POWERS OF THE VERTEX COVER IDEALS

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Abstract. We describe a combinatorial condition on a graph which guarantees that all powers of its vertex cover ideal are componentwise linear. Then motivated by Eagon and Reiner's Theorem we study whether all powers of the vertex cover ideal of a Cohen-Macaulay graph have linear free resolutions. After giving a complete characterization of Cohen-Macaulay cactus graphs (i.e., connected graphs in which each edge belongs to at most one cycle) we show that all powers of their vertex cover ideals have linear resolutions.

Introduction

Let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ with the edge set $E(G)$, and let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. Two monomial ideals are associated to G, the *edge ideal* $I(G)$ generated by all monomials x_ix_j with $\{x_i, x_j\} \in E(G)$, and the *vertex cover ideal* $J(G)$ generated by monomials $\prod_{x_i \in C} x_i$ for all minimal vertex covers C of G. We recall that a *minimal vertex cover* of G is a subset $C \subset V(G)$ such that each edge has at least one vertex in C and no proper subset of C has the same property. The vertex cover ideal $J(G)$ is the Alexander dual of the edge ideal of G, i.e. $J(G) = I(G)^\vee = \bigcap_{\{x_i,x_j\} \in E(G)} (x_i, x_j)$.

A homogeneous ideal $I \subset R$ is called *componentwise linear* if for each d, the ideal (I_d) generated by all forms of degree d in I has a linear resolution, see [9]. A criteria for an ideal being componentwise linear is given in [1]. Francisco and Van Tuyl showed that the vertex cover ideal of a chordal graph is componentwise linear, and recently Herzog, Hibi and Ohsugi studied the powers of these ideals, and they conjectured that all powers of the vertex cover ideal of a chordal graph are componentwise linear, see [5, 10]. We are interested in the following question: Do there exist non-chordal graphs such that all powers of their vertex cover ideals are componentwise linear? This kind of problem is studied by Nevo and Peeva for the powers of some edge ideals, see [17, 18].

On the other hand, in general it is hard to prove that a class of ideals has linear resolutions, and even having an ideal with a linear resolution, does not guarantee to have the same property for its powers, see [20]. Therefore it is natural to study the problem for the vertex cover ideals of graphs, and investigate some combinatorial properties of graphs which are sufficient to have minimal linear resolutions for ideals $J(G)^k$. In this point of view, bipartite and chordal graphs have been studied in [8, 10]. A graph G is called (sequentially) *Cohen-Macaulay* if the quotient ring $R/I(G)$ is (sequentially) Cohen-Macaulay over every field K. As a consequence of Eagon and Reiner's Theorem, we have

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that a graph is Cohen-Macaulay if and only if its vertex cover ideal has a linear resolution. One can raise this question: given an arbitrary Cohen-Macaulay graph, what can be said about the powers of its vertex cover ideal? Does each power of the vertex cover ideal of a Cohen-Macaulay graph have a linear resolution? As a consequence of [10, Theorem 3.7] we know that all powers of the vertex cover ideal of a Cohen-Macaulay chordal graph have linear resolutions. Also it is shown that all powers of the vertex cover ideal of a Cohen-Macaulay bipartite graph are componentwise linear, see [8]. In fact in Theorem 2.3 and some results of [14, 15] mentioned in the last part, it is shown that all powers of the vertex cover ideal a Cohen-Macaulay bipartite graph are weakly polymatroidal ideals and so they have linear resolutions.

As Herzog, Hibi and Zheng described in [11] classifying Cohen-Macaulay graphs in general is as hard as classifying all Cohen-Macaulay simplicial complexes. However finding some combinatorial conditions equivalent to Cohen-Macaulay property in graphs has been extensively studied by several authors in [4, 7, 11, 22]. The problem is solved for two well-known families of graphs, chordal and bipartite graphs, and it was shown that in these graphs the Cohen-Macaulay property is independent of the field K, see [7, 11].

The paper is structured as follows. In § 1 we investigate a combinatorial condition involving the number of free vertices in the clique complex of G that guarantees that all powers of $J(G)$ have linear quotients (and hence componentwise linear resolutions), see Theorem 1.3. In § 2 we study the ideals arising from a cactus graph. We give a combinatorial characterization of Cohen-Macaulay cactus graphs, see Theorem 2.3, and as a consequence we see that in cactus graphs the Cohen-Macaulay property is independent of the field K. Using this characterization in § 3 we show that all powers of the vertex cover ideal of a Cohen-Macaulay cactus graph are weakly polymatroidal and thus have linear resolutions, see Theorem 3.3.

1. Componentwise linear vertex cover ideals

A *vertex cover* C of G is a subset of $V(G)$ which meets every edge of G. A minimal vertex cover of G is a vertex cover C that there is no subset $C' \subset C$ such that C' is a vertex cover of G. The *vertex cover ideal* of G is the Alexander dual of the edge ideal of G, $J(G) = I(G)^\vee = \bigcap_{\{x_i,x_j\}\in E}(x_i, x_j)$. For any monomial ideal I, we denote by $G(I)$ the minimal set of generators of I, and we denote by $\deg_{x_i} f$ the exponent of the variable x_i in the monomial f. For two monomials f and g we say $f >_{lex} g$ if and only if the left-most nonzero entry in the sequence $(\deg_{x_1} f - \deg_{x_1} g, \dots, \deg_{x_n} f - \deg_{x_n} g)$ is positive. In the following by $>$, we mean $>_{lex}$.

We recall that the monomial ideal $I \subset R$ has *linear quotients*, if there exists a system of minimal generators $f_1, f_2, \dots, f_m$ of I such that the colon ideal $(f_1, \dots, f_{i-1}) : f_i$ is generated by a subset of $\{x_1, \dots, x_n\}$ for all i. Ideals with linear quotients were introduced by Herzog and Takayama in [12]. A class of ideals enjoying the nice property of having linear quotients is the class of weakly polymatroidal ideals introduced by Hibi and Kokubo in [13] for ideals generated in the same degree (and later in [15] for ideals not necessarily generated in one degree).

Definition 1.1. A monomial ideal $I \subset R$ is *weakly polymatroidal* if for any two monomials $f = x_1^{a_1} \cdots x_n^{a_n}$ and $g = x_1^{b_1} \cdots x_n^{b_n}$ in $G(I)$ with $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $\ell > t$ such that $x_t(g/x_\ell) \in I$.

