Weak truth table degrees of structures

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Abstract

We study the weak truth table (wtt) degree spectra of first-order relational structures. We prove a dichotomy among the possible wtt degree spectra along the lines of Knight's upward-closure theorem for Turing degree spectra. We prove new results contrasting the wtt degree spectra of finite- and infinite-signature structures. We show that, as a method of defining classes of reals, the wtt degree spectrum is, except for some trivial cases, strictly more expressive than the Turing degree spectrum.

1 Introduction

A *first-order relational structure*, henceforth simply a *structure*, is a tuple of the form $\mathfrak{A} = \big(A, (R_k^{\mathfrak{A}})_{k\in I}\big)$, where A is a nonempty set (called the *universe* of $\mathfrak{A}$), I is some set used for indexing, and each $R_k^{\mathfrak{A}}$ is a set of tuples from A of a common arity $\mathsf{ar}(R_k)$—that is, $R_k^{\mathfrak{A}} \subseteq A^{\mathsf{ar}(R_k)}$. We are interested in those $\mathfrak{A}$ for which the universe A is ω and the indexing set I is either ω or a finite set. Unless otherwise specified, we assume that $I = \omega$. We also assume that the sequence $(\mathsf{ar}(R_0), \mathsf{ar}(R_1), \ldots)$, called the *signature* of $\mathfrak{A}$, is computable. By padding with empty relations if necessary, we make the assumption (convenient in some calculations below) that $\mathsf{ar}(R_k) \leq k/2$ for all k. When I is a finite set, we say that $\mathfrak{A}$ has *finite signature*.

We are interested in the computational content of a structure $\mathfrak{A}$. To give this a more precise meaning, we identify $\mathfrak{A}$ with its *atomic diagram* $D(\mathfrak{A}) = \{\langle k, \vec{u}\rangle : \vec{u} \in R_k^{\mathfrak{A}}\}$. Since this $D(\mathfrak{A})$ is a set of natural numbers, it can be assigned a degree of complexity in the usual computability-theoretic sense. Recall that a *reducibility* is reflexive, transitive, binary relation $\leq_r$ on 2^ω. Such a $\leq_r$ induces an equivalence relation $\equiv_r$ on 2^ω, by $A \equiv_r B \iff [A \leq_r B$ and $B \leq_r A]$. We let $(\mathcal{D}_r, \leq)$ denote the partially ordered structure whose universe is the set of all $\equiv_r$-equivalence classes, and

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whose order is induced by $\leq_r$. The elements of $\mathcal{D}_r$ are called *r-degrees*. A structure $\mathfrak{A}$ is said to have r-degree $\deg_r(\mathfrak{A})$, where

$$\deg_r(\mathfrak{A}) = \deg_r(D(\mathfrak{A})) = \{B \subseteq \omega : B \equiv_r D(\mathfrak{A})\}.$$

In most cases, $\deg_r(\mathfrak{A})$ is not invariant under isomorphism—that is, if $\mathfrak{B}$ is an isomorphic copy of $\mathfrak{A}$, it is possible that $\deg_r(\mathfrak{B}) \neq \deg_r(\mathfrak{A})$. Define the *$r$-degree spectrum* of $\mathfrak{A}$ to be:

$$\mathrm{DgSp}_r(\mathfrak{A}) = \{\mathbf{b} \in \mathcal{D}_r : (\exists \mathfrak{B} \cong \mathfrak{A})[\deg_r(\mathfrak{B}) = \mathbf{b}]\}.$$

In this paper, we concentrate our attention on the cases where $\leq_r$ is either *Turing reducibility* ($\leq_T$) or *weak truth table reducibility* ($\leq_{wtt}$). *Truth table reducibility* ($\leq_{tt}$) also appears. We assume some familiarity with $\leq_T$, $\leq_{wtt}$, and $\leq_{tt}$, and anchor our notation to texts such as Lerman [10] and Soare [14]. Considerable effort has already gone into studying $\mathrm{DgSp}_T(\mathfrak{A})$, and, recently, authors have begun studying other sorts of degree spectrum. For example, Soskov and Soskova [15, 16] have examined the enumeration degree spectrum $\mathrm{DgSp}_e(\mathfrak{A})$, and Greenberg–Knight [5] have lifted the Turing degree spectrum into the setting of higher recursion theory. Chisholm et al. [2] recently examined the tt and wtt *degree spectrum of a relation*—a notion distinct from, but related to, the *degree spectrum* studied here.

Although our new results concern the wtt degree spectrum, we draw inspiration from, and analogies with, the past few decades' research on $\mathrm{DgSp}_T(\mathfrak{A})$. The reader can find much more information on $\mathrm{DgSp}_T(\mathfrak{A})$ gathered in the text of Ash and Knight [1] and in the shorter survey article of Knight [8].

We begin in §1.1 with a discussion of some known theorems about $\mathrm{DgSp}_T(\mathfrak{A})$, and their relation to our new results about $\mathrm{DgSp}_{wtt}(\mathfrak{A})$. In §§2, 3, 5, and 6, we look at these new results and their proofs. The longest of these proofs, that of Theorem 3.6, comprises §4.

1.1 Background and overview

We begin with a brief overview of our new results, together with the questions and the known theorems—mainly about the Turing degree spectrum—that inspired them. We hope that this will, in one swoop, motivate and expose the work in the rest of the paper. Most of the results in this section are stated in a simplified or weakened form in order to emphasize the main idea over the details. In each case we indicate where, in the sections that follow, to find the stronger version and its proof.

For a fixed reducibility $\leq_r$, our questions about r-degree spectra fall into one or more of the following broad classes.

Main questions. I. Given a particular structure $\mathfrak{A}$, what can we say about $\mathrm{DgSp}_r(\mathfrak{A})$?
II. Given a particular class of structures (for example, the models of some fixed theory), what can we say about their r-degree spectra?
III. Given a class $\mathcal{C} \subseteq 2^\omega$ of reals, is it possible to write $\mathcal{C} = \bigcup \mathrm{DgSp}_r(\mathfrak{A})$ for some structure $\mathfrak{A}$? If so, what more can we say of such an $\mathfrak{A}$?

Questions of the third variety give a useful point of comparison between the Turing and wtt degree spectra, and between these and other methods of defining a class of reals. (For instance, given a structure $\mathfrak{A}$, the collection $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{A})$ is always a $\Sigma^{1,\mathfrak{A}}_1$ class.) A good first step in our study of the wtt degree spectrum is to check that it is not the same object as the Turing degree spectrum. In fact, except for some trivial cases, there are strictly more classes of reals that can be defined by a wtt degree spectrum than by a Turing degree spectrum.

Theorem 1.1. *(i) If $\mathfrak{A}$ is a structure, then either $\mathrm{DgSp}_T(\mathfrak{A})$ consists of a single Turing degree, or there is a structure $\mathfrak{B}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{B})$ coincides with $\mathrm{DgSp}_T(\mathfrak{A})$ in the sense that $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{B}) = \bigcup \mathrm{DgSp}_T(\mathfrak{A})$. In fact, we may take $\mathfrak{B}$ to be a graph.*
(ii) There is a structure $\mathfrak{A}$ with finite signature such that $\mathrm{DgSp}_T(\mathfrak{A})$ is not a singleton, and $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{A}) \neq \bigcup \mathrm{DgSp}_T(\mathfrak{A})$.
(iii) There is a structure $\mathfrak{A}$ with finite signature such that $\mathrm{DgSp}_T(\mathfrak{A})$ is not a singleton, and $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{A}) \neq \bigcup \mathrm{DgSp}_T(\mathfrak{B})$ for any structure $\mathfrak{B}$.

Parts (i) and (ii) are immediate from Propositions 6.3 and 6.5 below. Part (iii) can be deduced from Part (ii) and Theorem 1.3 below. The next step is to ask for a characterisation of the wtt degree spectra which coincide with a Turing degree spectrum. It can be more intuitive to frame such questions in terms of classes of degrees, rather than of reals. We make frequent use of the following definitions.

Definition 1.2. Let $\mathcal{C} \subseteq \mathcal{D}_r$ be a class of r-degrees, and fix a degree $\mathbf{a} \in \mathcal{D}_r$. Write $\mathcal{D}_r(\geq \mathbf{a}) = \{\mathbf{b} \in \mathcal{D}_r : \mathbf{b} \geq \mathbf{a}\}$. We say that $\mathcal{C}$ *contains the cone above* $\mathbf{a}$ if $\mathcal{D}_r(\geq \mathbf{a}) \subseteq \mathcal{C}$. We say, on the other hand, that $\mathcal{C}$ *avoids the cone above* $\mathbf{a}$ if $\mathcal{D}_r(\geq \mathbf{a}) \cap \mathcal{C} = \emptyset$. A nonempty class $\mathcal{C} \subseteq \mathcal{D}_r$ of r-degrees is called *upward closed* if, for any degree $\mathbf{a} \in \mathcal{C}$, the class $\mathcal{C}$ contains the cone above $\mathbf{a}$.

The following dichotomy theorem was proved by Knight [9].

Theorem 1.3 (Knight)**.** *Let $\mathfrak{A}$ be any structure. Either $\mathrm{DgSp}_T(\mathfrak{A})$ is upward closed, or $\mathrm{DgSp}_T(\mathfrak{A})$ is a singleton.*

We give the original, more detailed formulation, along with a sketch of a proof, below, as Theorem 2.3. As we shall see, $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is a singleton if and only if $\mathrm{DgSp}_T(\mathfrak{A})$ is a singleton; as a consequence, any wtt degree spectrum that coincides with a Turing degree spectrum is itself upward closed. We now present a new dichotomy for $\mathrm{DgSp}_{wtt}(\mathfrak{A})$, similar to Theorem 1.3, which gives a necessary condition for the wtt degree spectrum to be upward closed.

Theorem 1.4. *Let $\mathfrak{A}$ be any structure. Either $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above some degree $\mathbf{a}$, or $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ avoids the cone above some degree $\mathbf{a}$.*

Note that only one of the two alternatives in Theorem 1.4 can hold, since any two degrees $\mathbf{a}_1, \mathbf{a}_2$ have a common upper bound in the wtt degrees—namely their join $\mathbf{a}_1 \vee \mathbf{a}_2$. Note also that, although Theorem 1.4 could easily be deduced from certain large cardinal hypotheses[1], we actually prove a stronger result by specifying a bound on $\mathbf{a}$ (Theorem 3.6 and Corollary 3.7 below) within ZFC.

In §3 below we construct a structure $\mathfrak{A}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ avoids a cone but is not a singleton. This shows that Theorem 1.4 cannot, without some extra conditions, be extended to a perfect analogue of Theorem 1.3. We now suggest some candidate conditions:

[1]Namely, if $a^\sharp$ exists for all reals a, then a wtt version of Martin's Cone Lemma [11] gives the desired cones.

Question 1.5. *(a) Is it the case that, if* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed, then* $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{A}) = \bigcup \mathrm{DgSp}_T(\mathfrak{B})$ *for some* $\mathfrak{B}$*? (b) Is it the case that, if* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *contains a cone, then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed?*

We answer question (a) in the negative. In fact, it is easy to see from the proof of Proposition 6.5 below that the $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ of Theorem 1.1(ii) and (iii) is upward closed. Although we do not have a full answer to question (b), we do succeed in finding *examples* of a structure $\mathfrak{A}$ for which $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is upward closed. In §5 we list some additional conditions on a structure $\mathfrak{A}$ give an affirmative answer to questions (a) and (b) for that $\mathfrak{A}$.

