On the jumps of the degrees below an r.e. degree*

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Abstract

We consider the set of *jumps below* a Turing degree, given by $\mathsf{JB}(\mathbf{a}) = \{\mathbf{x}' : \mathbf{x} \leq \mathbf{a}\}$, with a focus on the problem: Which r.e. degrees $\mathbf{a}$ are uniquely determined by $\mathsf{JB}(\mathbf{a})$? Initially, this is motivated as a strategy to solve the rigidity problem for the partial order $\mathcal{R}$ of r.e. degrees. Namely, we show that if every high$_2$ r.e. degree $\mathbf{a}$ is determined by $\mathsf{JB}(\mathbf{a})$, then $\mathcal{R}$ cannot have a nontrivial automorphism. We then defeat the strategy—at least in the form presented—by constructing pairs $\mathbf{a}_0, \mathbf{a}_1$ of distinct r.e. degrees such that $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$ within any possible jump class $\{\mathbf{x} : \mathbf{x}' = \mathbf{c}\}$. We give some extensions of the construction and suggest ways to salvage the attack on rigidity.

1 Introduction

The general setting for this paper is the study of the relationships between a degree $\mathbf{a}$ and its jump $\mathbf{a}'$ and, more generally, between $\mathbf{a}$ and the degrees $REA(\mathbf{a})$, i.e. those recursively enumerable in and above $\mathbf{a}$. The question that concerns us here is to what extent a degree, or more specifically, an r.e. degree $\mathbf{a}$ is determined by the jumps of, or degrees REA in, degrees $\mathbf{x}$ near $\mathbf{a}$. In particular, we were motivated by a conjecture about these relationships that would have implied the rigidity of the r.e. degrees, $\mathbf{R}$. The conjecture was inspired by the hope of combining two important results. One by Soare and Stob [1982] tells us that, under certain conditions, we can find degrees REA in a given $\mathbf{a}$ but not in another $\mathbf{b}$. The second is the constellation of Jump Interpolation Theorems of Robinson [1971]. These theorems generally say that any simple statement about the

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ordering of r.e. degrees and and their jumps (e.g. extension-of-embeddings results) not shown false by an obvious property of the r.e. degrees and their jumps can be realized. We give specific versions of these results that we need for our analysis.

Theorem 1.1 (Soare–Stob)**.** *If* $\mathbf{0} < \mathbf{a} \in \mathbf{R}$*, then there is a degree* $\mathbf{c}$ *which is* $REA(\mathbf{a})$ *but not r.e.*

Theorem 1.2 (Robinson Jump Interpolation)**.** *If* $\mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbf{R}$*,* $\mathbf{e} \not\leq \mathbf{c} < \mathbf{d}$*,* $\mathbf{z} \geq \mathbf{c}'$ *and is* $REA(\mathbf{d})$*, then there is an* $\mathbf{f} \in \mathbf{R}$ *with* $\mathbf{c} < \mathbf{f} < \mathbf{d}$*,* $\mathbf{e} \not\leq \mathbf{f}$*, and* $\mathbf{f}' = \mathbf{z}$*.*

Our thought was to combine and extend these ideas (and their standard relativizations) so as to characterize some r.e. degrees in terms of the jumps of degrees below them or equivalently (see Lemma 1.5) in terms of the degrees r.e. in them and above $\mathbf{0}'$. If we could do this for enough r.e. degrees, we knew that we could prove the rigidity of $\mathbf{R}$. We begin with the definition of our primary object of study.

Definition 1.3. If $\mathbf{a}$ is a Turing degree, the class $\mathsf{JB}(\mathbf{a})$, *jumps below* $\mathbf{a}$, consists of the jumps of degrees below $\mathbf{a}$:

$$\mathsf{JB}(\mathbf{a}) = \{\mathbf{x}' : \mathbf{x} \leq \mathbf{a}\}.$$

As it is phrased in terms of degrees rather than sets, this definition does not lend itself to the usual methods of priority and diagonalization. Before getting into the details of how JB is used, we give several alternative definitions; as far as constructing examples is concerned, the most important uses the following, standard language.

Definition 1.4. If V is an r.e. set with enumeration $(V_s)_{s\in\omega}$ and A is any set, then V^A is the set $\{n : (\exists \sigma \subseteq A)\langle n, \sigma\rangle \in V\}$. For each s, we define V_s^A likewise, with V_s in place of V.

Note that the class of sets r.e. in A is equal to $\{W_e^A : e \in \omega\}$.