For each simple graph G, the simplicial complex on $V(G)$ whose faces are the cliques of G is called the *clique complex* of G denoted by $\Delta(G)$. A vertex $v \in V(G)$ is a *free* vertex if it belongs just to one clique of G. We denote by $\deg_G(u)$ the number of the adjacent vertices to u in G. For more background, we refer to [6, 19, 22] for the combinatorial point of view. We denote by $\mathcal{F}(\Delta(G))$ the set containing of all facets of $\Delta(G)$ with free vertices. The main result of this section, Theorem 1.3, explains having enough free vertices in a graph is sufficient to have componentwise linear vertex cover ideals. These graphs are not necessarily Cohen-Macaulay and they may not have linear resolutions, but they have componentwise linear vertex cover ideals.

The proof of componentwise linearity of these ideals depends on describing a nice labeling on the vertices of their corresponding graphs. Then we fix the natural lexicographic order on the variables corresponding to the vertices of G. This term order allows us to check the condition of weakly polymatoidals for our ideals which implies having linear quotients.

Example 1.2. The vertex cover ideal of the graph P_4 on the vertices v_1, v_2, v_3, v_4 with the edges v_1v_2, v_2v_3, v_3v_4 is weakly polymatroidal. We consider a new labeling on the vertices to have first non-free vertices v_2, v_3 and then free vertices v_1, v_4 of P_4. Then we set $x_1 := v_2,\ x_2 := v_3,\ x_3 := v_1$ and $x_4 := v_4$. The edges of P_4 are x_3x_1, x_1x_2, x_2x_4, and the vertex cover ideal is $J(P_4) = (x_1x_2, x_1x_4, x_2x_3)$. Now an easy computation shows that $J(P_4)$ is weakly polymatroidal.

Theorem 1.3. *Let G be a graph with $|\bigcup_{F\in\mathcal{F}(\Delta(G))} F| \geq |V(G)| - 1$. Then all powers of the vertex cover ideal of G have linear quotients.*

Proof. Let $F_1, \dots, F_m$ be the facets of $\Delta(G)$ with free vertices. We have $V(G) = F_1 \cup \cdots \cup F_m$ or $V(G) \setminus \{y\} = F_1 \cup \cdots \cup F_m$ for some $y \in V(G)$. In order to prove that $J(G)^k$ has linear quotients, we show $J(G)^k$ is weakly polymatroidal with respect to the order

$$y < y_m < \cdots < y_1 < x_n < \cdots < x_1$$

on the variables (corresponding to the vertices of G), where $y_1, \dots, y_m$ are the free vertices of G and $y, x_1, \dots, x_n$ are non-free vertices of G.

First we note that given any minimal vertex cover C of G and for any facet $F_i \in \mathcal{F}(\Delta(G))$, there exists a unique vertex $u_i \in F_i$ such that $u_i \notin C$ and $F_i \setminus \{u_i\} \subseteq C$.

Now let $f = f_1 \cdots f_k$ and $g = g_1 \cdots g_k$ be two elements in the minimal generating set of $J(G)^k$ such that $\deg_{z'} f = \deg_{z'} g$ for any variable $z' > z$ and $\deg_z f > \deg_z g$. Thus there exists t such that $z \notin \operatorname{supp}(g_t)$. Note that $z \neq y$, otherwise $\deg_{z'} f = \deg_{z'} g$ for any variable $z' \neq y$ and $\deg_y f > \deg_y g$ which implies that g divides f, which is a contradiction by our assumption that f and g belong to the minimal monomial set of generators of $J(G)^k$. Then in order to check the condition of weakly polymatroidals we consider the following cases:

Case 1. $z = x_j$ for some j: Let F_i be the facet of $\Delta(G)$ containing x_j. Since $\operatorname{supp}(g_t)$ is a minimal vertex cover of G, we have $F_i \setminus \{x_i\} \subseteq \operatorname{supp}(g_t)$. Therefore by substituting any free vertex y_ℓ of $F_i \cap \operatorname{supp}(g_t)$ with x_j we get again a vertex cover of G (not necessarily minimal). Hence

$$g' = g_1 \cdots g_{t-1}(x_j g_t / y_\ell) g_{t+1} \cdots g_k \in J(G)^k.$$

Case 2. $z = y_j$ for some j: Let F_i be the facet of $\Delta(G)$ containing y_j. First note that our assumption on z (that $\deg_{x_i} f = \deg_{x_i} g$ and $\deg_{y_s} f = \deg_{y_s} g$ for all i and all $s < j$), implies that the sets

$$\{f_m :\ x_i \notin \operatorname{supp}(f_m) \textit{ for some } i,\ \textit{ or } y_s \notin \operatorname{supp}(f_m) \textit{ for some } s < j\}$$

and
$$\{g_m :\ x_i \notin \operatorname{supp}(g_m) \text{ for some } i,\ \text{ or } y_s \notin \operatorname{supp}(g_m) \text{ for some } s < j\}$$
have the same cardinality. This implies that there exists a free vertex $y_\ell \in F_i$ with $\ell > j$ since $\deg_{y_j} f > \deg_{y_j} g$. Therefore the subset $(\operatorname{supp}(g_t) \setminus \{y_\ell\}) \cup \{y_j\}$ is again a vertex cover of G and so
$$g' = g_1 \cdots g_{t-1}(y_j g_t / y_\ell) g_{t+1} \cdots g_k$$
belongs to $J(G)^k$, as desired. □

In [11] it is shown that a chordal graph G is Cohen-Macaulay if and only if $V(G)$ is the disjoint union of the facets of $\Delta(G)$ with free vertices. Hence $\bigcup_{F\in\mathcal{F}(\Delta(G))} F = V(G)$. The following result is an extension of [10, Theorem 2.7]. Applying [15, Corollary 1.4] we have

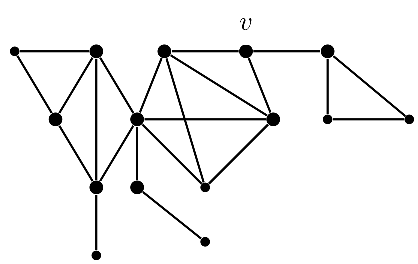
Corollary 1.4. *All powers of the vertex cover ideal of a graph G with the property that $|V(G)| - 1 \leq |\bigcup_{F\in\mathcal{F}(\Delta(G))} F|$ have componentwise linear quotients. In particular, they are componentwise linear.*

Recall that the *complete* graph K_n on $\{x_1, \dots, x_n\}$ is a finite graph with $\{x_i, x_j\} \in E(G)$ for all $1 \leq i < j \leq n$. Let G be a graph on the vertex set $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ such that $\{x_i, x_j\} \in E(G)$ for all $i < j$ and $\{y_i, y_j\} \notin E(G)$ for all $i < j$. Then G is called a star graph based on K_n. As a consequence of Theorem 1.3 we get [10, Theorem 2.3].

