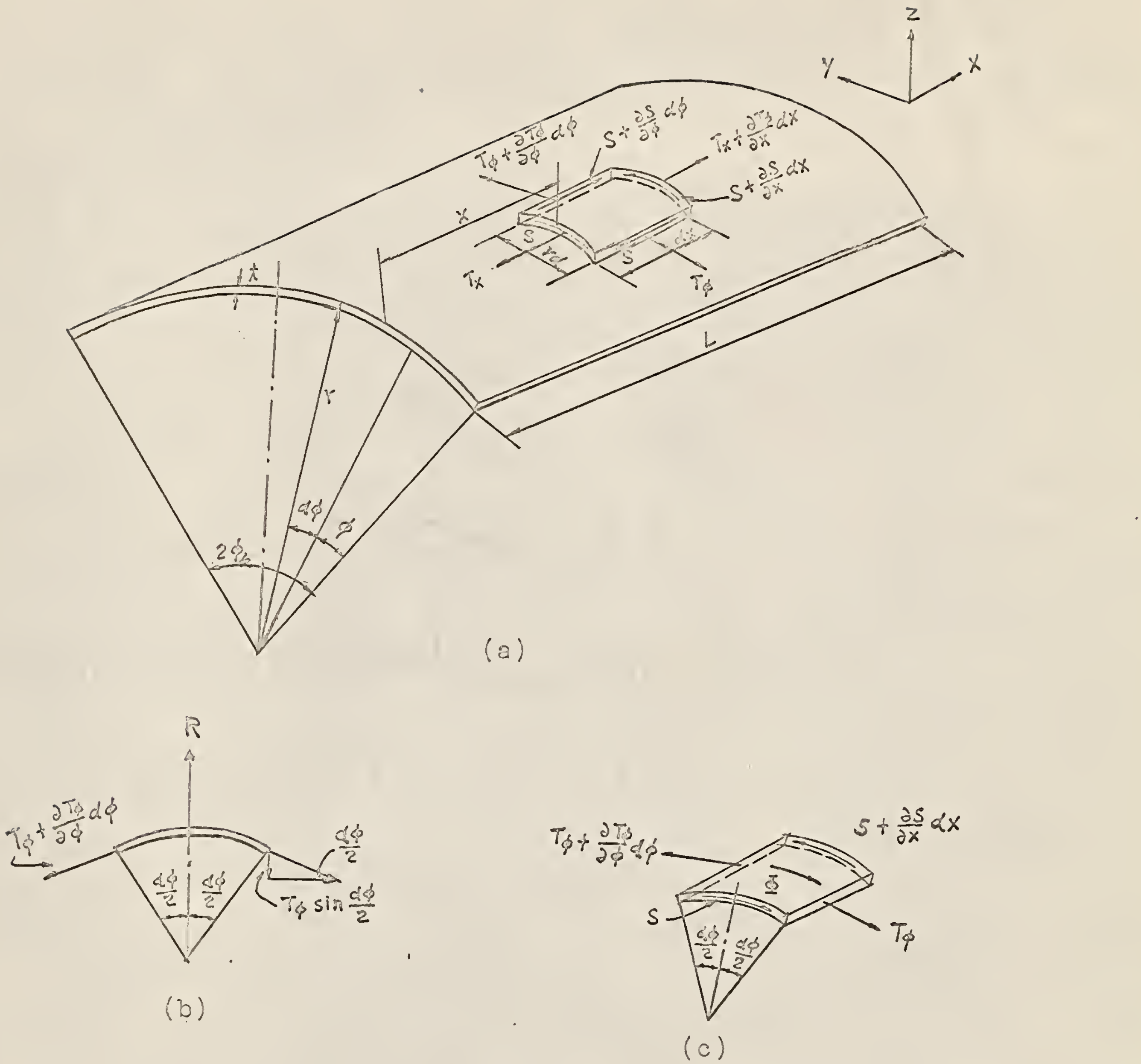
$$= 0 \text{ -- } 1a$$

taking $\sin\frac{d\phi}{2} = \frac{d\phi}{2}$, Eq. 1a becomes

$$(T_\phi + T_\phi + \frac{\partial T_\phi}{\partial \phi} d\phi)\, dX \frac{d\phi}{2} - Rrd\phi\, dX = 0 \text{ ----------- } 1b$$

Ignoring the higher order differential term $\frac{1}{2} \frac{T}{\quad} d$ and canceling $dXd\phi$, we obtain

$$T_\phi - Rr = 0$$



where

T_ϕ = the direct force component in the transverse direction.

T_x = the direct force component in the longitudinal direction.

s = the tangential shearing force.

L = length of the shell between supports.

r = radius of the shell.

t = thickness of the shell.

x = longitudinal distance measured from the left support.

ϕ = angle measured from the right edge of the shell.

ϕ_k = angle substended by the edge of the shell measured from the centerline axis.

R = surface load acting normal to the shell.

Φ = surface load acting parallel to the shell.

Fig. 9. Membrane forces per unit length acting upon a differential element.

or

$$T_\phi = Rr \text{ --------------------------------- } 1c$$

In the tangential direction (Fig. 9c)

$$(S + \frac{\partial s}{\partial X} dX)\, rd\phi - Srd\phi + (T_\phi + \frac{\partial T_\phi}{\partial \phi} d\phi) \cos \frac{d\phi}{2} dX$$

$$- T_\phi \cos \frac{d\phi}{2} dX - \Phi\, rd\phi\, dX = 0 \text{ ------------------- } 2a$$

taking $\cos \frac{d\phi}{2} = 1$, Eq. 2a may be written as

$$\frac{\partial s}{\partial X} rd\phi\, dX + \frac{\partial T}{\partial \phi} d\phi\, dX - \Phi\, rd\phi dX = 0$$

or

$$\frac{\partial s}{\partial X} r + \frac{\partial T_\phi}{\partial \phi} - \Phi r = 0 \text{ ---------------------- } 2b$$

Integrating Eq. 2b to obtain S, we get

$$S = - \frac{1}{r} \int \frac{\partial T_\phi}{\partial \phi} dX + \int \Phi\, dX + f_1\, (\phi) \text{ --------- } 2c$$

In the longitudinal direction (Fig. 8a)

$$(T_X + \frac{\partial T_X}{\partial X} dX)\, rd\phi - T_X rd\phi + (S + \frac{\partial s}{\partial \phi} d\phi)\, dX - SdX = 0 \text{ -- } 3a$$

Simplifying Eq. 3a we find that

$$\frac{\partial T_X}{\partial X} rd\phi\, dX + \frac{\partial s}{\partial \phi} d\phi\, dX = 0$$

or

$$\frac{\partial T_X}{\partial X} r + \frac{\partial s}{\partial \phi} = 0 \text{ ------------------------------ } 3b$$

Integrating Eq. 3b to obtain T_X, gives

$$T_X = -\frac{1}{r}\int \frac{\partial s}{\partial \phi} + f_2(\phi) \text{ ------------------------------------ } 3c$$

The forces T_ϕ , S and T_X can be obtained if we assign proper values to R and Φ . The arbitrary functions $f_1(\phi)$ and $f_2(\phi)$ in equations 2c and 3c represent functions of the variable ϕ, and their values depend on the boundary conditions.

In general, the load on the shell is assumed symmetrical about the crown in the transverse direction and is assumed to vary as $\sin \frac{n\pi X}{L}$ or $\cos \frac{n\pi X}{L}$ in the longitudinal direction (n = 1, 3, 5, 7, etc.) The reason for using a sinusoidal load instead of an ordinary uniform load is that the second principal step in shell analysis involving the application of corrective edge loads cannot be accomplished unless these edge loads are sinusoidal in shape. Therefore, to achieve compatibility of stresses produced by both edge and surface loads, the surface load is also assumed to be sinusoidal. This sinusoidal load can be represented by a series which is called "Fourier series." The Fourier series for a uniform load equals:

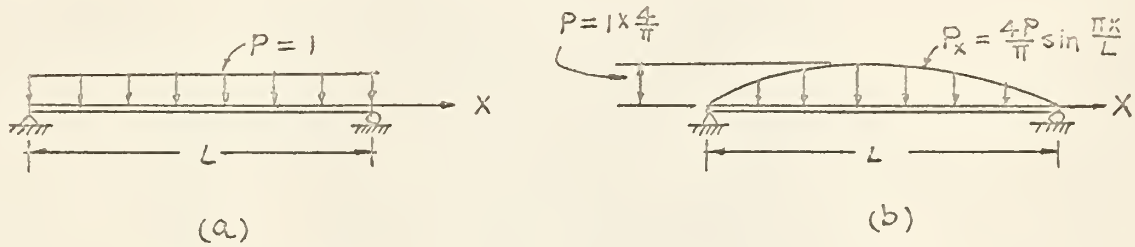
$$P_X = \frac{4P}{\pi}\left(\sin \frac{\pi X}{L} + \frac{1}{3}\sin \frac{3\pi X}{L} + \frac{1}{5}\sin \frac{5\pi X}{L} + \text{------} + \frac{1}{n}\sin \frac{n\pi X}{L}\right) \text{ ------------------------------------ } 4a$$

or

$$P_X = \frac{4P}{\pi}\sum_{n=1,3,5} \frac{1}{n}\sin \frac{n\pi X}{L} \text{ ------------------------------ } 4b$$

For common practical design purposes, only the first term of the series is used.

In order to compare the flexural effects of both types of loads, two simple beams supporting sinusoidal and uniform loads are shown in Fig. 10a.



Shear (uniform): $V_X = 0 \text{ or } L = \frac{P}{2} = \frac{1}{2} = 0.5$

Shear (sinusoidal): $V_X = 0 \text{ or } L = \frac{PL}{\pi} \times \frac{4}{\pi} \cos \frac{\pi X}{L} = \frac{1}{\pi} \times \frac{4}{\pi}$

$= 0.405$

Moment (uniform): $M_X = \frac{L}{2} = \frac{1}{8} PL^2 = 0.125L^2$

Moment (sinusoidal): $M_X = \frac{L}{2} = \frac{PL^2}{\pi^2} \times \frac{4}{\pi} \sin \frac{\pi X}{L} = \frac{L^2}{\pi^2} \times \frac{4}{\pi}$

$= 0.128L^2$

Deflection (uniform): $Y_X = \frac{L}{2} = \frac{5}{384} \frac{PL^4}{EI} = 0.013 \frac{L^4}{EI}$

Deflection (sinusoidal): $Y_X = \frac{L}{2} = \frac{P}{EI} \frac{L^4}{\pi^4} \times \frac{4}{\pi} \sin \frac{\pi X}{L} \times \frac{1}{\pi^4}$

$\times \frac{4}{\pi} \times \frac{P}{EI} = 0.013 \frac{L^4}{EI}$

Fig. 10. A simple beam subjected to (a) uniform load and (b) sinusoidal load.

Results of this comparison indicate that good agreement is obtained for moments and deflections of beams with only the first term of the Fourier series. This same trend holds for the shell.

