A self-contained provability calculus for Γ_0

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Abstract. Beklemishev introduced an ordinal notation system for the Feferman-Schütte ordinal Γ_0 based on the *autonomous expansion* of provability algebras. In this paper we present the logic **BC** (for *Bracket Calculus*). The language of **BC** extends said ordinal notation system to a strictly positive modal language. Thus, unlike other provability logics, **BC** is based on a purely modal signature that gives rise to an ordinal notation system instead of modalities indexed by some ordinal given *a priori*. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. The presented logic is proven to be equivalent to $\mathbf{RC}_{\Gamma_0}$, that is, to the strictly positive fragment of $\mathbf{GLP}_{\Gamma_0}$.

Keywords: Provability logic · Proof theory · Ordinal analysis.

1 Introduction

In view of Gödel's second incompleteness theorem, we know that the consistency of any sufficiently powerful formal theory cannot be established using purely 'finitary' means. Since then, the field of proof theory, and more specifically of ordinal analysis, has been successful in measuring the non-finitary assumptions required to prove consistency assertions via computable ordinals. Among the benefits of this work is the ability to linearly order natural theories of arithmetic with respect to notions such as their 'consistency strength' (e.g., their Π^0_1 ordinal) or their 'computational strength' (their Π^0_2 ordinal). Nevertheless, the assignment of these proof-theoretic ordinals to formal theories depends on a choice of a 'natural' presentation for such ordinals, with well-known pathological examples having been presented by Kreisel [24] and Beklemishev [8].[3] This raises the question of what it means for something to be a natural ordinal notation system, or even if such a notion is meaningful at all.

[3] The Π^1_1 ordinal of a theory is another measure of its strength and does not have such sensitivity to a choice of notation system. However, there are some advantages to considering Π^0_1 ordinals, among others that they give a finer-grained classification of theories.

One possible approach to this problem comes from Beklemishev's ordinal analysis of Peano arithmetic (**PA**) and related theories via their *provability algebras.* Consider the Lindenbaum algebra of the language of arithmetic modulo provability in a finitary theory U such as primitive recursive arithmetic (**PRA**) or the weaker *elementary arithmetic* (**EA**). For each natural number n and each formula φ, the n-consistency of φ is the statement that all Σ_n consequences of $U+\varphi$ are true, formalizable by some arithmetical formula $\langle n\rangle\varphi$ (where φ is identified with its Gödel number). In particular, $\langle 0\rangle\varphi$ states that φ is consistent with U. An *iterated consistency assertion,* also called *worm,* is then an expression of the form $\langle n_1\rangle \ldots \langle n_k\rangle\top$, where $\top$ is some fixed tautology.

The operators $\langle n\rangle$ and their duals $[n]$ satisfy Japaridze's provability logic **GLP** [22], a multi-modal extension of the Gödel-Löb provability logic **GL** [12]. As Beklemishev showed, the set of worms is well-ordered by their *consistency strength* $<_0$, where $A <_0 B$ if $A \to \langle 0\rangle B$ is derivable in **GLP**. Moreover, this well-order is of order-type ε_0, which characterizes the proof-theoretical strength of **PA**. This tells us that proof-theoretic ordinals already appear naturally within Lindenbaum algebras of arithmetical theories.

Beklemishev also observed that this process can be extended by considering worms with ordinal entries. Extensions of **GLP**, denoted $\mathbf{GLP}_\Lambda$, have been considered in cases where Λ is an ordinal [3,14,18] or even an arbitrary linear order [6]. Proof-theoretic interpretations for $\mathbf{GLP}_\Lambda$ have been developed by Fernández-Duque and Joosten [17] for the case where Λ is a computable well-order. Nevertheless, we now find ourselves in a situation where an expression $\langle\lambda\rangle\varphi$ requires a system of notation for the ordinal λ. Fortunately we may 'borrow' this notation from finitary worms and represent λ itself as a worm. Iterating this process we obtain the *autonomous worms,* whose order types are exactly the ordinals below the Feferman-Schütte ordinal Γ_0. By iterating this process we obtain a notation system for worms which uses only parentheses, as ordinals (including natural numbers) can be iteratively represented in this fashion. Thus the worm $\langle 0\rangle\top$ becomes `()`, $\langle 1\rangle\top$ becomes `(())`, $\langle\omega\rangle\top$ becomes `((()))`, etc.

These are Beklemishev's *brackets,* which provide a notation system for Γ_0 without any reference to an externally given ordinal [3]. However, it has the drawback that the actual computation of the ordering between different worms is achieved via a translation into a traditional ordinal notation system. Our goal is to remove the need for such an intermediate step by providing an autonomous calculus for determining the ordering relation (and, more generally, the logical consequence relation) between bracket notations. To this end we present the *bracket calculus;* our main result is that our calculus is sound and complete with respect to the intended embedding into $\mathbf{GLP}_{\Gamma_0}$.

2 The Reflection Calculus

Japaridze's logic **GLP** gained much interest due to Beklemishev's proof-theoretic applications [2]; however, from a modal logic point of view, it is not an easy system to work with. To this end, in [13,4,5] Beklemishev and Dashkov introduced the system called *Reflection Calculus*, **RC**, that axiomatizes the fragment of $\mathbf{GLP}_\omega$ consisting of implications of strictly positive formulas. This system is much simpler than $\mathbf{GLP}_\omega$ but yet expressive enough to maintain its main proof-theoretic applications. In this paper we will focus exclusively on reflection calculi, but the interested reader may find more information on the full **GLP** in the references provided.

