WKL_0 and induction principles in model theory

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Abstract

We examine the reverse mathematics of aspects of basic classical and effective model theory, including: existence of homogeneous and saturated models, different type-amalgamation properties, the preorder of models under elementary embeddability, and existence of indiscernibles. Most theorems are equivalent to RCA_0, WKL_0, or ACA_0. Some, however, exhibit peculiar strengths such as $\neg\mathsf{WKL}_0 \vee \mathsf{ACA}_0$ and $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$ over RCA_0.

1 Introduction

We consider the reverse mathematics of basic model theory. The corresponding study in effective mathematics, called interchangeably *effective*, *recursive*, or *computable model theory*, is well developed at this point, and the subject of surveys [7, 11] and monographs [1]. While Simpson and others have long since formalized the basics of first-order logic in second-order arithmetic, only recently have researchers such as Harris, Hirschfeldt, Lange, and Shore begun the wholesale formalization of model-theoretic theorems. Most of these theorems turn out to be equivalent to one of RCA_0, WKL_0, or ACA_0—three of the familiar Big Five systems—or to an induction principle such as $\mathsf{I}\Sigma^0_2$. Some theorems fall into other, previously unknown complexity classes. For example, Hirschfeldt, Shore, and Slaman [10] isolated new classes by considering the existence theorem for atomic models and type omitting theorems; the author [2] found a model-theoretic statement equivalent over RCA_0 to $\mathsf{ACA}_0 \vee \neg\mathsf{WKL}_0$; and in the present paper, we introduce a family of statements equivalent to $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$. Still other theorems reveal new classes not directly through their statements but through a careful study of their proofs. This was the case for the hierarchies of genericity principles $\Pi^0_n\mathsf{G}$ and $\Pi^0_n\mathsf{GA}$ found by Hirschfeldt, Lange, and Shore [9].

In this paper, we focus on existence theorems for countable homogeneous models (related to work in [9]), existence theorems for countable saturated models, theorems concerning elementary embeddings (building on [2]), theorems concerning type amalgamation properties (again related to [9]), and some other well-known theorems such as the existence of order

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indiscernibles. We separate our results into five categories along these lines and summarize them separately in §2.1, §2.2, §2.3, §2.4, and §2.5, respectively.

Most of the theorems we analyze have the expected complexities of $\mathsf{RCA_0}$, $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, or, echoing [2], $\neg\mathsf{WKL_0} \vee \mathsf{ACA_0}$. Breaking the pattern are several more unusual theorems; the most striking is a statement equivalent to the disjunction $\mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$ over $\mathsf{RCA_0}$ (see Theorems 2.24 and 2.14.) We know of only one other natural statement with this complexity: Friedman, Simpson, and Yu [4] have shown that $\mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$ holds if and only if any iteration f^n of a continuous function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is itself continuous. In our case, the theorem is provable by an induction argument (using $\mathsf{I\Sigma^0_2}$) or by a compactness argument (using $\mathsf{WKL_0}$). But neither of these is the optimal proof from a reverse-mathematical standpoint—rather, the optimal proof appears simply to choose between them based on what axioms are available.

1.1 Conventions and organization

All definitions are in the language of second-order arithmetic. Unless otherwise specified, all infinite sets are countable, all reasoning is carried out in $\mathsf{RCA_0}$, and all theorems are theorems of $\mathsf{RCA_0}$. We use the symbols $(M, \mathcal{S})$ to represent a model of $\mathsf{RCA_0}$, where M and $\mathcal{S}$ are the first- and second-order parts, respectively. We assume familiarity with basic notions of model theory and reverse mathematics. The reader may refer to Chang and Keisler [3] and Simpson [20] for background on model theory and on reverse mathematics, respectively.

In subsection §1.2 we give a quick account of how concepts from model theory are formalized in the language of second-order arithmetic. In subsection §1.3 we give some useful characterizations of the principles $\mathsf{ACA_0}$, $\mathsf{WKL_0}$, $\mathsf{I\Sigma^0_2}$, and $\mathsf{B\Sigma^0_2}$. Section §2 presents our main results, organized thematically into smaller subsections §2.1 through §2.5. Although §2 includes some proofs, the majority are too long and are instead deferred variously to sections §3 through §7. Section §3 begins with an introductory part summarizing a method introduced in [2], and then moves on to an 'Applications' subsection §3.1. Each section among §4 through §6 describes a new construction or class of constructions, and is divided into four parts: first, an unnumbered introductory part which describes the construction and its goals in inexact terms; second, a 'Construction' subsection giving the details; third, a 'Verification' subsection where we check basic properties (such as completeness and consistency of a theory); and fourth, an 'Applications' subsection where the construction is used to prove theorems from section §2. Section §7 follows this pattern but has two 'Applications' subsections to accommodate some small twists on the construction.

1.2 Formalizing model theory

A *language* L is a sequence of relation symbols and function symbols together with their arities. An *L-formula* and *L-sentence* are defined as usual. Rules for deduction and a sequent calculus can be formalized—see Simpson [20, section II.8]. An *L-theory* is a set of L-sentences. A *consistent L-theory* is one not entailing the contradiction $\neg x = x$. A *complete L-theory* is an L-theory containing either ϕ or $\neg\phi$ for every L-sentence ϕ. An *L-structure* is a sequence of elements $a_0, a_1, \ldots$ (its *domain*) together with a complete consistent $L \cup \{a_0, \ldots\}$-theory (its *elementary diagram*) containing the set $\{a_i \neq a_j : i \neq j\}$. When no confusion arises we omit L and talk simply of *formulas*, *theories*, etc.

Fix a language L and an L-theory T. A *model of* T is a structure whose elementary diagram contains T. T is *satisfiable* if it has a model. An *n-type of T* is a set $p(x_0, \ldots, x_{n-1})$ of L-formulas with variables in $\{x_0, \ldots, x_{n-1}\}$ such that $\{\phi(c_{i_0}, \ldots, c_{i_{k-1}}) : \phi(x_{i_0}, \ldots, x_{i_{k-1}})$ is in $p(x_0, \ldots, x_{n-1})\}$ is a complete consistent $L \cup \{c_0, \ldots, c_{n-1}\}$-theory, where $c_0, \ldots, c_{n-1}$ are new constant symbols. We often shorten $p(x_0, \ldots, x_{n-1})$ to p. We also often drop the n and refer to p as simply a *type*.

An n-type p of T is *principal* if there is a formula $\phi \in p$ such that p is the only n-type of T containing ϕ. Otherwise, p is *nonprincipal.* If $\bar{a}$ is a sequence of n elements of a model $\mathcal{A}$ of T, then $\mathsf{tp}^{\mathcal{A}}(\bar{a})$ is defined as the set of all n-ary formulas such that $\mathcal{A} \models \phi(\bar{a})$. Note that $\mathsf{tp}^{\mathcal{A}}(\bar{a})$ is an n-type. If p is a type and $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = p$ for some $\bar{a}$, we say that $\mathcal{A}$ *realizes* p and that $p(\bar{a})$ *holds.* Otherwise, $\mathcal{A}$ *omits* p.

We now consider some model-theoretic notions that do not admit a unique formulation in second-order arithmetic—or rather, they have several formulations which classically are considered equivalent and interchangeable, but which are not provably equivalent in $\mathsf{RCA_0}$.

Definition 1.1. Fix a complete theory T and a model $\mathcal{A}$ of T.

1. $\mathcal{A}$ is *atomic* if every type realized by $\mathcal{A}$ is principal.
2. $\mathcal{A}$ is *prime* if it embeds elementarily into every model of T.
3. $\mathcal{A}$ is *1-point homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$ and every element u, there is an element v such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown u) = \mathsf{tp}^{\mathcal{A}}(\bar{b}^\frown v)$. (Here '$^\frown$' denotes concatenation of tuples.)
4. $\mathcal{A}$ is *1-homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$ and every tuple $\bar{u}$, there is a tuple $\bar{v}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown \bar{u}) = \mathsf{tp}^{\mathcal{A}}(\bar{b}^\frown \bar{v})$.
5. $\mathcal{A}$ is *strongly 1-homogeneous* if for every pair $\bar{a}, \bar{b}$ of tuples such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{b})$, there is an automorphism of $\mathcal{A}$ which maps each entry of $\bar{a}$ to the corresponding entry of $\bar{b}$.
6. $\mathcal{A}$ is *homogeneous* if for every finite sequence of tuples $\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}_i) = \mathsf{tp}^{\mathcal{A}}(\bar{b}_i)$ for all $i < n$, and every sequence of tuples $\bar{u}_0, \ldots, \bar{u}_{n-1}$, there is a sequence $\bar{v}_0, \ldots, \bar{v}_{n-1}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}_i{}^\frown \bar{u}_i) = \mathsf{tp}^{\mathcal{A}}(\bar{b}_i{}^\frown \bar{v}_i)$ for all $i < n$.
7. $\mathcal{A}$ is *saturated* if, for every tuple $\bar{a}$ from its domain, the model $(\mathcal{A}, \bar{a})$ realizes every type of the theory $\mathsf{tp}^{\mathcal{A}}(\bar{a})$.
8. $\mathcal{A}$ is *universal* if every model of T embeds elementarily into $\mathcal{A}$.

Items 1 and 2 are classically equivalent; as are 3, 4, 5, and 6. Furthermore, 7 classically implies 8. None of these equivalences or implications is provable from $\mathsf{RCA_0}$; their precise strengths are explored variously in Hirschfeldt, Shore, and Slaman [10], Hirschfeldt, Lange, and Shore [9], and Harris [8].

1.3 The basics of $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, $\mathsf{I\Sigma^0_2}$, and $\mathsf{B\Sigma^0_2}$

Each of our new results involves one of the following well-known axioms: Weak König's Lemma, the Arithmetic Comprehension Axiom, Σ^0_2 induction, and Σ^0_2 bounding. When combined with $\mathsf{RCA_0}$, these form the axiom systems $\mathsf{WKL_0}$, $\mathsf{ACA_0}$, $\mathsf{RCA_0}+\mathsf{I\Sigma^0_2}$, and $\mathsf{RCA_0}+\mathsf{B\Sigma^0_2}$, respectively. In this subsection we define and give some alternate characterizations of each of these principles. The uninterested reader may skip it and refer back as needed.

Definition 1.2. The Arithmetic Comprehension Axiom is axiom scheme: For each arithmetical formula $\phi(x)$ is an arithmetical formula in the language of second-order arithmetic with a free first-order variable x and an arbitrary set as a parameter, there is a set C such that $\phi(x) \leftrightarrow x \in C$. We use $\mathsf{ACA_0}$ to denote $\mathsf{RCA_0}$+Arithmetic Comprehension Axiom.

Simpson [20] and others have compiled impressive lists of natural statements equivalent to $\mathsf{ACA_0}$ over $\mathsf{RCA_0}$. We content ourselves with just the computability-theoretic principle given as item (ii) of the following lemma.

Lemma 1.3. *The following are equivalent over* RCA_0*:*

(i) ACA_0

(ii) *For every set Z, there is a second set K^Z consisting of all e such that $\Phi_e(e)$ converges, where Φ_e is the e-th Turing machine.*

Proof. See Simpson [20, Ex. VIII.1.12]. □

The set K^Z is called the *Turing jump of Z*. Lemma 1.3 is commonly used for proving ACA_0 from some other principle. It reduces the task from showing the existence of infinitely many sets—one for each arithmetical formula with set parameters—to that of showing the existence of a single, well-understood set K^Z, with Z ranging over $\mathcal{S}$.

Definition 1.4. Weak König's Lemma is the statement: Every infinite binary tree has an infinite path. We use WKL_0 to denote RCA_0+Weak König's Lemma.

WKL_0 is strong enough to carry out certain compactness arguments that do not work in RCA_0 alone. In fact, WKL_0 is equivalent over RCA_0 to many well-known facts, among them numerous compactness theorems. The following lemma lists a few useful characterisations of WKL_0; much longer lists can be found in Simpson [20].

Lemma 1.5. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *The Compactness Theorem for first-order logic: If T is a set of first-order sentences and every finite subset of T is satisfiable, then T is satisfiable.*

(iii) *The Σ^0_1 separation principle: If $\phi(x,s)$ and $\psi(x,s)$ are quantifier-free formulas in the language of second-order arithmetic with set parameters, and $(\forall x \forall s \forall t)[\neg\phi(x,s) \vee \neg\phi(x,t)]$, then there is a set C such that $(\exists s)\phi(x,s)$ implies $x \in C$, and $(\exists s)\psi(x,s)$ implies $x \notin C$.*

Proof. For (i $\leftrightarrow$ ii), see Simpson [20, Thm IV.3.3]. For (i $\leftrightarrow$ iii), see [20, Lem IV.4.4]. □

We make use of all three equivalent statements (i), (ii), (iii) in this paper: We use Weak König's Lemma in its original form in §3, in the form of the Σ^0_1 separation principle in §6 and §7, and the first-order Compactness Theorem throughout. We now introduce a few definitions that make the Σ^0_1 separation principle easier to work with.

Definition 1.6.

1. A *disjoint Σ^0_1 pair* is a sequence $\langle U_s, V_s\rangle_{s\in M}$ of pairs $U_s, V_s \subseteq M$ with the following properties:
   - Each U_s and V_s is finite, with $\max(U_s \cup V_s) < s$.
   - $U_s \cap V_s = \emptyset$ for every s.
   - $U_s \subseteq U_{s+1}$ and $V_s \subseteq V_{s+1}$ for every s.
2. Given a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$, a set $C \subseteq M$ is called a *separating set for $\langle U_s, V_s\rangle_s$* if, for every s, we have $U_s \subseteq C \subseteq (M - V_s)$. If no such C exists, then $\langle U_s, V_s\rangle_s$ is called an *inseparable Σ^0_1 pair.*

The Σ^0_1 separation principle can be phrased in these terms:

Theorem 1.7. $\mathsf{RCA}_0 \vdash$ (The Σ^0_1 separation principle) $\leftrightarrow$ (There is no inseparable Σ^0_1 pair).

We now turn to induction and bounding principles.

Definition 1.8. The Σ^0_2 *induction scheme* is the axiom scheme: For each Σ^0_2 formula $\phi(n)$ in the language of second-order arithmetic with one free first-order variable n and an arbitrary set as a parameter, the formula $(\phi(0) \wedge (\forall n)\phi(n) \to \phi(n+1)) \to (\forall n)\phi(n)$ holds. We use $\mathsf{I}\Sigma^0_2$ to represent the Σ^0_2 induction scheme.

Note that, because set parameters are allowed, this $\mathsf{I}\Sigma^0_2$ is not the same as the $\mathsf{I}\Sigma_2$ studied in the setting of first-order Peano arithmetic. Note also that Simpson [20] uses the notation Σ^0_2-IND where we would write $\mathsf{I}\Sigma^0_2$. Like the other principles under consideration, $\mathsf{I}\Sigma^0_2$ can be phrased in a number of equivalent ways:

Lemma 1.9. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{I}\Sigma^0_2$

(ii) $\mathsf{L}\Pi^0_2$*: If ψ is a Π^0_2 formula, and there is an n such that $\psi(n)$ holds, then there is a least such n.*

(iii) *If $\langle D_1 \subseteq D_2 \subseteq \ldots\rangle$ is an increasing sequence of sets (coded as a single set) such that, for each n, D_n finite implies that D_{n+1} is finite, then either D_n is finite for all n, or D_n is infinite for all n.*

(iv) *If $\langle D_1 \subseteq D_2 \subseteq \ldots\rangle$ is an increasing sequence of sets (coded as a single set) such that, for each n, D_n finite implies that D_{2n} is finite, then either D_n is finite for all n, or D_n is infinite for all n.*

Proof. The equivalence (i $\leftrightarrow$ ii) is well-known; a proof can be adapted from the first-order case, found in Hajek and Pudlak [6]. The directions (i $\to$ iii) and (iii $\to$ iv) are immediate.