To obtain the membrane stresses due to surface loads, two types of loading are investigated as follows:

A. Dead load membrane stresses.

From Fig. 11c, we find that

$$R = - P_d \cos(\phi_k - \phi) \sin \frac{n\pi X}{L}$$

and

$$\Phi = - P_d \sin(\phi_k - \phi) \sin \frac{n\pi X}{L}.$$

Substituting the value of R in Eq. 1a, we obtain

$$T_\phi = - P_d\, r \cos(\phi_k - \phi) \sin \frac{n\pi X}{L} \text{ ----- } \quad 5a$$

which, when substituted in Eq. 2c, gives

$$S = - \frac{1}{r} \int - P_d\, r \frac{\partial}{\partial \phi} \cos(\phi_k - \phi) \sin \frac{n\pi X}{L}\, dX + \int P_d$$

$$\sin(\phi_k - \phi) \sin \frac{n\pi X}{L}\, dX + f_1(\phi) \text{ ------------------- } \quad 6a$$

Differentiating Eq. 6a with respect to and integrating the resulting expressions with respect to X,

$$S = P_d\, r \left(\frac{1}{r}\right) \left\{ \left[\sin(\phi_k - \phi) \left(- \frac{L}{n\pi} \cos \frac{n\pi X}{L}\right)\right] + \left[\sin(\phi_k - \phi) \right.\right.$$

$$\left.\left. \left(- \frac{L}{n\pi} \cos \frac{n\pi X}{L}\right)\right]\right\} + f_1(\phi)$$

or

$$S = - P_d\, r \left(\frac{L}{r}\right) \left(\frac{2}{n\pi}\right) \left[\sin(\phi_k - \phi) \cos \frac{n\pi X}{L}\right] + f_1(\phi) \text{ ---- } \quad 6b$$

When symmetrically loaded, S = 0 when $X = \frac{L}{2}$ and n = 1, it is found from Eq. 6b that

$$f_1(\phi) = 0$$

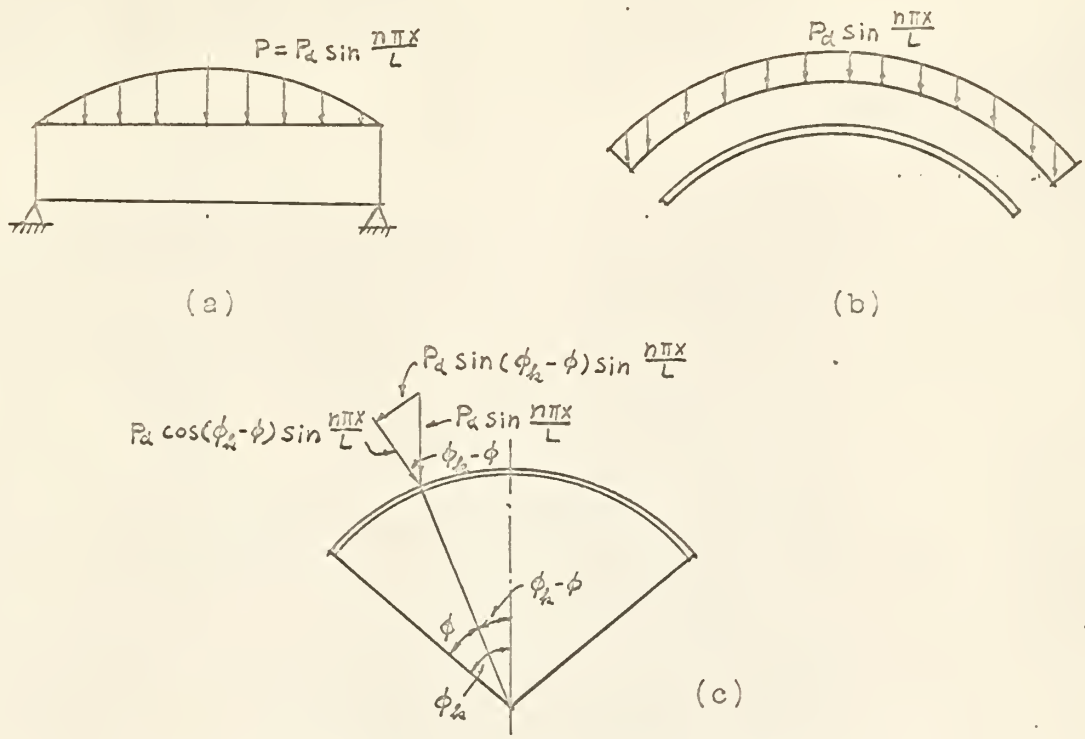


Fig. 11. A simply supported barrel shell subject to dead load.

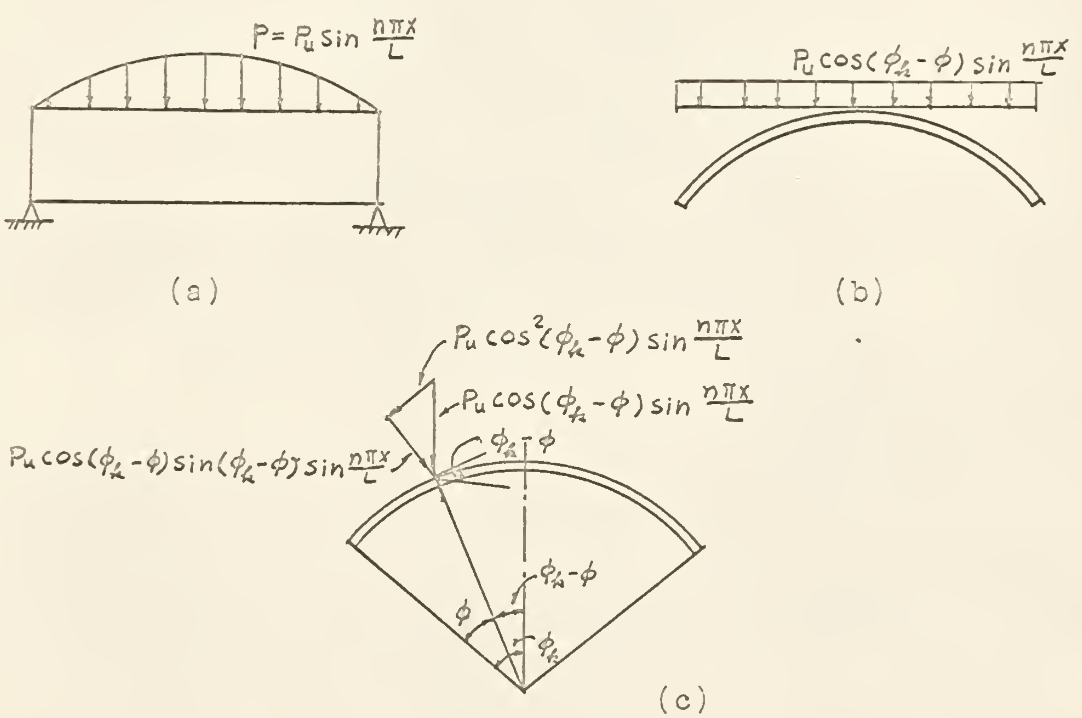


Fig. 12. A simply supported barrel shell subject to uniform live load.

Hence,

$$S = - P_d \ r \ (\frac{L}{r}) \ (\frac{2}{n\pi}) \left[\sin \ (\phi_k - \phi) \ \cos \frac{n\pi X}{L} \right] \text{------------} \quad 6c$$

Substituting Eq. 6c in Eq. 3c, we get

$$T_X = - \frac{1}{r} \int - P_d \ r \ (\frac{L}{r}) \ (\frac{2}{n\pi}) \left[\sin \ (\phi_k - \phi) \ \cos \frac{n\pi X}{L} \ dX \right]$$

$$+ f_2 \ (\phi) \text{ --- } \quad 7a$$

and after integrating with respect to X, Eq. 7a becomes

$$T_X = P_d \ r \ (\frac{L}{r^2}) \ (\frac{2}{n\pi}) \left[\cos \ (\phi_k - \phi) \ (- \frac{L}{n\pi} \ \sin \frac{n\pi X}{L}) \right]$$

$$+ f_2 \ (\phi)$$

or

$$T_X = - P_d \ r \ (\frac{L}{r})^2 \ (\frac{2}{n^2 \pi^2}) \ \cos \ (\phi_k - \phi) \ \sin \frac{n\pi X}{L} + f_2 \ (\phi) \text{--} \quad 7b$$

For a simply supported shell, $T_X = 0$ at the support when $X = 0$ or $X = L$, Eq. 7b yields

$$f_2 \ (\phi) = 0.$$

Thus:

$$T_X = - P_d \ r \ (\frac{L}{r})^2 (\frac{2}{n^2 \pi^2}) \ \cos \ (\phi_k - \phi) \ \sin \frac{n\pi X}{L} \text{ ----------} \quad 7c$$

B. Uniform live load membrane stresses.

From Fig. 12c, it is seen that

$$R = - P_u \ \cos^2 \ (\phi_k - \phi) \ \sin \frac{n\pi X}{L}$$

and

$$\Phi = P_u \cos(\phi_k - \phi) \sin(\phi_k - \phi) \sin\frac{n\pi X}{L}.$$

With the value of R, Eq. 1a becomes

$$T_\phi = - P_u r \cos^2(\phi_k - \phi) \sin\frac{n\pi X}{L} \text{ -------------------- } 8a$$

Substituting the T value in Eq. 2c yields

$$S = -\frac{L}{r}\int - P_u r \frac{\partial}{\partial\phi} \cos^2(\phi_k - \phi) \sin\frac{n\pi X}{L} dX + \int P_u \cos$$

$$(\phi_k - \phi) \sin(\phi_k - \phi) \sin\frac{n\pi X}{L} dX + f_1(\phi) \text{ ----------- } 9a$$

Differentiating Eq. 9a with respect to ϕ and integrating the resulting expressions with respect to X, we find

$$S = P_u r \left(\frac{L}{r}\right)\left\{\int 2\cos(\phi_k - \phi)\sin(\phi_k - \phi)\sin\frac{n\pi X}{L} dX + \right.$$

$$\left.\int \cos(\phi_k - \phi)\sin(\phi_k - \phi)\sin\frac{n\pi X}{L}\right\} + f_1(\phi)$$

or

$$S = - P_u r \left(\frac{L}{r}\right) \frac{3}{n\pi} \cos(\phi_k - \phi) \sin(\phi_k - \phi) \cos\frac{n\pi X}{L}$$

$$+ f_1(\phi) \text{ -- } 9b$$

Because of symmetry, S = 0 at $X = \frac{L}{2}$, it is found by Eq. 9b that

$$f_1(\phi) = 0$$

Therefore:

$$S = - P_u r \left(\frac{L}{r}\right) \left(\frac{3}{n\pi}\right) \cos(\phi_k - \phi) \sin(\phi_k - \phi) \cos\frac{n\pi X}{L} \text{ — } 9c$$