Similar to $\mathbf{GLP}_\Lambda$, the signature of $\mathbf{RC}_\Lambda$ contains modalities of the form $\langle\, \alpha \,\rangle$ for $\alpha \in \Lambda$. However, since this system only considers strictly positive formulas, the signature does not contain negation, disjunction or modalities $[\, \alpha \,]$. Thus, the set of formulas in this signature is defined as follows:

Definition 1. *Fix an ordinal Λ. By $\mathbb{F}_\Lambda$ we denote the set of formulas built-up by the following grammar:*

$$\varphi := \top \mid p \mid (\varphi \wedge \psi) \mid \langle\, \alpha \,\rangle\varphi \quad \textit{for } \alpha \in \Lambda.$$

Next we define a consequence relation over $\mathbb{F}_\Lambda$. For the purposes of this paper, a *deductive calculus* is a pair $\mathbf{X} = (\mathbb{F}_\mathbf{X}, \vdash_\mathbf{X})$ such that $\mathbb{F}_\mathbf{X}$ is some set, the *language* of $\mathbf{X}$, and $\vdash_\mathbf{X} \subseteq \mathbb{F}_\mathbf{X} \times \mathbb{F}_\mathbf{X}$. We write $\varphi \cong_\mathbf{X} \psi$ for $\varphi \vdash_\mathbf{X} \psi$ and $\psi \vdash_\mathbf{X} \varphi$. We will omit the subscript $\mathbf{X}$ when this does not lead to confusion, including in the definition below, where $\vdash$ denotes $\vdash_{\mathbf{RC}_\Lambda}$.

Definition 2. *Given an ordinal Λ, the calculus $\mathbf{RC}_\Lambda$ over $\mathbb{F}_\Lambda$ is given by the following set of axioms and rules:*

Axioms:

1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$;
2. $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;
3. $\langle\, \alpha \,\rangle\langle\, \alpha \,\rangle\varphi \vdash \langle\, \alpha \,\rangle\varphi$;
4. $\langle\, \alpha \,\rangle\varphi \vdash \langle\, \beta \,\rangle\varphi$ *for* $\alpha > \beta$;
5. $\langle\, \alpha \,\rangle\varphi \wedge \langle\, \beta \,\rangle\psi \vdash \langle\, \alpha \,\rangle\big(\varphi \wedge \langle\, \beta \,\rangle\psi\big)$ *for* $\alpha > \beta$.

Rules:

1. *If* $\varphi \vdash \psi$ *and* $\varphi \vdash \chi$, *then* $\varphi \vdash \psi \wedge \chi$;
2. *If* $\varphi \vdash \psi$ *and* $\psi \vdash \chi$, *then* $\varphi \vdash \chi$;
3. *If* $\varphi \vdash \psi$, *then* $\langle\, \alpha \,\rangle\varphi \vdash \langle\, \alpha \,\rangle\psi$;

For each $\mathbf{RC}_\Lambda$-formula φ, we can define the signature of φ as the set of ordinals occurring in any of its modalities.

Definition 3. *For any $\varphi \in \mathbb{F}_\Lambda$, we define the* signature *of φ, $\mathcal{S}(\varphi)$, as follows:*
1. $\mathcal{S}(\top) = \mathcal{S}(p) = \varnothing$;
2. $\mathcal{S}(\varphi \wedge \psi) = \mathcal{S}(\varphi) \cup \mathcal{S}(\psi)$;
3. $\mathcal{S}(\langle\, \alpha \,\rangle\varphi) = \{\alpha\} \cup \mathcal{S}(\varphi)$.

With the help of this last definition we can make the following observation:

Lemma 1. *For any $\varphi, \psi \in \mathbb{F}_\Lambda$:*
1. If $\mathcal{S}(\psi) \neq \varnothing$ *and* $\varphi \vdash \psi$, *then* $\max \mathcal{S}(\varphi) \geq \max \mathcal{S}(\psi)$;
2. If $\mathcal{S}(\varphi) = \varnothing$ *and* $\varphi \vdash \psi$, *then* $\mathcal{S}(\psi) = \varnothing$.

Proof. By an easy induction on the length of the derivation of $\varphi \vdash \psi$.

The reflection calculus has natural arithmetical [17], Kripke [13,5], algebraic [11] and topological [7,14,20,21] interpretations for which it is sound and complete, but in this paper we will work exclusively with reflection calculi from a syntactical perspective. Other variants of the reflection calculus have been proposed, for example working exclusively with worms [1], admitting the transfinite iteration of modalities [19], or allowing additional conservativity operators [9,10].

3 Worms and the consistency ordering

In this section we review the consistency ordering between worms, along with some of their basic properties.