Now we show that (iv) implies (ii). Suppose that ψ is a Π^0_2 formula given by $\psi(i) \Leftrightarrow (\forall x \exists y)\phi(i,x,y)$, where ϕ is Σ^0_0. For each $n \geq 1$, define

$$D_n = \{\langle i,s,t\rangle : \ i < \log_2 n \text{ and } t \text{ is least s.t. } (\forall x < s)(\exists y < t)\phi(i,x,y)\}.$$

These D_n form an increasing chain of sets, D_1 is empty, and, whenever D_n is finite and $\psi(\lfloor \log_2 n \rfloor)$ does not hold, we have D_{2n} finite as well; on the other hand, if $\psi(\lfloor \log_2 n \rfloor)$ holds, then D_{2n} is infinite. Now suppose that there is no least i satisfying ψ. Then (iv) implies D_n is finite for all n, and, in particular, that no i satisfies ψ is empty. ☐

Although they are relatively complicated to state, their use of sets in place of formulas makes (iii) and (iv) easier to use for some constructions when we work in a model of $\neg\mathsf{I}\Sigma^0_2$—see, for example, the constructions in §7. We also use the original formulation (i) of $\mathsf{I}\Sigma^0_2$ several times in §2.2. We make no further mention of (ii).

Definition 1.10. The Σ^0_2 *bounding principle* is the axiom scheme: For each Π^0_1 formula $\phi(i,x)$ in the language of second-order arithmetic with two free first-order variables i, x and an arbitrary set as a parameter, the formula

$$((\forall i < n)(\exists x)\phi(i,x)) \to (\exists x_0)(\forall i < n)(\exists x < x_0)\phi(i,x)$$

holds. We use $\mathsf{B}\Sigma^0_2$ to represent the Σ^0_2 bounding principle.

As with Σ^0_2 induction, we hasten to point out that $\mathsf{B}\Sigma^0_2$ is not the same as the principle $\mathsf{B}\Sigma_2$ studied in first-order arithmetic. We also point out one alternate characterization:

Lemma 1.11. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{B}\Sigma^0_2$

(ii) *For each Π^0_1 formula $\psi(i,x)$ with an arbitrary set as a parameter,*

$$((\forall i < n)(\exists x)\psi(i,x)) \to (\exists \text{ a tuple } \langle x_0,\ldots,x_{n-1}\rangle)\psi(0,x_0) \wedge \cdots \wedge \psi(n-1,x_{n-1})$$

holds.

2 Main Results

Our results are organized into five subsections. The first two deal with existence theorems for homogeneous and saturated models, respectively; the third, with type amalgamation properties and the relations between them; the fourth, with elementary embeddings and prime and universal models; and the fifth, with the strength of the existence theorem for indiscernibles.

2.1 Existence theorems for homogeneous models

Consider the following well-known fact of classical model theory.

Theorem 2.1 (Weak homogeneous model existence theorem. Classical). *If T is a complete consistent countable theory, then T has a countable homogeneous model.*

The word *Weak* is meant to distinguish this theorem from a stronger version which does not require T to be complete. What is the strength of Theorem 2.1 over RCA_0? In Definition 1.1, we gave a number of different formalizations of the term *homogeneous* in the language of second-order arithmetic. On the face of it it looks as though the corresponding versions of the existence theorem may have wildly different strengths. Lange in her thesis showed the following:

Theorem 2.2 (Lange [14]). $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow$ *Every complete consistent theory has a 1-point homogeneous model.*

In fact, three of the four versions of homogeneity from Definition 1.1 give a statement of equivalent strength:

Theorem 2.3. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *Every complete consistent theory has a 1-point homogeneous model.*

(iii) *Every complete consistent theory has a 1-homogeneous model.*

(iv) *Every complete consistent theory has a strongly 1-homogeneous model.*

A proof of Theorem 2.3 is implicit in Lange's proof of Theorem 2.2. We give an alternate proof and some extensions of (i $\leftrightarrow$ iv) in §3. Our first new result extends (i $\leftrightarrow$ iv) by introducing restrictions on the types of T:

Theorem 2.4. $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow$ *Every complete consistent theory with only principal types has a strongly 1-homogeneous model.*

Proof. The $\rightarrow$ direction is immediate from Theorem 2.3. The $\leftarrow$ direction is proved as Proposition 5.6 below. □

On the other hand, Hirschfeldt, Lange, and Shore [9] have shown that if one first specifies the type spectrum of the required model, following Goncharov [5] and Peretyatkin [19], one ends up with a large number of nonequivalent statements.[1] As well, we do not know much about the strength of Theorem 2.1 when we use the fourth, remaining formalization of homogeneity from Definition 1.16, except that, using Theorem 2.4 and results from [9], it is provable from $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Question 2.5. What is the strength over RCA_0 of the statement, 'Every complete consistent theory has a homogeneous model in the sense of Definition 1.1'? Is it equivalent to $\mathsf{RCA}_0+\mathsf{B}\Sigma^0_2$?

[1]For example, they find one equivalent to $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ over RCA_0; one provable in $\Pi^0_1\mathsf{GA}$ but not in RCA_0; and one provable in $\Pi^0_1\mathsf{GA}$ and equivalent to $\mathsf{I}\Sigma^0_2$ over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

2.2 Existence theorems for saturated models

We have already given a definition of *saturated* in second-order arithmetic as part of Definition 1.1. We begin this subsection with a second, weaker notion.

Definition 2.6. Let T be a complete theory, and $\mathcal{A}$ a model of T. We say that $\mathcal{A}$ is *$\emptyset$-saturated* if it realizes every type of T.

The following characterization of saturated models, well-known in the classical setting, also holds in RCA_0. It will be helpful in the work that follows.

Lemma 2.7. *Let T be a complete theory, and $\mathcal{A}$ a model of T. Then $\mathcal{A}$ is saturated if and only if $\mathcal{A}$ is both $\emptyset$-saturated and 1-homogeneous.*

Proof. First we show the 'only if' direction. Suppose that $\mathcal{A}$ is saturated. It is immediate from the definition that $\mathcal{A}$ is $\emptyset$-saturated as well. To see that $\mathcal{A}$ is 1-homogeneous, choose any three tuples $\bar{a}, \bar{b}, \bar{u}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}) = \mathsf{tp}^{\mathcal{A}}(\bar{u})$. Let $p = \mathsf{tp}^{(\mathcal{A},\bar{a})}(\bar{b})$ be the type of $\bar{b}$ over the enriched structure $(\mathcal{A}, \bar{a})$; since $\mathcal{A}$ is saturated, there is a tuple $\bar{v}$ such that $\mathsf{tp}^{(\mathcal{A},\bar{u})}(\bar{v}) = p$. Hence $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown\bar{b}) = \mathsf{tp}^{\mathcal{A}}(\bar{u}^\frown\bar{v})$, so $\mathcal{A}$ is 1-homogeneous.

Next we deal with the 'if' direction. Suppose that $\mathcal{A}$ is $\emptyset$-saturated and 1-homogeneous. Let $\bar{a}$ be any tuple, and let $p(\bar{y})$ be any type of the theory $\mathsf{tp}^{\mathcal{A}}(\bar{a})$. Replace the constants $\bar{a}$ in p with new variables $\bar{x}$ to get a type $p'(\bar{x}^\frown\bar{y})$ of T. This p' is realized by some tuple $\bar{u}^\frown\bar{v}$ from $\mathcal{A}$, with $\mathsf{tp}^{\mathcal{A}}(\bar{u}) = \mathsf{tp}^{\mathcal{A}}(\bar{a})$. Hence, by 1-homogeneity, there is a tuple $\bar{b}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}^\frown\bar{b}) = p'$, as desired. □

Now consider the following well-known theorem.

Theorem 2.8 (Weak saturated model existence theorem. Classical)**.** *If T is a complete consistent theory with only countably many types, then T has a countable saturated model.*

As we did for the homogeneous case at the start of §2.1, we ask for the reverse-mathematical strength of Theorem 2.8. And as in the homogeneous case, we must begin by formalizing the statement in second-order arithmetic. We have already settled on a suitable notion of saturation; our next worry is the notion of *countably many types.*

Definition 2.9. Fix a complete consistent theory T.

1. A *sequence of types* of T is a coded sequence $X = \langle p_0, p_1, \ldots \rangle$ such that each p_i is a type of T. X is a *sequence of all types of T* if every type of T is equal to some p_i.
2. We say T *has countably many types* if it has a sequence of all types.

Even given this definition, there are a number of different ways to formalize and analyze Theorem 2.8. We begin with the most basic:

Theorem 2.10. *The following are equivalent over* RCA_0*.*

(i) WKL_0

(ii) *Every complete consistent theory with countably many types has a saturated model.*

Proof. The (i $\to$ ii) direction follows from Corollary 3.4 below. The (ii $\to$ i) direction is immediate from Proposition 6.5 below. □

The proof of the (ii $\to$ i) direction works by assuming $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, and constructing a complete consistent theory T with two types p and q that can never be realized in the same model. We can rule out this obstruction by requiring that a theory's types have one of the following amalgamation properties.

Definition 2.11. Fix a complete consistent theory T and a sequence $X = \langle q_0, \dots \rangle$ of types of T.

1. We say X has the *pairwise full amalgamation property* if, for every type $p(\bar{x})$ and every pair $q_i(\bar{x}, \bar{y})$, $q_j(\bar{x}, \bar{z})$ of types in X extending p, there is a type $r(\bar{x}, \bar{y}, \bar{z})$ in X extending both q_i and q_j.
2. We say X has the *finite full amalgamation property* if, for every type $p(\bar{x})$ and every tuple $\langle i_0, \dots, i_{n-1} \rangle$ of indices such that $q_{i_k}(\bar{x}, \bar{y}_k)$ extends p for each $k < n$, there is a type $r(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$ in X extending each q_{i_k}.

Proposition 2.12. *Suppose T is a complete theory with a saturated model. Then T has countably many types, and every enumeration of all types of T has the pairwise full amalgamation property.*

Proof. Fix a saturated model $\mathcal{A}$ of T. We can enumerate the tuples $\langle \bar{a}_k \rangle_{k \in M}$ in $\mathcal{A}$, and hence enumerate the types $\langle p_k \rangle_{k \in M}$ realized in $\mathcal{A}$ by $p_k =$ the type realized by $\bar{a}_k$. Call this enumeration X. Clearly X is an enumeration of all types of T, so by definition, T has countably many types.

Now let Y be any enumeration of all types of T. To see that Y has the pairwise full amalgamation property, consider any type $p(\bar{x})$ and any two types $q_0(\bar{x}, \bar{y}), q_1(\bar{x}, \bar{z})$ of T extending p. Since $\mathcal{A}$ is $\emptyset$-saturated, it realizes q_0 and q_1, say with tuples $\bar{a}^\frown \bar{b}$ and $\bar{u}^\frown \bar{v}$, respectively, where $|\bar{a}| = |\bar{u}| = |\bar{x}|$ and $|\bar{b}| = |\bar{v}| = |\bar{y}|$. Since $\bar{a}$ and $\bar{u}$ realize the same type p, and since $\mathcal{A}$ is 1-homogeneous by Lemma 2.7, there is a tuple $\bar{c}$ such that $\mathsf{tp}(\bar{a}, \bar{c}) = \mathsf{tp}(\bar{u}, \bar{v}) = q_1$. Let $r(\bar{x}, \bar{y}, \bar{z}) = \mathsf{tp}(\bar{a}, \bar{b}, \bar{c})$. Then r extends $q_0(\bar{x}, \bar{y}) \cup q_1(\bar{x}, \bar{z})$. Hence we conclude that every enumeration of all types of T has the pairwise full amalgamation property. $\square$

In classical model theory, the converse of Proposition 2.12 is usually proved by a compactness argument; in the present setting, such a proof requires $\mathsf{WKL_0}$. In effective model theory, the converse is instead usually proved, following Millar [17, 15] and Morley [18], by a finite injury argument. This requires $\mathsf{I\Sigma^0_2}$. Hence we arrive at the following:

Proposition 2.13. $\mathsf{RCA_0} + (\mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}) \vdash$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Remarkably, if we include $\mathsf{B\Sigma^0_2}$ as an assumption, Proposition 2.13 admits a reversal.

Theorem 2.14. $\mathsf{RCA_0} + \mathsf{B\Sigma^0_2} \vdash (\mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}) \leftrightarrow$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Proof. The $\rightarrow$ direction is a weakening of Proposition 2.13. The $\leftarrow$ direction is proved as Proposition 7.17 below. $\square$

An obvious question is whether $\mathsf{B\Sigma^0_2}$ can be dropped in the statement of Theorem 2.14. We answer this question in the negative in Corollary 2.19 below. Our answer uses recent results about the combinatorial principle $\Pi^0_1\mathsf{GA}$, which states, roughly: For every sequence $\mathcal{D}$ of dense uniformly Π^0_1 subsets of $2^{<\mathbb{N}}$, there is a sequence $\sigma_0, \sigma_1, \ldots \in 2^{<\mathbb{N}}$ whose pointwise limit exists and is $\mathcal{D}$-generic. (Refer to [9] for a rigorous definition.) In terms of reverse-mathematical strength, this principle falls somewhere between $\mathsf{I\Sigma^0_1}$ and $\mathsf{I\Sigma^0_2}$, and is incomparable with $\mathsf{B\Sigma^0_2}$.

Theorem 2.15 (Hirschfeldt, Lange, Shore [9])**.** (i) $\mathsf{RCA_0} + \mathsf{B\Sigma^0_2} \nvdash \Pi^0_1\mathsf{GA}$

(ii) $\mathsf{RCA_0} + \mathsf{I\Sigma^0_2} \vdash \Pi^0_1\mathsf{GA}$

(iii) $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \Pi^0_1\mathsf{GA} \vdash \mathsf{I}\Sigma^0_2$

A further result in [10] is that the principle $\Pi^0_1\mathsf{G}$, which is stronger than $\Pi^0_1\mathsf{GA}$, has a certain conservation property over RCA_0. From this we deduce:

Theorem 2.16. $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} \nvdash \mathsf{WKL}_0 \vee \mathsf{B}\Sigma^0_2$

Proof. Immediate from the observation in [10, section 4] that $\Pi^0_1\mathsf{G}$ is restricted Π^1_2 conservative over RCA_0, and from the fact that $\Pi^0_1\mathsf{G}$ implies $\Pi^0_1\mathsf{GA}$. (Both $\Pi^0_1\mathsf{G}$ and *restricted* Π^1_2 *conservative* are defined in [10].) □

As mentioned above, the converse of Proposition 2.12 can be proved using $\mathsf{I}\Sigma^0_2$. In fact, the weaker axiom $\Pi^0_1\mathsf{GA}$ is already enough to prove a similar theorem:

Theorem 2.17 (Hirschfeldt, Lange, and Shore [9])**.** $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} \vdash$ *If T is a complete consistent theory and X is a sequence of types with the pairwise full amalgamation property, then T has a 1-homogeneous model which realizes exactly the types in X.*

Hence we derive:

Corollary 2.18. $\mathsf{RCA}_0 + (\mathsf{WKL}_0 \vee \Pi^0_1\mathsf{GA}) \vdash$ *Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model.*

Proof. WKL_0 proves the given statement by Theorem 2.10. $\Pi^0_1\mathsf{GA}$ proves the statement by Proposition 2.17 and Lemma 2.7. □

This allows us to prove that the assumption of $\mathsf{B}\Sigma^0_2$ cannot be dropped from the statement of Theorem 2.14:

Corollary 2.19. $\mathsf{RCA}_0 \nvdash$ *(Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model)*$\rightarrow (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. By Theorem 2.16 we may fix a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \Pi^0_1\mathsf{GA} + \neg\mathsf{B}\Sigma^0_2 + \neg\mathsf{WKL}_0$. Then by Theorem 2.17 there is a theory $T \in \mathcal{S}$ as in the corollary statement, but $(M, \mathcal{S})$ is neither a model of WKL_0 (by assumption) nor of $\mathsf{I}\Sigma^0_2$ (since $\mathsf{I}\Sigma^0_2$ implies $\mathsf{B}\Sigma^0_2$). □

On the other hand, these results suggest the following, weaker question, to which we do not know the answer.