Then by substituting Eq. 9c in Eq. 3c, the following value of T_X can be obtained.

$$T_X = -\frac{L}{r}\int -P_u\, r\, (\frac{L}{r})\, (\frac{3}{n\pi})\, \frac{\partial}{\partial\phi} \cos(\phi_k - \phi)\, \sin(\phi_k - \phi)$$

$$\cos\frac{n\pi X}{L}\, dX + f_2(\phi) \text{ --------------------------------- } 10a$$

Differentiating Eq. 10a with respect to ϕ gives

$$T_X = P_u\, r\, (\frac{3}{n\pi})\, \frac{L}{r^2}\int [\cos^2(\phi_k - \phi) - \sin^2(\phi_k - \phi)]\, \cos\frac{n\pi X}{L}$$

$$dX + f_2(\phi)$$

and integrating T_X with respect to X, we get

$$T_X = -P_u\, r\, (\frac{3}{n^2\pi^2})\, (\frac{L}{r})^2 [\cos^2(\phi_k - \phi) - \sin^2(\phi_k - \phi)$$

$$\sin\frac{n\pi X}{L}] + f_2(\phi) \text{ ------------------------------------- } 10b$$

For simple supports, $T_X = 0$ at $X = 0$ or $X = L$. Hence, from Eq. 10b

$$f_2(\phi) = 0$$

Thus:

$$T_X = -P_u\, r\, (\frac{3}{n^2\pi^2})\, (\frac{L}{r})^2 [\cos^2(\phi_k - \phi) - \sin^2(\phi_k - \phi)$$

$$\sin\frac{n\pi X}{L}] \text{ -- } 10c$$

Equations 5a, 6c, 7c, 8a, 9c and 10c may be rearranged by taking $n = 1$ and written as follows:

$$T_\phi = P_d\, r \times (-) \text{ coefficient} \times \sin\frac{\pi X}{L} \text{ ------------------- } 5b$$

$$S = P_d\, r\, [(\frac{L}{r}) \times (-) \text{ coefficient}]\, \cos\frac{\pi X}{L} \text{ --------------- } 6d$$

$$T_X = P_d\, r\, [(\frac{L}{r})^2 \times (-) \text{ coefficient}]\, \sin\frac{\pi X}{L} \text{ ------------- } 7d$$

$$T_\phi = P_u\, r \times (-) \text{ coefficient} \times \sin \frac{\pi X}{L} \text{ ------------------ 8b}$$

$$S = P_u\, r \left[\left(\frac{L}{r}\right) \times (-) \text{ coefficient} \right] \cos \frac{\pi X}{L} \text{ --------------- 9d}$$

$$T_X = P_u\, r \left[\left(\frac{L}{r}\right)^2 \times (-) \text{ coefficient} \right] \sin \frac{\pi X}{L} \text{ -------------- 10d}$$

For the purpose of completeness, the displacements, ΔV and ΔH, in the tangential and radial direction, due to the surface loads are listed as follows:

$$\Delta V = - P_d\, r \frac{r}{Et} \left(\frac{L}{r}\right)^4 \frac{2}{n^4 \pi^4} \left[1 + 2 n^2 \pi^2 \left(\frac{r}{L}\right)^2 \right] \sin (\phi_k - \phi)$$

$$\sin \frac{n \pi X}{L} \text{ -- 11a}$$

$$\Delta H = - P_d\, r \frac{r}{Et} \left(\frac{L}{r}\right)^4 \frac{2}{n^4 \pi^4} \left[1 + 2 n^2 \pi^2 \left(\frac{r}{L}\right)^2 + \frac{n^4 \pi^4}{2} \left(\frac{r}{L}\right)^4 \right]$$

$$\cos (\phi_k - \phi) \sin \frac{n \pi X}{L} \text{ ---------------------------- 12a}$$

and

$$\Delta V = - P_u\, r \frac{r}{Et} \left(\frac{L}{r}\right)^4 \frac{6}{n^4 \pi^4} \cos (\phi_k - \phi) \sin (\phi_k - \phi)$$

$$\left[2 + \left(\frac{n \pi r}{L}\right)^2 \right] \sin \frac{n \pi X}{L} \text{ -------------------------- 13a}$$

$$\Delta H = - P_u\, r \frac{r}{Et} \left(\frac{L}{r}\right)^4 \frac{6}{n^4 \pi^4} \left\{ \left[\cos^2 (\phi_k - \phi) - \sin^2 (\phi_k - \phi) \right. \right.$$

$$\left. \left. \times \left[2 + \left(\frac{n \pi X}{L}\right)^2 + \frac{1}{6} \left(\frac{n \pi r}{L}\right)^4 \cos (\phi_k - \phi) \right\} \sin \frac{n \pi X}{L} \text{ -- 14a}$$

These displacement equations are derived from the bending theory which will be discussed in the next section. The formulae for the displacements are inserted here because they are produced by surface loads. Their simplified forms may be written with (n = 1):

$$\Delta V = P_d\ r\ \frac{L^4}{r^3tE} \left[\left(\frac{2r}{\pi L}\right)^2 + \frac{2}{\pi^4} + \left(\frac{r}{L}\right)^4 \times (+)\ \text{coefficient}\right]$$

$$\sin \frac{\pi X}{L} \text{ -- } 11b$$

$$\Delta H = P_d\ r\ \frac{L^4}{r^3tE} \left[\left(\frac{r}{L}\right)^4 \times (+)\ \text{coefficient}\right] \sin \frac{\pi X}{L} \text{ -------- } 12b$$

and

$$\Delta V = P_u\ r\ \frac{L^4}{r^3tE} \left[1 + \frac{1}{2} \left(\frac{\pi r}{L}\right)^2 + \frac{1}{12} \left(\frac{\pi r}{L}\right)^4 \right] \times (+)\ \text{coefficient}$$

$$\times \sin \frac{\pi X}{L} \text{ -- } 13b$$

$$\Delta H = P_u\ r\ \frac{L^4}{r^3tE} \left\{ \left(\frac{r}{t}\right)^4 \times (+)\ \text{coefficient} + \left[1 + \frac{1}{2} \left(\frac{\pi r}{L}\right)^2 \right.\right.$$

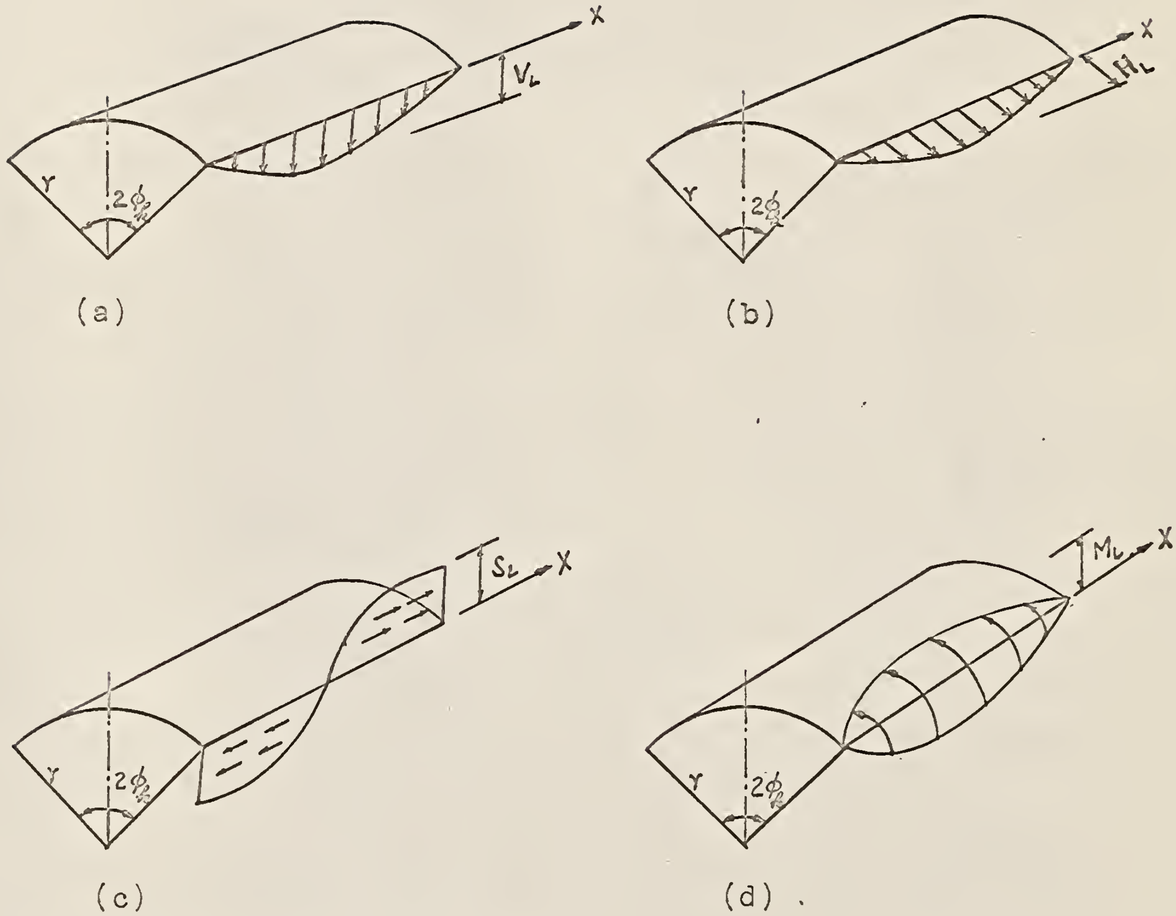
$$\left.\left. + \frac{1}{12} \left(\frac{\pi r}{L}\right)^4 \right] \times (-)\ \text{coefficeint} \right\} \sin \frac{\pi X}{L} \text{ ----------- } 14b$$

Equations 5b, 6d, 7d, 8b, 9d, 10d, 11b, 12b, 13b and 14b mentioned above are the design formulae for the simply-supported barrel shell. The coefficients of the formulae depend on the variable $(\phi_k - \phi)$ and they are given in Table 1B of the ASCE Manual, No. 31.