Definition 4. *Fix an ordinal Λ. The set of* worms *in $\mathbb{F}_\Lambda$, $\mathbb{W}_\Lambda$, is recursively defined as follows: 1. $\top \in \mathbb{W}_\Lambda$; 2. If $A \in \mathbb{W}_\Lambda$ and $\alpha < \Lambda$, then $\langle \alpha \rangle A \in \mathbb{W}_\Lambda$. Similarly, we inductively define for each $\alpha \in \Lambda$ the set of worms $\mathbb{W}_\Lambda^{\geq\alpha}$ where all ordinals are at least α: 1. $\top \in \mathbb{W}_\Lambda^{\geq\alpha}$; 2. If $A \in \mathbb{W}_\Lambda^{\geq\alpha}$ and $\beta \geq \alpha$, then $\langle \beta \rangle A \in \mathbb{W}_\Lambda^{\geq\alpha}$.*

Definition 5. *Let $A = \langle\xi_1\rangle \ldots \langle\xi_n\rangle\top$ and $B = \langle\zeta_1\rangle \ldots \langle\zeta_m\rangle\top$ be worms. Then, define $AB = \langle\xi_1\rangle \ldots \langle\xi_n\rangle\langle\zeta_1\rangle \ldots \langle\zeta_m\rangle\top$. Given an ordinal λ, define $\lambda \uparrow A$ to be $\langle\lambda + \xi_1\rangle \ldots \langle\lambda + \xi_n\rangle\top$.*

Often we will want to put an extra ordinal between two worms, and we write $B\langle\lambda\rangle A$ for $B(\langle\lambda\rangle A)$. Next, we define the consistency ordering between worms.

Definition 6. *Given an ordinal Λ, we define a relation $<_0$ on $\mathbb{W}_\Lambda$ by $B <_0 A$ if and only if $A \vdash \langle 0\rangle B$. We also define $B \leq_0 A$ if $B <_0 A$ or $B \cong A$.*

The ordering $\leq_0$ has some nice properties. Recall that if A is a set (or class), a *preorder* on A is a trasitive, reflexive relation $\preccurlyeq \subseteq A \times A$. The preorder $\preccurlyeq$ is *total* if, given $a, b \in A$, we always have that $a \preccurlyeq b$ or $b \preccurlyeq a$, and *antisymmetric* if whenever $a \preccurlyeq b$ and $b \preccurlyeq a$, it follows that $a = b$. A total, antisymmetric preorder is a *linear order.* We say that $\langle A, \preccurlyeq\rangle$ is a *pre-well-order* if $\preccurlyeq$ is a total preorder and every non-empty $B \subseteq A$ has a minimal element (i.e., there is $m \in B$ such that $m \preccurlyeq b$ for all $b \in B$). A *well-order* is a pre-well-order that is also linear. Note that pre-well-orders are not the same as well-quasiorders (the latter need not be total). Pre-well-orders will be convenient to us because, as we will see, worms are pre-well-ordered but not linearly ordered.

Theorem 1. *For any ordinal Λ, the relation $\leq_0$ is a pre-well-order on $\mathbb{W}_\Lambda$.*

Note that $\leq_0$ fails to be a linear order merely because it is not antisymmetric. To get around this, one may instead consider worms modulo provable equivalence. Alternately, as Beklemishev has done [3], one can choose a canonical representative for each worm.

Definition 7 (Beklemishev Normal Form). *A worm $A \in \mathbb{W}$ is defined recursively to be in* BNF *if either*

1. $A = \top$, *or*
2. $A := A_k\langle \alpha \rangle A_{k-1}\langle \alpha \rangle \ldots \langle\alpha\rangle A_0$ *with*
   - $\alpha = \min \mathcal{S}(A)$;
   - $k \geq 1$;
   - $A_i \in \mathbb{W}^{\geq \alpha+1}_A$, *for* $i \leq k$;

   such that $A_i \in$ BNF *and* $A_i \vdash_{\mathbf{RC}_{\Gamma_0}} \langle \alpha + 1 \rangle A_{i+1}$ *for each* $i < k$.

This definition essentially mirrors that of Cantor normal forms for ordinals. The following was proven in [3].

Theorem 2. *Given any worm A there is a unique $A' \in$ BNF such that $A \cong A'$.*

4 Hyperexponential notation for Γ_0

Ordinal numbers are canonical representatives of well-orders; we assume some basic familiarity with them, but a detailed account can be found in a text such as [23]. In particular, since the set of worms modulo equivalence yields a well-order, we can use ordinal numbers to measure their order-types. More generally, if $\mathfrak{A} = \langle A, \preccurlyeq \rangle$ is any pre-well-order, for $a \in A$ we may define an ordinal $o(a) = \sup_{b \prec a}(o(b) + 1)$, where by convention $\sup \varnothing = 0$, representing the *order-type* of a; this definition is sound since $\mathfrak{A}$ is pre-well-ordered. The rank of $\mathfrak{A}$ is then defined as $\sup_{a \in A}(o(a) + 1)$.

The following lemma is useful in characterizing the rank function [15].

Lemma 2. *Let $\langle A, \preccurlyeq \rangle$ be a well-order. Then $o \colon A \to \mathsf{Ord}$ is the unique function such that*

1. $x \prec y$ *implies that* $o(x) < o(y)$,
2. *if* $\xi < o(x)$ *then* $\xi = o(y)$ *for some* $y \in A$.