Lemma 1.5. *For every degree* $\mathbf{y}$ *and every r.e. degree* $\mathbf{a}$ *with* $A \in \mathbf{a}$*, the following are equivalent:*

1. $\mathbf{y} = \mathbf{x}'$ *for some* $\mathbf{x} \leq \mathbf{a}$*.*
2. $\mathbf{y} = \mathbf{x}'$ *for some r.e.* $\mathbf{x} \leq \mathbf{a}$*.*
3. $\mathbf{y} \geq \mathbf{0}'$ *and* $\mathbf{y}$ *is* $REA(\mathbf{a})$*.*
4. $\mathbf{y} = \deg(W_e^A \oplus 0')$ *for some* e*.*

Proof. The implications ($2 \Rightarrow 1$) and ($3 \Rightarrow 4$), and ($4 \Rightarrow 3$) are immediate. The implication ($1 \Rightarrow 3$) follows from the monotonicity of the jump operator and the fact that if $X \leq_T A$ and Y is r.e. in X then Y is r.e. in A. The final implication ($3 \Rightarrow 2$) follows from Theorem 1.2. $\square$

Therefore, as long as $\mathbf{a}$ is r.e., the expression in Definition 1.3 can be replaced with $\mathsf{JB}(\mathbf{a}) = \{\mathbf{z} : \mathbf{0}' \leq \mathbf{z}\ \&\ \mathbf{z} \in \mathbf{REA}(\mathbf{a})\}$ or with $\mathsf{JB}(\mathbf{a}) = \{\deg(W_e^A \oplus 0') : e \in \omega\}$. It is this last formulation that we use in our priority constructions. As usual, $\mathbf{REA}(\mathbf{a})$ denotes the class of degrees $REA(\mathbf{a})$.

As Soare and Stob [1982] point out, relativizing Theorem 1.1 to any incomplete *high* degree $\mathbf{h}$ (i.e. $\mathbf{h}' = \mathbf{0}''$) and taking $\mathbf{0}'$ to play the role of $\mathbf{a}$, one sees that, for any incomplete high degree $\mathbf{h}$, there is a $\mathbf{c}$ $REA(\mathbf{0}')$ which is not $REA(\mathbf{h})$. Thus $\mathsf{JB}(\mathbf{0}') \neq \mathsf{JB}(\mathbf{a})$ for any incomplete r.e. $\mathbf{a}$ and so $\mathbf{0}'$ is determined within $\mathbf{R}$ by $\mathsf{JB}(\mathbf{0}')$. (If $\mathbf{a}$ is not high, then it is trivial that $\mathsf{JB}(\mathbf{0}') \neq \mathsf{JB}(\mathbf{a})$ as $\mathbf{0}'' \notin \mathsf{JB}(\mathbf{a})$ but $\mathbf{0}'' \in \mathsf{JB}(\mathbf{0}')$.) Our goal was to extend this to other degrees, in the hopes that we could characterize enough r.e. degrees $\mathbf{a}$ in terms of $\mathsf{JB}(\mathbf{a})$ to provide an automorphism basis that would be fixed under all automorphisms of $\mathbf{R}$ and use this to prove its rigidity.

A slightly different relativized version of Theorem 1.1 is as follows:

Corollary 1.6 (Soare and Stob)**.** *For any r.e.* $\mathbf{a}$ *and* $\mathbf{b}$ *with* $\mathbf{a} \not\leq \mathbf{b}$, *there is a* $\mathbf{c}$ $REA(\mathbf{a})$ *which is not* $REA(\mathbf{b})$.

Proof. If $\mathbf{b} < \mathbf{a}$ then this is the straightforward relativization of Theorem 1.1 as just used for $\mathbf{h} < \mathbf{0}'$. Otherwise $\mathbf{a}$ itself is the desired $\mathbf{c}$. □

The theme of the Robinson Interpolation Theorems is that anything not ruled out by simple relations between degrees and their jumps should be realizable. Along these lines we hoped to prove that the witness $\mathbf{c}$ in Corollary 1.6 (which is to be REA in $\mathbf{a}$ but not in $\mathbf{b}$) could be taken to lie between $\mathbf{0}'$ and $\mathbf{a}'$ at least for many degrees. Clearly this is not possible if these degrees are low, i.e. $\mathbf{a}' = \mathbf{b}' = \mathbf{0}'$ but there are many other candidates. What constitutes "enough degrees" here is driven by the desire to get an automorphism basis for $\mathbf{R}$ whose elements $\mathbf{a}$ would be determined by the degrees $REA(\mathbf{a})$ and above $\mathbf{0}'$, i.e. by $\mathsf{JB}(\mathbf{a})$. In particular, we state our original Conjecture for the class $\mathbf{H}_2 = \{\mathbf{x} \in \mathbf{R} | \mathbf{x}'' = \mathbf{0}'''\}$ of *high*$_2$ r.e. degrees which is known to be an automorphism basis for $\mathbf{R}$ (see §2).