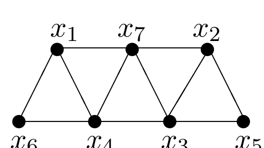
Corollary 1.5. *All powers of the vertex cover ideal of a star graph based on K_n are componentwise linear.*

Proof. We have $V(G) = F_1 \cup \cdots \cup F_m \cup F_{m+1}$, where $F_\ell = \{y_\ell\} \cup N(y_\ell)$ for $\ell = 1, \dots, m$ and $F_{m+1} = \{x_i :\ N(x_i) \subseteq \{x_1, \dots, x_n\}\}$. Note that y_ℓ is a free vertex in the facet F_ℓ for $\ell = 1, \dots, m$, and in the case that F_{m+1} is nonempty, each vertex in F_{m+1} is a free vertex. Therefore the condition of Theorem 2.3 holds which completes the proof. □

Example 1.6. Let G be the graph depicted in the following picture. The free vertices have been marked by smaller nodes, and the vertex v is the only vertex which does not belong to any facet F with a free vertex. Thus by Theorem 1.3 all powers of the vertex cover ideals of G are componentwise linear.



Example 1.7. Let G be the chordal graph drawn below. The graph G is neither a star graph nor a Cohen-Macaulay graph. But we have $|\bigcup_{F\in\mathcal{F}(\Delta(G))} F| = |V(G)| - 1$. We consider the labeling described in Theorem 1.3. Therefore all powers of the vertex cover ideals of G are componentwise linear.



2. Cohen-Macaulay cactus graphs

In this section we give a combinatorial characterization for Cohen-Macaulay edge ideals among cactus graphs. A connected graph is a cactus graph if each edge belongs to at most one cycle. Note that the facets of $\Delta(G)$ in a cactus graph are 3-cycles and some edges. Sequentially Cohen-Macaulay cactus graphs are classified in [16, Theorem 2.8].

Theorem 2.1. *A cactus graph G is sequentially Cohen-Macaulay if and only if for each cycle C_m, $m \neq 3, 5$, one of the following holds:*

(i) *C_m has a common vertex u with some clique F of G, where F has a free vertex $u' \neq u$;*
(ii) *C_m has a common vertex u with some cycle C_5 such that for $u \in V(C_5) \cap V(C_m)$ and $v, w \in N_{C_5}(u)$ we have $\deg_G(v) = \deg_G(w) = 2$.*

A sequentially Cohen-Macaulay graph is Cohen-Macaulay if and only if it is unmixed, (see [19, 22]). Thus in order to classify Cohen-Macaulay cactus graphs we need to determine unmixed graphs among those sequentially Cohen-Macaulay graphs. First we prove the following technical lemma on sequentially Cohen-Macaulay cactus graphs. A vertex v of G is called *an adjacent vertex to a cycle C_m* if v is adjacent to a vertex $u \in V(C_m)$.

Lemma 2.2. *Let G be a sequentially Cohen-Macaulay cactus graph. Then one of the following statements holds:*

(a) *G has a free vertex;*
(b) *There is a cycle C_5 with no adjacent vertex of degree greater than two.*

Proof. The proof is by induction on the number of the vertices of G. If $|V(G)| \leq 5$, then G has a free vertex or G is indeed a 5-cycle. By contrary assume that G is a sequentially Cohen-Macaulay cactus graph with the smallest number of vertices which does not fulfill the conditions of theorem. Since G does not have a free vertex, by Dirac's Theorem G is not a chordal graph and so it has a chordless cycle of length greater than three. If there exists a cycle C_m with $m \neq 5$, then by Theorem 2.1(ii) there exists a cycle C_5 on the vertices $x_1, \ldots, x_5$ adjacent to C_m with $x_1 \in V(C_5) \cap V(C_m)$ and $\deg_G(x_2) = \deg_G(x_5) = 2$. By our (contrary) assumption we have $\deg_G(x_3) > 2$ and $\deg_G(x_4) > 2$.

Let G' be the contraction of the vertices of C_m in G. Since G is a cactus graph, G' is again a cactus graph with fewer vertices. Next we show that G' is sequentially Cohen-Macaulay. Any cycle C_m in G' with $m \neq 3, 5$, is indeed a cycle in G. Therefore there exists a cycle C_5 adjacent to C_m with desired property as Theorem 2.1(ii), since G has no free vertex. Note that the contraction of C_m does not remove any vertex of C_5 and this process does not decrease the degree of any vertex in $V(G) \setminus V(C_m)$. Hence G' has no free vertex. Moreover, contraction does not change the degrees of the vertices of degree two in the induced subgraph on the vertices $V(G) \setminus V(C_m)$. Therefore the condition (ii) of Theorem 2.1 holds for each cycle C_m in G'. So G' has a 5-cycle which is also in G, with the property desired in (b), a contradiction.