INTERNAL STRESSES DUE TO CORRECTIVE LINE LOAD SOLVED BY THE BENDING THEORY

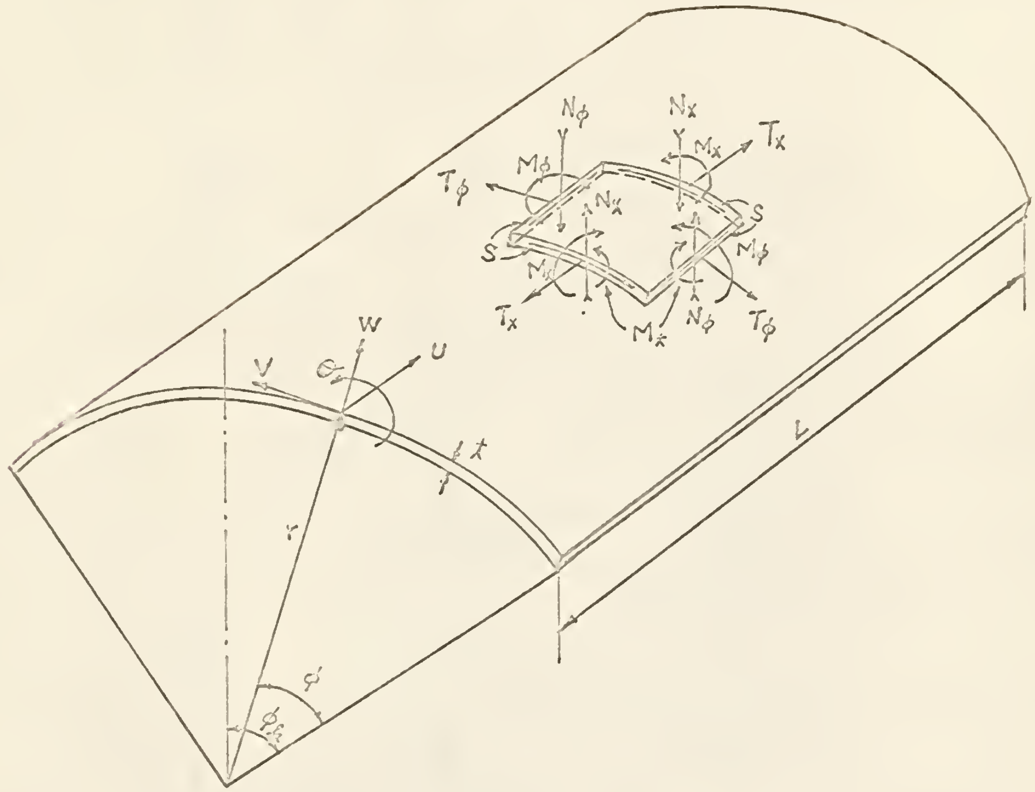
The previous membrane analysis gives transverse and shearing forces as well as displacements occurring along the longitudinal edge. In other words, the statical requirements of the boundary conditions of the shell are unfulfilled. To satisfy the requirements of statics, line loads must be applied.

For a single barrel with free edges, subject to a symmetrical load, the corrective line loads will consist of a tangential transverse force, a radial shearing force, a longitudinal force and a moment normal to the edge. The first two mentioned forces have been resolved into vertical, V_L, and horizontal, H_L, line loads as shown in Fig. 13a and 13b; the other two, shear force, S, and moment, M_L, are shown in Fig. 13c and 13d. These line forces must be equal and opposite in direction to those boundary forces, so that the boundary conditions of the free edges are consistent. The applied line loads produce not merely direct forces and bending moments but also deformations as well (Fig. 14). In general, only T_ϕ, T_X, S and M_ϕ will be considered. The value of the other bending moments and force components are omitted because they are insignificant. The determination of the internal forces produced by the line loads makes use of the exact solution of differential equations (bending theory). As mentioned before, it is not the aim of this paper to present this theory, but for the sake of the design example given in the following section, equations of T_ϕ, T_X, S and M_ϕ, which were found by the bending theory, are as follows:



(a) Due to vertical line load.
(b) Due to horizontal line load.
(c) Due to shear line load.
(d) Due to moment line load.

Fig. 13. The line loads acting on the longitudinal edge.



where

M_ϕ = bending moment on the radial face.

M_x = bending moment on the transverse face.

M_t = torsional moment.

N_ϕ = the radial shearing force on the radial face.

N_x = the radial shearing force on the longitudinal face.

U = the displacement of the shell in the longitudinal direction.

V = the displacement of the shell in the tangential direction.

W = the displacement of the shell in the radial direction.

Q = rotation of the shell.

Fig. 14. Internal stresses produced by the line loads.

(1) Vertical line load, V_L:

$T_X = V_L [(\frac{L}{r})^2 \text{ x coefficient (1)}] \sin \frac{\pi X}{L}$ ---------------- 15a

$S = V_L [\frac{L}{r} \text{ x coefficient (2)}] \cos \frac{\pi X}{L}$ ---------------- 15b

$T_\phi = V_L \text{ x coefficient (3) x } \sin \frac{\pi X}{L}$ ---------------- 15c

$M_\phi = V_L [r \text{ x coefficient (4)}] \sin \frac{\pi X}{L}$ ---------------- 15d

(2) Horizontal line load, H_L:

$T_X = H_L [(\frac{L}{r})^2 \text{ x coefficient (5)}] \sin \frac{\pi X}{L}$ ---------------- 16a

$S = H_L [\frac{L}{r} \text{ x coefficient (6)}] \cos \frac{\pi X}{L}$ ---------------- 16b

$T_\phi = H_L \text{ x coefficient (7) x } \sin \frac{\pi X}{L}$ ---------------- 16c

$M_\phi = H_L [r \text{ x coefficient (8)}] \sin \frac{\pi X}{L}$ ---------------- 16d

(3) Shear line load, S_L:

$T_X = S_L [(\frac{L}{r})^2 \text{ x coefficient (9)}] \sin \frac{\pi X}{L}$ ---------------- 17a

$S = S_L [\frac{L}{r} \text{ x coefficient (10)}] \cos \frac{\pi X}{L}$ ---------------- 17b

$T_\phi = S_L \text{ x coefficient (11) x } \sin \frac{\pi X}{L}$ ---------------- 17c

$M_\phi = S_L [r \text{ x coefficient (12)}] \sin \frac{\pi X}{L}$ ---------------- 17d

(4) Moment line load, M_L:

$T_X = \frac{M_L}{r} [(\frac{L}{r})^2 \text{ x coefficient (13)}] \sin \frac{\pi X}{L}$ ---------------- 18a

$S = \frac{M_L}{r} [\frac{L}{r} \text{ x coefficient (14)}] \cos \frac{\pi X}{L}$ ---------------- 18b

$T_\phi = \frac{M_L}{r} \text{ x coefficient (15) x } \sin \frac{\pi X}{L}$ ---------------- 18c

$M_{\phi} = M_L$ x coefficient (16) x $\sin \frac{\pi X}{L}$ -------------------- 18d

The equations of the deformations ΔV and ΔH in the W and V directions were presented previously.

The coefficients for determining internal forces and moments of the above formulae are given in Tables 2A and 2B of ASCE Manual, No. 31. Tables 2A and 2B are developed for the cases when symmetrical loads are applied simultaneously at both longitudinal edges, when $\frac{r}{L}$ is less than 0.6 and when n is taken as 1. As $\frac{r}{L}$ increases beyond 0.6, analysis requires only one application of the line loads at the near edge and neglect of the far edge effects. When $\frac{r}{L}$ is 0.6 or less, the line loads applied along one edge create significant stresses and displacements at the opposite edge. To eliminate the far edge effects, another set of loads has to be applied at the far edge; this, in turn, creates inconsistent forces at the near edge. This is the reason for applying line loads on both edges as in Tables 2A and 2B, so that the repetitive corrections are eliminated. Tables 3A and 3B cover shells whose ratio $\frac{r}{L}$ exceeds 0.6, and for which the first two terms of the Fourier expansion of the loading have been computed, one with n = 1 and one with n = 3.

DESIGN OF A SIMPLY-SUPPORTED SHELL ROOF

A simply-supported, single barrel shell roof is designed as an example.

Dimensions of the shell:

Length of shell, $L = 83$ ft.

Radius of shell, $r = 33$ ft.

Half of shell angle, $\phi_k = 40$ deg.

Thickness of shell, $t = 4$ in.*

Width of shell, $2R \sin \phi_k = 42.5$ ft.

Rise of shell $= 2R \sin^2 \frac{2\phi k}{4} = 7.72$ ft.

Design data:

Dead load (measured along the curve), $P_d = \frac{4}{12} \times 150 = 50$ psf.

Live load (horizontal projection), $P_u = 30$ psf.

$fc = 3000$ psi.

$n = 9$.

$fs = 20{,}000$ psi.

$V = 600$ psi.

$\frac{L}{r} = \frac{83}{33} = 2.518$

$(\frac{L}{r})^2 = 6.3$

---

*The minimum shell thickness is determined by the reinforcement, the cover required and the maximum size of aggregate. Based on the A.C.I. building code, the following minimum thickness would be obtained.

Reinforcement cover 2 x 3/4 inch	= 1.50 inch
Double mesh reinforcement 2 x 2 x 3/8 inch	= 1.50 inch
Minimum spacing between reinforcement	= 1.00 inch
	4.00 inch

The minimum deformed bar size is #3 and hence for 3/4 inch gravel the minimum required thickness remains about 4 inch.

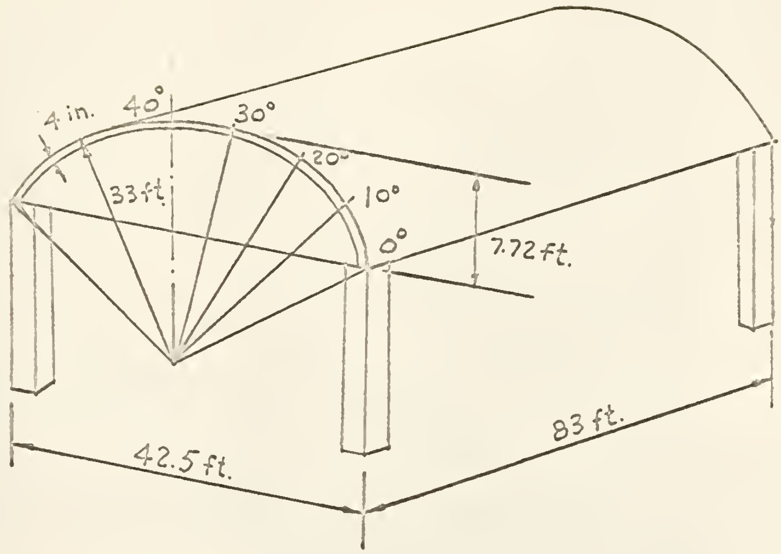


Fig. 15. A simply-supported single barrel shell.

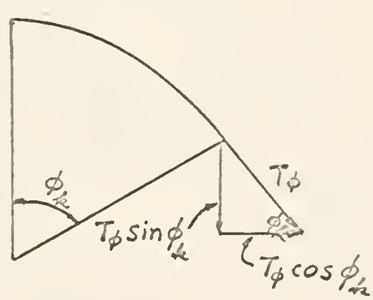


Fig. 16. Components of transverse force.

$\frac{r}{t} = \frac{33 \times 12}{4} = 99$ (say 100)

$\frac{r}{L} = \frac{33}{83} = 0.398$ (say 0.4)

Code:

ACI Building Code (318-63).

Design Procedure. The design example is a simply-supported single barrel shell. By simply-supported, we mean that the shell, supported by two end diaphragms, offers no flexural resistance. Since the shell structure is symmetrical, only one-half of it will be considered. The design procedures follow the analysis procedures on p. 9.