Question 2.20. Is the statement, 'Every complete consistent theory with countably many types and whose types have the pairwise full amalgamation property has a saturated model' equivalent to $\mathsf{WKL}_0 \vee \Pi^0_1\mathsf{GA}$ over RCA_0?

2.3 Type amalgamation, WKL_0, and induction

Recall from Definition 2.11 the *pairwise full amalgamation property* and the *finite full amalgamation property.* We now list four more properties in the same family.

Definition 2.21 (Hirschfeldt, Lange, and Shore [9])**.** Fix a complete consistent theory T and a sequence $X = \langle q_0, \ldots \rangle$ of types of T.

1. We say X has the *1-point full amalgamation property* if for every n-type $p(\bar{x})$ in X and every pair of $(n+1)$-types $q_0(\bar{x}, y)$, $q_1(\bar{x}, z)$ in X extending p, there is an $(n+2)$-type $r(\bar{x}, y, z)$ in X extending both q_0 and q_1.

2. We say X has the *1-point free amalgamation property* if for every n-type $p(\bar{x})$ in X and every 1-type $q(y)$ in X, there is an $(n+2)$-type $r(\bar{x}, y)$ in X extending both p and q.

3. We say X has the *pairwise free amalgamation property* if for every pair $p(\bar{x})$, $q(\bar{y})$ of types in X, there is a type $r(\bar{x}, \bar{y}, \bar{z})$ in X extending both q_i and q_j.

4. We say X has the *finite free amalgamation property* if for every tuple $\langle i_0, \ldots, i_{n-1}\rangle$ of indices such that the variables of $q_{i_k}(\bar{y}_k)$ are pairwise disjoint, there is a type $r(\bar{y}_0, \ldots, \bar{y}_{n-1})$ in X extending each q_{i_k}.

These amalgamation properties are based on those used by Goncharov [5] and Peretyatkin [19] in studying homogeneous models in effective mathematics. We are interested in the special case where X is the sequence of all types of T; the situation for more general X is explored in [9]. We introduce six predicates which take as their argument a set X, and which abbreviate the six kinds of amalgamation property. The following serves as a prototype:

- $\mathsf{1PT\ FREE}(X) \Leftrightarrow X$ is a sequence of all types of a complete consistent theory T with the 1-point free amalgamation property.

The predicates $\mathsf{1PT\ FULL}(X)$, $\mathsf{PW\ FREE}(X)$, $\mathsf{PW\ FULL}(X)$, $\mathsf{FIN\ FREE}(X)$, and $\mathsf{FIN\ FULL}(X)$ are defined analogously for the 1-point full, pairwise free, pairwise full, finite free, and finite full amalgamation properties, respectively.

Theorem 2.22. (i) $\mathsf{WKL_0} \vdash (\forall X)\mathsf{1PT\ FREE}(X) \to \mathsf{FIN\ FULL}(X)$.

(ii) $\mathsf{RCA_0} + \mathsf{I\Sigma^0_2} \vdash (\forall X)\mathsf{1PT\ FREE}(X) \to \mathsf{FIN\ FREE}(X)$.

(iii) $\mathsf{RCA_0} + \mathsf{I\Sigma^0_2} \vdash (\forall X)\mathsf{1PT\ FULL}(X) \to \mathsf{FIN\ FULL}(X)$.

Proof. Item (i) is immediate by the Compactness Theorem. (And in fact, this proof does not require the 1-point free amalgamation property as an assumption.) Items (ii) and (iii) are each proved by a straightforward induction. □

Theorem 2.23. (i) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{1PT\ FULL}(X) \to \mathsf{PW\ FREE}(X)] \to \mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$.

(ii) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{PW\ FULL}(X) \to \mathsf{FIN\ FREE}(X)] \to \mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2}$.

(iii) $\mathsf{RCA_0} \vdash (\forall X)[\mathsf{FIN\ FREE}(X) \to \mathsf{1PT\ FULL}(X)] \to \mathsf{WKL_0}$.

Proof. Item (i) is proved as Proposition 7.11 below. Item (ii) is proved as Proposition 7.15. Item (iii) is proved as Proposition 6.4. □

Theorem 2.24. *The table in Figure 1 has the following property. If a principle P is listed in the row corresponding to an amalgamation property A and the column corresponding to an amalgamation property B, then*

$$\mathsf{RCA_0} \vdash P \leftrightarrow (\forall X)[A(X) \to B(X)].$$

If the cell in row A and column B is greyed out, then $\mathsf{RCA_0} \vdash (\forall X)[A(X) \to B(X)]$ *immediately from the definitions.*

Proof. For every cell in row A and column B which is not greyed out, the implication $A \to B$ is weaker than one or more implications mentioned in Theorem 2.22 and stronger than one mentioned in Theorem 2.23. It is straightforward in each case to compare the facts from these two theorems and arrive at the promised result. □

	1PT FREE	PW FREE	FIN FREE	1PT FULL	PW FULL	FIN FULL
1PT FREE		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	WKL_0	WKL_0	WKL_0
PW FREE			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	WKL_0	WKL_0	WKL_0
FIN FREE				WKL_0	WKL_0	WKL_0
1PT FULL		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$		$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$	$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$
PW FULL			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$			$\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$
FIN FULL						

Figure 1: See Theorem 2.24 for a description.

2.4 Elementary embeddings and universal models

Here we consider certain existence theorems for elementary embeddings between models, and for models which have elementary embeddings between them.

Theorem 2.25. WKL_0 *proves the following. Suppose T is a complete theory, and $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ is a countable sequence of models of T. Then there is a model $\mathcal{B}$ of T such that each $\mathcal{A}_j$ embeds elementarily into $\mathcal{B}$.*

Proof. See §3.1 below. □

Recall from Definition 1.1 the notion of a *universal* model. Theorem 2.25 has an immediate corollary in terms of universal models:

Corollary 2.26. $\mathsf{WKL}_0 \vdash$ *If T is a complete theory and there is a listing $\langle \mathcal{A}_0, \ldots \rangle$ of all models of T up to isomorphism, then T has a universal model.*

We can also guarantee the existence of a universal model by looking at the number of n-types:

Theorem 2.27. WKL_0 *proves the following. Suppose that T is a complete theory, and $f : M \to M$ is a function such that $f(n)$ is greater than the number of n-types of T for all n. Then T has a universal model.*

Proof. See §3.1. □

It is easy to see that the relation of elementary embeddability is reflexive and transitive—that is, it forms a preorder on models of T. Our next result shows that the conjunction $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ is equivalent over RCA_0 to a peculiar but natural statement about this preorder. A closely-related statement, weaker on its face but also equivalent to $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$, can be found in [2].

Theorem 2.28. *The following are equivalent over* RCA_0*:*

(i) $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$

(ii) *If T is a theory which has infinitely many n-types for some n, then any partial order can be embedded into the preorder of models under elementary embedding.*

Proof. The (i → ii) direction is proved as Proposition 3.6 below. For the (ii → i) direction, we prove the contrapositive. Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA}_0 + \neg(\mathsf{WKL}_0 + \neg\mathsf{ACA}_0)$. In other words, $(M, \mathcal{S})$ is a model either of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, or of ACA_0. If it is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, then there is a complete consistent theory $T \in \mathcal{S}$ with infinitely many 1-types but only one model up to isomorphism. (See [2]; alternatively, use the T constructed in §4 below.) Otherwise, if it is a model of ACA_0, a classical construction due to Ehrenfeucht can be carried out to obtain a complete consistent theory $T \in \mathcal{S}$ with infinitely many 1-types and exactly three models up to isomorphism. (This construction can be found in Chang and Keisler [3].) □

2.5 Indiscernibles

Here we list one more consequence of the constructions in this paper.

Definition 2.29. Fix a language L, a complete consistent L-theory T and a model $\mathcal{A}$ of T. We say a sequence $\langle a_0, a_1, \ldots \rangle$ of distinct elements of $\mathcal{A}$ is *a sequence of indiscernibles* if, for every strictly increasing tuple $\langle i_0, \ldots, i_{n-1} \rangle$ of numbers, and for every n-ary L-formula ϕ, we have

$$\mathcal{A} \models \phi(a_0, \ldots, a_{n-1}) \text{ if and only if } \mathcal{A} \models \phi(a_{i_0}, \ldots, a_{i_{n-1}}).$$

The classical existence theorem for indiscernibles is:

Theorem 2.30 (Classical)**.** *Every complete consistent countable theory has a countable model with a sequence of indiscernibles.*

Indiscernibles have been studied in recursive model theory by Kierstead and Remmel [12, 13]. Among their results is the following bound on the classical existence theorem's complexity:

Theorem 2.31 (Kierstead and Remmel [13])**.** *There is a decidable theory for which every decidable model has a sequence of indiscernibles, but no decidable model has a sequence of indiscernibles which is hyperarithmetic.*

When reasoning in second-order arithmetic, one might therefore suspect Theorem 2.30 to be strictly stronger than Δ^0_1-CA_0. However, we find that this is not the case. In fact, every decidable theory has a low model with a low sequence of indiscernibles.

Theorem 2.32. *The following are equivalent over* RCA_0*:*

(i) WKL_0

(ii) *Every complete consistent theory has a model with a sequence of indiscernibles.*

Proof. To see the (i $\to$ ii) direction, simply notice that WKL_0 is strong enough to carry out the classical proof of (ii) by way of the Compactness Theorem 1.5. The (ii $\to$ i) direction is proved as Proposition 4.7 below. □

3 Models and embeddings from a tree of Henkin constructions

Fix a model $\langle M, \mathcal{S} \rangle$ of WKL_0, and suppose that $T \in \mathcal{S}$ is a complete theory. In this first unnumbered subsection, we describe a general method for representing models as trees of Henkin-style diagrams, and give an idea of how it is to be used. This replicates a similar description from [2]. Afterwards, in §3.1, we use the method to prove several new results.

Definition 3.1. Fix a language L and a complete L-theory T.

- Let L' be the expanded language $L \cup \{c_0, c_1, \ldots\}$, where each c_i is a new constant symbol. Let $\langle \phi_s \rangle_s$ be a one-to-one enumeration of all L'-sentences. Define a $2^{<M}$-indexed sequence $\langle D_\sigma \rangle_{\sigma \in 2^{<M}}$ of sets of L'-sentences by

$$D_\sigma = \{\phi_s : s < |\sigma| \text{ and } \sigma(s) = 1\} \cup \{\neg\phi_s : s < |\sigma| \text{ and } \sigma(s) = 0\}.$$

Define a sequence $\langle W_s \rangle_{s \in M}$ of sets of L'-sentences by recursion:

$$W_0 = \emptyset$$
$$W_{s+1} = \begin{cases} W_s \cup \{\phi_s \to \psi(c_{2k+1})\} & \text{if } \phi_s \text{ is of the form } (\exists x)\psi(x), \text{ where} \\ & 2k+1 \text{ is the least odd index such} \\ & \text{that } c_{2k+1} \text{ is not mentioned in } W_s \text{ or in } D_\sigma \\ & \text{for any } \sigma \text{ of length } \leq s. \\ W_s & \text{if } \phi_s \text{ is not of this form.} \end{cases}$$

The *tree of odd Henkin diagrams* is the tree $\mathcal{H} \subseteq 2^{<M}$ given by

$$\mathcal{H} = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \text{ is consistent}\}.$$

- Given an infinite path β in $\mathcal{H}$, let $D_\beta = \bigcup_{s \in M} D_{\beta \restriction s}$. Then D_β is a complete, consistent L'-theory. Define an equivalence relation E on the constants $\{c_0, c_1, \ldots\}$ by $c_i E c_j$ iff $D_\beta \vdash c_i = c_j$. Denote the E-equivalence class of c_i by $[c_i]_E$, and let $\langle b_0, b_1, \ldots \rangle$ be the one-to-one listing of all E-equivalence classes given by

$$b_m = [c_{i_m}]_E, \text{ where } i_m \text{ is least s.t. } c_{i_m} \notin b_k \text{ for all } k < m.$$

  Let $\mathcal{B}$ be the L-structure such that, for any L-formula ϕ,

$$\mathcal{B} \models \phi(b_0, \ldots, b_{n-1}) \iff D_\beta \vdash \phi(c_{i_0}, \ldots, c_{i_{m-1}}).$$

  Then $\mathcal{B}$ is a model of T. We say that $\mathcal{B}$ is the *Henkin model encoded by* β.

Our simplest constructions using $\mathcal{H}$ work as follows. Fix a theory T, let $\mathcal{H}$ be the tree of odd Henkin diagrams, and let P be a property desired of a model. We specify a subtree $\mathcal{H}^*$ of $\mathcal{H}$ by writing a set Φ_P of L'-sentences and letting $\mathcal{H}^*$ equal

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_P \text{ is consistent}\}.$$

Typically Φ_P is designed to ensure that any model encoded by a path of $\mathcal{H}^*$ has property P. We then show that $\mathcal{H}^*$ is an infinite tree. An appeal to Weak König's Lemma yields a model of T with the property P.

Some examples of such Φ_P are:

- A set Φ_H which ensures the model is strongly 1-homogeneous. (Proposition 3.2)
- A set Φ_S which ensures the model is $\emptyset$-saturated. (Proposition 3.3)
- The union $\Phi_H \cup \Phi_S$, which ensures the model is saturated using Lemma 2.7. (Corollary 3.4)
- Given a model $\mathcal{A}$ of T, a set $\Phi_{\mathcal{A}}$ which ensures that $\mathcal{A}$ embeds elementarily into the new model. (Theorem 2.25, proved below. A similar set appears in the proof of Proposition 3.6.)
- Given a model $\mathcal{A}$ of T, a set which ensures that either the new model embeds elementarily into $\mathcal{A}$, or ACA_0 holds. (Used in [2]. A similar set is in the proof of Proposition 3.6.)

Sometimes we construct not one but a whole sequence $\langle \mathcal{B}_0, \mathcal{B}_1, \ldots \rangle$ of models with some property such as being pairwise non-isomorphic. We do this by considering the set $\{\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \mathcal{H}^{<M} :$ each σ_i has length $|\sigma_i| = n\}$ with the ordering $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \prec \langle \tau_0, \ldots, \tau_{m-1} \rangle$ if $n \leq m$ and $\sigma_i \subseteq \tau_i$ for every $i < n$. Any path through this tree encodes a sequence $\langle \mathcal{B}_0, \ldots \rangle$ of models of T. For an example of this method, see the proof of Proposition 3.6 below.

3.1 Applications

Our first use of the tree of odd Henkin diagrams is to prove one direction of Theorem 2.3. An alternate proof of the same direction is implicit in Lange [14, Proof of Thm 4.3.1].