In the case that there exists no C_m with $m \neq 5$, we have a cycle C_5 in G, since G is not chordal. We consider G' as the contraction of the vertices of C_5 in G. Then the same argument completes the proof. □

Theorem 2.3. *For a cactus graph G, let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ with some free vertices and $G_1, \ldots, G_n$ be the 5-cycles of the induced subgraph on the vertices $V(G) \setminus V(F_1 \cup \cdots \cup F_m)$ with no adjacent vertex of degree greater than two in G, and $L_1, \ldots, L_t$*

be the edges in the induced subgraph on the vertices

$$V(G) \setminus V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_n)$$

such that each L_i belongs to some 4-cycle in G. Then the following statements are equivalent:

(a) *G is Cohen-Macaulay;*
(b) *$V(G)$ is the disjoint union of the vertices of $F_1, \ldots, F_m, G_1, \ldots, G_n, L_1, \ldots, L_t$.*

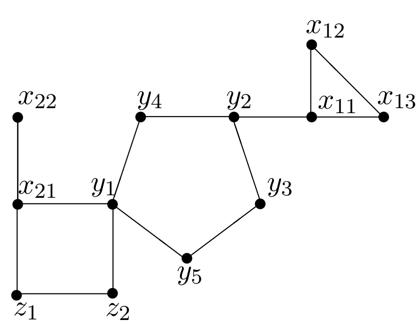
In order to formulate the main result of this section we first fix our notation.

Notation. With the hypothesis of Theorem 2.3 (with no loss of generality) we assume that $F_i = \{x_{i1}, \ldots, x_{ik_i}\}$, where $x_{ib_i}, \ldots, x_{ik_i} = u_i$ are the free vertices of F_i for all i. Consider a labeling on the vertices of the 5-cycle G_j as $y_{j1}, y_{j2}, \ldots, y_{j5}$ such that the degrees of y_{j3}, y_{j4}, y_{j5} are two for all j. Assume that z_{i1}, z_{i2} are the vertices of L_i for all i.

Example 2.4. Let G be the cactus graph depicted in the following figure. The labeling given on the vertices of G is of the form which we described above. Note that each minimal vertex cover C of G is of the form

$$C = \{x_{1i}, x_{1j}, x_{2k}, y_r, y_s, y_t, z_m\}$$

for some $1 \leq i \leq j \leq 3$, $1 \leq k \leq 2$, $1 \leq r \leq s \leq t \leq 5$ and $m = 1$ or 2. Note that acccording to Theorem 2.3 this graph is Cohen-Macaualy.



Before giving the proof of the theorem we need the following technical lemma (in which we use the similar notation as Theorem 2.3).

Lemma 2.5. *Let G be a Cohen-Macaulay cactus graph and*

$$B = V(G) \setminus V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_n \cup L_1 \cup \cdots \cup L_t).$$

Then $G|_B$, the induced subgraph G on B is a Cohen-Macaulay cactus graph.

Proof. The induced subgraph $G|_B$ is again a cactus graph. First we show that $G|_B$ is sequentially Cohen-Macaulay. By Theorem 2.1 it is enough to show that each cycle C_m in $G|_B$ with $m \neq 3, 5$ has some adjacent free vertex or some adjacent 5-cycle fulfilling the condition (ii) of Theorem 2.1. Each cycle C_m in $G|_B$ with $m \neq 3, 5$ is indeed a cycle in G. Therefore it has an adjacent free vertex v or an adjacent 5-cycle C as (ii). If $v \in F_i$ for some i, then the vertex of C_m adjacent to v is not in G' and so C_m is not a cycle in G', a contradiction. Let $V(C) = \{x_1, \ldots, x_5\}$, $x_1 \in V(C) \cap V(C_m)$ and $\deg(x_2) = \deg(x_5) = 2$. Since C_m is a cycle in $G|_B$ and $x_1 \in V(G|_B)$, we have $C \neq G_j$ for all j. Moreover the vertices x_2 and x_5 are in $V(G|_B)$, since $\deg(x_2) = \deg(x_5) = 2$. If x_3 is not in $V(G|_B)$, then x_2 is a vertex of degree one adjacent to C_m. If x_4 is not in $V(G|_B)$, then x_5 is a vertex of degree one adjacent to C_m, as desired. Therefore one of the statements (i) or (ii) of Theorem 2.1 holds for C_m and so $G|_B$ is sequentially Cohen-Macaulay.

Now we show that $G|_B$ is unmixed. The set

$$M_1 = \{y_{11}, y_{12}, y_{13}, \dots, y_{n1}, y_{n2}, y_{n3}, z_{11}, \dots, z_{t1}\} \cup V(F_1 \setminus \{u_1\} \cup \cdots \cup F_m \setminus \{u_m\})$$

is a minimal vertex cover of the induced subgraph on $V(G) \setminus B$. For any minimal vertex cover X of $G|_B$,

$$M = X \cup M_1$$

is a vertex cover of G, since $\{u_i, b\}$, $\{y_{j4}, b\}$ and $\{y_{j5}, b\}$ are not in $E(G)$ for all i, j, and all $b \in B$. Moreover, all neighbors of z_{i2} are already in M_1. Let $M' \subseteq M$ be a minimal vertex cover of G. Assume that the 4-cycle consisting L_i has the vertices $z_{i1}, z_{i2}, z_{i3}, z_{i4}$, where $z_{i3}, z_{i4} \in V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_n)$ for each i. If $z_{i3} \in V(F_\ell)$, then $z_{i3} \in M'$, since u_ℓ and z_{i3} are adjacent. If $z_{i\ell} \in V(G_\ell)$, then $z_{i\ell} \in M'$, since one of the vertices $y_{\ell 4}, y_{\ell 5}$ which are not in M are adjacent to z_{i3}. Hence $z_{i3} \in M'$. Similarly $z_{i4} \in M$. Since $\{z_{i1}, z_{i2}\} \in E(G)$ and $z_{i1} \notin M_1$ we have $z_{i2} \in X$. Hence $M' = M$ which implies that $G|_B$ is unmixed. $\square$

Now we are ready to state the proof of the theorem.