Conjecture 1.7. *If* $\mathbf{a}, \mathbf{b} \in \mathbf{H}_2$ *and* $\mathbf{a} \not\leq \mathbf{b}$, *then there is a* $\mathbf{c} \geq \mathbf{0}'$ *which is* $REA(\mathbf{a})$ *but not* $REA(\mathbf{b})$. *As usual this should also be true relativized to any* $\mathbf{x}$.

In §2 we show that this conjecture would imply the rigidity of $\mathbf{R}$ and so proving it would have answered many of the most important questions about $\mathbf{R}$. Of course, if $\mathbf{a}$ and $\mathbf{b}$ have different jumps then, trivially, $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ so the real questions only arise when $\mathbf{a}' = \mathbf{b}'$. Known results can be used to show that we can at times distinguish between some $\mathbf{a}$ and $\mathbf{b}$ (with the same jump) by distinguishing between $\mathsf{JB}(\mathbf{a})$ and $\mathsf{JB}(\mathbf{b})$. The simplest examples follow easily from Theorems 1.1 and 1.2:

Proposition 1.8. *If* $\mathbf{h} \in \mathbf{R}$ *is high then there is an incomplete high r.e.* $\mathbf{g}$ *with* $\mathsf{JB}(\mathbf{h}) \neq \mathsf{JB}(\mathbf{g})$. *In fact, we may find such a* $\mathbf{g}$ *which is incomparable with* $\mathbf{h}$.

Proof. As mentioned above, relativizing Theorem 1.1 to $\mathbf{h}$ and applying it to $\mathbf{0}'$ gives a $\mathbf{c}$ $REA(\mathbf{0}')$ and not $REA(\mathbf{h})$. Applying Theorem 1.2 (or just the jump theorem of Sacks [1963] gives us an r.e. $\mathbf{k}$ such that $\mathbf{k}' = \mathbf{c}$. Of course, as $\mathbf{k}$ is not high, $\mathbf{h} \not\leq \mathbf{k}$. Applying Theorem 1.2 again gives us an incomplete high r.e. $\mathbf{g} \geq \mathbf{k}$ with $\mathbf{h} \not\leq \mathbf{g}$. It is now clear that $\mathbf{c} \in \mathsf{JB}(\mathbf{g})$ but $\mathbf{c} \notin \mathsf{JB}(\mathbf{h})$ and so $\mathbf{g} \not\leq \mathbf{h}$ as well. $\square$

We can extend and then apply a result of Arslanov, Lempp and Shore [1996, Proposition 1.13] to get the same result for the high$_2$ r.e. degrees.

Proposition 1.9 (Arslanov, Lempp and Shore)**.** *If* $\mathbf{c} < \mathbf{h}$ *are r.e.,* $\mathbf{c}$ *is low, i.e.* $\mathbf{c}' = \mathbf{0}'$*, and* $\mathbf{h}$ *is high, then there is an* $\mathbf{a} < \mathbf{h}$ *which is* $REA(\mathbf{c})$ *but not r.e.*

Corollary 1.10. *If* $\mathbf{x} \in \mathbf{R}$ *is high$_2$ then there is an r.e. degree* $\mathbf{g}$ *with* $\mathbf{x}' = \mathbf{g}'$ *such that* $\mathsf{JB}(\mathbf{x}) \neq \mathsf{JB}(\mathbf{g})$*. In fact, we may find such a* $\mathbf{g}$ *which is incomparable with* $\mathbf{x}$*.*

Proof. First note that the Proposition can be improved to allow $\mathbf{c}$ to be *low*$_2$, i.e. $\mathbf{c}'' = \mathbf{0}''$: Given such a $\mathbf{c}$ and a high $\mathbf{h} > \mathbf{c}$ apply Theorem 1.2 to get an r.e. $\mathbf{d}$ with $\mathbf{d} < \mathbf{c} < \mathbf{h}$ and $\mathbf{d}' = \mathbf{c}'$. Now relativize the Proposition to $\mathbf{d}$ and note that $\mathbf{c}$ is low relative to $\mathbf{d}$ while $\mathbf{h}$ is still high relative to $\mathbf{d}$ as $\mathbf{h}' = \mathbf{0}'' = \mathbf{d}''$. Thus we have an $\mathbf{a}$ which is $REA(\mathbf{c})$ but not r.e. in $\mathbf{d}$ and so certainly not r.e.