Here is another remarkable limitation on the Turing degree spectrum, essentially proved in Knight [9].

Theorem 1.6 (Knight)**.** *Suppose* $\mathfrak{A}$ *is a structure,* $(\mathbf{e}_n)_{n\in\omega}$ *is a sequence of Turing degrees, and* $\mathrm{DgSp}_T(\mathfrak{A}) \subseteq \bigcup_{n\in\omega} \mathcal{D}_T(\geq \mathbf{e}_n)$*. Then there is an* $n_0 \in \omega$ *such that* $\mathrm{DgSp}_T(\mathfrak{A}) \subseteq \mathcal{D}_T(\geq \mathbf{e}_{n_0})$*.*

One of our new theorems, proved in §5 below, gives a similar-looking result for wtt degree spectra of structures with finite signature.

Theorem 1.7. *Suppose* $\mathfrak{A}$ *is a structure with finite signature,* $(\mathbf{e}_n)_{n\in\omega}$ *is a sequence of wtt degrees, and* $\mathrm{DgSp}_{wtt}(\mathfrak{A}) \subseteq \bigcup_{n\in\omega} \mathcal{D}_{wtt}(\geq \mathbf{e}_n)$*. Then there is an* $n_0 \in \omega$ *such that* $\mathbf{e}_{n_0} = \mathbf{0}$*.*

The most direct analogue of Theorem 1.7 does not hold in the Turing case; for example, an early paper of Richter [12] constructs, for each Turing degree $\mathbf{a} > \mathbf{0}$, a partially-ordered set $P = (\omega, \preceq)$ such that $\mathrm{DgSp}_T(P) = \mathcal{D}_T(\geq \mathbf{a})$.

Another known result is that every nonsingleton Turing degree spectrum is the Turing degree spectrum of a graph. A highly effective construction can be found in the paper of Hirschfeldt–Khoussainov–Shore–Slinko [6].

Theorem 1.8 (H–K–S–S)**.** *If* $\mathfrak{A}$ *is a structure and* $\mathrm{DgSp}_T(\mathfrak{A})$ *is not a singleton, then there is a graph* $G = (\omega, E^G)$ *such that* $\mathrm{DgSp}_T(G) = \mathrm{DgSp}_T(\mathfrak{A})$*.*

Deliberately ignoring the singleton case, we say that the theory of graphs is *universal for Turing degree spectra*. One might ask whether the theory of graphs is similarly universal for wtt degree spectra. Sadly, it is not. We can see this by taking a structure $\mathfrak{B}$ and a wtt degree $\mathbf{a} > \mathbf{0}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{B}) \subseteq \mathcal{D}_{wtt}(\geq \mathbf{a})$ (a suitable $\mathfrak{B}$ is constructed in Proposition 6.2 below), and invoking Theorem 1.7 with $\mathbf{e}_n = \mathbf{a}$ for all n. We leave open the question of whether a suitable analogue can be found when we consider only structures with finite signature.

Question 1.9. *Is there a fixed, finite* $n \in \omega$ *such that, if* $\mathfrak{A}$ *is a structure with finite signature, then there is a structure* $\mathfrak{B}$ *on alphabet* $(R_0, \ldots, R_{n-1})$ *such that* $\mathrm{DgSp}_{wtt}(\mathfrak{B}) = \mathrm{DgSp}_{wtt}(\mathfrak{A})$*?*

2 Knight's dichotomy for Turing degree spectra

We have already mentioned, as Theorem 1.3, a result of Knight stating that, for a structure $\mathfrak{A}$, the spectrum $\mathrm{DgSp}_T(\mathfrak{A})$ is either a singleton or upward closed. Because it motivates our definitions and results in §3, we now give a more detailed formulation, as Theorem 2.3; and because it serves as a prototype for the proofs of Lemmas 4.1 and 4.2, we also sketch a proof. The following definitions will be used frequently.

Notation 2.1. (i) We use the word *permutation* to mean a bijection from ω to ω.

(ii) Given a set S and a permutation π, we say that π *fixes* S if $\pi \restriction S = id_S$.

(iii) Given a permutation π and structures $\mathfrak{A}, \mathfrak{B}$, we write $\pi : \mathfrak{A} \cong \mathfrak{B}$ to mean that π is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(iv) Given numbers $x, z \in \omega$, we write $[x, z)$ to denote the interval $\{y \in \omega : x \leq y < z\}$. We write $[x, \infty)$ to denote the set $\{y \in \omega : x \leq y\}$. Following the usual convention, each natural number $x \in \omega$ is identified with the interval $[0, x)$.

Definition 2.2. A structure $\mathfrak{A}$ is called *trivial* if there exists a finite set $S \subset \omega$ such that any permutation π fixing S is an automorphism of $\mathfrak{A}$. We say that S *witnesses the triviality of* $\mathfrak{A}$.

For example, any graph (ω, E) with only finitely (or cofinitely) many edges in E is trivial. A linear order $(\omega, \preceq)$, on the other hand, is never trivial. To see this, given any finite nonempty set S, choose two distinct elements $a, b \notin S$; then the permutation which transposes a and b and fixes all other elements is not an automorphism of $(\omega, \preceq)$.

If S is a finite set witnessing the triviality of a structure $\mathfrak{A}$, π is a permutation, and $\mathfrak{B}$ is the isomorphic copy of $\mathfrak{A}$ given by $\pi : \mathfrak{A} \cong \mathfrak{B}$, then we can compute the atomic diagram of $\mathfrak{B}$ using that of $\mathfrak{A}$ and the restricted map $\pi \restriction S$. Since $\pi \restriction S$ is a finite set, this implies that $\mathfrak{B} \leq_T \mathfrak{A}$; a symmetric argument also gives $\mathfrak{A} \leq_T \mathfrak{B}$. A trivial structure therefore has only a single degree in its Turing degree spectrum. In particular, it is easy to see that any trivial structure with finite signature has $\{\mathbf{0}\}$ as its Turing degree spectrum.

On the other hand, suppose $\mathfrak{A}$ is *not* trivial. Then we can list (noneffectively) an infinite collection of pairs $\{\{a_i, b_i\}\}_i$, pairwise disjoint, where the transposition of any $\{a_i, b_i\}$ is not an automorphism of $\mathfrak{A}$. By transposing simultaneously any nonempty subcollection of these pairs $\{a_i, b_i\}$, we again get a permutation which is not an automorphism of $\mathfrak{A}$. Thus there are $2^{\aleph_0}$-many different atomic diagrams of structures isomorphic to $\mathfrak{A}$. By the pigeonhole priciple, the degree spectrum $\mathrm{DgSp}_T(\mathfrak{A})$ has cardinality $2^{\aleph_0}$ as well.

Therefore, no Turing degree spectrum can have cardinality strictly between 1 and $2^{\aleph_0}$: in classifying structures into the *trivial* and the *not trivial*, we uncover a significant gap among the possible Turing degree spectra. The gap is actually much wider, however, as Knight showed in [9].

Theorem 2.3 (Knight)**.** *If $\mathfrak{A}$ is a structure, then*

(1) $\mathfrak{A}$ is not trivial if and only if $\mathrm{DgSp}_T(\mathfrak{A})$ is upward closed in the Turing degrees;

(2) $\mathfrak{A}$ is trivial if and only if $\mathrm{DgSp}_T(\mathfrak{A})$ is a singleton.

We sketch a proof; for a detailed version, the reader should refer to [9].

Definition 2.4. If $\mathfrak{A}$ is a structure and $X, Y \subseteq \omega$ are sets of natural numbers, then we define the *restricted diagram* $\mathfrak{A} \restriction^X_Y$ to be the restriction of $D(\mathfrak{A})$ to those relations indexed by X and those elements in Y, that is,

$$\mathfrak{A} \restriction^X_Y (\langle k, \vec{u} \rangle) = \begin{cases} D(\mathfrak{A})(\langle k, \vec{u} \rangle) \text{ if } k \in X \text{ and } u_i \in Y \text{ for each } i \\ \uparrow \text{otherwise.} \end{cases}$$

This $\mathfrak{A}\restriction^X_Y$ is seen as a structure with universe Y and alphabet $\{R_i : i \in X\}$. In practice, X and Y will usually be initial segments of ω. When X contains all of $\mathfrak{A}$'s relations, we sometimes write $\mathfrak{A}\restriction_Y$ for $\mathfrak{A}\restriction^X_Y$.

Proof of 2.3 (sketch). We have already established Part (2) and the 'if' direction of Part (1) through our discussion of the cardinality of $\mathrm{DgSp}_T(\mathfrak{A})$.

We now show the 'only if' direction of Part (1). Suppose that $\mathfrak{A}$ is not trivial, and fix any set $C \in 2^\omega$ such that $C \geq_T \mathfrak{A}$. We exhibit a permutation π such that, if $\mathfrak{B}$ is the unique structure with $\pi : \mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{B} \equiv_T C$. We get $C \leq_T \mathfrak{B}$ by coding the elements of C directly into $\mathfrak{B}$; to ensure that $C \geq_T \mathfrak{B}$, we build π effectively in C and use the fact that $\mathfrak{B} \leq_T \mathfrak{A} \oplus \pi$.

Construction. The permutation π is built computably in C as the pointwise limit of a sequence $(\pi_s)_s$ of permutations, alongside which we build a sequence $(m_s)_s$ of natural numbers to act as restraints. Begin with $\pi_0 = id_\omega$ and $m_s = 0$.

At each stage s, suppose that we have already defined π_s and m_s, and that $\mathfrak{B}_s$ is the unique structure such that $\pi_s : \mathfrak{A} \cong \mathfrak{B}_s$. Because $\mathfrak{A}$ is not trivial, there is a permutation ρ which fixes the interval $[0, m_s)$ and which is not an automorphism of $\mathfrak{B}_s$. In fact, it is easy to see that there is such a ρ fixing $[0, m_s) \cup [m_s + k, \infty)$ for some k. From here it is easy to see that there is a ρ fixing $[0, m_s) \cup [m_s + k, \infty)$ which is not an automorphism of $\mathfrak{B}_s\restriction^k_{m_s+k}$; choose the least such k.

Make a list $(G_0, G_1, \ldots, G_{n-1})$ of all possible images of $\mathfrak{B}_s\restriction^k_{m_s+k}$ under a permutation of $[0, m_s + k)$ fixing $[0, m_s)$. Find the least $k^* \in \omega$ such that there exist $i, j < n$ with $G_i\restriction^{k^*}_{m_s+k^*}$ and $G_j\restriction^{k^*}_{m_s+k^*}$ unequal, but isomorphic through a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$.

Using some fixed computable enumeration of ordered pairs of finite atomic diagrams, choose i, j as above with $\langle G_i\restriction^{k^*}_{m_s+k^*}, G_j\restriction^{k^*}_{m_s+k^*}\rangle$ coming as early as possible in the enumeration. There exist permutations ρ_0, ρ_1, each fixing $[0, m_s) \cup [m_s + k, \infty)$, such that $\rho_0 : \mathfrak{B}_s\restriction^k_{m_s+k} \cong G_i$ and $\rho_1 : \mathfrak{B}_s\restriction^k_{m_s+k} \cong G_j$. If $s \notin C$, let $\tau = \rho_0 \circ \pi_s$; if $s \in C$, let $\tau = \rho_1 \circ \pi_s$. Find the least $x \in \omega$ such that $\tau(x) \geq m_s + k^*$, and let $y = \tau(x)$. Let σ_s be the permutation which transposes y and $m_s + k^*$, and fixes all other elements. Define the next π_{s+1} by $\pi_{s+1} = \sigma_s \circ \tau$, and define $m_{s+1} = m_s + k^* + 1$. This completes the construction.