Proof of Theorem 2.3. (a) $\Rightarrow$ (b): The proof is by induction on the number of the vertices of G. Let

$$B = V(G) \setminus V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_n \cup L_1 \cup \cdots \cup L_t).$$

By contrary assume that $B \neq \emptyset$. The induced subgraph $G|_B$ is again a Cohen-Macaulay cactus graph by Lemma 2.5. So by induction hypothesis

$$B = V(H_1 \cup \cdots \cup H_s \cup K_1 \cup \cdots \cup K_r \cup P_1 \cup \cdots \cup P_h)$$

where $H_1, \dots, H_s$ are the facets of $\Delta(G|_B)$ with some free vertices, and $K_1, \dots, K_r$ are the 5-cycles of the induced subgraph on the vertices $B \setminus V(H_1 \cup \cdots \cup H_s)$ with no adjacent vertex of degree greater than two in G, and $P_1, \dots, P_h$ are the edges belonging to some 4-cycle in $G|_B$. Let v_i be a free vertex of H_i in $G|_B$. For each 5-cycle K_j with vertices $y_{j1}, y_{j2}, \dots, y_{j5}$ assume that the degrees of y_{j3}, y_{j4}, y_{j5} are two in $G|_B$ and z_{i1}, z_{i2} are the vertices of P_i for all i. Then

$$M_2 = \{y_{1,1}, y_{1,2}, y_{1,3}, \dots, y_{r,1}, y_{r,2}, y_{r,3}, z_{1,1}, \dots, z_{h,1}\} \cup V(H_1 \setminus \{v_1\} \cup \cdots \cup H_s \setminus \{v_s\})$$

is a minimal vertex cover of $G|_B$ of size $|B| - s - 2r - h$. Since

$$Z = M_1 \cup M_2$$

is a minimal vertex cover of G and G is unmixed, each minimal vertex cover of G consists of $n - (m + s + 2k + 2r + t + h)$ vertices.

Let Y be a minimal vertex cover of G. If $F_i \subset Y$, then $Y \setminus \{u_i\}$ is a vertex cover of G for each free vertex $u_i \in F_i$. Therefore $|Y \cap V(F_i)| = |F_i| - 1$ for all minimal vertex covers Y of G and for all i. Since Y is a minimal vertex cover of G, $|Y \cap V(G_j)| = 3$. Otherwise there exists a vertex of degree two in G_j such that its neighbors are all in Y, a contradiction (by the minimality assumption on Y). Assume that the 4-cycle corresponding to L_i has the vertices $z_{i1}, \dots, z_{i4}$, where $z_{i1}, z_{i2} \in L_i$. Since at least one of the vertices z_{i3}, z_{i4} is in Y, we have $|Y \cap V(L_i)| = 1$.

Moreover, $|Y \cap V(H_i)| \geq |H_i| - 1$, $|Y \cap V(K_j)| \geq 3$ and $|Y \cap V(P_\ell)| \geq 1$ for all i, j, ℓ. Since $|Y \cap B| = |B| - s - 2r - h$, we have $|Y \cap V(H_i)| = |H_i| - 1$, $|Y \cap V(K_j)| = 3$ and $|Y \cap V(P_\ell)| = 1$ for all i, j, ℓ.

Now we show that $V(F_i) \cap V(F_j) = \emptyset$ for all i, j. By contradiction assume that $w \in V(F_{i_1}) \cap \cdots \cap V(F_{i_\ell})$. Let $v_{i_j} \in V(F_{i_j}) \setminus Y$ be a free vertex for $j = 1, \ldots, \ell$. Then $(Y \cup \{v_{i_1}, \ldots, v_{i_\ell}\}) \setminus \{w\}$ is a minimal vertex cover of G consisting of $|Y| + \ell - 1$ vertices, a contradiction.

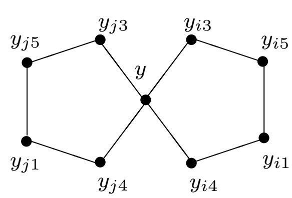
Now, we claim that $r = 0$. By contrary assume that K_1 is a 5-cycle with the vertices $y_1, \ldots, y_5$ belonging to $B \setminus V(H_1 \cup \cdots \cup H_s)$. Therefore, K_1 has a pair of adjacent vertices of degree greater than two in G such that at least one of them has some neighbor in $V(G) \setminus B$ or in $V(H_1 \cup \cdots \cup H_s)$. Assume that y_2, y_3 are some vertices of degree greater than two in G. Consider $a \in N_G(y_2)$ and $b \in N_G(y_3)$, where $a \in (V(G) \setminus B) \cup V(H_1 \cup \cdots \cup H_s)$ and $b \neq y_2, y_5$. Since G is a cactus graph, $a \neq b$ and they are not adjacent. Also $\{y_1, a\}, \{y_1, b\}$ are not in $E(G)$. Therefore, there exists a minimal vertex cover X of G with $X \subset V(G) \setminus \{a, b, y_1\}$. Therefore $\{y_2, y_3, y_4, y_5\} \subset X$ which is a contradiction by our observation that $|K_1 \cap X| = 3$. Hence we have $r = 0$.

By induction hypothesis we have $V(H_i) \cap V(H_j) = \emptyset$ for all i, j. Since H_1 is a non-facet of $\Delta(G)$ or a facet with no free vertex, (for each free vertex δ of H_1 in $\Delta(G|_B)$, there is $a \in V(G) \setminus B$ with $\{\delta, a\} \in E(G)$). Let $\delta_1, \ldots, \delta_\ell$ be the free vertices of H_1 and $v_1, \ldots, v_d$ be the non-free vertices of H_1. Set $A = \{a_1, \ldots, a_\ell, b_1, \ldots, b_d\}$, where $a_l \in V(G) \setminus B$ is an adjacent vertex to δ_ℓ (for each ℓ) and $b_i \in B \setminus H_1$ is an adjacent vertex to v_i. Since G is a cactus graph, each pair of the vertices in A are non-adjacent. So there exists a minimal vertex cover X of G with $X \subseteq V(G) \setminus A$. In this case $H_1 \subseteq X$ which is a contradiction by the fact that $|X \cap H_1| = |H_1| - 1$. Therefore $s = 0$ which implies that $B = \emptyset$, since two vertices of P_1 should be in $V(H_1 \cup \cdots \cup H_s) \cup V(K_1 \cup \cdots \cup K_r)$ which is empty.

Next we show that for all i, j, we have

$$V(G_i) \cap V(G_j) = \emptyset. \tag{2.1}$$

By contrary assume that $V(G_i) \cap V(G_j) \neq \emptyset$ for some i, j. Since G is a cactus graph, we have $|V(G_i) \cap V(G_j)| = 1$, say $V(G_i) \cap V(G_j) = \{y\}$, as depicted in the following picture.