Proposition 3.2. $\mathsf{WKL_0} \vdash$ *Every complete consistent theory has a strongly 1-homogeneous model.*

Proof. Let $(M, \mathcal{S})$ be a model of $\mathsf{WKL_0}$, and fix a complete consistent theory $T \in \mathcal{S}$. Define sequence of finite sets $\Phi_{H,0} \subseteq \Phi_{H,1} \subseteq \cdots$ of L'-sentences:

$$\begin{aligned} \Phi_{H,0} &= \emptyset \\ \Phi_{H,s+1} &= \begin{cases} \Phi_{H,s} \cup \{\phi_s \to \psi(r ^\frown c_{2\langle \bar{p},q,\bar{r}\rangle})\} \text{ if } \phi_s \text{ is of the form } \psi(\bar{p} ^\frown q) \wedge (\exists x)\psi(\bar{r} ^\frown x) \\ \text{with } \psi \text{ an } L\text{-formula, each } \bar{p}, q, \bar{r} \text{ taken from } \{c_i\}_{i \in M}, \\ \Phi_{H,s} \text{ if } \phi_s \text{ is not of this form.} \end{cases} \end{aligned}$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{H,|\sigma|} \text{ is consistent}\}.$$

First we check that $\mathcal{H}^*$ is infinite. Fix a model $\mathcal{A}$ of T and a level s of $\mathcal{H}^*$. It is easy to see that there is some assignment of constants $c_i^{\mathcal{A}}$ such that $(\mathcal{A}, c_i^{\mathcal{A}}) \models T \cup W_{|\sigma|} \cup \Phi_{H,s}$, and furthermore that this $(\mathcal{A}, c_i^{\mathcal{A}})$ satisfies some D_σ with $|\sigma| = s$. It follows that σ is in $\mathcal{H}^*$. Apply Weak König's Lemma to get a path β in $\mathcal{H}^*$, and let $\mathcal{B}$ be the model encoded by β.

Now we argue that $\mathcal{B}$ is strongly 1-homogeneous. For this we use an effective back-and-forth argument; we show only the 'forth' direction, the 'back' direction being similar. Let $\bar{a}, \bar{b}$ be any pair of tuples such that $\mathsf{tp}^{\mathcal{B}}(\bar{a}) = \mathsf{tp}^{\mathcal{B}}(\bar{b})$. Let $\bar{d}, \bar{e}$ be tuples of constants in $\{c_0, \ldots\}$ such that $a_i = [d_i]_E$ and $b_i = [e_i]_E$ for each i. Let u be the least-indexed element of $\mathcal{A}$ not in $\bar{a}$, and let j be an index such that $u = [c_j]_E$. Now let $k = 2\langle \bar{d}, c_j, \bar{e}\rangle$ and $v = [c_k]_E$. Then $\mathsf{tp}^{\mathcal{B}}(\bar{a} ^\frown u) = \mathsf{tp}^{\mathcal{B}}(\bar{b} ^\frown \bar{v})$. Notice that the procedure for finding v from $\bar{a}, \bar{b}$, and u is effective, so that we can iterate the construction in a model of $\mathsf{RCA_0}$. □

Proposition 3.3. $\mathsf{WKL_0} \vdash$*Every complete consistent theory with countably many types has a $\emptyset$-saturated model.*

Proof. Let $(M, \mathcal{S})$ be a model of $\mathsf{WKL_0}$, and fix a complete consistent theory $T \in \mathcal{S}$ with an enumeration of all types $X = \langle p_0, \ldots \rangle$. Let $\langle \bar{d}_0, \ldots \rangle$ be a sequence of tuples of constants in $\{c_{2i} : i \in M\}$, where each $\bar{d}_j$ has the same arity as p_j and where no constant c_{2i} appears twice. Define a sequence of finite sets of L^*-sentences:

$$\Phi_{S,s} = \{\phi_t(\bar{d}_j) : j, t < s,\ \phi_t(\bar{x}) \in p_j(\bar{x})\}.$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{S,|\sigma|} \text{ is consistent}\}.$$

It can be checked as in the proof of Proposition 3.2 that $\mathcal{H}^*$ is infinite. Use Weak König's Lemma to get the model $\mathcal{B}$ encoded by some path β in $\mathcal{H}^*$. The resulting $\mathcal{B}$ is $\emptyset$-saturated, since each type p_j in X is realized by the tuple of elements interpreting $\bar{d}_j$. □

Corollary 3.4. $\mathsf{WKL_0} \vdash$*Every complete consistent theory with countably many types has a saturated model.*

Proof. Fix $(M, \mathcal{S}) \models \mathsf{WKL_0}$, a complete consistent theory $T \in \mathcal{S}$, and an enumeration $X = \langle p_0, \ldots \rangle$ of all types of T. Let $\Phi_{H,s}$ and $\Phi_{S,s}$ be sets of sentences as in the proofs of Proposition 3.2 and Proposition 3.3, respectively, except with each $\Phi_{H,s}$ using only every fourth constant c_{4i}, and each $\Phi_{S,s}$ using only every fourth c_{4i+2}. Let

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{H,|\sigma|} \cup \Phi_{S,|\sigma|} \text{ is consistent}\}.$$

Once again, we may check that $\mathcal{H}^*$ is an infinite tree. By Weak König's lemma, there is a model $\mathcal{B}$ of T encoded by some path β through $\mathcal{H}^*$. This $\mathcal{B}$ is both 1-homogeneous and $\emptyset$-saturated, and hence is saturated by Lemma 2.7. $\square$

This method is also used to prove the results from section §2.4, which focus on the existence and nonexistence of elementary embeddings. We begin with the following:

Proof of Theorem 2.25. Let $(M, \mathcal{S})$ be a model of $\mathsf{WKL_0}$, and fix a complete consistent theory $T \in \mathcal{S}$ with a sequence of models $\langle \mathcal{A}_0, \ldots \rangle$. For simplicity, assume each $\mathcal{A}_i$ shares the same domain $A = \{a_0, a_1, \ldots\}$. For each $i \in M$, define a sequence of finite sets of L^*-sentences:

$$\Phi_{\mathcal{A}_i,s} = \{\phi_t(c_{2\langle i,k_0\rangle}, \ldots, c_{2\langle i,k_{n-1}\rangle}) : k_0, \ldots, k_{n-1}, t < s, \text{ and } \mathcal{A}_i \models \phi_t(a_{k_0}, \ldots, a_{k_{n-1}})\}.$$

Let $\mathcal{H}^*$ be the subtree of $\mathcal{H}$ given by:

$$\mathcal{H}^* = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \cup \Phi_{\mathcal{A}_0,|\sigma|} \cup \cdots \cup \Phi_{\mathcal{A}_{|\sigma|},|\sigma|} \text{ is consistent}\}.$$

As in the proof of Proposition 3.2, we can check that $\mathcal{H}^*$ is an infinite tree. Use Weak König's Lemma to get a model $\mathcal{B}$ encoded by some path in $\mathcal{H}^*$. We claim that every $\mathcal{A}_i$ embeds elementarily into $\mathcal{B}$. To see this, it is enough to notice that whenever $\langle b_{j_0}, \ldots, b_{j_{n-1}} \rangle$ is the tuple of elements of $\mathcal{B}$ corresponding to the tuple of constants $\langle c_{2\langle i,0\rangle}, \ldots, c_{2\langle i,n-1\rangle} \rangle$, we have $\mathsf{tp}^{\mathcal{B}}(b_{j_0}, \ldots, b_{j_{n-1}}) = \mathsf{tp}^{\mathcal{A}}(a_{i,0}, \ldots, a_{i,n-1})$. $\square$

Next we wish to prove Theorem 2.27. The following will be helpful.

Lemma 3.5. $\mathsf{WKL_0}$ *proves the following. If T is a complete theory and $\mathcal{A}, \mathcal{B}$ are models of T with domains $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$, respectively, and there is a function $f : M \to M$ such that for every n there is a tuple $\langle i_0, \ldots, i_{n-1} \rangle$ such that $i_j \leq f(j)$ for all j and such that*

$$\mathsf{tp}^{\mathcal{B}}(b_0, \ldots, b_{n-1}) = \mathsf{tp}^{\mathcal{A}}(a_{i_0}, \ldots, a_{i_{n-1}}),$$

then there is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$.

Proof. Let $(M, \mathcal{S})$ be a model of $\mathsf{WKL_0}$, and fix $T, \mathcal{A}, \mathcal{B}, f \in \mathcal{S}$ as in the hypothesis. We build a tree $\mathcal{T} \in \mathcal{S}$ such that any path through $\mathcal{T}$ can be used to define an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$ in a Δ^0_1 way. We then argue that $\mathcal{T}$ is infinite, and obtain the desired path using Weak König's Lemma. Let

$$\mathcal{U} = \{\sigma \in M^{<M} : \sigma(i) \leq f(i) \text{ for all } i < |\sigma|, \text{ and } \sigma(i) \neq \sigma(j) \text{ whenever } i \neq j\}.$$

Then $\mathcal{U}$ is a tree, and the infinite paths through $\mathcal{U}$ are exactly the injections $h : M \to M$ such that $h(n) \leq f(n)$ for all n. For each n, let $\{\phi^{(n)}_0, \phi^{(n)}_1, \ldots\}$ be an enumeration of all n-ary L-formulas. We define $\mathcal{T}$ to be the following subtree of $\mathcal{U}$:

$$\mathcal{T} = \{\sigma \in \mathcal{U} : (\forall i, n < |\sigma|)[\mathcal{B} \models \phi^{(n)}_i(b_0, \ldots, b_{n-1}) \text{ iff } \mathcal{A} \models \phi^{(n)}_i(a_{\sigma(0)}, \ldots, a_{\sigma_{n-1}})]\}.$$

If α is an infinite path of $\mathcal{T}$, then the function $g : \{b_0, \ldots\} \to \{a_0, \ldots\}$ given by $g(b_i) = a_{\alpha(i)}$ is an elementary embedding.

It remains to check that $\mathcal{T}$ is infinite. Fix any $n \in M$. By hypothesis, there is a tuple $(a_{i_0}, \ldots, a_{i_{n-1}})$ such that $i_j \leq f(j)$ for each j, and such that $\mathsf{tp}^{\mathcal{B}}(b_0, \ldots, b_{n-1}) = \mathsf{tp}^{\mathcal{A}}(a_{i_0}, \ldots, a_{i_{n-1}})$. Then the string σ of length n with $\sigma(j) = i_j$ is in $\mathcal{T}$. Hence $\mathcal{T}$ has at least n elements, as required. $\square$

Proof of Theorem 2.27. Fix a model $(M, \mathcal{S})$ of WKL_0, and fix $T, f \in \mathcal{S}$ such that T is a complete consistent theory, and $f : M \to M$ is a function such that $f(n)$ is greater than the number of n-types of T for all n. We must show that T has a universal model. To do this, we define a sequence $X = \langle p_0, \ldots \rangle$ of types of T such that every n-type is equal to p_i for some $i < 2^{f(0)} + 2^{f(1)} + \cdots + 2^{f(n)}$. We then let $\mathcal{A}$ be the model constructed as in the proof of Corollary 3.4 above, and use Lemma 3.5 to argue that $\mathcal{A}$ is universal.

For each $n \in M$, let $(\phi_t^{(n)})_t$ be an enumeration of all n-ary L-formulas. We describe how to build a tuple $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ of n-types which includes every n-type of T. Let $q_{k,0} = \emptyset$ for all k. If $\langle q_{0,s}, \ldots, q_{2^{f(n)}-1,s} \rangle$ is defined, let $q_{k,s+1} = q_{k,s} \cup \{\phi_s\}$ for exactly half of all k such that $T \nvdash \bigwedge q_{k,s} \to \neg\phi_s$; let $q_{k,s+1} = q_{k,s} \cup \{\neg\phi(s)\}$ for all other k. Clearly each $q_k = \bigcup_s q_{k,s}$ is an n-type of T, and the tuple $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ exists by Δ^0_1 comprehension. To see that $\langle q_0, \ldots, q_{2^{f(n)}-1} \rangle$ contains all n-types, it is enough to notice that each n-type $p = \{\psi_0, \psi_1, \ldots\}$ contains at most $f(n)$ distinct ψ_m such that $T \nvdash \bigwedge_{i<m} \psi_i \to \psi_m$.

Now iterate this method for all $n \in M$ to produce a sequence $X = \langle p_0, p_1, \ldots \rangle$ of types of T such that the first $2^{f(0)}$-many are a list of all 0-types,[2] the next $f(1)$-many are a list of all 1-types, and so on. Then X is an enumeration of all types of T; let $\mathcal{A}$ be the model produced in the proof of Corollary 3.4 using this X. Using the bound $2^{f(0)} + \cdots + 2^{f(n)}$ and the mapping from $\bar{p}, q, \bar{r}$ to $c_{\langle \bar{p}, q, \bar{r} \rangle}$ in the definition of $\Phi_{H,s}$, we can define a function $g : M \to M$ as in the hypothesis of Lemma 3.5. We conclude by that Lemma that $\mathcal{A}$ is a universal model of T. □

Note that Lemma 3.5 can also be used to get a shorter, less explicit proof of Theorem 2.25. Moving on: this section's final result constructs not one, but a sequence of models. Its proof is based on a construction found in [2] and partially duplicates a theorem from [2].

Proposition 3.6. $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ *proves the following. Fix a complete theory T which has infinitely many n-types for some n. If $(P, \leq)$ is a partial order with $P = \{p_0, p_1, \ldots\}$, then there is a sequence $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ of models of T such that $p_i \leq p_j$ if and only if $\mathcal{A}_i$ embeds elementarily into $\mathcal{A}_j$.*

Proof. Let $(M, \mathcal{S})$ be a model of $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$. By Lemma 1.3, we may fix a set $Z \in \mathcal{S}$ whose Turing jump K^Z is not in $\mathcal{S}$. Fix a complete consistent theory $T \in \mathcal{S}$ a partial order $(P, \leq) \in \mathcal{S}$ with $P = \{p_0, p_1, \ldots\}$ with a number n as in the theorem statement.

Consider the set $\mathcal{H}^\dagger = \{\langle \sigma_0, \ldots, \sigma_{k-1} \in \mathcal{H}^{<M} :$ each σ_i has length $|\sigma_i| = k\}$ with the ordering $\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \prec \langle \tau_0, \ldots, \tau_{\ell-1} \rangle$ if $k \leq \ell$ and $\sigma_i \subseteq \tau_i$ for every $i < k$. This $\mathcal{H}^\dagger$ is an infinite tree, any path of which encodes a sequence $\langle \mathcal{B}_0, \ldots \rangle$ of models of T. What's more, $\mathcal{H}^\dagger$ can be encoded homeomorphically as a binary branching tree in a Δ^0_1 way. Similar to other proofs in this section, we define an infinite subtree of $\mathcal{H}^\dagger$ such that any $\langle \mathcal{B}_0, \ldots \rangle$ encoded by one of its paths satisfies the theorem, and then apply WKL_0.

We have two sorts of requirement to meet. First, given i, j such that $p_i \leq p_j$, we must ensure that $\mathcal{B}_i$ embeds elementarily into $\mathcal{B}_j$. Second, given i, j such that $p_i \not\leq p_j$, we must ensure that $\mathcal{B}_i$ does not embed elementarily into $\mathcal{B}_j$. We address these two requirements separately, and then show how to combine the strategies to prove the theorem.