In order to compute the ordinals $o(A)$, let us recall a notation system for Γ_0 using *hyperexponentials* [16]. The class of all ordinals will be denoted Ord, and ω denotes the first infinite ordinal. Recall that many number-theoretic operations such as addition, multiplication and exponentiation can be defined on the class of ordinals by transfinite recursion. The ordinal exponential function $\xi \mapsto \omega^\xi$ is of particular importance for representing ordinal numbers. When working with order types derived from reflection calculi, it is convenient to work with a slight variant of this exponential.

Definition 8 (Exponential function). *The* exponential function *is the normal function $e \colon \mathsf{Ord} \to \mathsf{Ord}$ given by $\xi \mapsto -1 + \omega^\xi$.*

The function e is an example of a *normal function,* i.e. $f \colon \mathsf{Ord} \to \mathsf{Ord}$ which is strictly increasing and *continuous,* in the sense that if λ is a limit then $f(\lambda) = \sup_{\xi<\lambda} f(\xi)$. When $f \colon X \to X$, it is natural and often useful to ask whether f has *fixed points,* i.e., solutions to the equation $x = f(x)$. In particular, normal functions have many fixed points:

Proposition 1. *Every normal function on* Ord *has arbitrarily large fixed points.*

The first ordinal α such that $\alpha = \omega^\alpha$ is the limit of the ω-sequence $(\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots)$, and is usually denoted ε_0. Every $\xi < \varepsilon_0$ can be written in terms of 0 using only addition and the function $\omega \mapsto \omega^\xi$ via its Cantor normal form. The hyperexponential function is then a natural transfinite iteration of the ordinal exponential which remains normal after each iteration.

Definition 9 (Hyperexponential functions). *The* hyperexponential functions $(e^\zeta)_{\zeta \in \mathsf{Ord}}$ *are the unique family of normal functions that satisfy*

1. $e^1 = e$,
2. $e^{\alpha+\beta} = e^\alpha \circ e^\beta$ *for all* α *and* β, *and*
3. *if* $(f^\xi)_{\xi \in \mathsf{Ord}}$ *is a family of functions satisfying 1 and 2, then for all* $\alpha, \beta \in \mathsf{Ord}$, $e^\alpha \beta \leq f^\alpha \beta$.

Fernández-Duque and Joosten proved that the hyperexponentials are well-defined [16]. If $\alpha > 0$ then $e^\alpha \beta$ is always *additively indecomposable* in the sense that $\xi, \zeta < e^\alpha \beta$ implies that $\xi + \zeta < e^\alpha \beta$; note that zero is additively indecomposable according to our definition. In [15] it is also shown that the function $\xi \mapsto e^\xi 1$ is itself a normal function, hence it has a least non-zero fixed point: this fixed point is the Feferman-Schütte ordinal, Γ_0. Just like ordinals below ε_0 may be written using 0, addition, and ω-exponentiation, every ordinal below Γ_0 may be written in terms of 0, 1, addition and the function $(\xi, \zeta) \mapsto e^\xi \zeta$.

Theorem 3. *Let* A, B *be worms and* α *be an ordinal. Then,*

1. $o(\top) = 0$,
2. $o(B\langle 0 \rangle A) = o(A) + 1 + o(B)$, *and*
3. $o(\alpha \uparrow A) = e^\alpha o(A)$.

Remark 1. We will not discuss notation systems based on the Veblen hierarchy $(\phi_\xi)_{\xi \in \mathsf{Ord}}$, but a fairly simple translation from one notation to the other is given in [16]. Beklemishev [3] gives an explicit computation of o in terms of the standard Veblen functions.

Finally we mention a useful property of o proven in [15], where $\max A$ is the greatest ordinal appearing in A.

Lemma 3. *Let* $A \neq \top$ *be a worm and* μ *an ordinal. Then,*

1. *if* $\mu \leq \max A$, *then* $o(\langle \mu \rangle \top) \leq o(A)$, *and*
2. *if* $\max A < \mu$, *then* $o(A) < o(\langle \mu \rangle \top)$.

5 Beklemishev's bracket notation system for Γ_0

Before we introduce the full bracket calculus, let us review Beklemishev's notation system from [3].

Definition 10. *By $\mathbb{W}_{()}$ we denote the smallest set such that: 1. $\top \in \mathbb{W}_{()}$; 2. if a, $b \in \mathbb{W}_{()}$, then $(a)b \in \mathbb{W}_{()}$.*

By convention we shall write $()a$, for $a \in \mathbb{W}_{()}$ to the denote $(\top)a \in \mathbb{W}_{()}$.

We can define a translation $* : \mathbb{W}_{()} \to \mathbb{W}$ in such a way that an element $a \in \mathbb{W}_{()}$ will denote the ordinal $o(a^*)$:

1. $\top^* = \top$
2. $\big((a)b\big)^* = \langle\, o(a^*)\,\rangle b^*$.

Therefore, we can also define $o^* : \mathbb{W}_{()} \to \Gamma_0$ as $o^*(a) = o(a^*)$.

Next we make some observations about how the ordinals represented by worms in $\mathbb{W}_{()}$ can be bounded in terms of the maximum number of nested brackets occurring in them. With this purpose, we introduce the following two definitions.

Definition 11. *For $a \in \mathbb{W}_{()}$, we define the nesting of a, $\mathsf{N}(a)$, as the maximum number of nested brackets. That is:*

1. $\mathsf{N}(\top) = 0$;
2. $\mathsf{N}((a)b) = \max\big(\mathsf{N}(a) + 1, \mathsf{N}(b)\big)$.