Let $A = \{i\colon\ y \in V(G_i)\}$ and $V(G_i) = \{y, y_{i2}, y_{i3}, y_{i4}, y_{i5}\}$ for each $i \in A$. Since $\deg(y) > 2$, we can assume that $\deg(y_{i3}) = \deg(y_{i4}) = \deg(y_{i5}) = 2$ for all $i \in A$.

Since $V(G) = V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_n \cup L_1 \cup \cdots \cup L_t)$, for each vertex $v \in N_G(y) \setminus (\cup_{j \in A} G_j)$ there exists some i such that v belongs to F_i or the 4-cycle corresponding to L_i. Let

$$B = \{v\colon\ v \in N_G(y) \cap V(F_1 \cup \cdots \cup F_m \cup L_1 \cup \cdots \cup L_t)\}.$$

Assume that $F_{i_1}, \ldots, F_{i_\ell}$ are all facets with some common vertex with $N_G(y)$. Let v_{i_j} be a free vertex of F_{i_j} for $j = 1, \ldots, \ell$. Assume that y belongs to the 4-cycles $L'_{j_1}, \ldots, L'_{j_s}$, where L'_{j_ℓ} is the 4-cycle containing the edge L_{j_ℓ}. Then B contains some non-free vertices of $F_{i_1}, \dots, F_{i_\ell}$ and also the neighbors of y in $L'_{j_1}, \dots, L'_{j_s}$. Consider the subset

$$C = \{v_{i_1}, \dots, v_{i_\ell}, z_{j_1}, \dots, z_{j_s}\},$$

where for all ℓ, z_{j_ℓ} is a vertex of L'_{j_ℓ} which is not adjacent to y.

Since G is a cactus graph, no vertex of C is adjacent to G_j for $j \in A$. Note that each vertex z_{j_ℓ} is of degree two and so it is not adjacent to any vertex of C. Also v_{i_j} is not adjacent to any vertex of C, since v_{i_j} is a free vertex. Therefore, each pair of the vertices in C are not adjacent and so there exists a minimal vertex cover X of the induced subgraph on the vertices $V(G) \setminus (\cup_{i \in A} V(G_i))$ such that

$$X \subset V(G) \setminus (C \cup (\cup_{i \in A} V(G_i))) \ .$$

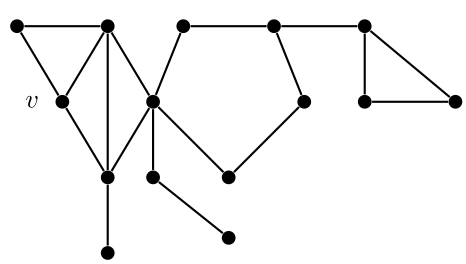
Then all non-free vertices of $F_{i_1}, \dots, F_{i_\ell}$ and the neighbors of y in $L'_{j_1}, \dots, L'_{j_s}$ should be in X which implies that $B \subset X$. Let

$$Y_1 = X \cup \{y\} \cup (\cup_{i \in A} \{y_{i1}, y_{i3}\}) \quad \text{and} \quad Y_2 = X \cup (\cup_{i \in A} \{y_{i1}, y_{i3}, y_{i4}\}).$$

Since y_{i4} and y_{i5} are non-adjacent vertices of degree two, and z_{j_ℓ} is just adjacent to some neighbors of u belonging to X, Y_1 is a minimal vertex cover of G. We have $N_G(u) = B \cup (\cup_{i \in A} \{y_{i3}, y_{i4}\})$ which implies that Y_2 is a minimal vertex cover of G. But Y_1 and Y_2 have the same size if and only if $k = 1$, a contradiction.

(b) $\Rightarrow$ (a): Let that $X \subset V(G)$ be a minimal vertex cover of G. Then we have $|X \cap V(F_i)| = |F_i| - 1$ and $|X \cap V(G_j)| = 3$ for all i, j. Let L'_i be the 4-cycle on the vertices $z_{i1}, z_{i2}, z_{i3}, z_{i4}$, where z_{i1} and z_{i2} are of degree two. Since $\{z_{i3}, z_{i4}\} \in E(G)$, at least one of these vertices belongs to X. Assume that $z_{i3} \in X$. Since $N(z_{i2}) = \{z_{i1}, z_{i3}\}$ and X is a minimal vertex cover, we can not have both z_{i1}, z_{i2} in X which implies that $|X \cap L_i| = 1$. Therefore $|X| = n - 2k - m - t$ for each vertex cover X of G and so G is unmixed. Let C_m be a cycle in G with $m \neq 3, 5$. Then C_m has a common vertex v with some F_i or G_j, since $V(G)$ is the disjoint union of the vertices of $F_1, \dots, F_m, G_1, \dots, G_n, L_1, \dots, L_t$. If $v \in V(F_i)$ for some i, then (i) holds in Theorem 2.1. If $v \in V(G_j)$, then the vertices of G_j adjacent to v are of degree two, since G_j has no adjacent vertex of degree greater than two in G which implies (ii) in Theorem 2.1. $\square$

Example 2.6. Let G be the graph depicted in the following picture. Then $G \backslash \{v\}$ and $G \backslash (\{v\} \cup N(v))$ are both cactus graphs which are Cohen-Macaulay by Theorem 2.3. Thus G is sequentially Cohen-Macaulay (see, e.g., [21, Lemma 2.4]). One can easily check that G is also unmixed and hence Cohen-Macaulay.



3. Vertex cover ideal of a Cohen-Macaulay cactus graph

The vertex cover ideals of Cohen-Macaulay graphs are studied for bipartite graphs and chordal graphs, see [8, 10, 5, 14]. In all cases there are some combinatorial properties of G which guarantee being Cohen-Macaulay. Moreover all powers of the vertex cover ideal of G are weakly polymatroidal with respect to the natural lexicographical ordering on the variables associated to the vertices of G. More precisely, $J(G)^k$ is weakly polymatroidal for each k when G is of one of the following types:

(1) Cohen-Macaulay bipartite graph (see [8] and [15, Theorem 2.2]);
(2) Cohen-Macaulay chordal graph (see [10, Theorem 2.7] and [14, Theorem 1.7]).