Step 1. In the membrane analysis surface loads, dead and live, induce three forces throughout the shell: T_X, the force in the longitudinal direction; T_ϕ, the force in the transverse direction; and S, the tangential shear force. For maximum values, T_X and T_ϕ are found at mid-span or at $X = \frac{L}{2}$, while S is found at the supports where X = 0 or X = L. The coefficients are taken from Table 1B on page 14 of the ASCE Manual No. 31.

For dead load, when $X = \frac{L}{2}$, Eq. 5b becomes

$$T_\phi = P_d \, r \times (-) \text{ coef.}$$

or

$$T_\phi = \frac{4}{\pi}^{*} \times 50 \times 33 \times (-) \text{ coef.}$$
$$= 2,100 \times (-) \text{ coef. lb. per ft.}$$

---

*See p. 15, Eq. 4b.

At $\phi = 40^\circ$ $T_\phi = 2{,}100 \times 1.000 = -2{,}100$ lb. per ft.

$\phi = 30^\circ$ $T_\phi = 2{,}100 \times -0.9848 = -2{,}065$ lb. per ft.

$\phi = 20^\circ$ $T_\phi = 2{,}100 \times -0.9397 = -1{,}970$ lb. per ft.

$\phi = 10^\circ$ $T_\phi = 2{,}100 \times -0.8660 = -1{,}820$ lb. per ft.

$\phi = 0^\circ$ $T_\phi = 2{,}100 \times -0.7660 = -1{,}610$ lb. per ft.

When X = 0 or X = L, Eq. 6d gives

$$S = P_d \, r \, (\frac{L}{r}) \times (-) \text{ coef.}$$

or

$$S = \frac{4}{\pi} \times 50 \times 33 \times 2.518 \times (-) \text{ coef.}$$

$$= 5{,}270 \times (-) \text{ coef.}$$

At $\phi = 40^\circ$ $S = 5{,}270 \times 0 = 0$

$\phi = 30^\circ$ $S = 5{,}270 \times -0.1105 = -584$ lb. per ft.

$\phi = 20^\circ$ $S = 5{,}270 \times -0.2178 = -1{,}147$ lb. per ft.

$\phi = 10^\circ$ $S = 5{,}270 \times 0.3183 = -1{,}680$ lb. per ft.

$\phi = 0^\circ$ $S = 5{,}270 \times -0.4092 = -2{,}160$ lb. per ft.

When $X = \frac{L}{2}$, Eq. 7d can be written as

$$T_X = P_d \, r \left[(\frac{L}{r})^2 \times (-) \text{ coef.} \right]$$

or

$$T_X = \frac{4}{\pi} \times 50 \times 33 \times 63 \times (-) \text{ coef.}$$

$$= 13{,}230 \times (-) \text{ coef. lb. per ft.}$$

At $\phi = 40^\circ$ $T_X = 13{,}230 \times -0.2026 = -2{,}680$ lb. per ft.

$\phi = 30^\circ$ $T_X = 13{,}230 \times -0.1996 = -2{,}640$ lb. per ft.

$\phi = 20^\circ$ $T_X = 13{,}230 \times -0.1904 = -2{,}520$ lb. per ft.

$\phi = 10^\circ$ $T_X = 13{,}230 \times -0.1754 = -2{,}320$ lb. per ft.

$\phi = 0^\circ$ $T_X = 13{,}230 \times -0.1552 = -2{,}060$ lb. per ft.

For live load, when $X = \frac{L}{z}$, Eq. 8b yields

$$T_\phi = P_u \, r \times (-) \text{ coef.}$$

$$T_\phi = \frac{4}{\pi} \times 30 \times 33 \times (-) \text{ coef.}$$
$$= 1{,}260 \times (-) \text{ coef. lb. per ft.}$$

At $\phi = 40°$ $T_\phi = 1{,}260 \times -1.000 = -1{,}260$ lb. per ft.

$\phi = 30°$ $T_\phi = 1{,}260 \times -0.9698 = -1{,}220$ lb. per ft.

$\phi = 20°$ $T_\phi = 1{,}260 \times -0.8830 = -1{,}120$ lb. per ft.

$\phi = 10°$ $T_\phi = 1{,}260 \times -0.7500 = -945$ lb. per ft.

$\phi = 0°$ $T_\phi = 1{,}260 \times -0.5868 = -740$ lb. per ft.

When X = 0 or X = L, Eq. 9d becomes

$$S = P_u \, r \left[\left(\frac{L}{r}\right) \times (-) \text{ coef.} \right]$$

or

$$S = \frac{4}{\pi} \times 30 \times 33 \times 2.518 \times (-) \text{ coef.}$$
$$= 3.165 \times (-) \text{ coef. lb. per ft.}$$

At $\phi = 40°$ $S = 3{,}165 \times 0 = 0$ lb. per ft.

$\phi = 30°$ $S = 3{,}165 \times -0.1633 = -518$ lb. per ft.

$\phi = 20°$ $S = 3{,}165 \times -0.3069 = -973$ lb. per ft.

$\phi = 10°$ $S = 3{,}165 \times -0.4140 = -1{,}310$ lb. per ft.

$\phi = 0°$ $S = 3{,}165 \times -0.4702 = -1{,}490$ lb. per ft.

When $X = \frac{L}{z}$, Eq. 10d may be written

$$T_X = P_u \, r \left[\left(\frac{L}{r}\right)^2 \times (-) \text{ coef.} \right]$$

or

$$T_X = \frac{4}{\pi} \times 30 \times 33 \times 6.3 \times (-) \text{ coef.}$$
$$= 7{,}950 \times (-) \text{ coef. lb. per ft.}$$

At $\phi = 40^\circ$ $T_X = 7{,}950 \times -0.3040 = -2{,}420$ lb. per ft.

$\phi = 30^\circ$ $T_X = 7{,}950 \times -0.2856 = -2{,}270$ lb. per ft.

$\phi = 20^\circ$ $T_X = 7{,}950 \times -0.2329 = -1{,}850$ lb. per ft.

$\phi = 10^\circ$ $T_X = 7{,}950 \times -0.1520 = -1{,}210$ lb. per ft.

$\phi = 0^\circ$ $T_X = 7{,}950 \times -0.0528 = -420$ lb. per ft.

Step 2. From step 1, T_X, T_ϕ, and S were found at the unsupported edges. Corrective line loads of equal and opposite values must be applied, so that the edge conditions are consistent.

The unbalanced forces along the free edges, i.e., $\phi_k = 0^\circ$ are:

At $X = \frac{L}{z}$, $T_\phi = -1{,}610 - 740 = -2{,}350$ lb. per ft.

and at X = 0, S = -2,160 - 1,490 = -3,650 lb. per ft.

Then the vertical and horizontal components of the transverse force, T_ϕ, are obtained from Fig. 12,

$$V_L = 2{,}350 \times \sin\phi_k$$
$$= 2{,}350 \times 0.6428$$
$$= 1{,}510 \text{ lb. per ft.}$$

and

$$H_L = 2{,}350 \times \cos\phi_k$$
$$= 2{,}350 \times 0.7660$$
$$= 1{,}800 \text{ lb. per ft.}$$

Also, the shearing force is found as $S_L = 3{,}560$ lb. per ft.

The internal forces due to these corrective line forces are found by the equations on pages 25 and 26. The coefficients in those equations are taken from Table 2A of the manual where $\phi_k = 40^\circ$, $\frac{r}{t} = 100$ and $\frac{r}{L} = 0.4$.

1. Due to the vertical line load, V_L

Equations 15a to 15d may be rewritten as

$$T_X = V_L \times (\frac{L}{r})^2 \times \text{coef. (1)}$$
$$= 1{,}510 \times 6.3 \times \text{coef. (1)}$$
$$= 9{,}520 \times \text{coef. (1) lb. per ft.}$$

$$S = V_L \times (\frac{L}{r}) \times \text{coef. (2)}$$
$$= 1{,}510 \times 2.518 \times \text{coef. (2)}$$
$$= 3{,}800 \times \text{coef. (2) lb. per ft.}$$

$$T_\phi = V_L \times \text{coef. (3)}$$
$$= 1{,}510 \times \text{coef. (3) lb. per ft.}$$

$$M_\phi = V_L \times r \times \text{coef. (4)}$$
$$= 1{,}510 \times 33 \times \text{coef. (4)}$$
$$= 49{,}800 \times \text{coef. (4) lb.-ft. per ft.}$$

2. Due to the horizontal line load, H_L.

The corrected internal forces may be obtained from Eq. 16a to 16d, as follows:

$$T_X = H_L \times (\frac{L}{r})^2 \times \text{coef. (5)}$$
$$= 1{,}800 \times 6.3 \times \text{coef. (5)}$$
$$= 11{,}350 \times \text{coef. (5) lb. per ft.}$$

$$S = H_L \times \frac{L}{r} \times \text{coef. (6)}$$
$$= 1{,}800 \times 2.518 \times \text{coef. (6)}$$
$$= 4{,}530 \times \text{coef. (6) lb. per ft.}$$

$$T_\phi = H_L \times \text{coef. (7)}$$
$$= 1{,}800 \times \text{coef. (7) lb. per ft.}$$

$$M_\phi = H_L \times r \times \text{coef. (8)}$$

$= 1,800 \times 33 \times$ coef. (8)

$= 59,500 \times$ coef. (8) lb.-ft. per ft.

Table 1. Corrected internal forces and moments due to V_L.

ϕ_k in Degrees	T_X in lb./ft.	S in lb./ft.	T_ϕ in lb./ft.	M_ϕ in lb.-ft./ft.
	Coefficients			
0	+ 1.720	0	+ 0.643	0
10	- 2.814	- 2.749	- 0.749	- 0.0862
20	- 4.296	- 0.278	- 1.730	- 0.1580
30	+ 0.117	+ 0.913	- 1.512	- 0.1948
40	+ 2.485	0	- 1.221	- 0.2044
Multipliers	9520	3800	1510	49800
0	+ 164000	0	+ 970	0
10	- 26800	- 10450	- 1130	- 4300
20	- 40800	- 1060	- 2620	- 7870
30	+ 1112	+ 3470	- 2285	- 9700
40	+ 23700	0	- 1840	- 10200

3. Due to the shear line load, S_L.

From Eq. 17a to 17d, the corrected internal forces are found as below:

$$T_X = S_L \times \left(\frac{L}{r}\right)^2 \times \text{coef. (9)}$$

Table 2. Corrected internal forces and moments due to H_L.