Making $\mathcal{B}_i$ embed into $\mathcal{B}_j$. Fix i and j. Let $(\psi_s)_{s \in M}$ be an enumeration of all L-formulas. Define a subtree $\mathcal{H}^\dagger_0$ of $\mathcal{H}^\dagger$ by:

$$\begin{aligned} \mathcal{H}^\dagger_0 = \{\langle \sigma_0, \ldots, \sigma_{k-1} \rangle \in \mathcal{H}^\dagger :\ & \text{if } T \cup D_{\sigma_i} \vdash \phi_s(a_0, \ldots, a_{m-1}) \\ & \text{then } T \cup D_{\sigma_j} \nvdash \neg\phi_s(a_{2\langle i,j,0 \rangle}, \ldots, a_{2\langle i,j,m-1 \rangle})\}. \end{aligned}$$

If $\langle \mathcal{B}_0, \ldots \rangle$ is encoded by a path in $\mathcal{H}^\dagger_0$, define a mapping from $\mathcal{B}_i$ to $\mathcal{B}_j$ by taking each $[c_k]_E$ in $\mathcal{B}_i$ to $[c_{2\langle i,j,k \rangle}]_E$ in $\mathcal{B}_j$. This is a Δ^0_1-definable elementary embedding.

[2] That is, if $i < 2^{f(0)}$, then $p_i = T$.

Making $\mathcal{B}_i$ not embed into $\mathcal{B}_j$. Fix i and j. Our strategy is to ensure that the Turing jump K^Z is Δ^0_1-definable from any elementary embedding $\mathcal{B}_i \hookrightarrow \mathcal{B}_j$, and argue that $K^Z \notin \mathcal{S}$ implies no such embedding exists. We adapt the argument from [2]. Let $(\phi_s)_s$ be an enumeration of all n-ary L-formulas. For each pair $\sigma, \tau \in \mathcal{H}$ and each natural number t, define an L^*-sentence $\theta_{\sigma,t}$ as follows.

- If there is an $s < t$ such that $T \vdash (\exists \bar{x})\phi_s(\bar{x})$, such that $T \cup D_\sigma \cup W_{|\sigma|} \vdash \neg\phi_s(\bar{d})$ for each n-tuple $\bar{d}$ from among constants $\{c_0, \ldots, c_{t-1}\}$, then let $\theta_{\sigma,t} = \phi_s$ for the least such s.
- Otherwise, let $\theta_{\sigma,t} = \mathsf{Tr}$ be the formal 'true' predicate.

Notice that if $\theta_{\sigma,t}$ is defined as in the first alternative and $\sigma \subseteq \tau$ then $\theta_{\tau,t} = \theta_{\sigma,t}$. Notice also that, if f is a path in $2^{<M}$ and t is a number, since T has infinitely many n-types, there is an initial segment $\sigma \subseteq f$ such that $\theta_{\sigma,t}$ is defined as in the first alternative. Furthermore, we can find this initial segment effectively. Define a subtree $\mathcal{H}^\dagger_1$ of $\mathcal{H}^\dagger$ by:

$$\begin{aligned}\mathcal{H}^\dagger_1 = \{\langle \sigma_0, \ldots, \sigma_{k-1}\rangle \in \mathcal{H}^\dagger :\ & \text{if } \ell \in K^Z_{t-1} \text{ and } i,j,k < t \\ & \text{then } T \cup D_{\sigma_i} \nvdash \neg\theta_{\sigma_j,t}(c_{2n\langle i,j,\ell\rangle}, \ldots, c_{2n(\langle i,j,\ell\rangle+1)-2})\}.\end{aligned}$$

Let $\langle \mathcal{B}_0, \ldots\rangle$ be the sequence encoded by a path in $\mathcal{H}^\dagger_1$. Suppose for a contradiction that g is an elementary embedding from $\mathcal{B}_i$ to $\mathcal{B}_j$. Let ℓ and t be any pair such that $\ell \in K^Z_{\underline{at}\ t}$. Then we have $\mathcal{B}_i \models \theta([c_{2n\langle i,j,\ell\rangle}]_E, \ldots, [c_{2n(\langle i,j,\ell\rangle+1)-2}]_E)$ and $\mathcal{B}_j \models \neg\theta([\bar{d}])$ for all n-tuples $\bar{d}$ taken from $\{c_0, \ldots, c_{t-1}\}$ where $\theta = \theta_{\sigma,t}$ for some σ. Hence g maps one of $[c_{2n\langle i,j,\ell\rangle}]_E, \ldots, [c_{2n(\langle i,j,\ell\rangle+1)-2}]_E$ to a $[c_s]_E$ with $s > t$. This allows us to define a function which dominates the modulus function for K^Z. It follows by Δ^0_1 comprehension that K^Z is an element of $\mathcal{S}$, a contradiction.

Combining the strategies. We combine the two strategies in a straightforward way. Define a subtree $\mathcal{H}^\ddagger$ of $\mathcal{H}^\dagger$ by:

$$\begin{aligned}\mathcal{H}^\ddagger = \bigcap_{p_i \le p_j} \{\langle \sigma_0, \ldots, \sigma_{k-1}\rangle \in \mathcal{H}^\dagger :\ & \text{if } T \cup D_{\sigma_i} \vdash \phi_s(a_0, \ldots, a_{m-1}) \\ & \text{then } T \cup D_{\sigma_j} \nvdash \neg\phi_s(a_{2\langle i,j,0\rangle}, \ldots, a_{2\langle i,j,m-1\rangle})\} \\ \cap \bigcap_{p_i \not\le p_j} \{\langle \sigma_0, \ldots, \sigma_{k-1}\rangle \in \mathcal{H}^\dagger :\ & \text{if } \ell \in K^Z_{t-1} \text{ and } i,j,k < t \\ & \text{then } T \cup D_{\sigma_i} \nvdash \neg\theta_{\sigma_j,t}(c_{2n\langle i,j,\ell\rangle}, \ldots, c_{2n(\langle i,j,\ell\rangle+1)-2})\}.\end{aligned}$$

It is not difficult to see that $\mathcal{H}^\ddagger$ is infinite and that, if $\langle \mathcal{B}_0, \ldots\rangle$ is the sequence of models encoded by a path, then by the arguments above $\mathcal{B}_i$ embeds elementarily into $\mathcal{B}_j$ if and only if $p_i \le p_j$. We now obtain the desired $\langle \mathcal{B}_0, \ldots\rangle$ by applying $\mathsf{WKL_0}$. □

4 A Controlled failure of compactness

Recall from Lemma 1.5 that $\mathsf{WKL_0}$ is equivalent over $\mathsf{RCA_0}$ to the compactness theorem for first-order logic. The usual proof of the leftward direction of this equivalence begins by fixing a binary tree $\mathcal{T}$, and then building a complete theory T which satisfies the Compactness Theorem only if $\mathcal{T}$ has an infinite path. In this section, we give a construction that accomplishes roughly the same thing: it takes a tree and attempts to provide a counterexample to the compactness theorem. Yet this construction has certain advantages, namely, that it produces very intuitive models—in its most basic instance, it produces a theory where every singleton in every model is a definable set—and that it can be cleanly extended, as we do in §5.

The present section is laid out as follows. In §4.1, we detail a construction that transforms an infinite binary tree $\mathcal{T}$ into a complete theory T, and defines a certain sequence of unary predicates $\langle P_i \rangle_i$. Then, in §4.2, we show that, if $\mathcal{T}$ has no infinite path, the predicates P_i partition the universe of any model of T into infinitely many sets, each with the same cardinality. In particular, the set $\{\neg P_i(x) : i \in M\}$ of formulas is finitely satisfiable but not satisfiable.

To simplify the axioms and some steps of the verification, we build T indirectly as a reduct of another theory T^* on an expanded language. Our construction also has the odd feature that, for certain choices of binary tree $\mathcal{T}$, the theory T being built might be incomplete. It simplifies our analysis to assume from the start that $\mathcal{T}$ is an infinite tree with no infinite path, and, in particular, that $\mathcal{T}$ belongs to a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$.

4.1 Construction

Fix a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, an infinite binary tree $\mathcal{T} \in \mathcal{S}$ with no infinite paths, and a number $n \in M$. Let $\langle \tau_0, \tau_1, \ldots \rangle$ be a one-to-one listing of all terminal nodes of $\mathcal{T}$. Define a larger tree $\mathcal{T}_0$ by:

$$\mathcal{T}_0 = \mathcal{T} \cup \{\tau_i {}^\frown 0^s : \ i, s \in M\}.$$

Then $\mathcal{T}_0$ has no terminal nodes. Let $L = \langle R_\sigma : \sigma \in 2^{<M} \rangle$ be an infinite language of unary relations, and let $L^* = L \cup \langle c_{i,j} : i \in M, \ j < n \rangle$. Consider the following axiom schemes:

Ax I. $R_\emptyset(x)$.

Ax II. $R_\sigma(x) \to R_{\sigma {}^\frown 0}(x) \vee R_{\sigma {}^\frown 1}(x)$.

Ax III. $\neg(R_\sigma(x) \wedge R_{\sigma'}(x))$ whenever σ, σ' are incompatible strings.

Ax IV. $\neg R_\sigma(x)$ whenever σ is not an element of $\mathcal{T}_0$.

Ax V. $c_{i_0,j_0} \neq c_{i_1,j_1}$ whenever $i_0 \neq i_1$ or $j_0 \neq j_1$.

Ax VI. $R_\sigma(c_{i,j})$ whenever $\sigma \subseteq \tau_i$ and $j < n$.

Ax VII. $R_\sigma(x) \to \bigvee\limits_{i \in F, j<n} x = c_{i,j}$ whenever F is a finite set containing all i such that $\sigma \subseteq \tau_i$.

Axioms I–IV say that, whenever $\mathcal{A}$ is a model of the axioms and a is an element, the set $\{\sigma : \mathcal{A} \models R_\sigma(a)\}$ forms a path through $\mathcal{T}$. By the definition of $\mathcal{T}_0$, this set is uniquely determined by the unique index i such that $\mathcal{A} \models R_{\tau_i}(a)$. Axiom V says simply that all the constants $c_{i,j}$ are distinct, and axioms VI–VII guarantee that those elements a for which $\mathcal{A} \models R_{\tau_i}(a)$ are exactly those given by constants $c_{i,0}, \ldots, c_{i,n-1}$. Despite their indirect definition, the axioms of Ax VII are a Δ^0_1 set; to see this, notice that every node $\sigma \in \mathcal{T}$ either has only finitely many extensions in $\mathcal{T}$, or has infinitely many terminal extensions.

Define a sequence of predicates P_i by:

$$P_i(x) \iff R_{\tau_i}(x).$$

We finish the construction by letting T^* be the deductive closure of the Ax I–VII, and letting T be the reduct of T^* to the language L. At this point, it is far from clear that T^* and T exist in the second-order part of $(M, \mathcal{S})$; one of our main tasks in the verification below is to show that they do. This is accomplished below in Corollary 4.2.

4.2 Verification

We must verify that T is in the second-order part of $(M, \mathcal{S})$, that it is a complete, consistent theory, and that the predicates P_i partition the universe of any model as outlined above. Since we have not yet proved that T or T^* exist in $(M, \mathcal{S})$, we cannot bring to bear the usual model-theoretic tools, such as the Completeness Theorem. Instead, we must deal with formulas from first principles, by manipulating their syntax.

Lemma 4.1. *There is an algorithm which, given a conjunction of L^*-literals $\phi(\bar{x}, y)$, returns a quantifier-free L^*-formula $\psi(\bar{x})$ such that Axioms I–VII entail $\psi(\bar{x}) \leftrightarrow (\exists y)\phi(\bar{x}, y)$.*

Proof. Suppose that $\phi(\bar{x}, y)$ is a conjunction of L^*-literals, and let m be the length of $\bar{x} = \langle x_0, \ldots, x_{m-1}\rangle$. We may assume by Ax III, IV, and VI that no conjunct is of the form $R_\sigma(c_{i,j})$ or $\neg R_\sigma(c_{i,j})$; by Ax V, that none is of the form $c_{i_0,j_0} = c_{i_1,j_1}$ or $c_{i_0,j_0} \neq c_{i_1,j_1}$; by substituting variables, that none is of the form $t_0 = t_1$ for any terms z_0, z_1; and, by symmetry of $=$, that none is of the form $c_{i,j} \neq z$ for any variable z. (The remaining conjuncts are of the form $R_\sigma(z)$, $\neg R_\sigma(z)$, and $z \neq c_{i,j}$, where z is a variable and $c_{i,j}$ is a constant.)

Let $\phi_0(y)$ be the formula obtained by replacing with Tr every conjunct mentioning any x_k, $k < m$. Then ϕ_0 is a conjunction of literals of the forms Tr, $R_\sigma(y)$, $\neg R_\sigma(y)$, and $y \neq c_{i,j}$. If $\langle i, j\rangle$ is a pair such that $\tau_i \supseteq \sigma$ for each $R_\sigma(y)$ in ϕ_0, such that $\tau_i \not\supseteq \sigma$ for each $\neg R_\sigma(y)$ in ϕ_0, and such that $y \neq c_{i,j}$ is not in ϕ_0, then Axioms I–VII imply $\phi_0(c_{i,j})$; otherwise, they imply $\neg\phi_0(c_{i,j})$. We can check effectively—using the fact that $\mathcal{T}$ has no infinite path—whether there exist more that m distinct such pairs.

Case 1: There are distinct such pairs $\langle i_0, j_0\rangle, \ldots, \langle i_m, j_m\rangle$. Let $\psi(\bar{x})$ be the formula:

$$\psi(\bar{x}) \Leftrightarrow \phi(\bar{x}, c_{i_0,j_0}) \vee \cdots \vee \phi(\bar{x}, c_{i_m,j_m}).$$

The implication $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ is a tautology. We now show that Ax I–VII prove the converse statement $(\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$. Let $\phi_1(\bar{x})$ be the formula obtained from ϕ by replacing with Tr each conjunct mentioning y. Then $\phi(\bar{x}, y)$ is equivalent to the formula

$$\phi_1(\bar{x}) \wedge \phi_0(y) \wedge \bigwedge_{\ell \in E} x_\ell \neq y$$

for some set $E \subseteq \{0, \ldots, m-1\}$. Of course, $(\exists y)\phi(\bar{x}, y) \to \phi_1(\bar{x})$ is a tautology, and $\phi_0(c_{i_k,j_k})$ is true for each k by choice of i_k, j_k. As well, by the Pigeonhole Principle, Axiom V is enough to prove

$$\bigvee_{\ell \leq m} \bigwedge_{k<m} x_k \neq c_{i_\ell,j_\ell}.$$

Hence Ax I–VII can prove

$$(\exists y)\phi(\bar{x}, y) \to \bigvee_{\ell \leq m} \left(\phi_1(\bar{x}) \wedge \phi_0(c_{i_k,j_k}) \wedge \bigwedge_{k \notin E} x_\ell \neq c_{i_k,j_k}\right),$$

which is equivalent to the desired statement.

Case 2: There are no more than m distinct such pairs. Let $\langle i_0, \ldots, i_{\ell-1}\rangle$ be a list of all i such that $\tau_i \supseteq \sigma$ whenever $R_\sigma(y)$ is in ϕ_0 and $\tau_i \not\supseteq \sigma$ whenever $\neg R_\sigma(y)$ is in ϕ_0. Axioms II and IV prove the statement

$$\phi_0(y) \to \bigvee_{i<\ell} P_i(y).$$

Together with Axiom VII, this gives

$$\phi_0(y) \to \bigvee_{k<\ell} \bigvee_{j<n} y = c_{i_k,j}. \tag{1}$$

Now let $\psi(\bar{x})$ be the formula:

$$\psi(\bar{x}) \Leftrightarrow \bigvee_{k<\ell} \bigvee_{j<n} \phi(\bar{x}, c_{i_k,j}).$$

The implication $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ is a tautology. The converse implication $(\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$ follows from the displayed formula (1). □

Corollary 4.2. *The deductive closure T^* of Ax I–VII exists in $(M, \mathcal{S})$, and admits effective quantifier elimination. The reduct T of T^* to the language L exists in $(M, \mathcal{S})$.*

Proof. If ϕ is an L^*-sentence, we can apply the effective procedure from Lemma 4.1 iteratively to obtain a quantifier-free L^*-sentence ψ such that Ax I–VII entail $\phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. (See [2, Lem 1.6] or [10, Proof of Thm 2.3].) Since Ax I–VII decide every quantifier-free L^*-sentence—it is clear which constants satisfy which relations—it follows that Ax I–VII decide every L^*-sentence.