Verification. Because at each stage s the functions ρ_0, ρ_1 are permutations fixing $[0, m_s)$ and the bounds $(m_s)_s$ form an increasing sequence, the limit π is an injective partial function from ω into ω. The final transposition $(y, m_s + k^*)$ at each stage guarantees that π is total and surjective. Hence π is a permutation.

Let $\mathfrak{B}$ be the unique structure such that $\pi : \mathfrak{A} \cong \mathfrak{B}$. Using knowledge of $\mathfrak{B}$, we can recover the sequence $(m_s)_s$ and the set C inductively, as follows. Suppose that $(m_0, m_1, \ldots, m_s)$ are already known. Find the least $k^* \in \omega$ such that there is a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$ which is not an automorphism of $\mathfrak{B}\restriction^{k^*}_{m_s+k^*}$. This k^* is the same as the k^* from stage s of the construction. So we may compute $m_{s+1} = m_s + k^* + 1$.

Enumerate all possible images $(H_0, \ldots, H_n)$ of $\mathfrak{B}\restriction^{k^*}_{m_s+k^*}$ under a permutation fixing $[0, m_s) \cup [m_s + k^*, \infty)$, and, within the same fixed computable enumeration as before, choose the earliest pair $\langle H_i, H_j\rangle$ with $H_i \neq H_j$. Then $\mathfrak{B}\restriction^{k^*}_{m_s+k^*}$ is equal either to H_i, in which case $s \notin C$, or to H_j, in which case $s \in C$.

$\square$

This result is nice enough, and the construction effective enough, that one might wish to adapt it to the wtt case. As we have stated in §1.1, the most direct possible analogue—swapping wtt for T in the statement of the theorem—does not hold. Still, the ideas used in proving Theorem 2.3 are useful in the wtt case. We come back to this construction in proving Proposition 3.4 and Lemma 4.2 below.

3 A Dichotomy for the wtt degree spectrum

What follows will require notation from computability theory. To streamline the discussion, we fix an enumeration $(\varphi_e)_e$ of some (but not all) computable functions, and we introduce a nonstandard symbol $\hat{\Phi}_e$.

Definition 3.1. (i) We let $(\Phi_e)_e$ be the standard effective listing of computable functionals.

(ii) We are interested in those partial computable functions ψ with domain an inital segment of ω, and which are increasing on their domain. We let $(\varphi_e)_e$ be an effective listing of all such ψ.

(iii) Define the *sequence of all wtt-functionals* $(\hat{\Phi}_e)_e$ *operating on structures* as follows. Recall that we identify a structure $\mathfrak{A}$ with its atomic diagram $D(\mathfrak{A}) \subseteq \omega$. Given $\mathfrak{A}$ and natural numbers $x, s \in \omega$, if $\varphi_{e,s}(x)\downarrow$ and $\Phi^{\mathfrak{A}}_{e,s}(x)\downarrow$ while using queries only to $\mathfrak{A}\restriction^{\varphi_e(x)}_{\varphi_e(x)}$—that is, asking only oracle questions of the form '$\langle k, y_0, \ldots, y_n\rangle \in D(\mathfrak{A})$?' with each $k, x_i < \varphi_{e,s}(x)$, then $\hat{\Phi}^{\mathfrak{A}}_{e,s}(x)\downarrow = \Phi^{\mathfrak{A}}_e(x)$. Otherwise, $\hat{\Phi}^{\mathfrak{A}}_{e,s}(x)\uparrow$. If there is an s such that $\hat{\Phi}^{\mathfrak{A}}_{e,s}(x)\downarrow = y$, then we write $\hat{\Phi}^{\mathfrak{A}}_e(x) = y$. Otherwise, we write $\hat{\Phi}^{\mathfrak{A}}_e(x)\uparrow$. If $\hat{\Phi}^{\mathfrak{A}}_e(x)\downarrow \in \{0,1\}$ for every $x \in \omega$, then we identify $\hat{\Phi}^{\mathfrak{A}}_e$ with a subset of ω in the usual way.

An application of the s-m-n theorem shows that, for any X and $\mathfrak{A}$, we have $X \leq_{wtt} \mathfrak{A}$ if and only if there is an e such that $X = \hat{\Phi}^{\mathfrak{A}}_e$.

Now let us try to determine where the proof of Theorem 2.3 breaks down when we substitute $\leq_{wtt}$ for $\leq_T$. The cardinality argument for part (2) carries over unchanged:

Proposition 3.2. *A structure* $\mathfrak{A}$ *is trivial if and only if* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is a singleton.*

The construction for the 'only if' direction of Theorem 2.3(1) does not on its face give $\mathfrak{B} \leq_{wtt} C$, since there might not be a computable bound on the length of the searches used in choosing k. As well, we might not end up with $C \leq_{wtt} \mathfrak{B}$, since the sequence $(m_s)_s$, and hence the length of the searches used to compute C, might not have a computable bound.

We can do away with these objections in certain cases. If $\mathfrak{A}$ has finite signature, for instance, then surely $C \leq_{wtt} \mathfrak{B}$. If $\mathfrak{A} = (\omega, \leq^{\mathfrak{A}})$ is a linear order, then at each stage s of the construction we get $m_{s+1} \leq m_s + 2$, giving $\mathfrak{B} \leq_{wtt} C$. Hence $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is upward closed for any linear order $\mathfrak{A}$. We examine the finite-signature case more closely in §5.

It is also useful to consider degree-theoretic conditions on $\mathfrak{A}$.

Definition 3.3. We say that a set $A \in 2^\omega$ is *of* $\mathbf{0}$*-dominated degree* (also called *of hyperimmune-free degree*) if, for every total function $f \leq_T A$, there is a total computable function g such that $(\forall x)[f(x) \leq g(x)]$. Equivalently, we could replace '$f \leq_T A$' in this definition with '$\mathsf{graph}(f) \leq_{wtt} A$,' where $\mathsf{graph}(f) = \{\langle x, y\rangle : y = f(x)\}$.

From our point of view, structures of $\mathbf{0}$-dominated degree behave nicely.

Proposition 3.4. *If $\mathfrak{A}$ is not trivial and is of $\mathbf{0}$-dominated degree, then $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\deg_{wtt}(\mathfrak{A})$. In particular, if $\mathfrak{A}$ is computable and not trivial, then $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is all of $\mathcal{D}_{wtt}$.*

Proof. Suppose that $\mathfrak{A}$ is of $\mathbf{0}$-dominated degree, and fix any set $C \geq_{wtt} \mathfrak{A}$. Build $\mathfrak{B} \equiv_T C$ using the construction for Theorem 2.3. We use this construction to define two functions f and g. Let f be given by $f(s) = m_s$, and let $g(s) = m_s + \ell$, where ℓ is the greatest among all k used in steps $t \leq s$ of the construction. Then $g \leq_T \mathfrak{A}$, so there is a total computable function ψ such that $(\forall x)[g(x) \leq \psi(x)]$. Note that f is dominated by ψ in the same way.

In building $\mathfrak{B} \restriction^{\omega}_{m_s}$ from $\mathfrak{A} \oplus C$, we use only queries to $C \restriction s+1$ and to $\mathfrak{A} \restriction^{m_s+k}_{m_s+k}$. Since and $s+1$ and $m_s + k$ are no greater than $\psi(s)$, this means $\mathfrak{B} \leq_{wtt} C$. On the other hand, in recovering $C(s)$ from $\mathfrak{B}$, we use only queries to $\mathfrak{B} \restriction^{m_{s+1}}_{m_{s+1}}$. Since m_{s+1} is no greater than $\psi(s)$, this implies $C \leq_{wtt} \mathfrak{B}$, and hence $\mathfrak{B} \equiv_{wtt} C$. $\square$

One last approach is to consider a bounded version of triviality for structures. Recall from Definition 2.2 the notion of a finite set *witnessing the triviality* of a structure.

Definition 3.5. A structure $\mathfrak{A}$ is *w-trivial* if for each total computable function f there is a finite set S witnessing the triviality of the reduct $\mathfrak{A} \restriction^{f(|S|)}_{\omega}$.

It is immediate from the definitions that any trivial structure is also w-trivial. There do, however, exist structures which are w-trivial but not trivial. An easy example can be found in §6 below.

A structure $\mathfrak{A}$ that is w-trivial but not trivial must have $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ of size $2^{\aleph_0}$. Such a $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is nonetheless far from upward-closed within the wtt degrees, to the extent that there is a set X such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ avoids the cone above $\deg_{wtt}(X)$. In fact, we shall exhibit a whole family of such X in the form of a relativised $\Pi^{0,\mathfrak{A}}_1$ class. A structure that is not w-trivial, on the other hand, is amenable to a version of the proof of Theorem 2.3, which will be enough to show that its wtt degree spectrum does contain *some* upward cone. What we have stated is the following theorem.

Theorem 3.6. *Given a structure $\mathfrak{A}$:*

(1) If $\mathfrak{A}$ is not w-trivial then there is a set $B \leq_T \mathfrak{A}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\deg_{wtt}(B)$.

(2) If $\mathfrak{A}$ is w-trivial then there is a nonempty relativised $\Pi^{0,\mathfrak{A}}_1$ class $P \subseteq 2^{\omega}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ avoids the cone above $\deg_{wtt}(X)$ for every $X \in P$.

See §4 for a proof of this theorem. Again, there cannot be wtt degrees $\mathbf{a}, \mathbf{b}$ such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\mathbf{a}$ and avoids the cone above $\mathbf{b}$, since the intersection $\mathcal{D}_{wtt}(\geq \mathbf{a}) \cap \mathcal{D}_{wtt}(\geq \mathbf{b})$ is nonempty. Hence our classification of structures into the *w-trivial* and the *not w-trivial* admits a simple degree-theoretic characterisation—namely, the dichotomy of Theorem 1.4. With some additional effort, we can get a localised version:

Corollary 3.7. *Given a structure $\mathfrak{A}$:*

(1) $\mathfrak{A}$ is not w-trivial if and only if there is a set $C \geq_{wtt} \mathfrak{A}$, $C \equiv_T \mathfrak{A}$, such that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\deg_{wtt}(C)$.

(2) $\mathfrak{A}$ *is w-trivial if and only if there is a set* $C \geq_{wtt} \mathfrak{A}$*,* $C' \leq_{tt} \mathfrak{A}'$*, such that* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *avoids the cone above* $\deg_{wtt}(C)$*. (Here* $\mathfrak{A}'$ *is the Turing jump of the atomic diagram of* $\mathfrak{A}$*.)*

The proof will use the following relativised, truth-table version of the Low Basis Theorem of Jockusch–Soare [7].

Lemma 3.8. *Let A be a set of natural numbers. If P is a nonempty $\Pi^{0,A}_1$ class, then there is an element $X \in P$ such that $X' \leq_{tt} A'$.*

The proof of this lemma, omitted here, is a straightforward relativisation of the proof of the Superlow Basis Theorem due to Marcus Schaefer—see, for example, Downey and Hirschfeldt [4, Theorem 2.19.9].

Proof of Corollary 3.7. For (1), take B as in Theorem 3.6 and let $C = \mathfrak{A} \oplus B$.

