Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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To cite this article: N I Stoilova and J Van der Jeugt 2019 *J. Phys.: Conf. Ser.* **1194** 012102

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012102 doi:10.1088/1742-6596/1194/1/012102

Parabosons, parafermions and representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras

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Abstract. For a set of m parafermion operators and n paraboson operators, there are two nontrivial ways to unify them in a larger algebraic structure. One of these corresponds to the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. The other one is no longer a $\mathbb{Z}_2$-graded Lie superalgebra but a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, a rather different algebraic structure, denoted here by $\mathfrak{pso}(2m+1|2n)$. In a recent paper, the Fock spaces $\tilde{V}(p)$ of order p for $\mathfrak{pso}(2m+1|2n)$ were determined. In the current paper, we summarize some of the main properties of $\mathfrak{pso}(2m+1|2n)$ and its Fock spaces. In particular, we concentrate on the Fock space for $p=1$, and indicate how it reduces to an ordinary boson-fermion Fock space.

1. Introduction: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra

In quantum mechanics or quantum field theory, the creation and annihilation operators corresponding to classical particles are usually required to satisfy the quadratic relations of bosons or fermions. That is the reason why algebras with commutation relations and/or anti-commutation relations, i.e.

$$[x,y] = xy - yx, \qquad \{x,y\} = xy + yx, \tag{1.1}$$

are so important in mathematical physics. In (1.1), it is implicitly assumed that x and y are elements of some associative algebra, such that the right hand side is properly defined. It is obvious that in an associative algebra there are two ways to rewrite the trivial identity

$$xy + yx - xy - yx = 0 \tag{1.2}$$

by means of commutators and anti-commutators:

$$[x,y] = -[y,x] \qquad \text{or} \qquad \{x,y\} = \{y,x\}. \tag{1.3}$$

These bracket symmetries are the basic parts appearing in the definition of a Lie algebra or ($\mathbb{Z}_2$-graded) Lie superalgebra. For a Lie algebra $\mathfrak{g}$ with bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the bracket should indeed be a bilinear product satisfying the antisymmetry relation $[x,y] = -[y,x]$ and the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0. \tag{1.4}$$

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, graded by $\mathbb{Z}_2 = \{0, 1\}$, the bracket $[\![\cdot, \cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ should preserve the grading, be bilinear, and satisfy the supersymmetry relation and the super Jacobi identity [1]. The supersymmetry relation reads

$$[\![x, y]\!] = -(-1)^{\deg(x)\deg(y)}[\![y, x]\!]. \tag{1.5}$$

Herein, x and y are homogeneous elements of $\mathfrak{g}$ (i.e. belonging to $\mathfrak{g}_\alpha$ with $\alpha \in \{0, 1\}$), and $\deg(x) = \alpha$ when $x \in \mathfrak{g}_\alpha$. Note that the bracket corresponds to a commutator or anti-commutator, since it satisfies one of the two symmetries of (1.3). The super Jacobi identity is given by

$$(-1)^{\deg(x)\deg(z)}[\![x, [\![y, z]\!]]\!] + (-1)^{\deg(y)\deg(x)}[\![y, [\![z, x]\!]]\!] + (-1)^{\deg(z)\deg(y)}[\![z, [\![x, y]\!]]\!] = 0. \tag{1.6}$$

Most commonly, Lie algebras or Lie superalgebras are constructed by starting from an underlying associative algebra, and in that case the brackets are actually just commutators or anti-commutators as in (1.1). When the bracket is a commutator, the Jacobi identity is equivalent to the following trivial identity in an associative algebra:

$$xyz + yzx + zxy + xzy + zyx + yxz - xyz - yzx - zxy - xzy - zyx - yxz = 0. \tag{1.7}$$

Note that (1.7) is the counterpart of (1.2) for three elements: there are six ways to write the product of three elements x, y and z in an associative algebra, and (1.7) is the corresponding trivial identity. In how many ways can this identity (1.7) be rewritten purely by means of embedded commutators and anti-commutators? Up to permuting the elements, there are only four ways to do this [2], namely

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \\
&[x, \{y, z\}] + [y, \{z, x\}] + [z, \{x, y\}] = 0, \\
&[x, \{y, z\}] + \{y, [z, x]\} - \{z, [x, y]\} = 0, \\
&[x, [y, z]] + \{y, \{z, x\}\} - \{z, \{x, y\}\} = 0.
\end{aligned} \tag{1.8}$$

Clearly, the first of these appears as the ordinary Jacobi identity or as the Jacobi identity for Lie superalgebras (1.6) with all elements even (or two even and one odd element); the second one appears as (1.6) with all elements odd; the third one appears as (1.6) with x even and y and z odd; but the fourth one does not at all appear as one of the forms of (1.6). In fact, the fourth relation of (1.8) appears as the Jacobi identity of a new structure, a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebra, to be defined soon. In other words, to have a complete picture of algebras for which the bracket is a commutator or anti-commutator, covering all cases of (1.7), one should go beyond Lie algebras and Lie superalgebras and introduce so-called $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebras.