ϕ_k in Degrees	T_X in lb./ft.	S in lb./ft.	T_ϕ in lb./ft.	M_ϕ in lb.-ft./ft.
	Coefficients			
0	- 7.122	0	+ 0.766	0
10	+ 2.495	+ 0.6036	+ 1.272	+ 0.0665
20	+ 2.058	- 0.9139	+ 1.236	+ 0.1013
30	- 1.123	- 1.171	+ 0.606	+ 0.1093
40	- 2.664	0	+ 0.250	+ 0.1088
Multipliers	11350	4530	1800	59500
0	- 80800	0	+ 1380	0
10	+ 28400	+ 2730	+ 2290	+ 3960
20	+ 23400	- 4140	+ 2220	+ 6050
30	- 12780	- 5300	+ 1090	+ 6520
40	- 30300	0	+ 450	+ 6460

$T_X = 3{,}560 \times 6.3 \times$ coef. (9)

$= 22{,}400$ coef. (9) lb. per ft.

$S = S_L \times \frac{L}{r} \times$ coef. (10)

$= 3{,}560 \times 2.518 \times$ coef. (10)

$= 8{,}950 \times$ coef. (10) lb. per ft.

$T_\phi = S_L \times$ coef. (11)

$= 3{,}560 \times$ coef. (11)

$M_\phi = S_L \times r \times$ coef. (12)

$= 3{,}560 \times 33 \times$ coef. (12)

$= 117{,}500 \times$ coef. (12) lb.-ft. per ft.

Table 3. Corrected internal forces and moments due to S_L.

ϕ_k in Degrees	T_X in lb./ft.	S in lb./ft.	T_ϕ in lb./ft.	M_ϕ in lb.-ft./ft.
	Coefficients			
0	+ 1.3680	+ 0.4000	0	0
10	+ 0.4228	- 0.0471	+ 0.0728	0
20	+ 0.0023	- 0.1351	+ 0.0079	- 0.0017
30	- 0.1315	- 0.0851	- 0.0567	- 0.0039
40	- 0.1542	0	- 0.0818	- 0.0049
Multipliers	22400	8950	3560	117500
0	+ 30600	+ 3650	0	0
10	+ 8450	- 422	+ 260	0
20	+ 52	- 1210	+ 4	- 200
30	- 2950	- 760	- 202	- 458
40	- 3460	0	- 292	- 575

When the edge conditions are complicated by the presence of longitudinal edge beams or adjoining shells, equations 11b to 14b and 18a to 18d will be considered in addition to the above equations.

Step 3:

Adding the results of Steps 1 and 2, the final values of T_X,

T_ϕ, S and M_ϕ are obtained.

Table 4. Internal forces and moments in the shell.

No.	Loadings	Angles ϕ in Degrees: 0	10	20	30	40
		T_ϕ at $X = \frac{L}{Z}$ (Pounds Per Foot)				
1	Dead	- 1610	- 1820	- 1970	- 2065	- 2100
2	Live	- 740	- 945	- 1120	- 1220	- 1260
3	V_L	+ 970	- 1130	- 2620	- 2285	- 1840
4	H_L	+ 1380	+ 2290	+ 2220	+ 1090	+ 450
5	S_L	0	+ 260	+ 4	- 202	- 292
6		0	- 1345	- 3486	- 4682	- 5042
		S at X = 0 (Pounds Per Foot)				
1	Dead	- 2160	- 1680	- 1147	- 584	0
2	Live	- 1490	- 1310	- 973	- 518	0
3	V_L	0	- 10450	- 1060	+ 3470	0
4	H_L	0	+ 2730	- 4140	- 5300	0
5	S_L	+ 3650	- 422	- 1210	- 760	0
6		0	- 11132	- 8530	- 3692	0
		T_X at $X = \frac{L}{Z}$ (Pounds Per Foot)				
1	Dead	- 2060	- 2320	- 2520	- 2640	- 2680
2	Live	- 420	- 1210	- 1850	- 2270	- 2420
3	V_L	+ 164000	- 26800	- 40800	+ 1112	+ 23700
4	H_L	- 80800	+ 28400	+ 23400	- 12780	- 30300
5	S_L	+ 30600	+ 8450	+ 52	- 2950	- 3460
6		+ 111320	+ 6520	- 21718	- 20538	- 8560
		M_ϕ at $X = \frac{L}{Z}$ (Foot Pounds Per Foot)				
1	Dead					
2	Live					
3	V_L	0	- 4300	- 7870	- 9700	- 10200
4	H_L	0	+ 3960	+ 6050	+ 6520	+ 6460
5	S_L	0	0	- 200	- 458	- 575
6		0	- 340	- 2020	- 3638	- 4315

Since the line loads applied at the edges are assumed to be either sine curves or cosine curves, the internal forces at any other point in the shell may be found by proportioning the maximum value to the desired value. This can be done by simply multiplying the values of T_ϕ, and T_X and M_ϕ found in Table 4 by the sin $\frac{\pi X}{L}$ and S by the cos $\frac{\pi X}{L}$. Taking the distances at $\frac{X}{L}$ = 0, 1/8, 1/4, 3/8, and 1/2 from one end of the shell, T_ϕ, T_X, and M_ϕ and S are listed in Table 5.

Rows 6 of Table 4 give internal forces in the simply-supported barrel shell roof. These are obtained by summation of membrane forces and effects of line loads applied at both longitudinal edges. Variation of these forces is shown graphically in Fig. 17.

Step 4.

Checking the accuracy of the internal forces is necessary as a precaution against computation errors in Step 3. The sum of horizontal forces, T_X, across a transverse section of the barrel must equal zero, the sum of vertical components of the shearing force S at each support must equal one-half the total load on the shell, and the total internal longitudinal moment at midspan due to T_X forces must equal the external moment.

(1) Since the variation between ordinates is not linear, the summation of horizontal forces at midspan is made by means of Simpson's rule.

$$\text{Area} = \frac{\Delta L}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 \text{ ------ } + 4y_n)$$

in which y_0 to y_n are equally spaced ordinates of the curve enclosing the area, n is an even number, and ΔL is the distance

Table 5. Force components throughout the shell.

	ϕ_k in Degrees	0	1/8	1/4	3/8	1/2
		T_ϕ (Pounds Per Foot) at $\frac{X}{L}$.				
1	0	0	0	0	0	0
2	10	0	- 515	- 952	- 1242	- 1345
3	20	0	- 1338	- 2465	- 3220	- 3486
4	30	0	- 1790	- 3310	- 4320	- 4682
5	40	0	- 1928	- 3560	- 4650	- 5042
		S (Pounds Per Foot) at $\frac{X}{L}$				
1	0	0	0	0	0	0
2	10	- 11132	- 10360	- 8020	- 4320	0
3	20	- 8530	- 7870	- 6030	- 3265	0
4	30	- 3692	- 3410	- 2610	- 1410	0
5	40	0	0	0	0	0
		T_X (Pounds Per Foot) at $\frac{X}{L}$				
1	0	0	+ 42600	+ 78900	+ 102800	+ 111320
2	10	0	+ 2500	+ 4620	+ 5940	+ 6520
3	20	0	- 8320	- 15350	- 20000	- 21718
4	30	0	- 7860	- 14900	- 18100	- 20578
5	40	0	- 3280	- 6060	- 7910	- 8560
		M_ϕ (Pounds Foot Per Foot) at $\frac{X}{L}$				
1	0	0	0	0	0	0
2	10	0	- 130	- 240	- 314	- 340
3	20	0	- 775	- 1430	- 1865	- 2020
4	30	0	- 1392	- 2564	- 3358	- 3638
5	40	0	- 1650	- 3070	- 3980	- 4315

between ordinates. We find

$$L = \frac{s}{4} = \frac{r\phi_k}{4} = \frac{33 \times 40 \times \pi}{4 \times 180} = 5.75 \text{ ft.}$$

and s is the arc distance from shell edge to crown divided into four parts. Using the values of T_X in Table 4, the summation of

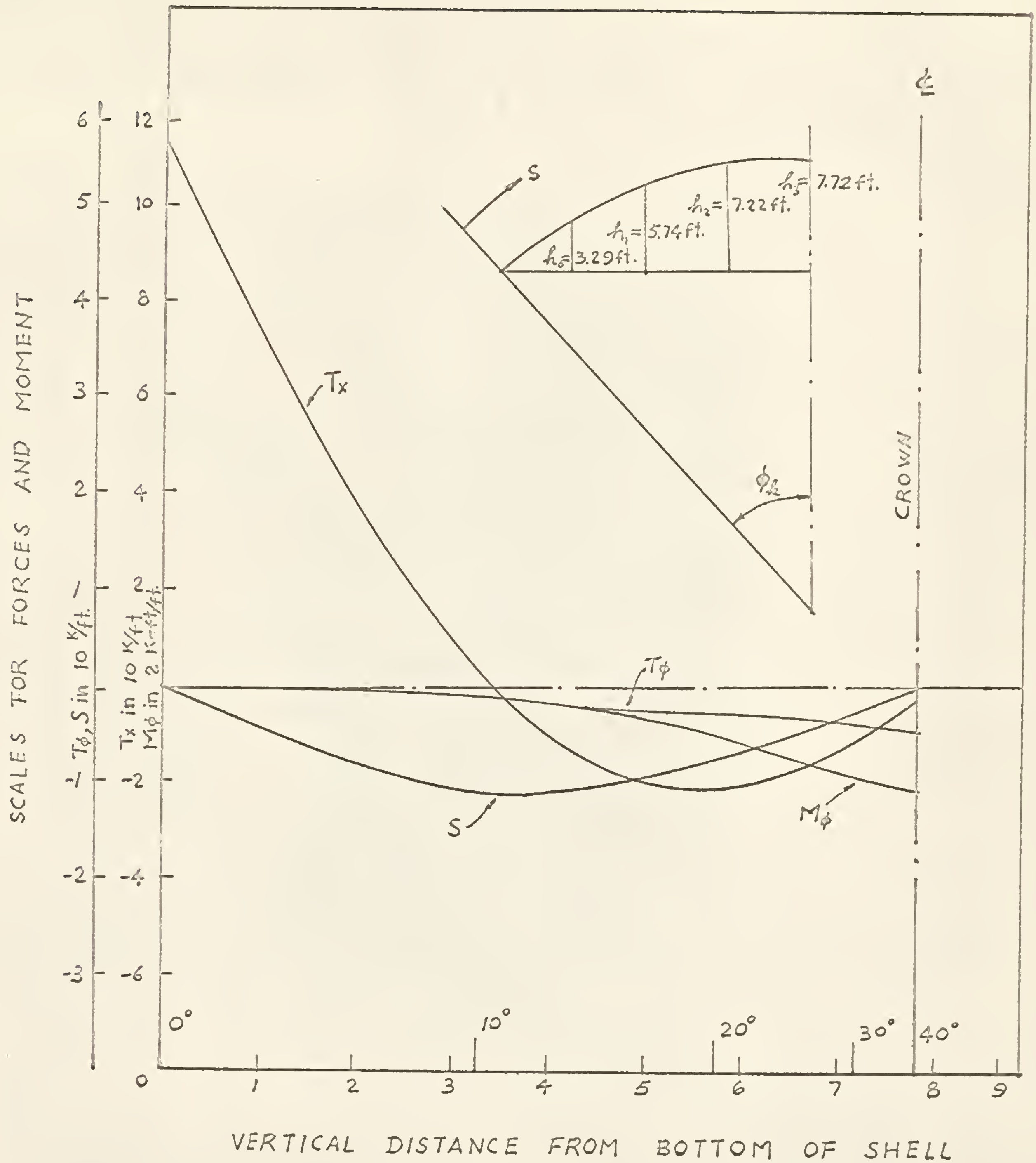


Fig. 17. Force distribution in shell.

longitudinal forces is:

$$H = \frac{5.75}{3} (111,320 + 4 \times 6,520 + 2 \times (-21,718) + 4 \times (-20,538) + (-8,560) = + 266,000 - 258,000 = 48,000 \text{ lb.}$$

The difference between positive and negative values is less than 2 percent of their arithmetic sum.