For (2), take P as in Theorem 3.6 and let $Q = \{\mathfrak{A} \oplus Y : Y \in P\}$. Then Q is a nonempty $\Pi^{0,\mathfrak{A}}_1$ class, and $X \in Q$ implies $\mathfrak{A} \leq_{wtt} X$. Apply the Lemma to Q. □

Note that it is not possible to replace $C' \leq_{tt} \mathfrak{A}'$ in Corollary 3.7(2) with the stronger condition $C \equiv_T \mathfrak{A}$. For, if $\deg_T(\mathfrak{A})$ is not **0**-dominated and consists of exactly one wtt-degree (e.g., one of the *strongly contiguous* c.e. degrees introduced by Downey [3]; such a degree must contain a w-trivial structure by Proposition 6.1 below), then it would be absurd for C and $\mathfrak{A}$ to share a Turing degree.

4 Proof of Theorem 3.6

Proof of Part (1).

We are to show that if $\mathfrak{A}$ is not w-trivial there is an isomorphic copy $\mathfrak{A}^*$ of $\mathfrak{A}$ such that $\mathfrak{A}^* \leq_T \mathfrak{A}$ and $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\deg_{wtt}(\mathfrak{A}^*)$. We do this in two steps. First, in Lemma 4.1, we give a condition on $\mathfrak{A}^*$ which implies that $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above $\deg_{wtt}(\mathfrak{A}^*)$. The second step, in Lemma 4.2, is to show that a suitable $\mathfrak{A}^*$ can be built computably in $\mathfrak{A}$.

Lemma 4.1. *Suppose $\mathfrak{A}^*$ is a structure and there is a total computable function g such that, for every $m \in \omega$, there exists a permutation fixing $[0, m) \cup [m + g(m), \infty)$ which is not an automorphism of $\mathfrak{A}^* \restriction^{g(m)}_{m+g(m)}$. Then $\mathrm{DgSp}_{wtt}(\mathfrak{A}^*)$ contains the cone above $\deg_{wtt}(\mathfrak{A}^*)$.*

Proof. Fix any $C \geq_{wtt} \mathfrak{A}^*$, and perform the construction for Theorem 2.3 with $\mathfrak{A}^*$ in place of $\mathfrak{A}$ to get a copy $\mathfrak{B} \cong \mathfrak{A}^*$. We claim that the construction gives $\mathfrak{B} \equiv_{wtt} C$. To see $\mathfrak{B} \leq_{wtt} C$, notice that, at each stage s, we have $k \leq g(m_s)$, and so π_{s+1} and m_{s+1} can be computed using queries only to $C \restriction s + 1$ and to $\mathfrak{A} \restriction^{g(m_s)}_{m_s+g(m_s)}$.

To see $C \leq_{wtt} \mathfrak{B}$, define a computable function h by $h(0) = m_0$, $h(s + 1) = g(h(s)) + 1$. Then $m_s \leq h(s)$ for all s. We can therefore recover $C(s)$ from $\mathfrak{B}$ using only queries to $\mathfrak{B} \restriction^{h(s)}_{h(s+1)}$. □

It is possible for a structure $\mathfrak{A}$ to have some isomorphic copies $\mathfrak{A}^*$ that satisfy the conditions of the above lemma and other isomorphic copies that do not. Our second lemma connects the existence of a suitable $\mathfrak{A}^*$ with the isomorphism-invariant property of not being w-trivial:

Lemma 4.2. *If $\mathfrak{A}$ is not w-trivial, then there is an isomorphic copy $\mathfrak{A}^* \cong \mathfrak{A}$ and a function g meeting the hypotheses of Lemma 4.1.*

Proof. Using the fact that $\mathfrak{A}$ is not w-trivial, fix a computable, increasing function f such that no finite set S witnesses the triviality of $\mathfrak{A} \upharpoonright_{\omega}^{f(|S|)}$. We use f to define a permutation π giving the desired structure $\mathfrak{A}^*$ by $\pi : \mathfrak{A} \cong \mathfrak{A}^*$. This π is constructed as the pointwise limit of a sequence $(\pi_s)_s$ of permutations.

We also define a computable, nondecreasing sequence of restraints $(m_s)_s$ by $m_0 = 0$, $m_{s+1} = m_s + f(m_s) + 1$. These m_s act as restraints in the construction of π_s.

Construction. We define the sequence $(\pi_s)_s$ by stages, beginning with $\pi_0 = id_\omega$. Suppose we have already defined π_s and wish to define π_{s+1}. Let $\mathfrak{A}_s^*$ be the unique structure such that $\pi_s : \mathfrak{A} \cong \mathfrak{A}_s^*$. By choice of f, there is a permutation ρ_s fixing $[0, m_s)$ which is not an automorphism of $\mathfrak{A}_s^* \upharpoonright_{\omega}^{f(m_s)}$. Recall our assumption from §1 that the arity $\mathsf{ar}(R_k)$ of a relation R_k does not exceed $k/2$. Hence we may assume that there is a set $T \subseteq [m_s, \infty)$ of size $|T| \leq f(m_s)$ such that ρ_s fixes the complement $\omega \setminus T$ pointwise. Let τ_s be a permutation fixing $[0, m_s)$ and mapping T into the interval $[m_s, m_s + f(m_s))$.

Take the least $x \in \omega$ such that $\tau_s \circ \pi_s(x) \geq m_{s+1} - 1$, and write $y_s = \tau_s \circ \pi_s(x)$. Let σ_s be the permutation transposing y_s and $m_{s+1} - 1$, and fixing all other numbers, and define

$$\pi_{s+1} = \sigma_s \circ \tau_s \circ \pi_s.$$

This completes the construction.

Verification. Let π be the pointwise limit of the $(\pi_s)_s$. Then π is an injective partial function from $\omega \to \omega$; we claim that π is a permutation. At each stage s, the interval $[0, m_s)$ is in the image of π_s, and for all $t \geq s$ we have $\pi_t^{-1} \upharpoonright m_s = \pi_s^{-1} \upharpoonright m_s$, so π is surjective. The addition of σ_s in the construction ensures that π is total.

Now let $\mathfrak{A}^*$ be the unique structure such that $\pi : \mathfrak{A} \cong \mathfrak{A}^*$, and for each s, let $g(s) = m_{s+1}$. Given any $s \in \omega$ we may define a permutation ψ_s by

$$\psi_s = (\sigma_s \circ \tau_s) \circ \rho_s \circ (\sigma_s \circ \tau_s)^{-1}.$$

Then ψ_s is not an automorphism of $\mathfrak{A}^* \upharpoonright_{s+g(s)}^{g(s)}$ and fixes $[0, m_s) \cup [m_s + f(m_s), \infty)$ pointwise, and hence fixes the smaller set $[0, s) \cup [s + g(s), \infty)$ pointwise as well. $\square$

This completes the proof of part (1).

Proof of Part (2).

Given a w-trivial structure $\mathfrak{A}$, we wish to construct a nonempty $\Pi_1^{0,\mathfrak{A}}$ class such that no member of P is wtt-below an isomorphic copy of $\mathfrak{A}$. Before providing the proof in full detail, we give a rough plan of how P will be made.

The class P will be defined through a sequence of restraints of the form '$X \in P \Rightarrow X(w) \neq y$,' with $w \in \omega$ and $y \in \{0, 1\}$. The set of restraints will be computably enumerable in $\mathfrak{A}$, so P will indeed be a $\Pi_1^{0,\mathfrak{A}}$ class. As well, each natural number w will be used in at most one of these constraints, so P will be nonempty. Each restraint will be the result of a diagonalisation against the eventuality $\hat{\Phi}_e^{\mathfrak{B}}(w) = X(w)$, for some $w \in \omega$, some wtt-functional $\hat{\Phi}_e$, and some possible isomorphic copy $\mathfrak{B}$ of $\mathfrak{A}$.

The challenge will be to diagonalise against all $\hat{\Phi}_e, \mathfrak{B}$ with only a countable supply of $w \in \omega$. We must play the w-triviality of $\mathfrak{A}$ against the computable bound φ_e used in $\hat{\Phi}_e$. In fact, for a fixed $\hat{\Phi}_e$, there is a strategy to diagonalise against $\hat{\Phi}_e^{\mathfrak{B}} = X$ for *all* $\mathfrak{B} \cong \mathfrak{A}$ while using only finitely many w. First we exhibit the basic strategy, for a single $\hat{\Phi}_e$, by proving a weaker result.

Proposition 4.3. *If $\mathfrak{A}$ is w-trivial and $\hat{\Phi}_e$ is a wtt-functional, then there is a nonempty class $P \subseteq 2^\omega$ such that, if $X \in P$, then $X \neq \hat{\Phi}_e^{\mathfrak{B}}$ for any isomorphic copy $\mathfrak{B} \cong \mathfrak{A}$.*

Proof. If φ_e is not total, then $\hat{\Phi}_e^{\mathfrak{B}}$ is not total, so any nonempty P will do. Assume, then, that φ_e is total. Recall our assumption in Definition 3.1 that φ_e is strictly increasing. We build P as the class of all elements of 2^ω satisfying a finite set of constraints of the form: '$X \in P \Rightarrow X(w) \neq y$'.

We consider all permutations π and structures $\mathfrak{B}$ such that $\pi : \mathfrak{A} \cong \mathfrak{B}$. If g is any total computable function, then there is a finite set $S \subseteq \omega$, say of cardinality $n = |S|$, such that $\pi \upharpoonright S$ uniquely determines the reduct $\mathfrak{B} \upharpoonright_\omega^{g(n)}$. What's more, for any $N \in \omega$, the further restriction $\mathfrak{B} \upharpoonright_N^{g(n)}$ can—as we allow π and $\mathfrak{B}$ to vary—take no more than $(N+1)^n$ different values: one for each partial function from $S \to N$.

Now suppose that $g(n)$ is large enough to admit a sequence

$$N_0 < N_1 < \cdots < N_n < N_{n+1} \leq g(n)$$

such that, for each $i \leq n$, we have $N_{i+1} \geq \varphi_e(N_i + (N_i+1)^n)$. Consider the intervals $[N_i, N_{i+1})$, for $i \leq n$. Since these intervals are pairwise disjoint, there are $n+1$ of them, and the set S has only n elements, for any particular choice of π and $\mathfrak{B}$, the Pigeonhole Principle gives an $i_0 \leq n$ such that π maps no element of S into $[N_{i_0}, N_{i_0+1})$. Then the restricted diagram $\mathfrak{B} \upharpoonright_{N_{i_0+1}}^{g(n)}$ is uniquely determined by its further restriction $\mathfrak{B} \upharpoonright_{N_{i_0}}^{g(n)}$, and so can—as we allow π and $\mathfrak{B}$ to vary, preserving $\pi(S) \cap [N_{i_0}, N_{i_0+1}) = \emptyset$—take no more than $(N_{i_0}+1)^n$ possible values. Enumerate these possible diagrams $D_0, D_1, \ldots, D_{\ell-1}$, with $\ell \leq (N_{i_0}+1)^n$.

Suppose that $\pi, \mathfrak{B}$ are such that $\pi(S) \cap [N_{i_0}, N_{i_0+1}) = \emptyset$, say with $\mathfrak{B} \upharpoonright_{N_{i_0+1}}^{g(n)} = D_j$, and note that

$$N_{i_0+1} > \varphi_e(N_{i_0} + (N_{i_0}+1)^n) \geq \varphi_e(N_{i_0} + \ell).$$

We can ensure that $X \in P \Rightarrow X \neq \hat{\Phi}_e^{\mathfrak{B}}$ by waiting for $\hat{\Phi}_e^{D_j}(N_{i_0}+j)$ to converge, and then adding the constraint: '$X \in P \Rightarrow X(N_{i_0}+j) \neq \hat{\Phi}_e^{\mathfrak{B}}(N_{i_0}+j)$'.