Therefore the theory T^* exists by Δ^0_1 comprehension, as does its reduct T. □

Lemma 4.3. *T^* is consistent.*

Proof. We begin defining a model $\mathcal{A}^*$ with universe $A = \langle a_{i,j} : i \in M \text{ and } j < n\rangle$ by specifying its *atomic* diagram, beginning with:

$\mathcal{A}^* \models a_{i,j} = c_{i,j}$, and $\mathcal{A}^* \models a_{i_0,j_0} \neq a_{i_1,j_1}$ whenever $\langle i_0, j_0\rangle \neq \langle i_1, j_1\rangle$, and $\mathcal{A}^* \models R_\sigma(a_{i,j})$ if and only if $\sigma \subseteq \tau_i$ or $\sigma = \tau_i {}^\frown 0^s$ for some s.

This atomic diagram satisfies each of Ax I–VII. Use the effective procedure for quantifier elimination given by Lemma 4.2 to assign a truth value to every $L^* \cup \{a_0, \ldots\}$-sentence for $\mathcal{A}^*$. To see that we end up with an elementary diagram—that is, a set free of inconsistencies and closed under entailment—notice first that, by the derivation of our effective procedure, every ϕ with quantifier depth 1 is assigned a truth value that is *semantically* correct from the atomic diagram. It follows by Δ^0_1 induction that every sentence's truth value is semantically true, giving the desired consistency and closure properties. (A formal development of the semantic side of first-order logic can be found in [20, section II.8].) □

Corollary 4.4. *T^* and T are complete, consistent theories.*

And at last we can check some less basic properties of T. Recall that n is a natural number fixed in §4.1 and used in defining the axioms of T^*.

Lemma 4.5. (i) *If $\mathcal{A}^*$ is a model of T^*, the sets $P_i^{\mathcal{A}^*} = \{a : \mathcal{A}^* \models P_i(a)\}$ partition its domain. Furthermore, each of these sets has size n.*

(ii) *If $\mathcal{A}^*$ is a model of T^*, each element is equal to some constant $c_{i,j}^{\mathcal{A}^*}$.*

(iii) *If $\mathcal{A}$ is a model of T, then the sets $P_i^{\mathcal{A}} = \{a : \mathcal{A} \models P_i(a)\}$ partition its domain into sets of size n.*

Proof. (i) Because $\mathcal{T}$ has no infinite path, Ax I–IV ensure that for each element a there is a unique terminal node τ_i of $\mathcal{T}$ such that $\mathcal{A}^* \models P_i(a)$. Hence the sets $P_i^{\mathcal{A}^*}$ partition the domain. If a is an element and τ_i is the corresponding terminal node, then by Axiom VII we know that $\mathcal{A}^* \models a = c_{i,j}$ for some $j < n$. It follows that $P_i^{\mathcal{A}^*}$ is equal to the set $\{c_{i,j}^{\mathcal{A}^*} : j < n\}$. By Axiom V, these $c_{i,j}^{\mathcal{A}^*}$ are all distinct, so $P_i^{\mathcal{A}^*}$ has size n.

(ii) Already proved as part of 4.5.

(iii) Each of Axioms I–IV uses only symbols from L, and so is contained in T. As in 4.5, this means the sets $P_i^{\mathcal{A}}$ partition the domain of $\mathcal{A}$. What's more, by 4.5 we know that the formula $(\exists^{=n} x) P_i(x)$ is contained in T^* and uses only symbols from L, and so is contained in T as well. It follows that each $P_i^{\mathcal{A}}$ has size n.

□

Lemma 4.6. *Suppose $\mathcal{A}$ is a model of T with domain A.*

(i) *There is a model $\mathcal{A}^*$ of T^* extending $\mathcal{A}$.*

(ii) *Any permutation of A taking each $P_i^{\mathcal{A}}$ back to $P_i^{\mathcal{A}}$ is an automorphism of $\mathcal{A}$.*

Proof. (i) Given $i \in M$, we may effectively find all n distinct elements a such that $\mathcal{A} \models P_i(a)$. Define $\mathcal{A}^*$ by letting $c_{i,0}^{\mathcal{A}^*}, \ldots, c_{i,n-1}^{\mathcal{A}^*}$ be a listing of these elements for each i. Extend to an elementary diagram as in the proof of Lemma 4.3.

(ii) Suppose that f is a permutation of the domain of $\mathcal{A}$ mapping each $P_i^{\mathcal{A}}$ back to $P_i^{\mathcal{A}}$. Let $\mathcal{A}_0^*$ be an extension of $\mathcal{A}$ as in part (i) above, and let $\mathcal{A}_1^*$ be another extension given by $c_{i,k}^{\mathcal{A}_1^*} = f(c_{i,k}^{\mathcal{A}_0^*})$ for each i, k. Then f is an isomorphism from $\mathcal{A}_0^*$ to $\mathcal{A}_1^*$.

□

4.3 Applications

The main application of this construction comes when we extend it in §5. For now, we now give a separate, immediate model-theoretic consequence.

Proposition 4.7. $\mathsf{RCA}_0 \vdash$ (*Every complete consistent theory has a model with a sequence of order indiscernibles*) $\rightarrow \mathsf{WKL}_0$.

Proof. We prove the contrapositive statement. Suppose $(M, \mathcal{S})$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, let T be the theory constructed in §4.1 with $n = 1$, and let $\mathcal{A}$ be any model of T. Suppose for a contradiction that there is a sequence of order indiscernibles with distinct elements a and b. Then by Lemma 4.54.5, there is a j such that $\mathcal{A} \models P_j(a)$ and $\mathcal{A} \models \neg P_j(b)$, a contradiction. □

We also note in passing that, with a few minor changes to the axioms and verification, the construction in §4.1 gives a theory whose every model is partitioned into countably many *infinite* sets, or sets of different sizes.

Corollary 4.8 ($\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$)**.** *Let f be a total function $f : M \to \{1, 2, \ldots\} \cup \{\aleph_0\}$, where $\aleph_0$ is a formal symbol denoting a countable infinity. There is a complete consistent theory T with a sequence of unary formulas $P_0(x), \ldots$ with the following properties: If $\mathcal{A}$ is a model of T with universe A, then the sets $P_m^{\mathcal{A}}$ form a partition of A, with $|P_m^{\mathcal{A}}| = f(m)$ for all m, and any permutation of A fixing each $P_m^{\mathcal{A}}$ is an automorphism of $\mathcal{A}$.*

5 1-Homogeneity vs strong 1-homogeneity

In this section, we produce an example of a theory T with only principal types, but with no strongly 1-homogeneous model. This theory is built by extending the construction in §4 above. As such, we again work within a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, and construct T indirectly as a reduct of a larger theory T^*.

We begin with an outline of the construction and its verification. Fix a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$. Recall from Definition 1.6 the notion of an *inseparable Σ_1^0 pair*. Using $\neg\mathsf{WKL}_0$ and Lemma 1.5, fix an inseparable Σ_1^0 pair $\langle U_s, V_s \rangle_s \in \mathcal{S}$. Let $L = \langle Q_s, B_s, R_\sigma \rangle_{s,\sigma}$ be the language where each Q_s and each R_σ is a unary relation symbol, and each B_s is a binary relation symbol. We design an L^*-theory T^* so that, if $\mathcal{A}^*$ is a model of T^*, $\mathcal{A}$ is the reduct of $\mathcal{A}^*$ to L, and A is the domain of $\mathcal{A}^*$, then the following hold.

(B1) T^* includes all the axioms listed in the construction of §4.1 with $n = 2$.

(B2) There is a sequence of L-formulas $P_0(x), P_1(x), \ldots$ such that the sets $P_i^{\mathcal{A}}$ form a partition of A. Furthermore, each set $P_i^{\mathcal{A}}$ consists exactly of the elements $c_{k,0}^{\mathcal{A}^*}$ and $c_{k,1}^{\mathcal{A}^*}$.

(B3) The elements $c_{0,0}^{\mathcal{A}^*}$ and $c_{0,1}^{\mathcal{A}^*}$ satisfy the same L-formulas. (In other words, $c_{0,0}^{\mathcal{A}^*}$ and $c_{0,1}^{\mathcal{A}^*}$ realize the same 1-type in $\mathcal{A}$.)

(B4) Any automorphism of $\mathcal{A}$ which maps $c_{0,0}^{\mathcal{A}^*}$ to $c_{0,1}^{\mathcal{A}^*}$ computes a separating set for $\langle U_s, V_s \rangle_s$. (Hence no such automorphism exists in $\mathcal{S}$.)

We now give a hint as to what the structures $\mathcal{A}^*$ and $\mathcal{A}$ look like. As mentioned in property (B2), there is a sequence $P_0, \ldots$ of unary predicates which partition A into sets of size 2, with each $P_s^{\mathcal{A}}$ consisting of the elements $c_{k,0}^{\mathcal{A}^*}$ and $c_{k,1}^{\mathcal{A}^*}$. The unary predicate Q_s holds of an element $a \in A$ if and only if $a = c_{k+1,0}^{\mathcal{A}^*}$ where $k \in U_{at\ s}$. The binary predicate B_s holds of a pair $a, b \in A$ if and only if both $a = c_{0,j}^{\mathcal{A}^*}$ and $b = c_{k+1,j}^{\mathcal{A}^*}$ are true when $k \in V_{at\ s}$ and some $j \in \{0, 1\}$.

5.1 Construction

Fix a model $(M, \mathcal{S})$ of $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$. Fix an infinite tree $\mathcal{T} \in \mathcal{S}$ with no infinite path. Let $L = \{Q_s, R_\sigma, B_s : s \in M, \sigma \in 2^{<M}\}$ be a relational language where each Q_s, R_σ is unary and each B_s is binary. Let $L^* = L \cup \{c_{i,j} : i \in M, j \in \{0, 1\}\}$, where each $c_{i,j}$ is a constant symbol. Consider the following axiom schemes:

Ax I.
⋮ — As in §4.1, with $n = 2$.
Ax VII.

Ax VIII. $Q_s(c_{k+1,0})$ if k enters U at stage s.

Ax IX. $\neg Q_s(c_{k,j})$ for all other choices of j, k, s.

Ax X. $B_s(c_{0,j}, c_{k+1,j})$ for each j, k, s such that k enters V at stage s.

Ax XI. $\neg B_s(c_{k_0,j_0}, c_{k_1,j_1})$ for all other choices of j_0, j_1, k_0, k_1, s.

We now give the intuition behind the axioms, in terms of the properties (B1–B2) listed near the beginning of this section. The first seven are exactly the axioms used in the construction of §4.1 above when $n = 2$, so (B1) is true. It follows by Lemma 4.54.5 that (B2) holds as well. Axioms VIII–XI give property (B4)—see Lemma 5.5 below. The remaining property (B3) holds because, roughly speaking, the axioms treat $c_{0,0}$ an $c_{0,1}$ symmetrically—see Lemma 5.2(iv) below for the details.

Use $\mathcal{T}$ and the relations R_σ to define a sequence of unary predicates P_i as in §4.1. Finish the construction by letting T^* be the deductive closure of the Ax I–XI and T the reduct of T^* to L; as in §4.1, it is not yet clear that T^* and T should exist in $(M, \mathcal{S})$. We deal with this early in the verification as part of Lemma 5.1.

5.2 Verification

We begin by listing some basic properties of T and T^* such as existence and completeness. The proofs are analogous to those in §4.

Lemma 5.1. (i) *There is an algorithm which, given a conjunction of L-literals $\phi(\bar{x}, y)$, returns a quantifier-free L^*-formula $\psi(\bar{x})$ such that Ax I–XI prove $\psi(\bar{x}) \leftrightarrow (\exists y)\phi(\bar{x}, y)$.*

(ii) *T^* exists in $(M, \mathcal{S})$ and has effective quantifier elimination.*

(iii) *T exists.*

(iv) *T^* is consistent. T is consistent. T is complete.*

Proof. (i) Similar to Lemma 4.1.

(ii) Follows from (i), similar to Corollary 4.2.

(iii) Follows from part (ii) and Δ_1^0 comprehension.

(iv) Similar to Lemma 4.3: Find the unique structure $\mathcal{A}^* \models T^*$ with universe $\{a_{i,j} : i \in M,\ j \in \{0,1\}\}$ such that $\mathcal{A} \models a_{i,j} = c_{i,j}$ for each i, j.

□

Next, some less basic properties.

Lemma 5.2. (i) *If $\mathcal{A}^*$ is a model of T^*, then the predicates P_i partition its domain into sets $P_i^{\mathcal{A}^*}$ of size 2. Furthermore, $P_i^{\mathcal{A}^*}$ is equal to $\{c_{i,0}^{\mathcal{A}^*}, c_{i,1}^{\mathcal{A}^*}\}$ for all i. Hence property (B2) holds.*

(ii) *If $\mathcal{A}$ is a model of T, then the sets $P_i^{\mathcal{A}}$ partition its domain into sets of size 2.*

(iii) *Every 1-type of T is principal.*

(iv) *Every type of T is principal.*

(v) *Every model of T is 1-homogeneous.*

Proof. (i) Similar to Lemma 4.5(i).

(ii) Similar to Lemma 4.5(iii).

(iii) Fix a 1-type $p(x)$ of T. By Lemma 5.2(ii), Lemma 5.1(iii), and the Completeness Theorem, there is a j such that p contains $P_j(x)$ and $T \vdash (\exists^{=2} y) P_j(y)$. So either $P_j(x)$ generates $p(x)$, or there is a $\phi(x)$ such that $\phi(x) \to P_j(x)$ is a tautology, p contains $(\exists^{=1} y)\phi(y)$, and $\phi(x)$ generates $p(x)$.

(iv) Fix an n-type $p(\bar{x}) = p(x_0, \ldots, x_{n-1})$ of T. Identifying variables if necessary, we may assume that $x_i \neq x_j$ is in $p(\bar{x})$ for every pair $i \neq j$. We know from Lemma 5.1 that for each $k < n$ there is an i_k such that $P_{i_k}(x_k)$ is in $p(\bar{x})$, and $T \vdash (\exists^{\leq 2} y) P_{i_k}(y)$. Let $\psi(\bar{x})$ denote the conjunction $P_{i_0}(x_0) \wedge \cdots \wedge P_{i_{n-1}}(x_{n-1})$. Then $\psi(\bar{x})$ is in $p(\bar{x})$, and $T \vdash (\exists^{\leq 2^n} \bar{x})\psi(\bar{x})$.

Using $\mathsf{I\Sigma}^0_1$, let $k \leq n$ be greatest such that there is a formula $\phi(\bar{x})$ with $T \vdash \phi(\bar{x}) \to \psi(\bar{x})$ and $T \vdash (\exists^{\leq k} \bar{x})\phi(\bar{x})$. We claim $\phi(\bar{x})$ generates p. For a contradiction, suppose that it does not, i.e., suppose there is $\theta \in p$ such that $T \vdash (\exists \bar{x})[\phi(\bar{x}) \wedge \neg\theta(\bar{x})]$. Then $T \vdash (\exists^{\leq k-1} \bar{x})[\phi(\bar{x}) \wedge \theta(\bar{x})]$, and $\phi(\bar{x}) \wedge \theta(\bar{x})$ is in p, contradicting the minimality of k.

(v) Immediate from (iv).