Definition 12. *We recursively define the function $h : \mathbb{N} \to \Gamma_0$ as follows:*

1. $h(0) = 0$;
2. $h(n+1) = e^{h(n)}1$.

Note that $\lim_{n\to\infty} h(n) = \Gamma_0$. In the following proposition we can find upper and lower bounds for any ordinal $o^*(a)$, with $a \in \mathbb{W}_{()}$, according to the nesting of a.

Proposition 2. *For $a \in \mathbb{W}_{()}$, if $\mathsf{N}(a) = n$, then $h(n) \leq o^*(a) < h(n+1)$.*

Proof. By induction on n. If $\boldsymbol{n = 0}$ then we must have $a = \top$, hence $h(0) = 0 = o^*(a) < 1 = h(1)$.

For $\boldsymbol{n = n' + 1}$, we have that $a = (a_0)\ldots(a_m)$ for some $m \in \omega$. Moreover,

1. $\mathsf{N}(a_i) \leq n'$ for $i,\ 0 \leq i \leq m$;
2. there is a_J such that $\mathsf{N}(a_J) = n'$.

Thus by the I.H. we get that $a^* = \langle\alpha_0\rangle\ldots\langle\alpha_k\rangle\top$ such that:

1. For each i, $\alpha_i < h(n'+1)$;
2. there is $\alpha_J \geq h(n')$.

By Lemma 3,

$$o(\langle h(n')\rangle\top) \leq o(a^*) < o(\langle h(n'+1)\rangle\top);$$

but by Theorem 3 $o(\langle h(n')\rangle\top) = e^{h(n')}1 = h(n)$, while $o(\langle h(n'+1)\rangle\top) = e^{h(n)}1 = h(n+1)$, as needed.

As a consequence of this last proposition, we get the following corollaries.

Corollary 1. *For $a \in \mathbb{W}_{()}$, if $\mathsf{N}(a) = n$, then $a^* \in \mathbb{W}_{h(n)}$.*

Corollary 2. *For a, $b \in \mathbb{W}_{()}$, $o^*(a) \geq o^*(b) \;\Rightarrow\; \mathsf{N}(a) \geq \mathsf{N}(b)$.*

Proof. We reason by contrapositive applying Proposition 2.

6 The Bracket Calculus

In this section we introduce the *Bracket Calculus*, denoted **BC**. This system is analogous to $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ and, as we will see later, both systems can be shown to be equivalent under a natural translation of **BC**-formulas into $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-formulas.

The main feature of **BC** is that it is based on a signature that uses purely modal notations instead of modalities indexed by ordinals. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. In this sense, we say that **BC** provides an autonomous provability calculus.

The set of **BC**-formulas, $\mathbb{F}_{()}$, is defined by extending $\mathbb{W}_{()}$ to a strictly positive signature.

Definition 13. *By $\mathbb{F}_{()}$ we denote the set of formulas built-up by the following grammar:*

$$\varphi := \top \mid p \mid \varphi \wedge \psi \mid (a)\,\varphi \quad \textit{for } a \in \mathbb{W}_{()}.$$

Similarly to **RC**, **BC** is based on *sequents*, i.e. expressions of the form $\varphi \vdash \psi$, where $\varphi, \psi \in \mathbb{F}_{()}$. In addition to this, we will also use $a \succeq b$, for $a, b \in \mathbb{W}_{()}$, to denote that either $a \vdash ()\,b$ or $a \vdash b$ are derivable. Analogously, we will use $a \succ b$ to denote that the sequent $a \vdash ()b$ is derivable.

Definition 14. ***BC*** *is given by the following set of axioms and rules:*

Axioms: 1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$; *2.* $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;
Rules:

1. *If* $\varphi \vdash \psi$ *and* $\varphi \vdash \chi$, *then* $\varphi \vdash \psi \wedge \chi$;
2. *If* $\varphi \vdash \psi$ *and* $\psi \vdash \chi$, *then* $\varphi \vdash \chi$;
3. *If* $\varphi \vdash \psi$ *and* $a \succeq b$, *then* $(a)\,\varphi \vdash (b)\,\psi$ *and* $(a)\,(b)\,\varphi \vdash (b)\,\psi$;
4. *If* $a \succ b$, *then* $(a)\,\varphi \wedge (b)\,\psi \vdash (a)\big(\varphi \wedge (b)\,\psi\big)$.

7 Translation and preservability

In this section we introduce a way of interpreting **BC**-formulas as $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-formulas, and prove that under this translation, both systems can derive exactly the same sequents.

Definition 15. *We define a translation τ between $\mathbb{F}_{()}$ and $\mathbb{F}_{\Gamma_0}$, $\tau : \mathbb{F}_{()} \to \mathbb{F}_{\Gamma_0}$, as follows:*

1. $\top^\tau = \top$;
2. $p^\tau = p$;
3. $(\varphi \wedge \psi)^\tau = (\varphi^\tau \wedge \psi^\tau)$;
4. $(\,(a)\,\varphi)^\tau = \langle\, o^*(a)\,\rangle \varphi^\tau$.