It therefore suffices to produce a computable g, a natural number n, and a sequence $N_0 < \cdots < N_{n+1} \leq g(n)$ behaving as above. Define a 2-ary computable function h by $h(x,0) = x$, $h(x,y+1) = \varphi_e(h(x,y) + (h(x,y)+1)^x)$, and let $g(x) = h(x,x+1)$. Then g is a total computable function, giving a suitable n through w-triviality. We get $N_0, \ldots, N_{n+1}$ by setting $N_i = h(n,i)$ for each $i \leq n+1$. $\square$

We can get a quick and interesting, though weak, result by iterating the above construction in a recklessly noneffective way:

Proposition 4.4. *If $\mathfrak{A}$ is w-trivial, then there is a set $X \in 2^\omega$ such that $X \not\leq_{wtt} \mathfrak{B}$ for any isomorphic copy $\mathfrak{B} \cong \mathfrak{A}$.*

Proof. The construction from Proposition 4.3 uses only finitely many witnesses w to diagonalise—namely, each w is taken from the interval $[N_0, N_{n+1})$. We can therefore perform the construction for each $\hat{\Phi}_e$, $e = 0, 1, \dots$ in turn, either doing nothing (if φ_e is not total) or running the procedure for Proposition 4.3 with the additional stipulation that N_0 be larger than any number thus far considered. $\square$

Note that this is already gives a fairly effective proof of Theorem 1.4. The full proof of Theorem 3.6, of course, will do better still. We now press on with Part (2)

Idea. The idea is to use the construction from Proposition 4.3 as the basic module for meeting the requirement:

$$\mathcal{R}_e : e \in \omega, X \in P, \mathfrak{B} \cong \mathfrak{A} \implies X \neq \hat{\Phi}_e^{\mathfrak{B}}.$$

The main obstacle is that the construction we have given is not uniform with respect to e: it treats a total φ_e differently from a nontotal φ_e, and, in the total case, it assumes knowledge of a suitable finite set S. To fix this, we will treat all φ_e as if they *might* be total, create an effective list $g_{\langle e,n,x\rangle}$ of uniformly computable functions to use in place of g, and, for each such $g_{\langle e,n,x\rangle}$, make a certain finite number of *guesses* as to what a suitable S might be. For each such S, we then diagonalise as in the basic module.

Each $g_{\langle e,n,x\rangle}$ will come equipped with a guess—namely, n—for the cardinality of an S witnessing the triviality of $\mathfrak{A} \restriction_\omega^{g_{\langle e,n,x\rangle}(x)}$. Although, as has already been mentioned, the number of guesses we need for S is finite, it far exceeds the bound $g_{\langle e,n,x\rangle}(x)$. This is a source of tension. We overcome this by defining a much faster-growing computable function $f_{\langle e,n\rangle}$ and make the wilder guess that S witnesses the triviality of $\mathfrak{A} \restriction_\omega^{f_{\langle e,n\rangle}(x)}$. Then we use w-triviality to argue that, for some x and n, there is indeed a suitable S of size n, and the bound $f_{\langle e,n\rangle}(x)$ is large enough to diagonalise for each guess at S.

Before giving the construction in full, we state and prove some helpful combinatorial lemmas.

Definition 4.5. We are given a structure $\mathfrak{A}$. Define the *growth function* G as a two-place function taking as arguments $M, N \in \omega \cup \{\omega\}$, and yielding the value

$$G_N^M = (\mu n \in \omega)[\exists S \subseteq N \text{ of size } n \text{ s.t. } S \text{ witnesses the triviality of } \mathfrak{A} \restriction_N^M],$$

or $G_N^M = \omega$ if there is no such n.

Here are a few easy and useful properties of the growth function.

Facts. (i) The one-place function $M \mapsto G_\omega^M$ is an automorphism invariant of $\mathfrak{A}$.

(ii) When $M, N \in \omega$ are finite, G_N^M is computable effectively in $\mathfrak{A}$ as a function of $\langle M, N\rangle$.

(iii) G is monotonic in the sense that, if we have $M, M^*, N, N^* \in \omega \cup \{\omega\}$, then $M \leq M^*$ and $N \leq N^*$ implies $G_N^M \leq G_{N^*}^{M^*}$.

(iv) For each M, $\lim_{s\to\omega} G_s^M = G_\omega^M$.

(v) $\mathfrak{A}$ is w-trivial if and only if $(\forall M \in \omega)(\forall N \in \omega \cup \{\omega\})[G_N^M$ is finite] and for all total computable f there is an n such that $G_\omega^{f(n)} \leq n$.

(vi) If $\mathfrak{A}$ is w-trivial and $F_0 \leq F_1 \leq \cdots$ is a pointwise-increasing sequence of total uniformly computable functions, then there exist natural numbers n, y such that $G_\omega^{F_n(y)} = n = G_\omega^{F_{n+1}(y)}$.

Proof. (i) Immediate.

(ii) Use brute force: for every subset $S \subseteq N$, check whether S witnesses the triviality of $\mathfrak{A}\restriction_N^M$.

(iii) If S witnesses the triviality of $\mathfrak{A}\restriction_{N^*}^{M^*}$, then S also witnesses the triviality of $\mathfrak{A}\restriction_N^M$.

(iv) Immediate.

(v) Immediate from the definition of w-trivial.

(vi) Define a total computable function ψ by $\psi(x) = F_{x+1}(x)$, and use Fact (v) to get a y such that $G_\omega^{\psi(y)} \leq y$. By Fact (iii), we have $0 \leq G_\omega^{F_0(y)} \leq \cdots \leq G_\omega^{F_{y+1}(y)} \leq n$. The result now follows from the following pigeonhole-type fact: If $\sigma : y+2 \to y$ is an increasing sequence, then there is an n such that $\sigma(n) = n = \sigma(n+1)$.

□

We have mentioned that, when guessing at suitable sets S to use for the diagonalisation strategy, we need only finitely many guesses. The following result makes this precise.

Lemma 4.6. *Suppose that $M \in \omega$, that $G_\omega^M = n$, and that $t \in \omega$ is large enough that $G_t^M = n$. Then there is a set $S \subseteq t$ of cardinality n witnessing the triviality of $\mathfrak{A}\restriction_\omega^M$, and furthermore we can identify from $\mathfrak{A}\restriction_t^M$ a list of sets $(S_0, S_1, \ldots, S_{M^n-1})$, such that $S = S_j$ for some $j < \ell$.*

Proof. Pick any $S \subseteq \omega$ of cardinality n which witnesses the triviality of $\mathfrak{A}\restriction_\omega^M$. Then $S \cap t$ must witness the triviality of $\mathfrak{A}\restriction_t^M$. By our assumption that $G_\omega^M = n$, we must have $|S \cap t| \geq n$. Since $|S| = n$, this implies that $S \subseteq t$.

We may naturally associate with each $j < M^n$ a sequence $\tau_j : n \to M$. We build a guess S_j by a sequence $\emptyset = S_j^{(0)} \subseteq \ldots \subseteq S_j^{(n)} = S_j$, where each $S_j^{(i)}$ has cardinality i. Suppose that we have already chosen $S_j^{(i)}$, and $i < n$. Since $|S_j^{(i)}| = i < n = G_t^M$, this $|S_j^{(i)}|$ does not witness the triviality of $\mathfrak{A}\restriction_t^M$. In some fixed computable enumeration, find the first permutation ρ fixing $S_j^{(i)} \cup [t, \infty)$ which is not an automorphism of $\mathfrak{A}\restriction_t^M$. Next, find the lexicographically-least sequence $\langle k, x_0, \ldots, x_{\mathsf{ar}(R_k)-1}\rangle$ for which it is *not* the case that

$$R_k^{\mathfrak{A}}(x_0, \ldots, x_{\mathsf{ar}(R_k)-1}) \text{ holds if and only if } R_k^{\mathfrak{A}}(\rho(x_0), \ldots, \rho(x_{\mathsf{ar}(R_k)-1})) \text{ holds.}$$

Clearly, S must contain at least one element of the set

$$U = \{x_0, \ldots, x_{\mathsf{ar}(R_k)-1}, \rho(x_0), \ldots, \rho(x_{\mathsf{ar}(R_k)-1})\} \setminus S_j^{(i)}.$$

Recalling our assumption from §1 that $\mathsf{ar}(R_k) \leq k/2$, this U has size at most $k \leq n$. We extend $S_j^{(i)}$ to $S_j^{(i+1)}$ by adding the $\tau_j(i)$-th smallest element of U (if $\tau_j(i) \geq |U|$, we just add the largest element of U).

We can see by induction that, for every i, there is a j such that $S_j^{(i)} \subseteq S$. In particular, there is a j such that $S_j = S$. □

Strategy. Our strategy uses a certain class of partial functions $g_{\langle e,n,x\rangle}$. We show how to use $g_{\langle e,n,x\rangle}$ before defining it explicitly; for the moment, suffice it to say that $g_{\langle e,n,x\rangle}$ is uniformly computable, and that, whenever φ_e is total, $g_{\langle e,n,x\rangle}$ is total, and for all x and y, there is enough space in the interval $[x, g_{\langle e,n,x\rangle}(y))$ to diagonalise against a single S of size n witnessing the triviality of $\mathfrak{A}\restriction_\omega^{g_{\langle e,n,x\rangle}(y)}$. From $g_{\langle e,n,x\rangle}$ we define a second class of functions:

$$f_{\langle e,n\rangle}(x) = \underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{n^x \text{ times}}(0).$$

Then $f_{\langle e,n\rangle}$ is uniformly computable and is total whenever φ_e is total, and there is enough space in the interval $[x, f_{\langle e,n\rangle}(x))$ to diagonalise against n^x-many different sets S of size n. Here are the essential steps we use to construct P. Note that we dovetail at step (1). In the first pass, we have $s = 0$.

(1) Start with a 3-tuple $s = \langle e, n, x\rangle$. The number e identifies the requirement $\mathcal{R}_e$ that we are trying to fulfil. The number n represents a guess at the size of a suitable set S against which to diagonalise. The number x is a parameter that ranges over ω.

(2) Wait for a stage t at which $f_{\langle e,n\rangle,t}(x)\downarrow$ and such that $G_t^{f_{\langle e,n\rangle}(x)} = n$. While we are waiting, return to step (1), this time using $s+1$ as the 3-tuple.

(3) Assume—possibly incorrectly—that $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Use the method of Lemma 4.6 to make a sequence $(S_j)_{j<x^n}$ of guesses at an S of size n witnessing the triviality of $\mathfrak{A}\restriction_\omega^{f_{\langle e,n\rangle}(x)}$.

(4) For each $j < x^n$, use the space in the interval

$$[\underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{j \text{ times}}(0), \underbrace{g_{\langle e,n,x\rangle} \circ \cdots \circ g_{\langle e,n,x\rangle}}_{j+1 \text{ times}}(0))$$

to diagonalise for S_j, adding restraints to P by the method of Proposition 4.3. If our assumption at step (3) was correct, then this will satisfy the requirement $\mathcal{R}_e$.

Definition of $g_{\langle e,x,n\rangle}$ and allocation of space for diagonalisation.