The original definition of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (LSA) goes back to [3, 4], in the context of so-called color Lie superalgebras. We turn here to the definition as given by Tolstoy [5]: as a linear space, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\mathfrak{g}$ is a direct sum of four graded components:

$$\mathfrak{g} = \bigoplus_{\boldsymbol{a}} \mathfrak{g}_{\boldsymbol{a}} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)} \tag{1.9}$$

where $\boldsymbol{a} = (a_1, a_2)$ is an element of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Homogeneous elements of $\mathfrak{g}_{\boldsymbol{a}}$ are denoted by $x_{\boldsymbol{a}}, y_{\boldsymbol{a}}, \ldots$, and $\boldsymbol{a}$ is called the degree, $\deg x_{\boldsymbol{a}}$, of $x_{\boldsymbol{a}}$. If $\mathfrak{g}$ admits a bilinear operation (the generalized Lie bracket), denoted by $[\![\cdot, \cdot]\!]$, satisfying the identities (grading, symmetry, Jacobi):

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] \in \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}, \tag{1.10}$$

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, x_{\boldsymbol{a}}]\!], \tag{1.11}$$

$$[\![x_{\boldsymbol{a}}, [\![y_{\boldsymbol{b}}, z_{\boldsymbol{c}}]\!]]\!] = [\![[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!], z_{\boldsymbol{c}}]\!] + (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, [\![x_{\boldsymbol{a}}, z_{\boldsymbol{c}}]\!]]\!], \tag{1.12}$$

where

$$\boldsymbol{a}+\boldsymbol{b}=(a_1+b_1,a_2+b_2)\in\mathbb{Z}_2\times\mathbb{Z}_2,\qquad \boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2, \tag{1.13}$$

then $\mathfrak{g}$ is called a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra.

Note that by (1.11), the bracket for homogeneous elements is either a commutator or an anti-commutator. Furthermore, observe that the fourth relation of (1.8) appears when $x\in\mathfrak{g}_{(1,1)}$, $y\in\mathfrak{g}_{(1,0)}$ and $z\in\mathfrak{g}_{(0,1)}$.

2. Algebras with parabosons and parafermions

The physical significance of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebras is not as strong as that of Lie algebras or Lie superalgebras. But recently there have been more contributions in which this new algebraic structure plays a role. In particular, a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra appears naturally as a symmetry algebra of the Lévy-Leblond equation [6], in the study of $N=2$ super Schrödinger algebras [7], or in generalizations of superconformal Galilei algebras [8].

For us, the important result is the observation that certain algebras with parabosons and parafermions are actually $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebras [2, 5, 9, 10]. We follow here the presentation of [11] for a description of "parastatistics algebras".

The parafermion algebra, introduced by Green [12], is generated by a system of m parafermion creation and annihilation operators $f_j^\pm$ $(j=1,\dots,m)$ satisfying

$$[[f_j^\xi,f_k^\eta],f_l^\epsilon]=|\epsilon-\eta|\delta_{kl}f_j^\xi-|\epsilon-\xi|\delta_{jl}f_k^\eta, \tag{2.1}$$

where $j,k,l\in\{1,2,\dots,m\}$ and $\eta,\epsilon,\xi\in\{+,-\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon-\xi$ and $\epsilon-\eta$). Similarly, the paraboson algebra is generated by a system of n pairs of parabosons $b_j^\pm$ satisfying

$$[\{b_j^\xi,b_k^\eta\},b_l^\epsilon]=(\epsilon-\xi)\delta_{jl}b_k^\eta+(\epsilon-\eta)\delta_{kl}b_j^\xi. \tag{2.2}$$

It is nowadays common knowledge that the parafermionic algebra determined by (2.1) is the orthogonal Lie algebra $\mathfrak{so}(2m+1)$ [13,14], and that the parabosonic algebra determined by (2.2) is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [15].