(2) The moment about the bottom edge of the shell is computed by using Simpson's rule in the following form:

$$\text{Moment} = \frac{\Delta L}{3} \times 2 \, (y_0 h_0 + 4y_1 h_1 + 2y_2 h_2 + 4y_3 h_3 \text{ ------ } + y_n h_n)$$

where y_0 to y_n are the magnitude of the forces T_X at the specified locations of h, and h_0 to h_n are the moment arms measured from the lower edge of the shell (see Fig. 17). Also, the moments on either side of the crown have to be included. The total internal moment is

$$M = \frac{5.75}{3} \times 2 \, [\, 0 + 4 \times 6,520 \times 3.29 + 2 \times (-21,718) \times 5.74 + 4 \times (-20,578) \times 7.22 + (-8,560) \times 7.72] = - 3,200,000 \text{ ft. lb.}$$

The moment due to external load may be obtained by using the mid-span moment equation shown on page 13.

$$M = - \frac{PL^2}{\pi^2} \sin \frac{\pi X}{L}$$

or

$$M_{X} = \frac{L}{2} = - \frac{PL}{\pi^2} \times \frac{4}{\pi}$$

Since P_u = 30 psf, P_d = 50 psf and the arc length of the shell = 8 x 5.75 ft., the external moment is

$$M = -\frac{(83)^2}{\pi^2} \times \frac{4}{\pi}(30 \times 42.5 + 50 \times 8 \times 5.75)$$

$$= -3{,}170{,}000 \text{ ft. lb.}$$

The difference of the two moments is about 1 percent.

(3) Summation of vertical components of the shearing force S at one end of shell is:

$$V = \frac{\Delta L}{3} \times 2\left[y_0 \sin(\phi_k - \phi_0) + 4y_1 \sin(\phi_k - \phi_1) + 2y_1 \sin(\phi_k - \phi_2) + \text{-----}\ y_n \sin(\phi_k - \phi_n)\right]$$

in which y_0 to y_n are the magnitudes of the shear forces and the total upward force is:

$$V = \frac{5.75}{3} \times 2[\ 0 + 4 \times 11{,}132 \times 0.5 + 2 \times 8{,}530 \times 0.342 + 4 \times 3{,}692 \times 0.1736] = -116{,}000 \text{ lb.}$$

Again, the equation of the total shear due to the external load is shown on page 13 and reproduced here as

$$V = -\frac{PL}{\pi} \times \frac{4}{\pi} \cos\frac{4\pi X}{L}$$

or

$$V = -\frac{83}{\pi} \times \frac{4}{\pi}[\ 30 \times 42.5 + 50 \times 46] = -122{,}000 \text{ lb.}$$

The difference of the two vertical forces is less than 5 percent.

Step 5.

The design of reinforcement for the shell roof is based on the final forces in Tables 4 and 5. In general, shell reinforcement is provided for three force conditions:

(a) Longitudinal reinforcement, this being required to overcome T_X.

(b) Transverse reinforcement, this being due to T_ϕ and M_ϕ.

(c) Diagonal reinforcement, this being to resist the combination of T_X, T_ϕ and S.

The determination and arrangement of these reinforcement patterns are as follows:

(a) Longitudinal reinforcement.

The longitudinal steel stress varies linearly from a maximum at the edge of the shell to zero at the neutral axis, i.e., $T_X = 0$.

From Table 4, at $X = \frac{L}{2}$ with $T_X = +\ 111{,}320$ lb. per ft. the required reinforcement is:

$$A_s = \frac{1{,}113{,}200}{20{,}000} = 5.56 \text{ sq. in. per ft.}$$

To provide the required steel area two layers of 1 inch round bars at 3 inch centers are used. Between the edge and the neutral axis, T_X scaled from Fig. 13 is about 52,000 lb. per ft. The required steel is:

$$A_s = \frac{52{,}000}{20{,}000} = 2.6 \text{ sq. in. per ft.}$$

which is supplied by one layer of 1 in. diameter bars at 3 1/2 in. centers. Above the neutral axis, in the compression zone, reinforcement is theoretically unnecessary but for the sake of safety 3/8 in. round bars at 1 ft. centers are provided. The steel area can be reduced by 30% at $X = \frac{L}{4}$ and by 50% at $X = \frac{L}{8}$. The arrangement of the longitudinal steel is shown in Fig. 21.

(b) Transversal reinforcement.

The required transverse reinforcement is computed for the moment M_ϕ and the transverse force T_ϕ. The procedure is similar to that used for ordinary reinforced concrete beams subject to combined stresses. From equation 5 on page 87 of the Reinforced Concrete Design Handbook,* the steel area is determined by

$$A_s = \frac{NE}{adi} \quad \text{-----------------------------------} \quad 17$$

where

N = axial load

E = the distance between the equivalent eccentric load and the resultant of tensile stresses

a = coefficients in the handbook

d = effective depth of a beam

$$i = \frac{1}{1-\frac{jd}{e}}$$

At the center of the span at the crown, with an effective depth of 3 in., an $M_\phi = 4{,}315$ ft.-lb. and a $T_\phi = -\ 5{,}042$ lb., A_s is found to be 1.075** sq. in. per ft. by Eq. 17. To furnish the area required, 1/2 in. round bars spaced at 2 in. centers are used. Because both M_ϕ and T_ϕ decrease towards the edge, 2/3 of the bars are terminated at $\phi = 20°$. Also, the bars may be reduced 30% at $X = \frac{L}{4}$ and 50%

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**Determined according to Example B of the Reinforced Concrete Design Handbook.

at $X = \frac{L}{8}$. The arrangement of the bars is shown in Fig. 20.

(c) Diagonal reinforcement.

To determine the steel necessary to resist the combined T_X, T_ϕ and S, the principal stresses have to be evaluated. Consider a small element of the shell on which T_X, T_ϕ and S are acting (see Fig. 18), the principal stresses can be solved by using Mohr's Circle as

$$T_{p1}, T_{p2} = \frac{T_X + T_\phi}{2} \pm \sqrt{\frac{(T_X - T_\phi)^2}{4} + S^2} \quad \text{------------------} \quad 18$$

in which T_{p1} and T_{p2} are the principal stresses. The direction of the principal stresses is given by

$$\tan 2\alpha = \frac{2S}{T_X - T_\phi} \quad \text{------------------------} \quad 19$$

The principal forces and plane of which they act are calculated and tabulated in Table 6.

The forces listed in the table may be conveniently plotted as in Fig. 19. From Fig. 19 it is seen that only the forces acting along the rim need to be considered. At $\phi = 10^\circ$, the required steel is $A_s = \frac{T_{p2}}{f_s} = \frac{11,132}{20,000} = 0.565$ sq. in. and at $\phi = 20^\circ$ the required steel is $A_s = \frac{8,530}{20,000} = 0.425$ sq. in. The required steel is provided by using 1/2 in. diameter bars placed at 45 degrees with the edge and spaced at 4 in. and 5 1/2 in. center to center. The arrangement of the bars is shown in Fig. 21.

Table 6. Principal stresses for the shell.

		0°	10°	20°	30°	40°
		At $X = \frac{L}{2}$				
1	T_X	+ 111320	+ 6520	- 21718	- 20578	- 8560
2	T_ϕ	0	- 1345	- 3486	- 4682	- 5042
3	S	0	0	0	0	0
4	$\frac{T_X+T_\phi}{2}$	+ 55660	+ 2588	- 12602	- 12630	- 6801
5	$\frac{(T_X-T_\phi)^2}{4}$	+ $31x10^8$	+ $15.5x10^8$	+ $80x10^6$	+ $62.5x10^6$	+ $31x10^6$
6	S^2	0	0	0	0	0
7	(5)+(6)	+ $5.566x10^4$	+ $3.94x10^3$	+ $8.95x10^3$	+ $7.9x10^3$	$1.76x10^3$
8	$T_{p1}=(4)-(7)$	0	- 1352	- 21552	- 20530	- 8561
9	$T_{p2}=(4)+(7)$	+ 111300	+ 6528	- 3652	- 4730	- 5041
10	$\frac{2S}{T_X-T_\phi}$	0	0	0	0	0
11	α	90°	0	0	0	90°
		$X = \frac{L}{4}$				
1	T_X	+ 78900	+ 4620	- 15350	- 14500	- 6060
2	T_ϕ	0	- 952	- 2465	- 3310	- 3560
3	S	0	- 8020	- 6030	- 2610	0
4	$\frac{T_X+T_\phi}{2}$	+ 39450	+ 1834	- 8908	- 8905	- 4810

Table 6. (concl.)

		0°	10°	20°	30°	40°
				$X = \frac{L}{4}$		
5	$\frac{(T_X - T_\phi)^2}{4}$	+ 1550x10⁶	+ 7.25x10⁶	+ 41.25x10⁶	+ 28x10⁶	+ 1.56x10⁶
6	S^2	0	+ 64x10⁶	+ 36.2x10⁶	+ 6.8x10⁶	0
7	(5)+(6)	+ 39.45x10³	+ 8.45x10³	+ 8.8x10³	+ 5.9x10³	+ 1.25x10³
8	T_{p1}=(4)-(7)	0	- 6616	- 17708	- 14805	- 3560
9	T_{p2}=(4)+(7)	+ 78900	+ 10284	+ 108	- 3005	- 6060
10	$\frac{2S}{T_X - T_\phi}$	0	- 2.88	+ 0.976	+ 0.467	0
11	α	90°	35°	22°	13°	90°
				$X = 0$		
1	T_X	0	0	0	0	0
2	T_ϕ	0	0	0	0	0
3	S	0	- 11132	- 8530	- 3692	0
4	T_{p1}	0	+ 11132	+ 8530	+ 3692	0
5	T_{p2}	0	- 11132	- 8530	- 3692	0
6	α	45°	45°	45°	45°	45°

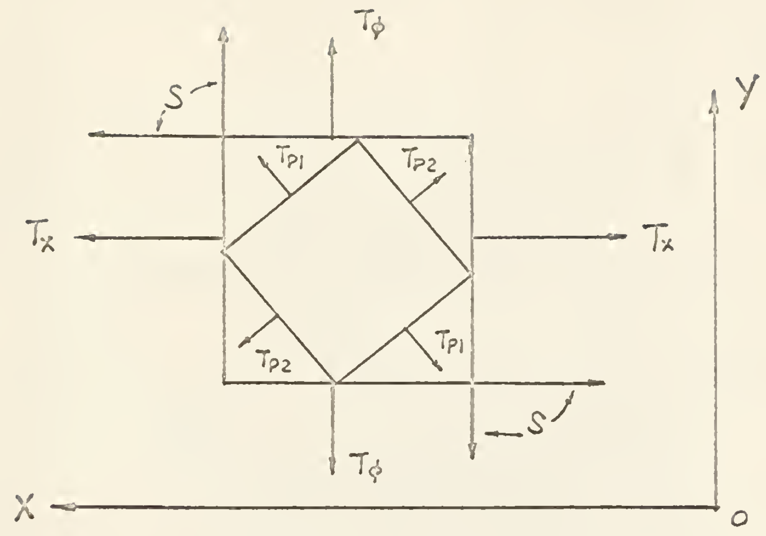

Fig. 18. Principal stresses acting on an element.