Define a sequence $(M_k)_k$ of natural numbers recursively by $M_0 = 0$ and $M_{k+1} = M_k + (M_k + 1)^k$. The intervals $[M_k, M_{k+1})$ form a partition of ω. For any total φ_e and any S of size $|S| \le k$, we could use the interval $[M_k, \varphi_e(M_{k+1}))$ as one of the $[N_i, N_{i+1})$ from the construction in Proposition 4.3, and diagonalise for the case $\pi(S) \cap [M_k, \varphi_e(M_{k+1})) = \emptyset$ by placing restraints on $X \cap [M_k, M_{k+1})$ for $X \in P$. To each 3-tuple $\langle e, n, x\rangle$ we assign a sequence of such intervals to use to meet requirement $\mathcal{R}_e$. We make this allocation methodical by defining a uniformly computable function $h_{\langle e,n,x\rangle}$:

$$h_{\langle e,n,x\rangle}(0) = M_{\langle e,n,x,i\rangle}, \quad \text{where } i \text{ is least such that } n \le \langle e,n,x,i\rangle$$

$$h_{\langle e,n,x\rangle}(y+1) = M_{\langle e,n,x,i\rangle}, \quad \text{where } i \text{ is least such that } \varphi_e(h_{\langle e,n,x\rangle}(y) + (h_{\langle e,n,x\rangle}(y)+1)^n) \le M_{\langle e,n,x,i\rangle}.$$

The intervals allocated to $\langle e, n, x\rangle$ are those of the form $[M_k, M_{k+1})$ such that $M_k = h_{\langle e,n,x\rangle}(y)$ for some y. Notice that $h_{\langle e,n,x\rangle}$ is total whenever φ_e is total. From here we can define the promised $g_{\langle e,n,x\rangle}$:

$$g_{\langle e,n,x\rangle}(x) = \underbrace{h_{\langle e,n,x\rangle} \circ \cdots \circ h_{\langle e,n,x\rangle}}_{n+1 \text{ times}}(x).$$

Verification. It remains to check that, for every e such that φ_e is total, there is a pair n, x such that $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Fix any e such that φ_e is total, and define a pointwise-increasing sequence of total uniformly computable functions $(F_n)_n$ recursively by $F_0 = id$ and $F_{n+1} = f_{\langle e,n\rangle} \circ F_n$. We can apply Fact (vi) to get a pair n, y such that $G_\omega^{F_{n+1}(y)} = n = G_\omega^{F_n(y)}$. Letting $x = F_n(y)$, this expression becomes $G_\omega^{f_{\langle e,n\rangle}(x)} = n = G_\omega^x$. Hence our strategy, when beginning with the triple $\langle e, n, x\rangle$, succeeds in satisfying $\mathcal{R}_e$.

This completes the proof of Theorem 3.6. □

5 Structures with finite signature

In this section, we examine the special case of a structure $\mathfrak{A}$ with finite signature $(R_0^{\mathfrak{A}}, \ldots, R_{n-1}^{\mathfrak{A}})$. As noted above, such an $\mathfrak{A}$ is w-trivial if and only if $\mathfrak{A}$ is trivial if and only if $\mathrm{DgSp}_T(\mathfrak{A}) = \{\mathbf{0}\}$; this, in turn, happens if and only if $\mathrm{DgSp}_{wtt}(\mathfrak{A}) = \{\mathbf{0}\}$. We may use Proposition 3.2 together with Theorem 3.6 to obtain a sharpened dichotomy in the finite-signature case:

Corollary 5.1. *Let $\mathfrak{A}$ be a structure with finite signature. Either $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains the cone above some degree $\mathbf{a}$, or $\mathrm{DgSp}_{wtt}(\mathfrak{A}) = \{\mathbf{0}\}$.*

Therefore, in restricting our structures to those with finite signature, we also restrict the possible wtt degree spectra. We shall see in Proposition 6.2 below that, for a structure with infinite signature, the wtt-degree spectrum may be contained within a single cone $\mathcal{D}_{wtt}(\geq \mathbf{a})$ with $\mathbf{a} > \mathbf{0}$. The following proposition shows that such a wtt degree spectrum is impossible for a structure with finite signature.

Proposition 5.2. *If $\mathfrak{A}$ has finite signature, then $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ is not contained in any cone of the form $\mathcal{D}_{wtt}(\geq \mathbf{e})$ with $\mathbf{e} > \mathbf{0}$.*

Our proof uses the following definition and lemma from basic model theory.

Definition 5.3. Let $\mathfrak{A}$ be a structure, let F be a finite set of elements of $\mathfrak{A}$, and let $I = (a_0, a_1, \ldots)$ be an infinite sequence of natural numbers without repetition. We say that I is a sequence of *quantifier-free order indiscernibles over* F if, for every pair of increasing sequences $(i_0 < \ldots < i_{n-1})$ and $(j_0 < \ldots < j_{n-1})$, the tuples $(a_{i_0}, \ldots, a_{i_{n-1}})$ and $(a_{j_0}, \ldots, a_{j_{n-1}})$ satisfy the same quantifier-free formulas with parameters from F.

Lemma 5.4. *Let $\mathfrak{A} = (\omega, R_0^{\mathfrak{A}}, \ldots, R_{n-1}^{\mathfrak{A}})$ be a structure with finite signature, and let m be a natural number.*

(i) There is an infinite sequence I of quantifier-free order indiscernibles over $\{0, \ldots, m-1\}$.

(ii) *There exists an infinite computable structure* $\mathfrak{C} = (\omega, R_0^{\mathfrak{C}}, \dots, R_{n-1}^{\mathfrak{C}})$ *and an increasing injection* $\rho : \omega \to \omega$ *such that* $\rho \upharpoonright m = id_m$ *and* ρ *embeds* $\mathfrak{C}$ *into* $\mathfrak{A}$.

Proof. Part (i) is an easy consequence of Ramsey's Theorem; see, for example, Shelah [13, Ch.1 §2 Theorem 2.4(1)]. We deduce part (ii) from part (i) as follows. Let $\mathfrak{A}$ and m be as in the statement of the Lemma, and let $I = (a_0, a_1, \dots)$ be the sequence given by part (i). Passing to a subsequence if necessary, we may assume I is increasing. Define $\rho : \omega \to \omega$ by $\rho(i) = i$ if $i < m$, and $\rho(j + m) = a_j$ for all j. Let $\mathfrak{C}$ be the unique structure such that ρ is an embedding of $\mathfrak{C}$ into $\mathfrak{A}$. Then $\mathfrak{C}$ is computable. □

We use this Lemma to prove Proposition 5.2 by a diagonalisation argument:

Proof of Proposition 5.2. Fix a structure $\mathfrak{A} = (\omega, R_0^{\mathfrak{A}}, \cdots, R_{n-1}^{\mathfrak{A}})$ and a set E of wtt degree $\mathbf{e} > \mathbf{0}$. We exhibit a permutation π such that, if $\mathfrak{B}$ is the unique structure such that $\pi : \mathfrak{B} \cong \mathfrak{A}$, then $E \not\leq_{wtt} \mathfrak{B}$. We build this π as the pointwise limit of a sequence $(\pi_e)_e$ of permutations, and alongside these we build a sequence $(m_e)_e$ of natural numbers to act as restraints.

Start with $\pi_0 = id_\omega$ and $m_0 = 0$.

Suppose that π_e and m_e have been defined. We define π_{e+1} and m_{e+1} as follows. Begin by letting $\mathfrak{B}_e$ be the unique structure such that $\pi_e : \mathfrak{B}_e \cong \mathfrak{A}$. Apply Lemma 5.4(ii) to the structure $\mathfrak{B}_e$ and the number m_e, and take the resulting structure $\mathfrak{C}_e$ and embedding ρ_e. Because $\mathfrak{C}_e$ is computable and E is not, there is an $x_e \in \omega$ such that either $\hat{\Phi}_e^{\mathfrak{C}_e}(x_e)\uparrow$ or $\hat{\Phi}_e^{\mathfrak{C}_e}(x_e)\downarrow \neq E(x_e)$. If $\varphi_e(x_e)\uparrow$, let $m_{e+1} = \max(m_e, x_e) + 1$; otherwise, let $m_{e+1} = \max(m_e, x_e, \rho_e(\varphi_e(x_e))) + 1$. Choose a permutation $\tau_e : \omega \to \omega$ such that $\tau_e \upharpoonright m_{e+1} = \rho_e \upharpoonright m_{e+1}$, and τ_e fixes $[0, m_e) \cup [m_{e+1}, \infty)$. Define $\pi_{e+1} = \tau_e \circ \pi_e$. Let π be the pointwise limit of $(\pi_e)_e$. This completes the construction.

Verification. The definition of π_{e+1} can be rewritten as $\pi_{e+1} = \tau_e \circ \tau_{e-1} \circ \cdots \circ \tau_0 \circ id_\omega$. Since each τ_e acts nontrivially only on the interval $[m_e, m_{e+1})$, and these intervals form a partition of ω, the limit π is a permutation. Let $\mathfrak{B}$ be the unique structure such that $\pi : \mathfrak{B} \cong \mathfrak{A}$; we claim that $\mathbf{e} \not\leq_{wtt} \mathfrak{B}$. Indeed, for each e, either $\varphi_e(x_e)\uparrow$, in which case $\hat{\Phi}_e^{\mathfrak{B}}(x_e)\uparrow$ by definition; or, for each $i \geq e + 1$, we have $\pi_i \upharpoonright m_{e+1} = \pi_{e+1} \upharpoonright m_{e+1}$, giving $\pi_i \upharpoonright \varphi_e(x_e) = \rho_e \upharpoonright \varphi_e(x_e)$, so that $\hat{\Phi}_e^{\mathfrak{B}}(x_e) = \hat{\Phi}_e^{\mathfrak{C}_e}(x_e) \neq E(x_e)$. □

Theorem 1.7 follows immediately.

Proof of Theorem 1.7. Dovetail the construction above, with $e = 0, 1, 2 \dots$. □

Finally, we mention some cases where the wtt degree spectrum is provably upward closed. We gave a brief argument in §3 that, if $\mathfrak{A}$ is a linear order, then the proof of Theorem 2.3 actually guarantees upward closure for $\mathrm{DgSp}_{wtt}(\mathfrak{A})$. This argument can now be formalised using Lemma 4.1 and applied to other examples.

Proposition 5.5. (i) *If* $\mathfrak{A} = (\omega, \leq^{\mathfrak{A}})$ *is a linear order, then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed.*

(ii) *If* $\mathfrak{A} = (\omega, E^{\mathfrak{A}})$ *is a structure where* $E^{\mathfrak{A}}$ *is an equivalence relation having more than one infinite class, then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed.*

(iii) *If* $\mathfrak{A} = (\omega, E^{\mathfrak{A}})$ *is a structure where* $E^{\mathfrak{A}}$ *is an equivalence relation having infinitely many nonsingleton classes, then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed.*

Proof. (i) Apply Lemma 4.1 to $\mathfrak{A}$, with $g(m) = m + 2$.

(ii) Let U_1 and U_2 be distinct infinite equivalence classes. Take the isomorphic copy $\mathfrak{A}^*$ specified by:

$$E^{\mathfrak{A}^*}(3x, 3y) \text{ holds} \iff E^{\mathfrak{A}}(x, y) \text{ holds;}$$
$$E^{\mathfrak{A}^*}(3x+1, z) \text{ holds} \iff z = 3y+1, \text{ or } (z = 3y \text{ and } y \in U_1);$$
$$E^{\mathfrak{A}^*}(3x+2, z) \text{ holds} \iff z = 3y+2, \text{ or } (z = 3y \text{ and } y \in U_2).$$

Then $\mathfrak{A}^*$ is isomorphic to $\mathfrak{A}$, and $\mathfrak{A}^* \equiv_{wtt} \mathfrak{A}$. Apply Lemma 4.1 to $\mathfrak{A}^*$, with $g(m) = m + 3$.