□

Now we wish to show that no model $\mathcal{A}$ of T is strongly 1-homogeneous. We begin by showing that T admits a restricted form of quantifier elimination, classically equivalent to model completeness.

Lemma 5.3. *Every L-formula is equivalent over T to an $\exists$ L-formula.*

Proof. Fix an L-formula $\phi(\bar{x})$. Using the quantifier elimination from Lemma 5.1(ii), fix a quantifier-free L^*-formula $\psi(\bar{x})$ such that $T^* \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Our goal is to find an $\exists$ formula $\sigma(\bar{x})$ in the language L such that $T^* \vdash \phi(\bar{x}) \leftrightarrow \sigma(\bar{x})$. Let $\psi[\bar{y}/\bar{c}]$ denote the L-formula obtained by replacing each occurrence of a constant $c_{m,j}$ in ψ, with a new variable $y_{m,j}$. ($\psi[\bar{y}/\bar{c}]$ has free variables $(\bar{x}, \bar{y})$.)

Now let $p(\bar{y})$ be any type containing $y_{m,j} = c_{m,j}$ for each m, j. Using Lemma 5.2(iv), choose an L-formula $\theta(\bar{y})$ which generates p. Define an L-formula $\sigma(\bar{x})$ by

$$\sigma(\bar{x}) \Leftrightarrow (\exists \bar{y})\theta(\bar{y}) \wedge \psi[\bar{y}/\bar{c}].$$

We claim $T \vdash \phi(\bar{x}) \leftrightarrow \sigma(\bar{x})$. The forward direction $\phi(\bar{x}) \to \sigma(\bar{x})$ is clearly in T^*, so it is in the reduct T as well. To see that the reverse direction $\sigma(\bar{x}) \to \phi(\bar{x})$ is in T, simply note that the sentence $(\forall \bar{x})(\phi(\bar{x}) \leftrightarrow \psi[\bar{y}/\bar{c}]$ is in p. □

The following two lemmas show that T has no strongly 1-homogeneous model.

Lemma 5.4. *The predicate $P_0(x)$ generates a principal 1-type of T. Hence property (B3) holds.*

Proof. It is clear from the axioms that, for every unary $\exists$ L-formula $\phi(x)$, either $T \vdash P_0(x) \to \phi(x)$ or $T \vdash P_0(x) \to \neg\phi(x)$. It follows by Lemma 5.3 that $P_0(x)$ generates a 1-type. □

Lemma 5.5. *Fix any model $\mathcal{A}$ of T.*

(i) *If f is an automorphism of $\mathcal{A}$ which swaps the two elements of $P_0^{\mathcal{A}}$, then there is a separating set for $\langle U_s, V_s\rangle_s$ which is Δ^0_1 definable from f.*

(ii) *There is no automorphism of $\mathcal{A}$ which swaps the two elements of $P_0^{\mathcal{A}}$.*

Proof. (i) Enumerate the elements of $\mathcal{A}$ as $a_{0,0}, a_{0,1}, \ldots, a_{i,0}, a_{i,1}, \ldots$, with $P_i^{\mathcal{A}} = \{a_{i,0}, a_{i_1}\}$ for every i. Suppose f is an automorphism of $\mathcal{A}$ such that $f(a_{0,0}) = a_{0,1}$. Define a set C to be all $k \in M$ such that f swaps the elements of $P_{k+1}^{\mathcal{A}}$, that is,

$$C = \{k : f(a_{k+1,0}) = f(a_{k+1,1})\}.$$

For every k, s such that $k \in U_s$, we must have $k \notin C$ by Axiom VIII; and for every k, s such that $k \in V_s$, we must have $k \in C$ by Axiom X. Hence C is a separating set for $\langle U_s, V_s\rangle_s$.

(ii) Follows from (i) and our choice of $\langle U_s, V_s\rangle_s$ as an inseparable Σ^0_1 pair.

□

5.3 Application

The following completes the proof of Theorem 2.4:

Proposition 5.6. $\mathsf{RCA}_0 \vdash$ (*Every complete theory with all types principal has a strongly 1-homogeneous model*) $\to \mathsf{WKL}_0$.

Proof. We show the contrapositive. Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, and let T be as in §5.1. Then T is complete, by Lemma 5.1; has all types principal, by Lemma 5.2(iv); and is not strongly 1-homogeneous, by Lemmas 5.4 and 5.5. □

6 A theory with the finite free amalgamation property, but without the 1-point full amalgamation property

In this section, we construct, in a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$, a theory with countably many types, and an enumeration of all types with the finite free amalgamation property but without the 1-point full amalgamation property. Our method is a very slight twist on Millar's [16] construction in effective mathematics of a decidable theory with exactly two decidable models up to recursive isomorphism, which was formalized in reverse mathematics in [2]. The changes from the version in [2] are minor: we add two new relations, a unary C and a binary E; we include axioms stating that E is an equivalence relation partitioning the domain into infinitely many infinite classes; and we require that E hold of a pair (x, y) whenever any other binary relation R_k holds of (x, y). Because the differences are so slight, we leave much of the verification as a sketch.

6.1 Construction

Work in a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$. Let $\langle U_s, V_s \rangle_s$ be an inseparable Σ^0_1 pair. Let $L = \{P_s, R_k, C, E : s, k \in M\}$ be the relational language where P_s, C are unary and E, R_k are binary for all k, s. Consider the following axiom schemes.

Ax I. E is an equivalence relation.

Ax II. $P_s(x) \to P_t(x)$, whenever $t \leq s$

Ax III. $R_k(x, y) \to (E(x, y) \wedge P_k(x) \wedge P_k(y) \wedge x_i \neq x_j)$.

Ax IV. $(E(x, y) \wedge P_s(x) \wedge P_s(y) \wedge x \neq y) \to R_k(x, y)$, whenever $k \in U_s$.

Ax V. $(P_s(x) \wedge P_s(y) \wedge x \neq y) \to \neg R_k(x, y)$, whenever $k \in V_s$.

Ax VI. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L_0-literals, where $L_0 = \{E, P_i, R_i, C : i < k\}$ for some k;
- $\phi(\bar{x}, y)$ is consistent with Ax I–V;
- $\phi(\bar{x}, y) \to \psi(\bar{x})$ is a tautology;
- For each atomic L_0-formula θ with variables taken from $\bar{x}$, either θ or $\neg\theta$ is a conjunct in ψ; similarly, each atomic L_0-formula with variables from $\bar{x}, y$ or its negation is a conjunct in ϕ.

Let T be the deductive closure of Ax I–VI. This completes the construction. Note that we have not shown T is an element of $\mathcal{S}$; this is accomplished as part of Lemma 6.1 below.

6.2 Verification

The following properties can each be verified in RCA_0 by altering the appropriate lemma from [2, §7.2]:

Lemma 6.1. (i) *T is an element of $\mathcal{S}$. T is complete.*

(ii) *T is consistent.*

(iii) *T has exactly two nonprincipal 1-types $q_0(x)$ and $q_1(x)$.*

(iv) *T has countably many types.*

(v) *If $\mathcal{A}$ is a model of T with elements a_0 and a_1 realizing $p_0(x)$, $p_1(x)$, respectively, then $\mathcal{A} \models \neg E(a_0, a_1)$.*

Let X be the enumeration of all types of T produced in Lemma 6.1(iv).

Lemma 6.2. *X has the finite free amalgamation property.*

Proof. Suppose that $\langle p_{i_0}, \ldots, p_{i_{n-1}} \rangle$ is a tuple of types in X, no two of which share a variable. Then it is easy to produce a type q extending

$$p_{i_0} \cup \cdots \cup p_{i_{n-1}} \cup \{\neg E(x, y) : x \text{ is a variable of } p_{i_j},\ y \text{ is a variable of } p_{i_k},\ j \neq k\}.$$

□

Lemma 6.3. *X does not have the 1-point full amalgamation property*

Proof. Let $q_0(y), q_1(z)$ be the distinct nonprincipal 1-types from Lemma 6.1(iii). Let $p(x)$ be the principal 1-type generated by $\neg P_0(x)$. Then there are 2-types $r_0(y) \supseteq p(x) \cup q_0(y) \cup \{E(x, y)\}$ and $r_1(z) \supseteq p(x) \cup q_1(z) \cup \{E(x, z)\}$. Suppose for a contradiction that X has the 1-point full amalgamation property. Then there is a 3-type $s(x, y, z)$ extending both r_1 and r_2. Let $\mathcal{A}$ be a model realizing s, say with $s(a, b, c)$ holding. Then $q_0(b)$ holds, $q_1(c)$ holds, and $\mathcal{A} \models E(b, c)$. But this is impossible by Lemma 6.1(v). □

6.3 Applications

Proposition 6.4. $\mathsf{RCA}_0 \vdash$ (*If X is an enumeration of all types of a complete consistent theory T and X has the finite free amalgamation property, then X has the 1-point full amalgamation property*) $\rightarrow \mathsf{WKL}_0$.

Proof. We prove the contrapositive. Suppose that $\neg\mathsf{WKL}_0$ holds, and let T be the theory constructed in §6.1, and let X be the sequence of all types described in the proof of Lemma 6.1(iv). We know from §6.2 that T is a complete consistent theory, and that X has the finite free amalgamation property but not the 1-point full amalgamation property. □

Proposition 6.5. $\mathsf{RCA}_0 \vdash$ (*Every complete consistent theory with countably many types has a saturated model*) $\rightarrow \mathsf{WKL}_0$.

Proof. Follows from Proposition 6.4 and Theorem 2.7. □

7 The case with neither WKL_0 nor Σ^0_2 induction

Our goal in this section is to complete the proofs of Theorem 2.14 and Theorem 2.23. We do this by constructing, within a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I\Sigma}^0_2$, a pair of complete consistent theories. The first (§7.3) is a theory with an enumeration of all types which has the 1-point full, but not the pairwise free, amalgamation property. This is enough to complete the proof of Theorem 2.23(i). The second (§7.4) is a theory with an enumeration of all types which has the pairwise full, but not the finite free, amalgamation property. This is enough to prove Theorem 2.23(ii) and, after we introduce Lemma 7.16 below, to complete the proof of Theorem 2.14.

The basic idea is as follows. Working within a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I\Sigma}^0_2$, let $\langle U_s, V_s\rangle_s$ be an inseparable Σ^0_1 pair, as given by Lemma 1.5, and let $\langle D_1 \subseteq D_2 \subseteq \cdots\rangle$ be a counterexample to $\mathsf{I\Sigma}^0_2$ as given by Lemma 1.9(iv). We use these $\langle U_s, V_s\rangle$ and D_i to define a theory T, along with a finite sequence $\langle p_0(x), \ldots, p_{n-1}(x)\rangle$ of nonprincipal 1-types. These p_i witness the failure of the appropriate amalgamation property in both of our theories; which amalgamation properties hold and which fail depends on the specifics of the sequence $\langle D_1 \subseteq \cdots\rangle$.

The construction is based loosely on the same paper of Millar's [16] as that in §6 above.

7.1 Construction

We work in a model $(M, \mathcal{S})$ of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I\Sigma}^0_2$. By Lemma 1.9, we may fix a coded sequence $D_1 \subseteq D_2 \subseteq \cdots$ of finite sets such that D_1 is finite, D_n finite implies D_{n+1} finite, and such that D_N is infinite for some N.

Let L be the relational language $L = (P_s, R^k_s, C_k)_{s\in M, k<N}$, where each P_s and C_k is unary, and each R^k_s is k-ary. Consider the following axiom schemes.

Ax I. $P_{s+1}(x) \rightarrow P_s(x)$.

Ax II. $R^k_s(x_0, \ldots, x_{k-1}) \rightarrow x_i \neq x_j$, whenever $i < j < k$.

Ax III. $R^k_s(x_0, \ldots, x_{k-1}) \rightarrow P_s(x_i)$, whenever $i < k$.

Ax IV. $\bigwedge_{i<k} P_s(x_i) \rightarrow R^k_\ell(x_0, \ldots, x_{k-1})$, whenever $s \in D_k$, ℓ is the m-th least element of D_k, and $m \in U_s$.

Ax V. $\bigwedge_{i<k} P_s(x_i) \to \neg R^k_\ell(x_0, \ldots, x_{k-1})$, whenever $s \in D_k$, ℓ is the m-th least element of D_k, and $m \in V_s$.

Ax VI. $\neg R^k_\ell$, whenever $\ell \notin D_k$.

Ax VII. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L_0-literals, where $L_0 = \{P_i, R_i, C_k : i < \ell\}$ for some ℓ;
- $\phi(\bar{x}, y)$ is consistent with Ax I–VI;
- $\phi(\bar{x}, y) \to \psi(\bar{x})$ is a tautology;
- For each atomic L_0-formula θ with variables taken from $\bar{x}$, either θ or $\neg\theta$ is a conjunct in ψ; similarly, each atomic L_0-formula with variables from $\bar{x}, y$ or its negation is a conjunct in ϕ.

Let T^{**} be the collection of all L-sentences in Ax I–VI, let T^* be the collection of all sentences in Ax I–VII, and let T be the deductive closure of T^*. This completes the construction. Notice that, although T^{**} is Δ^0_1 definable and therefore is an element of $\mathcal{S}$, we have not yet shown that either T^* or T is in $\mathcal{S}$; this is accomplished as part of Lemma 7.2 below.

We now explain the intuition behind these axioms. Axioms I–III are analogous to the first three axioms of §6. Axioms IV and V are similar to the fourth and fifth axioms of §6 and push the relations P_s and R^k_ℓ towards encoding a separating set for $\langle U_s, V_s\rangle_s$, but they apply only to numbers ℓ, s which are in the appropriate D_k. Axiom VI keeps the remaining R^k_ℓ from taking on too many possible values (which is necessary if we expect T to have only countably many types). Lastly, Axiom VII gives quantifier elimination (part of Lemma 7.2 below). Notice that the relations C_k appear only in instances of Axiom VII.

7.2 Verification

Our first task is to show that T is an element of $\mathcal{S}$ and is a complete, consistent theory. We begin with a simple, but technical, lemma.

Lemma 7.1. *Suppose that L_0 is a relational language and $\Phi = \{(\forall\bar{x})\theta_0, (\forall\bar{y})\theta_1, \ldots\}$ is a set of L_0-sentences, where each θ_n is quantifier-free and of the form $\psi_{0,n} \vee \psi_{1,n}$, where neither $\psi_{0,n}$ nor $\neg\psi_{0,n}$ is a tautology, and where no relation in $\psi_{0,n}$ appears in $\psi_{1,n}$ or in any θ_k, $k < n$. Then Φ is satisfiable, and there is a procedure that decides, given a quantifier-free L_0-formula ϕ, whether $\Phi \cup \{(\exists x)\phi\}$ is satisfiable.*

Proof. See [2, Lem 6.1]. □

This allows us to verify some basic facts about T:

Lemma 7.2. (i) *The sentences in T^{**} can be rewritten so as to meet the conditions on Φ in Lemma 7.1.*

(ii) *T^* is an element of $\mathcal{S}$. T^* has effective quantifier elimination.*

(iii) *T is an element of $\mathcal{S}$. T has effective quantifier elimination. T is complete.*

Proof. (i) It is not difficult to restate and reindex Axioms I–VI to get a sequence Φ as in the statement of Lemma 7.1. For example, if k, m, and $s > 0$ are such that $s \in D_k$ and $m \in U_s$, we can combine the appropriate instances of Ax II and IV into a single formula of the form:

$$\neg\left(\bigwedge_{i<n} P_s(x_i)\right) \vee \left(\bigwedge_{i<n} P_{s-1}(x_i) \wedge R^k_\ell(x_0, \ldots, x_{k-1})\right).$$

By Lemma 7.1, there is thus a procedure that decides whether a given quantifier-free L-formula ϕ is consistent with Axioms I–VI.