Greenberg and Messiah [16] showed that there are essentially four ways to combine parafermions and parabosons in one algebra. There are two trivial combinations, and two non-trivial combinations where the relative commutation relations between parafermions and parabosons are also expressed by means of triple relations. The first of these non-trivial ones are the "relative parafermion relations", determined by:

$$\begin{aligned}&[[f_j^\xi,f_k^\eta],b_l^\epsilon]=0,\qquad [\{b_j^\xi,b_k^\eta\},f_l^\epsilon]=0,\\ &[[f_j^\xi,b_k^\eta],f_l^\epsilon]=-|\epsilon-\xi|\delta_{jl}b_k^\eta,\qquad \{[f_j^\xi,b_k^\eta],b_l^\epsilon\}=(\epsilon-\eta)\delta_{kl}f_j^\xi.\end{aligned} \tag{2.3}$$

The second are the so-called "relative paraboson relations". In order to distinguish this from the operators satisfying (2.3), we will denote the parafermion operators by $\tilde{f}_j^\pm$ and the paraboson operators by $\tilde{b}_j^\pm$. So the operators $\tilde{f}_j^\pm$ among themselves still satisfy the triple relations (2.1), the operators $\tilde{b}_j^\pm$ still satisfy (2.2), but the relative relations are now determined by:

$$\begin{aligned}&[[\tilde{f}_j^\xi,\tilde{f}_k^\eta],\tilde{b}_l^\epsilon]=0,\qquad [\{\tilde{b}_j^\xi,\tilde{b}_k^\eta\},\tilde{f}_l^\epsilon]=0,\\ &\{\{\tilde{f}_j^\xi,\tilde{b}_k^\eta\},\tilde{f}_l^\epsilon\}=|\epsilon-\xi|\delta_{jl}\tilde{b}_k^\eta,\qquad [\{\tilde{f}_j^\xi,\tilde{b}_k^\eta\},\tilde{b}_l^\epsilon]=(\epsilon-\eta)\delta_{kl}\tilde{f}_j^\xi.\end{aligned} \tag{2.4}$$

The parastatistics algebra with relative parafermion relations, determined by (2.1), (2.2) and (2.3), was identified by Palev [17] as the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. The parastatistics algebra with relative paraboson relations, determined by (2.1), (2.2) and (2.4), remained unidentified for a long time, until it was observed to be a certain $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra [2, 5, 9, 10].

In [11], this $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra was denoted by $\mathfrak{pso}(2m+1|2n)$ – because of its resemblance with $\mathfrak{osp}(2m+1|2n)$ – and Fock spaces of $\mathfrak{pso}(2m+1|2n)$ were studied. In the current paper, we shall summarize some results of [11] and discuss some further examples.

First of all, let us recall the definition of $\mathfrak{pso}(2m+1|2n)$ starting from a matrix algebra [11]. It consists of all block matrices of the form

$$\left(\begin{array}{cc|c|cc} a & b & u & x & x_1 \\ c & -a^t & v & y & y_1 \\ \hline -v^t & -u^t & 0 & z & z_1 \\ \hline -y_1^t & -x_1^t & z_1^t & d & e \\ y^t & x^t & -z^t & f & -d^t \end{array}\right), \tag{2.5}$$

with a an $(m\times m)$-matrix, b and c skew symmetric $(m\times m)$-matrices, u and v $(m\times 1)$-matrices, x, y, x_1, y_1 $(m\times n)$-matrices, z and z_1 $(1\times n)$-matrices, d an $(n\times n)$-matrix, and e and f symmetric $(n\times n)$-matrices. The $\mathbb{Z}_2\times\mathbb{Z}_2$ grading for these matrices is fixed by

$$\left(\begin{array}{c|c|c} \mathfrak{g}_{(0,0)} & \mathfrak{g}_{(1,1)} & \mathfrak{g}_{(0,1)} \\ \hline \mathfrak{g}_{(1,1)} & 0 & \mathfrak{g}_{(1,0)} \\ \hline \mathfrak{g}_{(0,1)} & \mathfrak{g}_{(1,0)} & \mathfrak{g}_{(0,0)} \end{array}\right). \tag{2.6}$$

The relations (1.10)-(1.12) are satisfied for homogeneous elements of the form (2.5), with the bracket given in terms of matrix multiplication:

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = x_{\boldsymbol{a}}\cdot y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}}\cdot x_{\boldsymbol{a}}. \tag{2.7}$$

This matrix definition is closely related to that of $\mathfrak{osp}(2m+1|2n)$ [1], but differs from it by sign changes in the $\mathfrak{g}_{(0,1)}$ corner of the bottom left (and of course, also differs by the grading). In the same fashion as for $\mathfrak{osp}(2m+1|2n)$ algebra, one can introduce the following elements:

$$\begin{aligned} \tilde f_j^+ &= \sqrt{2}(e_{j,2m+1} - e_{2m+1,j+m}), \\ \tilde f_j^- &= \sqrt{2}(e_{2m+1,j} - e_{j+m,2m+1}); \quad (j=1,\ldots,m) \end{aligned} \tag{2.8}$$

$$\begin{aligned} \tilde b_k^+ &= \sqrt{2}(e_{2m+1,2m+1+n+k} + e_{2m+1+k,2m+1}), \\ \tilde b_k^- &= \sqrt{2}(e_{2m+1,2m+1+k} - e_{2m+1+n+k,2m+1}); \quad (k=1,\ldots,n), \end{aligned} \tag{2.9}$$

where e_{ij} is the matrix with zeros everywhere except a 1 on position (i,j). Then it is an easy exercise to check that for these elements the triple relations (2.1), (2.2) and (2.4) are satisfied. In the notation of Tolstoy [5] $\mathfrak{pso}(2m+1|2n)$ is denoted by $\mathfrak{osp}(1,2m|2n,0)$, and he showed:

Theorem 1 (Tolstoy) *The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g}$ defined by $2m+2n$ generators $\tilde f_j^\pm$ $(j=1,\ldots,m)$ and $\tilde b_k^\pm$ $(k=1,\ldots,n)$, where $\tilde f_j^\pm\in\mathfrak{g}_{(1,1)}$ and $\tilde b_k^\pm\in\mathfrak{g}_{(1,0)}$, subject to the relations* (2.1)*,* (2.2) *and* (2.4)*, is isomorphic to $\mathfrak{pso}(2m+1|2n)$.*

It will be useful to identify some subalgebras of $\mathfrak{g}=\mathfrak{pso}(2m+1|2n)$ which are themselves ordinary Lie algebras or Lie superalgebras. First, the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded subspaces of $\mathfrak{g}$ are spanned by:

$$\begin{array}{lcl} \mathfrak{g}_{(1,1)} & : & \tilde f_j^+, \quad \tilde f_j^- \\ \mathfrak{g}_{(1,0)} & : & \tilde b_j^+, \quad \tilde b_j^- \\ \mathfrak{g}_{(0,0)} & : & [\tilde f_j^\xi, \tilde f_k^\eta], \quad \{\tilde b_j^\xi, \tilde b_k^\eta\} \\ \mathfrak{g}_{(0,1)} & : & \{\tilde f_j^\xi, \tilde b_k^\eta\}, \end{array}$$

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where the indices run over the appropriate range. Then, with the same convention:

$$\begin{aligned}
&\mathrm{span}([\tilde f_j^\xi, \tilde f_k^\eta]) = \mathfrak{so}(2m)\\
&\mathrm{span}([\tilde f_j^\xi, \tilde f_k^\eta], \tilde f_j^\pm) = \mathfrak{so}(2m+1)\\
&\mathrm{span}(\{\tilde b_j^\xi, \tilde b_k^\eta\}) = \mathfrak{sp}(2n),
\end{aligned}$$

and hence

$$\mathfrak{g}_{(0,0)} = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n), \qquad \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)} = \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n). \tag{2.10}$$

Therefore, the "even subalgebra" of $\mathfrak{g}$ is the same as the even subalgebra of $\mathfrak{osp}(2m+1|2n)$. Thus the diagonal matrices of $\mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$ form the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. A basis of $\mathfrak{h}$ is given by the elements $h_i = e_{ii} - e_{i+m,i+m}$ $(i = 1, \ldots, m)$ and $h_{m+j} = e_{2m+1+j,2m+1+j} - e_{2m+1+n+j,2m+1+n+j}$ $(j = 1, \ldots, n)$. As for $\mathfrak{osp}(2m+1|2n)$, the dual basis for the dual space $\mathfrak{h}^*$ is denoted by ϵ_i $(i = 1, \ldots, m)$, δ_j $(j = 1, \ldots, n)$. Then it is easy to see that $\mathfrak{g} = \mathfrak{pso}(2m+1|2n)$ has the same root space decomposition as $\mathfrak{osp}(2m+1|2n)$ (but graded with respect to $\mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $\mathbb{Z}_2$).

Another important subalgebra is spanned by the following elements:

$$E_{jk} = \frac{1}{2}[\tilde f_j^+, \tilde f_k^-] \quad (j,k = 1, \ldots, m), \qquad E_{m+j,m+k} = \frac{1}{2}\{\tilde b_j^+, \tilde b_k^-\} \quad (j,k = 1, \ldots, n); \tag{2.11}$$

$$E_{j,m+k} = \frac{1}{2}\{\tilde f_j^+, \tilde b_k^-\}, \qquad E_{m+k,j} = \frac{1}{2}\{\tilde b_k^+, \tilde f_j^-\} \quad (j = 1, \ldots, m; k = 1, \ldots, n). \tag{2.12}$$

Let us also fix a $\mathbb{Z}_2$-grading for these elements, which is 0 for the elements (2.11) and 1 for the elements (2.12), and which is denoted by "dg" in order not to confuse it with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading "deg". The following relations are satisfied:

$$E_{ij}E_{kl} - (-1)^{\mathrm{dg}(E_{ij})\,\mathrm{dg}(E_{kl})} E_{kl}E_{ij} = \delta_{jk}E_{il} - (-1)^{\mathrm{dg}(E_{ij})\,\mathrm{dg}(E_{kl})}\delta_{il}E_{kj}. \tag{2.13}$$

These are the defining relations for the Lie superalgebra $\mathfrak{gl}(m|n)$. So $\mathfrak{pso}(2m+1|2n)$ contains $\mathfrak{gl}(m|n)$ as a subalgebra, and this will be important to construct its Fock representations.