Note that for $a \in \mathbb{W}_{()}$, $a^\tau = a^*$. From this and a routine induction, the following can readily be verified.

Lemma 4. *Given $\varphi \in \mathbb{F}_{()}$ and $\alpha \in \mathcal{S}(\varphi^\tau)$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\alpha = o^*(a)$.*

The following lemma establishes the preservability of **BC** with respect to $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$, under τ.

Lemma 5. *For any $\varphi, \psi \in \mathbb{F}_{()}$: $\varphi \vdash_{\boldsymbol{BC}} \psi \implies \varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$.*

Proof. By induction on the length of the derivation. We can easily check that the set of axioms of **BC** is preserved under τ. Likewise, the cases for a derivation ending on Rules 1 or 2 are straightforward. Thus, we only check Rules 3 and 4.

Regarding Rule 3, we need to prove that if $a \succeq b$ then both sequents $\langle o^*(a)\rangle\varphi^\tau \vdash \langle o^*(b)\rangle\psi^\tau$ and $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash \langle o^*(b)\rangle\psi^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$. We can make the following observations by applying the I.H.:

1. Since $a \succeq b$, we have that either $a^\tau \vdash \langle 0\rangle b^\tau$ or $a^\tau \vdash b^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$. Therefore, $o(a^\tau) \geq o(b^\tau)$. Since $o^*(a) = o(a^*) = o(a^\tau)$ and the same equality holds for b, we have that $o^*(a) \geq o^*(b)$.
2. We also have that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ and thus, by Rule 3 of $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ we obtain that $\langle o^*(a)\rangle\varphi^\tau \vdash \langle o^*(a)\rangle\psi^\tau$ and $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash \langle o^*(a)\rangle\langle o^*(b)\rangle\psi^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$.

On the one hand, by these two facts together with Axiom 4 we obtain that $\langle o^*(a)\rangle\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(b)\rangle\psi^\tau$. On the other hand, we can combine Axioms 4 and 3 to get that $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(b)\rangle\psi^\tau$.

We follow an analogous reasoning in the case of Rule 4. By the I.H. we have that $a^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0\rangle b^\tau$. Therefore $o^*(a) > o^*(b)$ and by Axiom 5, $\langle o^*(a)\rangle\varphi \wedge \langle o^*(b)\rangle\psi \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(a)\rangle\big(\varphi \wedge \langle o^*(b)\rangle\psi\big)$.

With the following definition we fix a way of translating $\mathbb{F}_{\Gamma_0}$-formulas into formulas in $\mathbb{F}_{()}$. However, since different words in $\mathbb{W}_{()}$ might denote the same ordinal, we need a normal form theorem for $\mathbb{W}_{()}$.

Definition 16. *We define $\mathsf{NF} \subset \mathbb{W}_{()}$ to be the smallest set of $\mathbb{W}_{()}$-words such that $\top \in \mathsf{NF}$ and for any $(a)b \in \mathbb{W}_{()}$, if $a, b \in \mathsf{NF}$ and $\big((a)b\big)^* \in \mathsf{BNF}$, then $(a)b \in \mathsf{NF}$.*

Every element of $\mathbb{W}_{()}$ has a unique normal form, as shown by L. Beklemishev in [3].

Theorem 4 (Beklemishev). *For each $\alpha \in \Gamma_0$ we can associate a unique $a_\alpha \in \mathsf{NF}$ such that $o^*(a_\alpha) = \alpha$.*

Proposition 3 (Beklemishev). *The ordering $\big(\mathsf{NF}, <_0\big)$ is a well-ordering of order type Γ_0.*

Now we are ready to translate $\mathbb{F}_{\Gamma_0}$-formulas into $\mathbb{F}_{()}$-formulas.

Definition 17. *We define a translation ι between $\mathbb{F}_{\Gamma_0}$ and $\mathbb{F}_{()}$, $\iota : \mathbb{F}_{\Gamma_0} \to \mathbb{F}_{()}$, as follows:*

1. $\top^\iota = \top$;
2. $p^\iota = p$;
3. $(\varphi \wedge \psi)^\iota = (\varphi^\iota \wedge \psi^\iota)$;
4. $(\langle\alpha\rangle\,\varphi)^\iota = (a_\alpha)\varphi^\iota$.

The following remark follows immediately from the definitions of τ and ι.

Remark 2. For any $\varphi \in \mathbb{F}_{\Gamma_0}$, $(\varphi^\iota)^\tau = \varphi$. In particular, if $A \in \mathbb{W}_{\Gamma_0}$ is a worm then A^ι is a worm and $o^*(A^\iota) = o((A^\iota)^*) = o((A^\iota)^\tau) = o(A)$.

With the next definition, we extend the nesting $\mathsf{N}(a)$ of $a \in \mathbb{W}_{()}$ to $\mathbb{F}_{()}$-formulas.

Definition 18. *For $\varphi \in \mathbb{F}_{()}$, we define the nesting of φ, $\mathsf{Nt}(\varphi)$, as the maximum number of nested brackets. That is:*

1. $\mathsf{Nt}(\top) = \mathsf{Nt}(p) = \mathsf{N}(\top)$;
2. $\mathsf{Nt}(\varphi \wedge \psi) = \max\big(\mathsf{Nt}(\varphi),\, \mathsf{Nt}(\psi)\big)$;
3. $\mathsf{Nt}((a)\,\varphi) = \max\big(\mathsf{N}((a)),\, \mathsf{Nt}(\varphi)\big) = \max\big(\mathsf{N}(a)+1,\, \mathsf{Nt}(\varphi)\big)$.