(ii) Follows from part (i) and Lemma 7.1. Similar to the proof of Corollary 4.2.

(iii) Follows from part (ii).

□

Lemma 7.3. *T is consistent.*

Proof. Since T has effective quantifier elimination, there is a procedure to check whether a given L-formula ϕ is consistent with Axioms I–VI. We can use this procedure to decide, given a finite L-structure $\mathcal{F}$ and an s such that $\mathcal{F}$ satisfies Axiom I and $\mathcal{F} \models \neg P_s(a)$ for each element a, whether $\mathcal{F}$ satisfies Axioms I–VI. Hence we can construct an enumeration $\mathbb{K} = \langle \mathcal{F}_0, \ldots \rangle$ of all finite L-structures satisfying Axioms I–VI and having such an s, together with a sequence $\langle s_0, s_1, \ldots \rangle$ where each s_i is the s for the corresponding $\mathcal{F}_i$. Then $\mathbb{K}$ meets the criteria listed in [2, Lem 6.5 and Lem 6.6]. It follows that $\mathbb{K}$ has a Fraïssé limit $\mathcal{A} \models T$, $\mathcal{A} \in \mathcal{S}$. □

We now prove a few results about the types of T.

Lemma 7.4. *Let N be the number fixed at the beginning of §7.1. Fix $k < N$. There is a 1-type $p_k(x)$ of T with $C_k(x) \in p_k(x)$ and $P_s(x) \in p_k(x)$ for every $s \in M$, and $\neg C_i(x)$ is in $p_k(x)$ for every $i \neq k$.*

Proof. A Fraïssé construction similar to the proof of Lemma 7.3. In this case, we allow at most one element a of every $\mathcal{F}$ to have $\mathcal{F} \models P_s(a)$ for all s. □

Lemma 7.5. *Recall that the set D_N is infinite by choice of N. There is an N-tuple $\langle p_0(x_0), \ldots, p_{N-1}(x_{N-1})\rangle$ of 1-types such that no N-type extends $p_0(x_0) \cup \cdots \cup p_{N-1}(x_{N-1})$.*

Proof. Let $p_0(x), \ldots, p_{N-1}(x)$ be the nonprincipal 1-types described in Lemma 7.4. Consider the tuple $\langle p_0(x_0), \ldots, p_{N-1}(x_{N-1})\rangle \in \mathcal{S}$. We claim that there is no N-type $q(x_0, \ldots, x_{N-1})$ extending $p_0(x_0) \cup \cdots \cup p_{N-1}(x_{N-1})$. Suppose for a contradiction that such a q does exist. Since whenever $k \neq \ell$ we have $p_k(x)$ containing $C_k(x)$ but $p_\ell(x)$ containing $\neg C_k(x)$, we know that q contains $x_k \neq x_\ell$ for all such k, ℓ. It follows by Ax IV and V that the set $\{s : q \text{ contains } R^N_s(x_0, \ldots, x_{N-1})\}$ is a separating set for $\langle U_s, V_s\rangle_s$, a contradiction. □

Lemma 7.6. *T has countably many types.*

Proof. We outline a procedure for enumerating types and argue that the enumeration is exhaustive. Note that, by effective quantifier elimination, it suffices to enumerate the quantifier-free parts of the types.

We use a dovetailing method. For each triple $\langle \ell, m, s\rangle$, we assume that D_ℓ is bounded above by s, and try to list all $(\ell + m)$-types $p(\bar{x}, \bar{y})$, where $\bar{x}$ has length ℓ and $\bar{y}$ has length m, such that p restricted to x_i is a nonprincipal 1-type for each x_i, and $\neg P_s(y_j)$ holds for each y_j. Beginning with P_0 and R^1_0, fill in the atomic diagram of $(\bar{x}, \bar{y})$ relation-by-relation in a way consistent with T. If D_ℓ is indeed bounded above by s, then $\neg R^k_t(\bar{z})$ necessarily holds for all $t > s$ and all $\bar{z}$ taken from $\bar{x}, \bar{y}$, so for relations and R^k_t, P_t with $t > s$, our diagrams are very straightforward. If our assumption was wrong and D_ℓ is not bounded above by s, we will find out, say at stage s_0; for all $t > s_0$ and all z taken from $\bar{x}, \bar{y}$, we let $\neg R_t(z)$ hold. Finally, close the enumeration under all possible renamings of variables.

Now suppose that $q(\bar{z})$ is any type of T. Using bounded Σ^0_1 comprehension to determine which entries of $\bar{z}$, if any, realize a nonprincipal 1-type. We can then find a 1-type $p(\bar{x}, \bar{y})$ of T and a bijection π from the entries of $(\bar{x}, \bar{y})$ to those of $\bar{z}$ such that $\bar{x}$ are the only variables of p whose restriction is a nonprincipal 1-type, and q is exactly $p(\pi(\bar{x}, \bar{y}))$. So q is covered by the enumeration. □

The final lemma of this subsection is used in showing that the types of T have some amalgamation properties—namely, in the special case described in §7.3, the 1-point full amalgamation property, and in §7.4, the pairwise full amalgamation property.

Lemma 7.7. *If $\mathcal{F}$ is a finite model of Axioms I–VI with domain F, then there is a $t \in M$ such that, for all subsets $G \subseteq F$, either:*

- *$D_{|G|}$ is bounded above by t; or*
- *$\mathcal{F} \models \neg P_t(a)$ for some $a \in G$.*

Proof. By $\mathsf{I\Sigma^0_1}$, we may partition F into two sets:

$$F_0 = \{a \in F : \mathcal{F} \models \neg P_s(a) \text{ for some } a\},$$

$$F_1 = \{a \in F : \mathcal{F} \models P_s(a) \text{ for all } a\}.$$

By Σ^0_1 bounding, we may fix an $s_0 \in M$ such that $\mathcal{F} \models \neg P_{s_0}(a)$ for all $a \in F_0$. If F_1 is empty, then s_0 is the desired t. Otherwise, write $F_1 = \{a_0, \dots, a_{k-1}\}$ without repetition, and consider D_k. If D_k were infinite, then

$$\{s : \mathcal{F} \models R^k_s(a_0, \dots, a_{k-1}) \text{ and } s \text{ is } k^{\text{th}} \text{ least in } D_k\}$$

would form a separating set for $\langle U_s, V_s\rangle_s$ by Axioms III and IV, a contradiction. Therefore D_k has some upper bound $s_1 \in M$. Now $t = \max(s_0, s_1)$ is as desired. □

7.3 The first application

Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA_0} + \neg\mathsf{WKL_0} + \neg\mathsf{I\Sigma^0_2}$. Obtain a theory T by performing the construction of §7.1 with the following extra constraint on the sequence $D_1 \subseteq D_2 \subseteq \cdots$: There is an N_0 such that D_{N_0} is finite but D_{2N_0} is infinite. To see that this is possible, let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence witnessing the failure of $\mathsf{I\Sigma^0_2}$ as in Lemma 1.9(iii), let N_0 be such that E_{N_0} is infinite, and define $D_k = \emptyset$ for all $k < N_0$, and let $D_{N_0+k} = E_k$ for all $k \in M$. Then the results of the Verification section §7.2 apply; let X be a sequence of all types of T.

Lemma 7.8. *Suppose that $p(\bar{x})$ is an m-type of T and that $q_0(\bar{x}, y)$ and $q_1(\bar{x}, z)$ are $(m+1)$-types of T extending p. Then there is $t^* \in M$ such that, for any string $\bar{a} = \langle a_0, \dots, a_{k-1}\rangle$ taken from the elements of $\bar{x}, y$, and z, one of the following holds:*

- *There are distinct i, j such that $(a_i = a_j)$ is in $q_0 \cup q_1$; or*
- *D_k is bounded above by t^*; or*
- *$\bigwedge_{i<k} P_d(a_i)$ is not in $q_0 \cup q_1$, where $d = \min\{s \in D_k : s \geq t^*\}$.*

Proof. Let X be the set of all tuples $\bar{a}$ taken from $\bar{x}$, y, and z such that $i \neq j$ implies that $(a_i = a_j)$ is not in $q_0 \cup q_1$. Form the subset

$$Y = \left\{\bar{a} \in X : (\exists s \in D_{|a|}) \left[\bigwedge_{i<k} P_s(a_i) \notin q_0 \cup q_1\right]\right\}. \tag{2}$$

By Σ^0_1 bounding, there is $t_0 \in M$ bounding all s needed in equation (2). Now let k_0 be the greatest length of any string in the complement $X - Y$. By the pigeonhole principle, there is a substring $\bar{b}$ of $\bar{a}$ of length $k_1 \geq k_0 - 1$ with all entries taken from either $\bar{x}^\frown y$ or $\bar{x}^\frown z$. Let t_0 and t_1 be the values of t given by Lemma 7.7 for q_0 and q_1, respectively. Then D_{k_1} is finite with upper bound $t^*_0 = \max(t_0, t_1)$. By construction of $\langle D_0, D_1, \dots\rangle$, it follows that D_{k_0} is also finite, say with upper bound t^*_1. Then $t^* = \max(t^*_0, t^*_1)$ is the desired t^*. □

Lemma 7.9. *X has the 1-point full amalgamation property.*

Proof. Suppose $p(\bar{x})$ is an m-type, and $q_0(\bar{x}, y)$ and $q_1(\bar{x}, z)$ are $(m+1)$-types extending p. Let t^* be the number given by Lemma 7.8 for the union $q_0 \cup q_1$. We extend $q_0 \cup q_1$ to a type $r(\bar{x}, \bar{y}, \bar{z})$ in three steps. First, compute U_{t^*} and V_{t^*}, and, using the effective quantifier elimination from Lemma 7.2, fill in the atomic formulas $R^k_s(\bar{a})$ for $s < t^*$ in a way consistent with Axioms I–VI. Next, for all $s > t^*$, fill in the remaining atomic formulas as $\neg P_s(a)$ and $\neg R^k_s(\bar{a})$. Lastly, complete the elementary diagram using the effective quantifier elimination given by Lemma 7.2. □

Lemma 7.10. *X does not have the pairwise free amalgamation property.*

Proof. Recall from the beginning of this subsection that N_0 is a natural number such that D_{N_0} is finite but D_{2N_0} is infinite. Let $\langle p_0, \ldots, p_{2N_0-1}\rangle$ be a sequence of 1-types as described in Lemma 7.5. It is straightforward to construct a pair of N_0-types $q_0(x_0, \ldots, x_{N_0-1})$ and $q_1(x_{N_0}, \ldots, x_{2N_0-1})$ extending $p_0(x_0) \cup \cdots \cup p_{N_0-1}(x_{N_0-1})$ and $p_{N_0}(x_{N_0}) \cup \cdots \cup p_{2N_0-1}(x_{2N_0-1})$, respectively. But there is no $2N_0$-type r extending $q_0 \cup q_1$. □

We are ready to prove the following part of Theorem 2.23:

Proposition 7.11. $\mathsf{RCA_0} \vdash (\mathsf{1PT\ FULL} \rightarrow \mathsf{PW\ FREE}) \rightarrow (\mathsf{WKL_0} \vee \mathsf{I\Sigma^0_2})$.

Proof. We show the contrapositive. Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA_0} + \neg\mathsf{WKL_0} + \neg\mathsf{I\Sigma^0_2}$. Let T and X be as described at the beginning of this subsection. By Lemmas 7.2, 7.3, 7.9, and 7.10, T is a complete consistent theory and X is an enumeration of all types with the 1-point full amalgamation property but without the pairwise free amalgamation property. □

7.4 The second application

Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA_0} + \neg\mathsf{WKL_0} + \neg\mathsf{I\Sigma^0_2}$. Again we obtain a theory T by the construction of §7.1, this time using a sequence $D_1 \subseteq D_2 \subseteq \cdots$ such that D_n finite implies D_{2n} finite, as in Lemma 1.9(iv). Let N be the number fixed in §7.1, and recall that D_N is infinite. $X = \langle p_0, p_1, \ldots\rangle$ be a sequence of all types such that, for each $k < N$, p_k is equal to the p_k described in Lemma 7.5. (To see this is possible, let X be the sequence of types produced by prepending the list $\langle p_0, \ldots, p_N\rangle$ from Lemma 7.5 onto the list of all types given by Lemma 7.6.)

Lemma 7.12. *Suppose that $p(\bar{x})$ is a type of T and that $q_0(\bar{x}, \bar{y})$ and $q_1(\bar{x}, \bar{z})$ are types of T extending p. Then there is $t^* \in M$ such that, for any string $\bar{a} = \langle a_0, \ldots, a_{k-1}\rangle$ taken from the elements of $\bar{x}, \bar{y}$, and $\bar{z}$, one of the following holds:*

- *There are distinct i, j such that $(a_i = a_j)$ is in $q_0 \cup q_1$; or*
- *D_k is bounded above by t^*; or*
- *$\bigwedge_{i<k} P_d(a_i)$ is not in $q_0 \cup q_1$, where $d = \min\{s \in D_k : s \geq t^*\}$.*

Proof. Similar to the proof of Lemma 7.8, except this time the Pigeonhole Principle tells us only that $k_1 \geq k_0/2$. Our more stringent requirement that D_n finite imply D_{2n} finite allows us to get a bound t^* by the same reasoning as before. □

Lemma 7.13. *X has the pairwise full amalgamation property.*

Proof. Similar to the proof of Lemma 7.9, using Lemma 7.12 in place of 7.8. □

Lemma 7.14. *X does not have the finite free amalgamation property.*

Proof. By choice of the initial segment $\langle p_0(x_0), \ldots, p_{N-1}(x_{N-1})\rangle$ and Lemma 7.5. $\square$

We are ready to prove the remaining part of Theorem 2.23.

Proposition 7.15. $\mathsf{RCA}_0 \vdash (\mathsf{PW\ FULL} \to \mathsf{FIN\ FREE}) \to (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. We show the contrapositive. Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \neg\mathsf{I}\Sigma^0_2$, and let T, X be as specified at the beginning of this subsection. Then by Lemmas 7.2, 7.3, 7.13, and 7.14, we know X is a sequence of all types of a complete consistent theory, and X has the pairwise full but not the finite free amalgamation property. $\square$

We now prove a simple lemma, and proceed to the final part of Theorem 2.14.

Lemma 7.16. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash$ (*If a complete consistent theory has an $\emptyset$-saturated model, then every enumeration of all its types has the finite free amalgamation property*).

Proof. Suppose that T^* is a complete consistent theory, $\mathcal{A}$ is an $\emptyset$-saturated model, $X^* = \langle p^*_0, \ldots\rangle$ is a sequence of all types, and $\langle i_0, \ldots, i_{n-1}\rangle$ is a tuple of indices. Each $p^*_{i_k} = p^*_{i_k}(\bar{x}_k)$ is realized by some tuple $\bar{a}$. By the characterization of $\mathsf{B}\Sigma^0_2$ found in Lemma 1.11, we may form a tuple $\langle \bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}}\rangle$ of tuples such that $p^*_{i_k}(\bar{a}_{j_k})$ holds for each $k < n$. Then the type $\mathsf{tp}^{\mathcal{A}}(\bar{a}_{j_0} {}^\frown \cdots {}^\frown \bar{a}_{j_{n-1}})$ extends every $p^*_{i_k}(\bar{x}_k)$, as required. $\square$

Proposition 7.17. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash$ (*If a complete consistent theory has a sequence of all types with the pairwise full amalgamation property, then it has an $\emptyset$-saturated model*) $\to (\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2)$.

Proof. Immediate from Proposition 7.15 and Lemma 7.16. $\square$

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