As a consequence of Eagon and Reiner's Theorem from [2], we have that a graph is Cohen-Macaulay if and only if its vertex cover ideal has a linear resolution. By the above consideration we expect that all powers of the vertex cover ideal of a Cohen-Macaulay graph have the same property, i.e. they have linear resolutions. We will choose an order on the vertices of a Cohen-Macaulay graph such that under this ordering $J(G)^k$ admits weakly polymatroidal condition for each k. Thus we pose the following question.

Question 3.1. *Do all powers of the vertex cover ideal of a Cohen-Macaulay graph have linear resolutions?*

In this section we study the above question for cactus graphs, and we show that in the case of cactus graphs the answer is yes. Indeed using the combinatorial characterization of Cohen-Macaulay cactus graph given in the previous section we show that all powers of the vertex cover ideals of these graphs have linear quotients.

Example 3.2. Returning to Example 2.4, before stating the main theorem of this section, we overview why the ideal $J(G)^2$ is weakly polymatroidal. First we consider the lexicographical ordering induced by the total ordering

$$x_{11} > x_{12} > x_{13} > x_{21} > x_{22} > y_1 > y_2 > y_3 > y_4 > y_5 > z_1 > z_2$$

on the variables (corresponding to the vertices of G). Consider the monomials

$$f = x_{11}^2 x_{12}^2 x_{21}^2 y_1^2 y_2^2 y_3^2 z_1^2 \quad \textit{and} \quad g = x_{11}^2 x_{12}^2 x_{21}^2 y_1 y_2 y_3^2 y_4 y_5 z_1 z_2$$

of $J(G)^2$ with $f >_{lex} g$. As we see, the variable y_1 is the greatest variable which has the higher exponent in f than g. Then we decompose g as $g = g_1 g_2$ in which $g_1 = x_{11}x_{12}x_{21}y_3y_4y_5z_2$ and $g_2 = x_{11}x_{12}x_{21}y_1y_2y_3z_1$. Then the fact that $\mathrm{supp}(g_2)$ is a minimal vertex cover of G together with $y_1 \notin \mathrm{supp}(g_2)$ implies that $N_G(y_1) = \{y_4, y_5\}$ is a subset of $\mathrm{supp}(g_2)$. Since y_2 and y_2 are adjacent we have $|\{y_2, y_3\} \cap C| \geq 1$. On the other hand $\{y_4, y_5\} \subseteq \mathrm{supp}(g_2)$ implies that only one of the vertices y_2 or y_3 belongs to C (since C is a minimal vertex cover of G). As we see $y_3 \in C$. Then $g_2' = y_1 g_2 / y_5$ is again in $J(G)$. Thus $g' = g_1 g_2'$ belongs to $J(G)^2$ which fulfills the condition of weakly polymatroidal.

Theorem 3.3. *All powers of the vertex cover ideal of a Cohen-Macaulay cactus graph are weakly polymatroidal. In particular they have linear resolutions.*

Proof. Let $V(G)$ be the disjoint union of $F_1, \ldots, F_m, G_1, \ldots, G_n, L_1, \ldots, L_t$ (with the same notation of Theorem 2.3). Consider the lexicographical ordering induced by the following total ordering on the variables (corresponding to the vertices of G):

$$x_{11} > \cdots > x_{1k_1} > \cdots > x_{m1} > \cdots > x_{mk_m} > y_{11} > \cdots > y_{15} >$$
$$\cdots > y_{n1} > \cdots > y_{n5} > z_{11} > z_{12} > \cdots > z_{t1} > z_{t2}$$

where $F_i = \{x_{i1}, \ldots, x_{ik_i}\}$ and $x_{ib_i}, \ldots, x_{ik_i}$ are the free vertices of F_i for all i, and $y_{j1}, y_{j4}, y_{j2}, y_{j3}, y_{j5}$ are the vertices of G_j such that the vertices y_{j3}, y_{j4}, y_{j5} are of degree two for all j, and z_{i1}, z_{i2} are the vertices of L_i of degree two for all i. First note that given any minimal vertex cover C of G and for any facet $F_i \in \mathcal{F}(\Delta(G))$, there exists a unique vertex $x_{ij} \in F_i$ such that $x_{ij} \notin C$ and $F_i \setminus \{x_{ij}\} \subseteq C$. More precisely, each minimal vertex cover of G has exactly $k_i - 1$ vertices of each F_i, three vertices of each G_j and one vertex of each L_i.

Now, let $f = f_1 \cdots f_k$ and $g = g_1 \cdots g_k$ be two elements in the minimal generating set of $J(G)^k$ such that $\deg_{z'} f = \deg_{z'} g$ for any variable $z' > z$ and $\deg_z f > \deg_z g$. Let g_j be a monomial such that $z \notin \operatorname{supp}(g_j)$. Then we consider the following cases. In each case we will find $g'_j >_{lex} g_j$ and

$$g = g_1 \cdots g_{j-1} g'_j g_{j+1} \cdots g_k$$

has desired properties (of weakly polymatroidal ideals) in $J(G)^k$. Hence $J(G)^k$ is a weakly polymatroidal ideal.

Case 1. $z = x_{i\ell}$ for some i, ℓ: In each minimal vertex cover C of G exactly one vertex of F_i is missed. Therefore $\ell \neq k_i$ (otherwise we will have $\deg_z(f) = \deg_z(g)$). Since $\ell \neq k_i$ the subset $(\operatorname{supp}(g_j) \setminus \{x_{ik_i}\}) \cup \{x_{i\ell}\}$ is again a minimal vertex cover of G and $g'_j = x_{i\ell} g_j / x_{ik_i} \in J(G)$.

Case 2. $z = y_{i1}$ for some i: Since $\operatorname{supp}(g_j)$ is a minimal vertex cover of G and $y_{i4}, y_{i5} \in N_G(y_{i1})$, we have $y_{i4}, y_{i5} \in \operatorname{supp}(g_j)$. On the other hand $|V(G_i) \cap \operatorname{supp}(g_j)| = 3$. Hence y_{i2} or y_{i3} belongs to $\operatorname{supp}(g_j)$. If $y_{i2} \in \operatorname{supp}(g_j)$, then we set $g'_j = g_j y_{i1} / y_{i4}$. Since $N_G(y_{i4}) = \{y_{i1}, y_{i2}\} \subseteq \operatorname{supp}(g'_j)$, we deduce that $\operatorname{supp}(g'_j)$ is a minimal vertex cover of G. If $y_{i3} \in \operatorname{supp}(g_j)$, then $g'_j = g_j y_{i1} / y_{i5}$ has desired property, since $N_G(y_{i5}) = \{y_{i1}, y_{i3}\} \subseteq \operatorname{supp}(g'_j)$ which implies that $\operatorname{supp}(g'_j)$ is a minimal vertex cover of G.