(iii) Build a permutation π by the following recursive procedure:

$$\pi(0) = 0$$

$$\pi(2x+1) = (\mu y)[y \text{ not in the image of } \pi \restriction 2x+1]$$

$$\pi(2x+2) = (\mu z)[z \text{ not } E^{\mathfrak{A}}\text{-equivalent to any } y \text{ in the image of } \pi \restriction 2x+2.$$

Let $\mathfrak{A}^*$ be the inverse image of $\mathfrak{A}$ under π, i.e., $\pi : \mathfrak{A}^* \cong \mathfrak{A}$. Then $\mathfrak{A}^* \leq_{wtt} \mathfrak{A}$. Apply Lemma 4.1 to $\mathfrak{A}$, with $g(x) = x + 6$.

□

Parts (*ii*) and (*iii*) can be combined into a single corollary:

Corollary 5.6. *Let $\mathfrak{A} = (\omega, E^{\mathfrak{A}})$ be an equivalence relation. Then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed if and only if $\mathfrak{A}$ is not trivial.*

The constructions for (*ii*) and (*iii*) in Proposition 5.5 are more typical than that for (*i*). By and large, Ramsey-type considerations make it difficult to meet the hypothesis of Lemma 4.1 without first rearranging a model's elements.

As one last example, we mention a large class of graphs $\mathfrak{A}$, each of which has an isomorphic copy $\mathfrak{A}^* \leq_{wtt} \mathfrak{A}$ to which we can apply Lemma 4.1. The proof is omitted.

Proposition 5.7. *If $\mathfrak{A} = (\omega, E^{\mathfrak{A}})$ is a graph, and if*

$$(\forall n)(\exists \text{ distinct } a_0, a_1, a_2, a_3 \geq n)(\forall x < n)[a_0 E^{\mathfrak{A}} a_1 \quad \wedge \quad \neg a_2 E^{\mathfrak{A}} a_3 \quad \wedge \bigwedge_{0 \leq i \leq 3} \neg a_i E^{\mathfrak{A}} x]$$

holds, then $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed. In particular, if $\mathfrak{A}$ has infinitely many nonsingleton components, then* $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ *is upward closed.*

6 Some specific examples

This section is devoted to a few elementary constructions each giving a partial answer to the question: *What sets of wtt degrees can form a wtt degree spectrum?*

Recall from Definition 3.3 that a set A is *of* $\mathbf{0}$*-dominated degree* if and only if, whenever f is a function such that $\mathsf{graph}(f) \leq_{wtt} A$, this f is dominated by a computable function. We say that a wtt-degree $\mathbf{a}$ is $\mathbf{0}$-dominated if its elements are of $\mathbf{0}$-dominated degree.

Proposition 6.1. *A wtt-degree* $\mathbf{a}$ *contains a structure that is w-trivial but not trivial if and only if* $\mathbf{a}$ *is not* $\mathbf{0}$*-dominated.*

Proof. The 'only if' direction is immediate from Theorem 3.6 and the observation in Proposition 3.4 that, if $\mathfrak{A}$ is $\mathbf{0}$-dominated, then $\mathrm{DgSp}_{wtt}(\mathfrak{A})$ contains a cone.

For the 'if' direction, suppose that $\mathbf{a}$ is not $\mathbf{0}$-dominated and fix a member $A \in \mathbf{a}$. Let f be a strictly increasing function that is not computably dominated, and such that $\mathsf{graph}(f) \leq_{wtt} A$. We construct a structure $\mathfrak{B} = (\omega, R_0^{\mathfrak{B}}, R_1^{\mathfrak{B}}, \ldots)$, with each $R_k^{\mathfrak{B}}$ unary, such that $\mathfrak{B}$ is w-trivial, $\mathfrak{B}$ is not trivial, and $\mathfrak{B} \equiv_{wtt} A$. For each k, define:

$$R_{2k}^{\mathfrak{B}} = \begin{cases} \{k\} & \text{if } k \text{ is in the image of } f, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$R_{2k+1}^{\mathfrak{B}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A \end{cases}$$

Then $\mathfrak{B} \equiv_{wtt} A$. To see that $\mathfrak{B}$ is w-trivial, let ψ be any increasing total computable function, and take n such that $\psi(n) < f(n)$. Let $S = \{k : 0 \leq k < n\}$. This S has cardinality n and witnesses the triviality of $\mathfrak{B} \restriction_{\omega}^{\psi(n)}$, as desired. $\square$

Our next construction gives a wide class of possible wtt degree spectra, and, as mentioned in §5 above, highlights an important difference between the finite- and infinite-signature cases.

Proposition 6.2. *For any wtt degree* $\mathbf{a}$*, there is a* $\mathfrak{B}$ *such that* $\mathrm{DgSp}_{wtt}(\mathfrak{B}) = \mathcal{D}_{wtt}(\geq \mathbf{a})$.

Proof. If $\mathbf{a} = \mathbf{0}$, then we can use any computable $\mathfrak{B}$ which is not trivial. So suppose that $\mathbf{a} > \mathbf{0}$, and fix a member $A \in \mathbf{a}$. Define $\mathfrak{B} = (\omega, R_0^{\mathfrak{B}}, R_1^{\mathfrak{B}}, \ldots)$, with each $R_k^{\mathfrak{B}}$ unary, as follows.

$$R_0^{\mathfrak{B}} = \{0, 2, 4, 6, \ldots\}$$

$$R_{k+1}^{\mathfrak{B}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A \end{cases}$$

Then A is wtt-below any isomorphic copy $\mathfrak{C}$ of $\mathfrak{B}$, since we can decide whether a given k is in A by checking whether $R_k^{\mathfrak{C}}(0)$ holds. On the other hand, if X is a set such that $X \geq_{wtt} A$, then X must be infinite and co-infinite, and so we may construct an isomorphic copy $\mathfrak{C}$ of $\mathfrak{B}$ such that $\mathfrak{C} \equiv_{wtt} X$ as follows:

$$R_0^{\mathfrak{C}} = X$$

$$R_{k+1}^{\mathfrak{C}} = \begin{cases} \omega \text{ if } k \in A \\ \emptyset \text{ if } k \notin A. \end{cases}$$

$\square$

Our next construction shows that, as a set of reals, every T degree spectrum not consisting of a single degree is equal to a wtt degree spectrum. Hence wtt degree spectra of nontrivial structures are at least as expressive, when considered as subsets of 2^{ω}, as T degree spectra of nontrivial structures.

Proposition 6.3. *If* $\mathfrak{A}$ *is a structure which is not trivial, then there is a graph* $H = (\omega, E^H)$ *such that* $\bigcup \mathrm{DgSp}_{wtt}(H) = \bigcup \mathrm{DgSp}_T(\mathfrak{A})$.

Proof. By Theorem 1.8, we may fix a graph $G = (\omega, E^G)$ with Turing degree spectrum $\mathrm{DgSp}_T(G) = \mathrm{DgSp}_T(\mathfrak{A})$. We may assume that G has no isolated points, that is, for all x there exists a y such that $(x, y) \in E^G$. We use G to build a new graph $H = (\omega, E^H)$ with the following properties:

(i) $\mathrm{DgSp}_T(H) = \mathrm{DgSp}_T(G)$

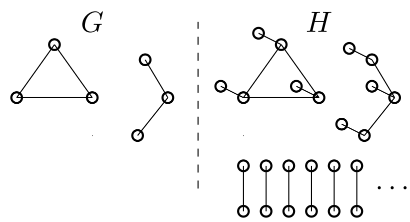
(ii) $\mathrm{DgSp}_{wtt}(H)$ is upward closed.

(iii) Given $X \in 2^\omega$ and a copy $K \cong H$, if $X \geq_T K$, then there is another copy $L \cong H$ such that $X \geq_{wtt} L$.

This is then the desired H by the following string of equivalences:

$$\begin{aligned}
X \in \bigcup \mathrm{DgSp}_T(\mathfrak{A}) \quad &\text{iff} \quad X \in \bigcup \mathrm{DgSp}_T(G), \quad \text{by choice of } G \\
&\text{iff} \quad X \in \bigcup \mathrm{DgSp}_T(H), \quad \text{by (i)} \\
&\text{iff} \quad X \geq_T K \text{ for some } K \cong H, \quad \text{since } \mathrm{DgSp}_T(H) \text{ is upward closed} \\
&\text{iff} \quad X \geq_{wtt} L \text{ for some } L \cong H, \quad \text{by (iii)} \\
&\text{iff} \quad X \in \bigcup \mathrm{DgSp}_{wtt}(H), \quad \text{by (ii)}
\end{aligned}$$

Construction. We transform G into the new graph H by appending exactly one new vertex to each vertex of G, and then adding a countable perfect matching. Pictorially, the transformation behaves like this:



We define the edge relation on H by cases, closing under symmetry:

- If $x = 4n$, $y = 4m$, and $(m, n) \in E^G$, then $(x, y) \in E^H$.
- If $x = 4n$ and $y = 4n + 1$, then $(x, y) \in E^H$.
- If $x = 4n + 2$ and $y = 4n + 3$, then $(x, y) \in E^H$.

We claim that this H satisfies conditions (i),(ii),(iii).

Verification of (i). Notice first that $H \equiv_T G$, and second that, if a copy $G_0 \cong G$ is transformed in the same manner as above into a graph H_0, then $H_0 \cong H$. Thus $\mathrm{DgSp}_T(G) \subseteq \mathrm{DgSp}_T(H)$. For the opposite inclusion, suppose that H_0 is an isomorphic copy of H. Define a set $A \subseteq \omega$ of vertices by:

$$A = \{x \in \omega : (\exists \text{ at least two distinct } y)[(x, y) \in E^{H_0}]\}.$$

Because G has no isolated points, the subgraph induced by H_0 on A is isomorphic to G. Define an injection $\rho : \omega \to \omega$ by

$$\rho(n) = \text{ the } n\text{-th element enumerated into A},$$

and let G_1 be the unique structure such that ρ is an embedding of G_1 into H. Then $G_1 \cong G$ and $G_1 \leq_T H$. We conclude by the upward-closure result of Theorem 2.3 that $\mathrm{DgSp}_T(G) \subseteq \mathrm{DgSp}_T(H)$.

Verification of (ii). For any n, the elements $a_0 = 4n+2$, $a_1 = 4n+3$, $a_2 = 4n+6$, and $a_3 = 4n+10$ satisfy the statement:

$$(\forall x < n)[a_0 E^H a_1 \quad \wedge \quad \neg a_2 E^H a_3 \quad \wedge \bigwedge_{0 \leq i \leq 3} \neg a_i E^H x].$$

Hence Proposition 5.7 implies that $\mathrm{DgSp}_{wtt}(H)$ is upward closed.

Verification of (iii). Suppose that K is is an isomorphic copy of H and that $X \geq_T K$, say by the computation $D(K) = \Phi_e^X$. We get the required L by the following 'padding' procedure. For each $n \in \omega$, let u_n be least such that $\Phi_e^{X \restriction u_n}$ computes the restricted diagram $K \restriction_n$, i.e., such that Φ_e^X computes $K \restriction_n$ with use u_n. Define a sequence $(v_n)_{n \in \omega}$ by $v_0 = u_0$, $v_{n+1} = v_n + 2u_n + 3$. We define the edge relation on L by the following cases, closing under symmetry:

- If $x = v_m$ and $y = v_n$, then $(x, y) \in E^L$ if and only if $(m, n) \in E^K$.
- If $v_m < x < v_{m+1} - 1$, then $(x, x+1) \in E^L$ if and only if $x - v_m$ is odd.