3. Fock representations $\tilde V(p)$ of $\mathfrak{pso}(2m+1|2n)$

The parastatistics Fock space of order p (p being a positive integer), for the relative paraboson relations, is an infinite-dimensional lowest weight representation $\tilde V(p)$ of the algebra $\mathfrak{pso}(2m+1|2n)$, and was constructed in [11]. $\tilde V(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of

$$\begin{aligned}
&\langle 0|0\rangle = 1, \qquad \tilde f_j^-|0\rangle = 0, \qquad \tilde b_j^-|0\rangle = 0, \qquad (\tilde f_j^\pm)^\dagger = f_j^\mp, \qquad (\tilde b_j^\pm)^\dagger = b_j^\mp,\\
&[\tilde f_j^-, \tilde f_k^+]|0\rangle = p\delta_{jk}\,|0\rangle, \qquad \{\tilde b_j^-, \tilde b_k^+\}|0\rangle = p\delta_{jk}\,|0\rangle,
\end{aligned} \tag{3.1}$$

and by irreducibility under the action of the algebra $\mathfrak{pso}(2m+1|2n)$ spanned by the elements $\tilde f_j^\pm$, $\tilde b_j^\pm$.

It is clear that the vacuum vector $|0\rangle$ is a lowest weight vector of weight $(-\frac{p}{2}, \ldots, -\frac{p}{2}|\frac{p}{2}, \ldots, \frac{p}{2})$ in the ϵ-δ-basis. So $\tilde V(p)$ can be constructed using a Verma module, the details of which are given in [11]. A consequence of this construction is that $\tilde V(p)$ has the same structure as the Fock representation $V(p)$ of $\mathfrak{osp}(2m+1|2n)$ determined in [18], and hence its basis vectors are labelled by the same Gelfand-Zetlin (GZ) patterns. These basis vectors consist of all possible

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GZ-patterns $|\mu)$ of $\mathfrak{gl}(m|n)$ with $\mu_{1r} \leq p$, where $r = m + n$, and the pattern is of the following triangular form [19]:

$$
|p;\mu) \equiv |\mu) \equiv |\mu)^r = \left| \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} \right)
$$

$$
= \left| \begin{array}{llllllll}
\mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\
\mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\
\mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\
\mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\
\mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\
\vdots & \iddots & & & & & & \\
\mu_{11} & & & & & & &
\end{array} \right) \tag{3.2}
$$

In order to be valid GZ-patterns, the integers $\mu_{ij} \in \mathbb{Z}_+$ should satisfy a number of conditions known as betweenness conditions and θ-conditions, given explicitly by [19, eq. (3.13)]. We shall speak of row r, row $r-1$, ..., row 1 of the pattern, going from top to bottom.

Before continuing with $\mathfrak{pso}(2m+1|2n)$, let us first make a detour to its LSA-counterpart $\mathfrak{osp}(2m+1|2n)$. The Fock representation $V(p)$ of the ordinary Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ has exactly the same basis vectors of GZ-patterns. In this case, the generators are the parafermions $f_j^\pm$ and parabosons $b_k^\pm$ satisfying the relative parafermion relations (2.3) (apart from the common triple relations (2.1) and (2.2)) – which is why they are denoted without the tilde-symbol. The explicit action of the generators $f_j^\pm$ $(1 \leq j \leq m)$ and $b_k^\pm$ $(1 \leq k \leq n)$ on the vectors $|\mu)$ of $V(p)$ (for $\mathfrak{osp}(2m+1|2n)$) was determined in [18]. The form of these actions is as follows (we describe here only those of the creation operators; those of the annihilation operators is similar):

$$
f_j^+|\mu) = \sum_{\mu'} C_{m+j,\mu,\mu'}\ |\mu'), \qquad b_k^+|\mu) = \sum_{\mu'} C_{k,\mu,\mu'}\ |\mu'). \tag{3.3}
$$

In the right hand side, all possible patterns $|\mu')$ appear such that

- μ' is still a valid GZ-pattern, i.e. $\mu'_{1r} \leq p$, and all betweenness conditions and θ-conditions are satisfied for the pattern $|\mu')$;
- for the action of f_j^+, the pattern of $|\mu')$ is obtained from that of $|\mu)$ by adding $+1$ to one position s of row i, i.e. $\mu'_{s,i} = \mu_{s,i} + 1$ for some s $(1 \leq s \leq i)$, and this for each row $i = r, r-1, \ldots, j$; all other entries of $|\mu)$ remain unchanged;
- for the action of b_k^+, the pattern of $|\mu')$ is similarly obtained from that of $|\mu)$ by adding $+1$ to one position s of row i, i.e. $\mu'_{s,i} = \mu_{s,i} + 1$ for some s $(1 \leq s \leq i)$, and this for each row $i = r, r-1, \ldots, m+k$; all other entries of $|\mu)$ remain unchanged.