The upcoming remark collects a useful observation concerning the nesting $\mathsf{Nt}(\varphi)$ of a formula φ and its subformulas. This fact can be verified by an easy induction.

Remark 3. For any $\varphi \in \mathbb{F}_{()}$ with $\varphi \neq p$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\mathsf{Nt}(\varphi) = \mathsf{Nt}(a)$. Moreover, if $\mathsf{Nt}(\varphi) \geq 1$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\mathsf{Nt}(\varphi) = \mathsf{Nt}(a) + 1$.

The following lemma relates the derivability in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ under τ, and the nesting of formulas in $\mathbb{F}_{()}$.

Lemma 6. *For any φ, $\psi \in \mathbb{F}_{()}$:*

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \implies \mathsf{Nt}(\varphi) \geq \mathsf{Nt}(\psi).$$

Proof. Suppose that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. If $\mathcal{S}(\psi^\tau) = \varnothing$ then it is easy to check that $\mathsf{Nt}(\psi) = 0$ and there is nothing to prove, so assume otherwise. Then, by Lemma 1.1, $\max\mathcal{S}(\varphi^\tau) \geq \max\mathcal{S}(\psi^\tau)$. Using Lemma 4, let $a \in \mathbb{W}_{()}$ be a subformula of φ such that $o^*(a) = \max\mathcal{S}(\varphi^\tau)$. Moreover, since $\mathcal{S}(\psi^\tau) = \varnothing$, then $\mathsf{Nt}(\psi) \geq 1$. Therefore, with the help of Remark 3 we can consider $b \in \mathbb{W}_{()}$, a subformula of ψ such that $\mathsf{Nt}(\psi) = \mathsf{N}(b) + 1$. If we had $\mathsf{N}(a) < \mathsf{N}(b)$ then it would follow from Corollary 2 that $o^*(a) < o^*(b)$, contradicting $\max\mathcal{S}(\varphi^\tau) \geq \max\mathcal{S}(\varphi^\tau)$. Thus $\mathsf{N}(a) \geq \mathsf{N}(b)$ and $\mathsf{Nt}(\varphi) \geq \mathsf{N}(a) + 1 \geq \mathsf{Nt}(\psi)$, as needed.

With the following theorem we conclude the proof of the preservability between **BC** and $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$.

Theorem 5. *For any φ, $\psi \in \mathbb{F}_{()}$:*

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \iff \varphi \vdash_{\boldsymbol{BC}} \psi.$$

Proof. The right-to-left direction is given by Lemma 5, so we focus on the other. Proceed by induction on $\mathsf{Nt}(\varphi)$. For the base case, assume $\mathsf{Nt}(\varphi) = 0$ and $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By a subsidiary induction on the length of the derivation of $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$, we set to prove $\varphi \vdash_{\mathbf{BC}} \psi$. If the derivation has length one it suffices to check $\mathbf{RC}_{\Gamma_0}$-Axioms 1 and 2, which is immediate. If it has length greater than one it must end in a rule. The case for $\mathbf{RC}_{\Gamma_0}$-Rule 1 follows by the I.H.. For $\mathbf{RC}_{\Gamma_0}$-Rule 2, we have that there is $\chi \in \mathbb{F}_{\Gamma_0}$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 2 and Lemma 6, we get that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi^\iota)^\tau$ and $(\chi^\iota)^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\mathsf{Nt}(\chi^\iota) = 0$. Thus, by the subsidiary I.H., $\varphi \vdash_{\mathbf{BC}} \chi^\iota$ and $\chi^\iota \vdash_{\mathbf{BC}} \psi$ and by $\mathbf{BC}$-Rule 2, $\varphi \vdash_{\mathbf{BC}} \psi$.

For the inductive step, let $\mathsf{Nt}(\varphi) = n + 1$. We proceed by a subsidiary induction on the length of the derivation. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by means of $\mathbf{RC}_{\Gamma_0}$-Axioms 1 and 2, then clearly $\varphi \vdash_{\mathbf{BC}} \psi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of $\mathbf{RC}_{\Gamma_0}$-Axiom 3, then we have that $\varphi^\tau := \langle o^*(a) \rangle \langle o^*(b) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(c) \rangle \chi^\tau$ for some $\chi \in \mathbb{F}_{()}$ and $a, b, c \in \mathbb{W}_{()}$ such that $o^*(a) = o^*(b) = o^*(c)$. Hence, there are $A, B, C \in \mathbb{W}$ such that $a^* = A$, $b^* = B$ and $c^* = C$, and so $A \vdash_{\mathbf{RC}_{\Gamma_0}} B$ and $B \vdash_{\mathbf{RC}_{\Gamma_0}} C$. Since $\mathsf{Nt}(w) < n + 1$ for $w \in \{a, b, c\}$, by the main I.H. we have that $a \vdash_{\mathbf{BC}} b$ and $b \vdash_{\mathbf{BC}} c$. Thus, we have the following $\mathbf{BC}$-derivation:

$$\dfrac{\dfrac{\dfrac{\chi \vdash \chi \qquad b \vdash c}{(b)\chi \vdash (c)\chi}\,\text{(Rule 3)} \qquad a \vdash b}{(a)\,(b)\chi \vdash (b)\,(c)\chi}\,\text{(Rule 3)} \qquad \dfrac{\chi \vdash \chi \qquad b \vdash c}{(b)\,(c)\chi \vdash (c)\chi}\,\text{(Rule 3)}}{(a)\,(b)\chi \vdash (c)\chi}\,\text{(Rule 2)}$$