That is, K is embedded into L by the mapping $m \mapsto v_m$, and the remaining elements of L form an infinite perfect matching. Since K itself contains an infinite perfect matching, L and K are isomorphic. Now we check that $L \leq_{wtt} X$. Given a number x, look at the computation of $\Phi_e^{X \restriction x}$ to find the least m such that $v_m > x$. We can use the computation of $\Phi_e^{X \restriction x}$ to recover both the restricted diagram $K \restriction_m$ and the sequence $(v_0, \ldots, v_{m-1})$. This information is enough to construct the restricted diagram $L \restriction_x$. □

We end with a construction of a wtt degree spectrum that, as a set of reals, does not coincide with any Turing degree spectrum. When combined with Proposition 6.3, this establishes the result promised in §1.1 that, as a means of specifying a set of reals, the wtt degree spectrum of a nontrivial structure is strictly more expressive than the Turing degree spectrum of a nontrivial structure. As usual, there is some tension between the complexity of the construction and the contrivedness of the object being built. The following class of structures appears to be a good compromise.

Definition 6.4. Let $\mathfrak{A} = (\omega, \underline{0}^{\mathfrak{A}}, S^{\mathfrak{A}}, E^{\mathfrak{A}})$ be a structure with $\underline{0}^{\mathfrak{A}}$ a unary relation, and $S^{\mathfrak{A}}, E^{\mathfrak{A}}$ binary relations. We say that $\mathfrak{A}$ is a *labelled graph* if the reduct $(\omega, E^{\mathfrak{A}})$ is a graph and the reduct $(\omega, \underline{0}^{\mathfrak{A}}, S^{\mathfrak{A}})$ is isomorphic to the natural numbers with zero and successor (with $\underline{0}^{\mathfrak{A}}$ and $S^{\mathfrak{A}}$ interpreted as a constant and a unary function, respectively). Given an element $n \in \omega$, let $\delta^{\mathfrak{A}}(n)$ be the neighbourhood of n in $(\omega, E^{\mathfrak{A}})$, i.e.,

$$\delta^{\mathfrak{A}}(n) = \{m \in \omega : (m, n) \in E^{\mathfrak{A}}\}.$$

For any natural number e, let $\underline{e}^{\mathfrak{A}}$ denote the unique e-th element:

$$\underline{e}^{\mathfrak{A}} = \underbrace{S^{\mathfrak{A}}(S^{\mathfrak{A}}(\cdots S^{\mathfrak{A}}}_{e \text{ times}}(\underline{0}^{\mathfrak{A}}))).$$

Proposition 6.5. *There is a labelled graph $\mathfrak{A}$ such that $\bigcup \mathrm{DgSp}_{wtt}(\mathfrak{A}) \neq \bigcup \mathrm{DgSp}_T(\mathfrak{A})$.*

Proof. Let $^\frown$ be the concatenation operator for strings, and let $\left(\tilde{\Phi}_e\right)_e$ be the enumeration of all wtt reductions given by:

$$\tilde{\Phi}_e^Y(x) = \begin{cases} \Phi_e^Y(x) \text{ if } \mathsf{use}\, \Phi_e^Y(x) < \varphi_e(x) \\ \uparrow \text{otherwise.} \end{cases}$$

We build $\mathfrak{A}$, together with a set $Z \subseteq \omega$, to satisfy the following requirements:

$\mathcal{P}$: $\mathfrak{A} \leq_T Z$.

$\mathcal{N}_e$: If $\mathfrak{B}$ is a labelled graph and $\mathfrak{B} = \tilde{\Phi}_e^Z$, then $\mathfrak{A} \not\cong \mathfrak{B}$.

The requirement $\mathcal{P}$ ensures that $\deg_T(Z) \in \mathrm{DgSp}_T(\mathfrak{A})$, while the requirements $\mathcal{N}_e$ together ensure that $\deg_{wtt}(Z) \notin \mathrm{DgSp}_{wtt}(\mathfrak{A})$.

Strategy. We build Z by initial segments $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq Z$. At each stage n, we specify σ_n and $\mathfrak{A}\restriction_n$. The reduct $(\omega, \underline{0}^{\mathfrak{A}}, S^{\mathfrak{A}})$ will be ordered in the most straightforward way, namely, $\underline{e}^{\mathfrak{A}} = e$ for all e.

We begin by declaring that each negative requirement $\mathcal{N}_e$ *has not acted.* At a stage of the form $n+1 = \langle e, x\rangle + 1$, if $\mathcal{N}_e$ has not yet acted, we may choose to fix the set $\delta^{\mathfrak{A}}(e)$ as either a finite or a cofinite set. The goal is to satisfy $\mathcal{N}_e$ by ensuring, if $\mathfrak{B}$ is labelled graph and $\mathfrak{B} = \tilde{\Phi}_e^Z$, that:

$$\left|\delta^{\mathfrak{A}}(e)\right| \neq \left|\delta^{\mathfrak{B}}(\underline{e}^{\mathfrak{B}})\right| \text{ or } \left|\omega \setminus \delta^{\mathfrak{A}}(e)\right| \neq \left|\omega \setminus \delta^{\mathfrak{B}}(\underline{e}^{\mathfrak{B}})\right|.$$

After we decide to fix $\delta^{\mathfrak{A}}(e)$, we say that $\mathcal{N}_e$ *has acted.* At the end of the stage, we define σ_{n+1} and the restricted diagram $\mathfrak{A}\restriction_{n+1}$ based on the decisions made at earlier stages for other neighbourhoods $\delta^{\mathfrak{A}}(i)$.

We meet $\mathcal{P}$ by coding the atomic diagram of $\mathfrak{A}$ directly into Z. For each n, σ_{n+1} will equal $\sigma_n \frown 0^s \frown 1 \frown 0^r \frown 1$ for some s to be specified below and a number r representing the atomic diagram $\mathfrak{A}\restriction_n$ by some fixed computable encoding.

Construction. At stage $n = 0$, we let $\sigma_0 = \emptyset$.

At each stage of the form $n+1 = \langle e, x\rangle + 1$, we try to fulfil requirement $\mathcal{N}_e$ as outlined above. If $\mathcal{N}_e$ has not yet acted, then use a $\mathbf{0}'$ oracle to extend σ_n, if possible, to a string $\tau = \sigma_n \frown 0^s$ such that $\tilde{\Phi}_{e,s}^{\tau}$ converges to give a large initial segment of an atomic diagram D, having at least $2n+1$ elements, of a $\mathfrak{B}$ as in $\mathcal{N}_e$. If $|\delta^D(\underline{e}^D)| \geq n+1$, then we fulfil the requirement $\mathcal{N}_e$ by declaring that $\delta^{\mathfrak{A}}(e)$ shall be a subset of $\{0, \dots, n-1\}$. Otherwise, the complement has size $|\delta^D(\underline{e}^D)| \geq n+1$, and so we fulfil $\mathcal{N}_e$ by declaring that $\omega \setminus \delta^{\mathfrak{A}}(e)$ shall be a subset of $\{0, \dots, n-1\}$. We then preserve the computation by letting $\sigma_{n+1} = \tau \frown 1 \frown 0^r \frown 1$, with r a number representing $\mathfrak{A}\restriction_n$. Declare that $\mathcal{N}_e$ *has acted.*

If $\mathcal{N}_e$ has acted at an earlier stage, or if no suitable τ exists, then $\mathcal{N}_e$ does not act at stage $n+1$, and we instead carry out the following procedure. Let $\mathfrak{B}$ be the (possibly partial) atomic diagram given by $\mathfrak{B} = \tilde{\Phi}_e^{\sigma_n \frown 0^\omega}$. One of four conditions must hold:

(i) There is a y such that $\tilde{\Phi}_e^{\sigma_n \frown 0^\omega}(y)\uparrow$.

(ii) $\mathfrak{B}$ contains more than one element of the form $\underline{e}^{\mathfrak{B}}$.

(iii) $\mathfrak{B}$ contains no element of the form $\underline{e}^{\mathfrak{B}}$.

(iv) The requirement $\mathcal{N}_e$ has already acted at an earlier stage.

If (i), then choose an extension $\tau = \sigma_n \frown 0^s$ long enough that, if ρ is a string extending τ, then $\tilde{\Phi}_e^{\rho}(y)\uparrow$. If (ii), choose $\tau = \sigma_n \frown 0^s$ long enough that, for some y, the atomic diagram $\tilde{\Phi}_e^{\tau}\restriction_y$ contains more than one $\underline{e}$. If (iii) or (iv), choose $\tau = \sigma_n$. In any case, let $\sigma_{n+1} = \tau \frown 1 \frown 0^r \frown 1$, with r a number representing $\mathfrak{A}\restriction_n$.

Verification. It is easy to see that $\mathcal{P}$ is satisfied: For each n, we can find an initial segment $\sigma \frown 1 \frown 0^r \frown 1$ of Z such that exactly $2n$ entries of σ are 1. Then we can use r to recover the restricted diagram $\mathfrak{A}\restriction_n$.

Now we check that $\mathcal{N}_e$ is fulfilled. If, at any stage $n+1$, we declared $\mathcal{N}_e$ *has acted*, then our diagonalisation strategy using $\delta^{\mathfrak{A}}(e)$ succeeds. So suppose that $\mathcal{N}_e$ never acts, and suppose, for a contradiction, that the requirement $\mathcal{N}_e$ is not met. Let $\mathfrak{B} = \tilde{\Phi}_e^Z$ be as in the statement of $\mathcal{N}_e$. Then there is an $n = \langle e, x \rangle$ and a y such that $\tilde{\Phi}_e^{\sigma_n} \restriction_y$ contains a well-defined e-th element. Let $\mathfrak{C} = \tilde{\Phi}_e^{\sigma_n \frown 0^\omega}$. Either $\mathfrak{C}$ contains a finite substructure D as in the construction, or $\mathfrak{C}$ is not total as a characteristic function, or $\mathfrak{C}$ contains more than one element of the form $\underline{e}^{\mathfrak{C}}$. Of these three possibilities, the first implies that $\mathcal{N}_e$ acts at stage $n+1$, a contradiction; the second puts us in case (i) of the construction; and the third, in case (ii). But in case (i), our choice of σ_{n+1} implies that $\mathfrak{B}$ is also not total as a characteristic function, also a contradiction; and in case (ii), our choice of σ_{n+1} implies that $\mathfrak{B}$ has multiple elements of the form $\underline{e}^{\mathfrak{B}}$ and hence is not a labelled graph, another contradiction. □

As an aside, we note that a labelled graph $\mathfrak{A} = (\omega, \underline{0}^{\mathfrak{A}}, S^{\mathfrak{A}}, E^{\mathfrak{A}})$ can be encoded into a single binary relation R with only a small loss of information. Namely:

$$\begin{aligned} (n,n) \in R \quad &\iff \quad n \in \underline{0}^{\mathfrak{A}}; \quad \text{and, for all pairs } n \neq m, \\ (n,m) \in R \quad &\iff \quad (n,m) \in S^{\mathfrak{A}} \text{ or } (n,m) \in E^{\mathfrak{A}} \text{ and } (m,n) \notin S^{\mathfrak{A}}. \end{aligned}$$

In this encoding, we lose the edges between consecutive elements $(n, S^{\mathfrak{A}}(n))$ of the labelled graph.

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