The coefficients $C_{j,\mu,\mu'}$ appearing in (3.3) have been computed in [18] and are products of $\mathfrak{gl}(m|n)$ Clebsch-Gordan coefficients and certain reduced matrix elements; see also the appendix of [11] for their explicit forms. So in a way the complete basis of $V(p)$ can be built by starting from the vacuum vector $|0)$ (which consists of all zeros in GZ-pattern), and by repeatedly acting by f_j^+ and b_k^+ in all possible ways.

The main result of [11] is that the Fock representation $\tilde{V}(p)$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{pso}(2m+1|2n)$, generated by the elements $\tilde{f}_j^\pm$ and $\tilde{b}_k^\pm$ satisfying (2.1), (2.2) and (2.4), is related to the $\mathfrak{osp}(2m+1|2n)$ representation $V(p)$ in the following way:

Theorem 2 *Let $V(p)$ be the vector space with orthonormal basis vectors $|\mu)$ with $\mu_{1r} \le p$, and let the action of the $\mathfrak{osp}(2m+1|2n)$ generators $f_j^\pm$ and $b_k^\pm$ be fixed and determined [18] by* (3.3). *Then $V(p)$ is also an irreducible $\mathfrak{pso}(2m+1|2n)$ module, $\tilde V(p) \cong V(p)$, where the action of its generators is given by:*

$$\begin{aligned} &\tilde f_j^\pm |\mu) = \pm(-1)^{\mu_{1r}+\mu_{2r}+\cdots+\mu_{rr}} f_j^\pm |\mu) \quad (j=1,\ldots,m); \\ &\tilde b_k^\pm |\mu) = b_k^\pm |\mu) \quad (k=1,\ldots,n). \end{aligned} \tag{3.4}$$

Thus the Fock representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{pso}(2m+1|2n)$ are completely understood. In the last section of this paper, we shall turn to an example and explain what happens when $p=1$.

4. Examples: $V(1)$ and $\tilde V(1)$

When the order of statistics p is equal to 1, the parastatistics Fock space $V(1)$ of $\mathfrak{osp}(2m+1|2n)$ and $\tilde V(1)$ of $\mathfrak{pso}(2m+1|2n)$ should simplify to ordinary boson-fermion Fock spaces. Although it is a simple special case of our more general result for arbitrary p, it is still instructive to work through this exercise and finally make the correspondence between the GZ-basis vectors of the previous section and the more common notation for boson-fermion states.

In order to make this identification, let us write out the betweenness conditions and θ-conditions of the GZ-patterns (3.2). For $p=1$, the basis vectors of the $\mathfrak{osp}(2m+1|2n)$ module $V(1)$ and the $\mathfrak{pso}(2m+1|2n)$ module $\tilde V(1)$ satisfy the following conditions:

$$\begin{array}{ll} 1. & \mu_{1,r} \in \{0,1\},\ \mu_{jr}-\mu_{j+1,r} \in \mathbb{Z}_+,\ 1 \le j \le m-1, \\ & \text{if } \mu_{mr}=0 \ \text{ then } \mu_{ir}=0,\ m+1 \le i \le r, \\ & \text{if } \ \mu_{mr}=1 \ \text{ then } \mu_{m+1,r} \in \mathbb{Z}_+ \ \text{ and } \ \mu_{ir}=0,\ m+2 \le i \le r; \\ 2. & \mu_{is}-\mu_{i,s-1} \equiv \theta_{i,s-1} \in \{0,1\}, \quad 1 \le i \le m;\ m+1 \le s \le r; \\ 3. & \text{if } \mu_{ms}=0, \ \text{ then } \mu_{is}=0,\ m+1 \le s \le r,\ m+1 \le i \le s, \\ & \text{if } \ \mu_{ms}=1, \ \text{ then } \mu_{m+1,s} \in \mathbb{Z}_+ \ \text{ and } \ \mu_{is}=0,\ m+1 \le s \le r,\ m+2 \le i \le s; \\ 4. & \mu_{is}-\mu_{i+1,s} \in \mathbb{Z}_+, \quad 1 \le i \le m-1;\ m+1 \le s \le r-1; \\ 5. & \mu_{i,j+1}-\mu_{ij} \in \mathbb{Z}_+ \ \text{ and } \mu_{i,j}-\mu_{i+1,j+1} \in \mathbb{Z}_+,\ 1 \le i \le j \le m-1; \\ 6. & \mu_{m+1,s} \le \mu_{m+1,s+1},\ m+1 \le s \le r-1. \end{array} \tag{4.1}$$