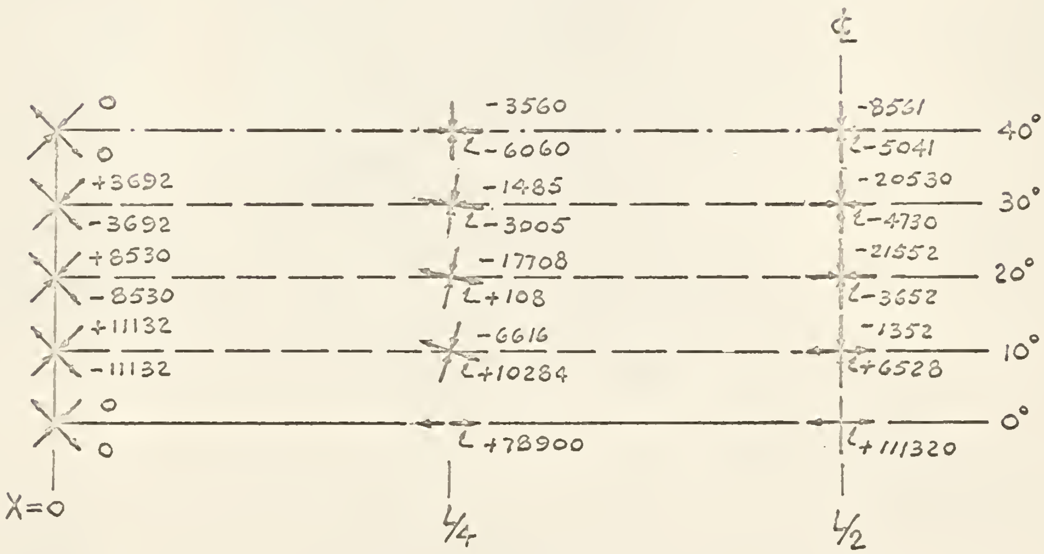

Fig. 19. Principal stress distribution.

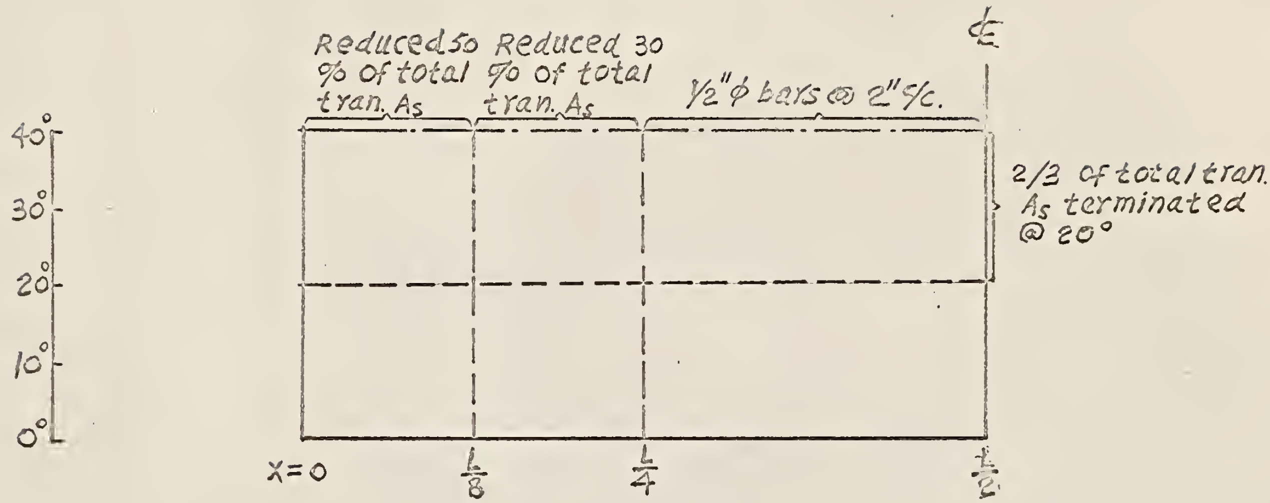

Fig. 20. Arrangement of transverse reinforcement.

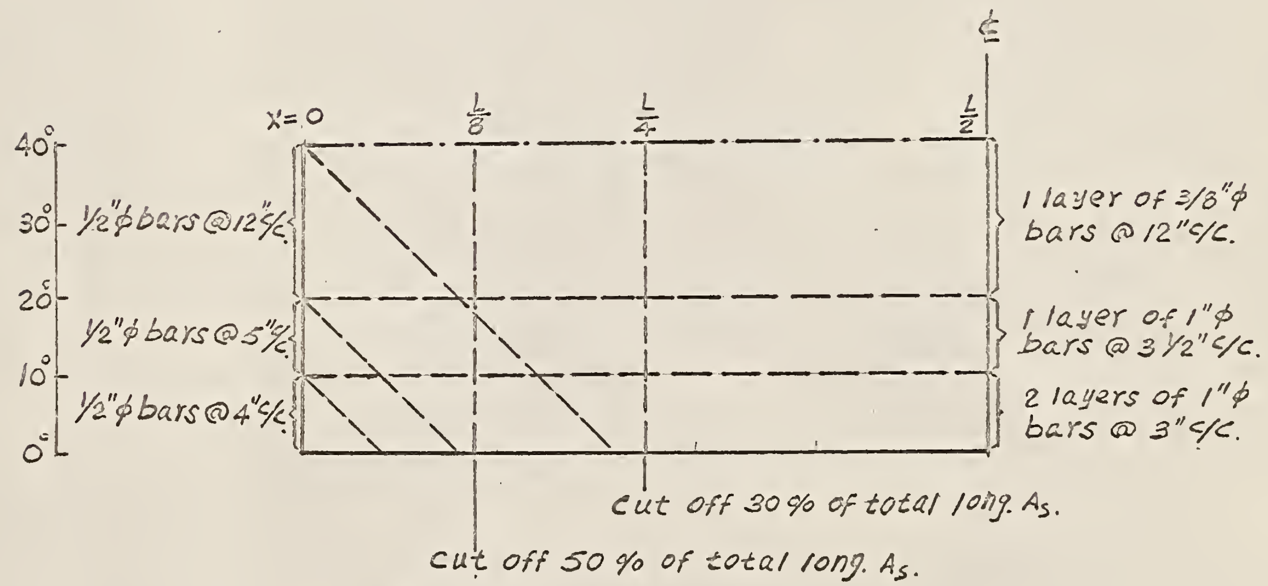

Fig. 21. Arrangement of longitudinal and diagonal reinforcement.

DISCUSSION OF RESULTS

In this example, the maximum difference of value between external and internal vertical forces, horizontal forces and moments, as obtained from Step 4, is less than 5 percent. This compares favorably with the results of a similar example with a length of 62 ft. and a radius of 31 ft. in the manual where the maximum difference was more than 5 percent. "Concrete Information" (3) stated, "Scaled ordinates, approximate integration given by Simpson's rule, and, to a minor extent, the incomplete satisfaction of edge conditions may result in differences amounting to as much as 10 percent in exceptional cases." It is seen that good agreement has been obtained in this example.

The clearances in the assumed 4 in. thickness of the shell can be checked by the summation of the sizes of the bars at the edge as found in Step 5:

2 x 1 in. round bars = 2 in.
1/2 in. round bars = 1/2 in.
1/2 in. round bars = 1/2 in.
Protection = 1 in.
4 in.

It is a safe practice to thicken the edges 1 in. in order to guard against the secondary stresses that may occur.

CONCLUSION

The ASCE Method presented herein is a very useful tool for cylindrical shell design, provided that the shells are subjected to uniformly distributed loads. The example illustrates that this method is efficient since it leads directly to the required structure. The procedures used in the design are based on familiar techniques and concepts and are relatively easy to follow. Even the complicated boundary conditions which create the difficult mathematical problems of shell design may be quickly determined by the tables and charts in the manual.

This method is not only an excellent one for use in designing single barrel shells with simple supports, but also for designing multiple barrel shells with either simple or several supports.

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A SIMPLY-SUPPORTED BARREL SHELL ROOF

by

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AN ABSTRACT

There are three theories in treating the problems of cylindrical shell structures, namely: the bending theory, the beam theory and the membrane theory. Each of these theories has its shortcoming: the bending theory involves complicated mathematical calculations; the beam theory is limited to long shells only; and the membrane theory is unable to predict the boundary conditions of shells. To overcome these weaknesses, ASCE Manual No. 31 presents a method of solving cylindrical shell problems by using a combination of the membrane theory and the bending theory.

The purpose of this report is to demonstrate the ASCE method in solving the problem of a simply-supported barrel shell. The principle used in this method in analyzing a shell problem is similar to the analysis of an ordinary indeterminate structure, that is, releasing some restraints to make the structure determinate, then applying reactions to bring the boundary back to its original position, and summing up the stresses found in the determinate state and those induced by the restraints to get the final stresses. The procedures of the ASCE method are:

1. Find the membrane stresses.

2. Correct the membrane stresses for the effects produced by the line loads acting along the free edge.

3. Add the corrections and the membrane stresses to obtain the final stresses.

By use of the membrane theory, the general equations for the membrane stresses can be obtained by writing the expressions for the equilibrium of a differential element in the shell. The resulting expressions are the three equations of stress in the longitudinal, tangential and radial directions. Bending

moments are not taken into consideration in this theory. Six design equations for the membrane stresses are then found from the general equations when the surface loads and proper boundary conditions are provided. The membrane stresses and deformations occurring along the free edges cause the boundary conditions to become inconsistent. To make the edge conditions compatible, line loads are applied. Using the bending theory, the equations for the corrective stresses and deformations due to the applied line loads can be obtained.

A design example is given, showing the application of the ASCE method. The tables and charts in the Manual will be used as an aid in working out the design.