Case 3. $z = y_{i2}$ for some i: Since $\operatorname{supp}(g_j)$ is a minimal vertex cover of G, we have $y_{i3}, y_{i4} \in \operatorname{supp}(g_j)$. Also y_{i1} or y_{i5} belongs to $\operatorname{supp}(g_j)$. If $y_{i1} \in \operatorname{supp}(g_j)$, then we set $g'_j = g_j y_{i2} / y_{i4}$. Then $\operatorname{supp}(g'_j)$ is a minimal vertex cover of G, since $N_G(y_{i4}) = \{y_{i1}, y_{i2}\} \subseteq \operatorname{supp}(g'_j)$. If $y_{i5} \in \operatorname{supp}(g_j)$, then set $g'_j = g_j y_{i2} / y_{i3}$. Since $N_G(y_{i3}) = \{y_{i2}, y_{i5}\} \subseteq \operatorname{supp}(g'_j)$, $\operatorname{supp}(g'_j)$ is a minimal vertex cover of G.

Case 4. $z = y_{i3}$ for some i: Each minimal vertex cover C contains one of the subsets

$$\{y_{i1}, y_{i2}, y_{i5}\}, \{y_{i2}, y_{i4}, y_{i5}\}, \{y_{i1}, y_{i2}, y_{i3}\}, \{y_{i1}, y_{i3}, y_{i4}\}, \{y_{i3}, y_{i4}, y_{i5}\}.$$

Since $\deg_{y_{i1}} f = \deg_{y_{i1}} g$ and $\deg_{y_{i2}} f = \deg_{y_{i2}} g$ we have

$$|\{f_\ell : \ \{y_{i1}, y_{i3}, y_{i4}\} \subset \operatorname{supp}(f_\ell)\}| = |\{g_\ell : \ \{y_{i1}, y_{i3}, y_{i4}\} \subset \operatorname{supp}(g_\ell)\}| \ .$$

and also

$$|\{f_\ell : \ \{y_{i2}, y_{i4}, y_{i5}\} \subset \operatorname{supp}(f_\ell)\}| = |\{g_\ell : \ \{y_{i2}, y_{i4}, y_{i5}\} \subset \operatorname{supp}(g_\ell)\}| \ .$$

So there exists some g_j with $\{y_{i1}, y_{i2}, y_{i5}\} \subseteq \operatorname{supp}(g_j)$. Now we set $g'_j = g_j y_{i3} / y_{i5}$.

Case 5. $z = y_{i4}$ or y_{i5} for some i: Since $\deg_{y_{i1}} f = \deg_{y_{i1}} g$, $\deg_{y_{i2}} f = \deg_{y_{i2}} g$ and $\deg_{y_{i3}} f = \deg_{y_{i31}} g$, the number of the components f_ℓ having $\{y_{i1}, y_{i3}, y_{i4}\}$ are equal to the number of the components g_ℓ having $\{y_{i1}, y_{i3}, y_{i4}\}$. Similarly the number of the components f_ℓ and g_ℓ which have $\{y_{i2}, y_{i4}, y_{i5}\}$, and the number of the components f_ℓ and g_ℓ

which have $\{y_{i1}, y_{i2}, y_{i3}\}$ are equal. So the degree of the variables y_{i4}, y_{i5} are equal in f and g, a contradiction.

Case 6. $z = z_{i1}$ or z_{i2} for some i: Assume that the z_{i3} and z_{i4} be the other vertices in the 4-cycle corresponding to L_i. Since $z_{i3}, z_{i4} \in V(F_1 \cup \cdots \cup F_m \cup G_1 \cup \cdots \cup G_k)$, we have $z_{i3}, z_{i4} >_{lex} z_{i1} >_{lex} z_{i2}$. In every vertex cover C of G exactly one of the vertices z_{i1}, z_{i2} appears. Moreover C contains one of the subsets

$$\{z_{i1}, z_{i3}\}, \{z_{i2}, z_{i4}\}, \{z_{i1}, z_{i3}, z_{i4}\}, \{z_{i2}, z_{i3}, z_{i4}\}.$$

Since f and g have the same degrees in the variables z_{i3} and z_{i4}, the number of f_ℓ with $\{z_{i1}, z_{i3}\} \subset \operatorname{supp}(f_\ell)$ is equal to the number of g_ℓ with $\{z_{i1}, z_{i3}\} \subset \operatorname{supp}(g_\ell)$. Also the number of f_ℓ with $\{z_{i2}, z_{i4}\} \subset \operatorname{supp}(f_\ell)$ is equal to the number of g_ℓ with $\{z_{i2}, z_{i4}\} \subset \operatorname{supp}(g_\ell)$. First assume that $z = z_{i2}$. These two facts imply that the number of the g_ℓ with $\{z_{i1}, z_{i3}, z_{i4}\} \subseteq \operatorname{supp}(g_\ell)$ is greater than the number of f_ℓ with this property and so $\deg_{z_{i1}}(g) > \deg_{z_{i1}}(f)$, a contradiction. So we have $z = z_{i1}$ and the number of the g_l's with $\{z_{i2}, z_{i3}, z_{i4}\} \subseteq \operatorname{supp}(g_\ell)$ is greater than the number of f_ℓ with this property. Now for some g_j with $\{z_{i2}, z_{i3}, z_{i4}\} \subseteq \operatorname{supp}(g_j)$, set $g'_j = g_j z_{i1}/z_{i2}$. Since $N_G(z_{i2}) = \{z_{i1}, z_{i3}\} \subset \operatorname{supp}(g_j)$ we have $\operatorname{supp}(g'_j)$ is a vertex cover of G. □

Example 3.4. The cactus graph given in Example 2.4 is Cohen-Macaulay by Theorem 3.3. With respect to the term order given in the proof of Theorem 2.3 we have that $J(G)^k$ is weakly polymatroidal ideals for each k.

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