In particular, all entries $\mu_{i,j}$ with $i \le m$ belong to $\{0,1\}$, the entries $\mu_{m+1,j}$ belong to $\mathbb{Z}_+$, and the entries $\mu_{i,j}$ with $i > m+1$ are all zero. Defining

$$\varphi_i = \sum_{j=1}^{i} \mu_{ji} - \sum_{j=1}^{i-1} \mu_{j,i-1} \qquad (i=1,2,\ldots,r) \tag{4.2}$$

then it follows from the above conditions that

$$\varphi_i \in \{0,1\} \text{ for } i \le m, \qquad \varphi_i \in \mathbb{Z}_+ \text{ for } m+1 \le i \le r. \tag{4.3}$$

Conversely, for every sequence $(\varphi_1,\ldots,\varphi_r)$ satisfying (4.3), i.e. for every element of $\{0,1\}^m \times \mathbb{Z}_+^n$, there is a unique GZ-pattern $|\mu)$ satisfying (4.1). Indeed, taking already into account that the $\mu_{i,j}$ with $i > m+1$ are all zero, the equations (4.2) are equivalent with

$$\mu_{1i}+\cdots+\mu_{ii} = \varphi_1+\cdots+\varphi_i, \qquad (i \le m) \tag{4.4}$$

$$\mu_{1j}+\cdots+\mu_{m+1,j} = \varphi_1+\cdots+\varphi_j. \qquad (j > m) \tag{4.5}$$

If the right hand side of (4.4) is equal to k, then $k \le i$, and the only solution for the entries of $|\mu)$ is $\mu_{1i} = \cdots = \mu_{k,i} = 1$ and $\mu_{k+1,i} = \cdots = \mu_{ii} = 0$. If the right hand side of (4.5) is

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012102 doi:10.1088/1742-6596/1194/1/012102

equal to k, there are two cases: (1) if $k \leq m$, the only solution for the entries on row j of $|\mu)$ is $\mu_{1j} = \cdots = \mu_{k,j} = 1$ and $\mu_{k+1,j} = \cdots = \mu_{mj} = \mu_{m+1,j} = 0$; (2) if $k > m$, the only solution is $\mu_{1j} = \cdots = \mu_{m,j} = 1$ and $\mu_{m+1,j} = k - m$. So for $p = 1$ there is a one-to-one correspondence between the vectors $|\mu) = |1;\mu)$ of $V(1)$ (or $\tilde{V}(1)$) and the sequences $(\varphi_1, \ldots, \varphi_r)$ satisfying (4.3). Let us therefore denote the vectors of $V(1) \cong \tilde{V}(1)$ by $|\varphi_1, \ldots, \varphi_r\rangle$. In order to have appropriate signs for the actions, we introduce the following signs in the correspondence:

$$|\varphi_1, \ldots, \varphi_r\rangle = (-1)^{k(k-1)/2}|1;\mu), \qquad \text{for } m \text{ even}; \tag{4.6}$$

$$|\varphi_1, \ldots, \varphi_r\rangle = (-1)^{\mu_{m+1,r}}(-1)^{k(k-1)/2}|1;\mu), \qquad \text{for } m \text{ odd}; \tag{4.7}$$

$$\text{where } k = \sum_{i=1}^{m} \mu_{i,r-1}. \tag{4.8}$$

Let us now write down the actions (3.3) and (3.4) explicitly in this new notation for the basis. In order to distinguish this from the general case with arbitrary p, the representatives of $f_i^\pm$, $b_j^\pm$ in $V(1)$ are denoted by $F_i^\pm$, $B_j^\pm$, and the representatives of $\tilde{f}_i^\pm$, $\tilde{b}_j^\pm$ in $\tilde{V}(1)$ are denoted by $\tilde{F}_i^\pm$, $\tilde{B}_j^\pm$. One finds:

$$\begin{aligned}
F_i^+|\varphi_1, \ldots, \varphi_r\rangle &= (1-\varphi_i)(-1)^{\sum_{j=1}^{i-1}\varphi_j}|\cdots, \varphi_{i-1}, \varphi_i + 1, \varphi_{i+1}, \cdots\rangle;\\
F_i^-|\varphi_1, \ldots, \varphi_r\rangle &= \varphi_i(-1)^{\sum_{j=1}^{i-1}\varphi_j}|\cdots, \varphi_{i-1}, \varphi_i - 1, \varphi_{i+1}, \cdots\rangle;\\
B_i^+|\varphi_1, \ldots, \varphi_r\rangle &= \sqrt{\varphi_{m+i}+1}(-1)^{\sum_{j=1}^{m}\varphi_j}|\cdots, \varphi_{m+i-1}, \varphi_{m+i} + 1, \varphi_{m+i+1}, \cdots\rangle;\\
B_i^-|\varphi_1, \ldots, \varphi_r\rangle &= \sqrt{\varphi_{m+i}}(-1)^{\sum_{j=1}^{m}\varphi_j}|\cdots, \varphi_{m+i-1}, \varphi_{m+i} - 1, \varphi_{m+i+1}, \cdots\rangle;
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
\tilde{F}_i^+|\varphi_1, \ldots, \varphi_r\rangle &= (1-\varphi_i)(-1)^{\sum_{j=1}^{i-1}\varphi_j}|\cdots, \varphi_{i-1}, \varphi_i + 1, \varphi_{i+1}, \cdots\rangle;\\
\tilde{F}_i^-|\varphi_1, \ldots, \varphi_r\rangle &= \varphi_i(-1)^{\sum_{j=1}^{i-1}\varphi_j}|\cdots, \varphi_{i-1}, \varphi_i - 1, \varphi_{i+1}, \cdots\rangle;\\
\tilde{B}_i^+|\varphi_1, \ldots, \varphi_r\rangle &= \sqrt{\varphi_{m+i}+1}|\cdots, \varphi_{m+i-1}, \varphi_{m+i} + 1, \varphi_{m+i+1}, \cdots\rangle;\\
\tilde{B}_i^-|\varphi_1, \ldots, \varphi_r\rangle &= \sqrt{\varphi_{m+i}}|\cdots, \varphi_{m+i-1}, \varphi_{m+i} - 1, \varphi_{m+i+1}, \cdots\rangle.
\end{aligned} \tag{4.10}$$

From these relations, it is clear that the representatives of $f_i^\pm$, $b_j^\pm$ in $V(1)$ satisfy

$$\{F_j^-, F_k^-\} = 0, \qquad \{F_j^+, F_k^+\} = 0, \qquad \{F_j^-, F_k^+\} = \delta_{jk}, \tag{4.11}$$

$$[B_j^-, B_k^-] = 0, \qquad [B_j^+, B_k^+] = 0, \qquad [B_j^-, B_k^+] = \delta_{jk}, \tag{4.12}$$

$$\{F_j^\eta, B_k^\zeta\} = 0. \tag{4.13}$$

They form a set of ordinary fermion and boson operators, mutually anticommuting. Similarly, the representatives of $\tilde{f}_i^\pm$, $\tilde{b}_j^\pm$ in $\tilde{V}(1)$ satisfy (4.11), (4.12) and

$$[\tilde{F}_j^\eta, \tilde{B}_k^\zeta] = 0. \tag{4.14}$$

So now they form a set of ordinary fermion and boson operators, mutually commuting. Note that in both cases the basis vectors of $V(1)$, respectively $\tilde{V}(1)$, could be identified as follows:

$$|\varphi_1, \ldots, \varphi_{m+n}\rangle = \frac{1}{\sqrt{\varphi_{m+1}! \cdots \varphi_r!}}(F_1^+)^{\varphi_1} \cdots (F_m^+)^{\varphi_m}(B_1^+)^{\varphi_{m+1}} \cdots (B_n^+)^{\varphi_{m+n}}|0\rangle, \tag{4.15}$$

$$|\varphi_1, \ldots, \varphi_{m+n}\rangle = \frac{1}{\sqrt{\varphi_{m+1}! \cdots \varphi_r!}}(\tilde{F}_1^+)^{\varphi_1} \cdots (\tilde{F}_m^+)^{\varphi_m}(\tilde{B}_1^+)^{\varphi_{m+1}} \cdots (\tilde{B}_n^+)^{\varphi_{m+n}}|0\rangle. \tag{4.16}$$

It follows that fermion operators and boson operators that mutually anti-commute provide a particular realization of $\mathfrak{osp}(2m+1|2n)$ [17], whereas fermion operators and boson operators that mutually commute provide a particular realization of $\mathfrak{pso}(2m+1|2n)$.

Acknowledgments
The authors were supported by the Joint Research Project "Lie superalgebras – applications in quantum theory" in the framework of an international collaboration programme between the Research Foundation – Flanders (FWO) and the Bulgarian Academy of Sciences. NIS was partially supported by Bulgarian NSF grant DN 18/1.

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