If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by using $\mathbf{RC}_{\Gamma_0}$-Axiom 4, then $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(b) \rangle \chi^\tau$. for some $\chi \in \mathbb{F}_{()}$ and $a, b, \in \mathbb{W}_{()}$ with $o^*(a) > o^*(b)$. Therefore, there are $A, B \in \mathbb{W}_{\Gamma_0}$ such that $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$, $a^* = A$ and $b^* = B$. Since $o^*(a) \geq o^*(()b)$, by Lemma 1, $\mathsf{Nt}(()b) \leq \mathsf{Nt}(a)$ and since $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$, we have that $\mathsf{Nt}(a) < \mathsf{Nt}(\varphi)$. Thus, by the main I.H. $a \vdash_{\mathbf{BC}} ()b$ and by $\mathbf{BC}$-Rule 3, $(a)\chi \vdash_{\mathbf{BC}} (b)\chi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of $\mathbf{RC}_{\Gamma_0}$-Axiom 5, then we have that $\varphi^\tau := \langle o^*(a) \rangle \chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau$ and $\psi^\tau := \langle o^*(a) \rangle \big(\chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau \big)$, for some $\chi_0, \chi_1 \in \mathbb{F}_{()}$ and $a, b \in \mathbb{W}_{()}$ with $o^*(a) > o^*(b)$. Therefore, there are $A, B \in \mathbb{W}_{\Gamma_0}$ such that $a^* = A$, $b^* = B$ and $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$. By Lemma 1 together with the main I.H. we obtain that $a \vdash_{\mathbf{BC}} ()b$ and by applying $\mathbf{BC}$-Rule 4, $(a)\chi_0 \wedge (b)\chi_1 \vdash (a)\big(\chi_0 \wedge (b)\chi_1\big)$. Regarding rules, $\mathbf{RC}_{\Gamma_0}$-Rule 1 is immediate and $\mathbf{RC}_{\Gamma_0}$-Rule 3 follows an analogous reasoning to that of Axiom 4. This way, we only check $\mathbf{RC}_{\Gamma_0}$-Rule 2. Assume $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by an application of $\mathbf{RC}_{\Gamma_0}$-Rule 2. Then, there is $\chi \in \mathbb{F}_{\Gamma_0}$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 2 together with Lemma 6 we obtain that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi^\iota)^\tau$ and $(\chi^\iota)^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\mathsf{Nt}(\chi) \leq n + 1$. By the subsidiary I.H. $\varphi \vdash_{\mathbf{BC}} \chi^\iota$ and $\chi^\iota \vdash_{\mathbf{BC}} \psi$ and hence, by $\mathbf{BC}$-Rule 2, $\varphi \vdash_{\mathbf{BC}} \psi$.

With this we obtain our main result: an autonomous calculus for representing ordinals below Γ_0.

Theorem 6. *For $a, b \in \mathsf{NF}$ define $a \lhd b$ if and only if $a \vdash_{\mathbf{BC}} ()b$. Then, $\lhd$ is a strict linear order of order-type Γ_0.*

Proof. By Theorem 5, $a \lhd b$ if and only if $a^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle b^\tau$ if and only if $o^*(a) < o^*(b)$. Moreover if $\xi < o^*(a)$ then by item 2 of Lemma 2 there is some $B <_0 a^\tau$ such that $\xi = o(B)$, hence in view of Remark 2, $\xi = o^*(B^\iota)$. Thus by Lemma 2, o^* is the order-type function on NF. That the range of o^* is Γ_0 follows from Proposition 2 which tells us that $o^*(a) < h(\mathsf{N}(a)+1) < \Gamma_0$ for all $a \in \mathbb{W}_{()}$, while if we define recursively $a_0 = \top$ and $a_{n+1} = (a_n)$, Theorem 3 and an easy induction readily yield $\Gamma_0 = \lim_{n\to\infty} h(n) = \lim_{n\to\infty} o^*(a_n)$.

8 Concluding remarks

Beklemishev's 'brackets' provided an autonomous notation system for Γ_0 based on worms, but did not provide a method for comparing different worms without first translating into a more traditional notation system. Our calculus **BC** shows that this is not necessary, and indeed all derivations may be carried out entirely within the brackets notation. To the best of our knowledge, this yields the first ordinal notation system presented as a purely modal deductive system.

Our analysis is purely syntactical and leaves room for a semantical treatment of **BC**. As before one may first map **BC** into $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ and then use the Kripke semantics presented in [5,13], but we leave the question of whether it is possible to define natural semantics that work only with **BC** expressions and do not directly reference ordinals.

Moreover, [15] suggests variants of the brackets notation for representing the Bachmann-Howard ordinal and beyond. Sound and complete calculi for these systems remain to be found.

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