Next, relativize this extension of the Proposition to our given $\mathbf{x}$ and apply it to $\mathbf{0}'$ (as $\mathbf{c}$) and $\mathbf{x}'$ (as $\mathbf{h}$). (As $\mathbf{x} \in \mathbf{H}_2$, $\mathbf{0}'$ is low$_2$ relative to $\mathbf{x}$ while $\mathbf{x}'$ is obviously high relative to $\mathbf{x}$.) This gives us an $\mathbf{a}$ with $\mathbf{0}' < \mathbf{a} < \mathbf{x}'$ such that $\mathbf{a}$ is $REA(\mathbf{0}')$ but not r.e. in $\mathbf{x}$. Now argue as for Proposition 1.8 using Theorem 1.2: First one gets an r.e., $\mathbf{k} \not\geq \mathbf{x}$ with $\mathbf{k}' = \mathbf{a}$. Then one gets an r.e. $\mathbf{g} \not\geq \mathbf{x}$ with $\mathbf{g} \geq \mathbf{k}$ and $\mathbf{g}' = \mathbf{x}'$ so that $\mathbf{a} \in \mathsf{JB}(\mathbf{g})$ but $\mathbf{a} \notin \mathsf{JB}(\mathbf{x})$. (Of course, this also implies that $\mathbf{g} \not\leq \mathbf{x}$.) $\square$

While these last results show that there are many degrees such that we can distinguish between them in terms of the JB operator, the main result of this paper is to exhibit (regrettably) a rather strong failure of any possible characterization of the r.e. degrees $\mathbf{a}$ in any particular *jump class*, i.e. those $\mathbf{a}$ with $\mathbf{a}' = \mathbf{c}$ for any $\mathbf{c}$ $REA(\mathbf{0}')$ based simply on $\mathsf{JB}(\mathbf{a})$.

Theorem 1.11 (Main Theorem)**.** *If* $\mathbf{c}$ *is* $REA(\mathbf{0}')$*, then*

1. *there is a pair* $\mathbf{a}_0, \mathbf{a}_1$ *of r.e. degrees such that* $\mathbf{a}_0' = \mathbf{a}_1' = \mathbf{c}$*,* $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$ *and* $\mathbf{a}_0 | \mathbf{a}_1$*; and*
2. *there is a pair* $\mathbf{b}_0, \mathbf{b}_1$ *of r.e. degrees such that* $\mathbf{b}_0' = \mathbf{b}_1' = \mathbf{c}$*,* $\mathsf{JB}(\mathbf{b}_0) = \mathsf{JB}(\mathbf{b}_1)$ *and* $\mathbf{b}_0 < \mathbf{b}_1$*.*

Part 1 of the Main Theorem is proved in §3, by an argument extending the usual $0''$ tree proof of the Sacks jump theorem. Part 2 we do not prove in full; instead, in §4 we outline how to modify the proof from §3 to get degrees which are comparable instead of incomparable.

Before presenting the proofs, we mention a couple of the natural questions raised by these results and proofs and discuss some methodological issues.

Question 1.12. Are there any incomplete r.e. degrees $\mathbf{a}$ characterized by $\mathsf{JB}(\mathbf{a})$, i.e. such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ for any r.e. $\mathbf{b} \neq \mathbf{a}$? If so, are there enough to constitute an automorphism base for $\mathbf{R}$ and could one then prove its rigidity?

In the other direction, there are several possible strengthenings of the Main Theorem. We mention one that would provide a negative answer to the previous question.

Question 1.13. Is there, for every r.e. $\mathbf{a}$, an r.e. $\mathbf{b} \neq \mathbf{a}$ such that $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$?

There are many possible variations on these questions some of which we discuss in §5 along with a couple of consequences of previous work which bear on them.

We also want to remark here on an unusual aspect of the construction for Theorem 1.11. While in several ways, it is quite similar to the usual proof of the Sacks jump theorem, satisfying the requirements to make $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$ seems to require an unusual organization of the priority tree (certainly unusual for $0''$ constructions such as the Sacks jump theorem). What would seem to be individual requirements for this goal (the $\mathcal{R}_{e,i}$ below) are divided up into infinitely many subrequirements (the $\mathcal{R}_{e,i,n}$ below). These subrequirements are spread across the tree (rather than, for example, along the paths below a node assigned to $\mathcal{R}_{e,i}$ as is often the case). In combination with other requirements, the subrequirements (to the right of the true path) can interfere with each other with the possible outcome of subverting the final satisfaction of the basic requirement (even along the true path). Our solution is to change the priority of these subrequirements in a dynamic way that depends on the actions of nodes to their left. In particular, nodes on the priority tree are assigned different requirements at different stages of the construction. While this is common in $0'''$ constructions it is unusual in $0''$ ones (as ours is). Moreover, the nodes are assigned requirements in a way that does not depend solely on the outcomes along the path leading to the node (and possibly external approximation procedures as well). This seems unusual (if not unique) even for $0'''$ arguments.

The first example of which we are aware of changes in the priority of requirements in a somewhat similar way occurs in some cases of the minimal pair construction in α-recursion theory (Shore [1978]). Another relatively early result that has variable priority assignments is Theorem 2 of Jockusch and Soare [1991]. They construct a low linear order with a predicate for infinitely far apart not isomorphic to a recursive one. An unusual construction with a requirement analogous to our $\mathcal{R}_{e,i}$ (to make various sets have r.e. degree) occurs in the $0'''$ construction for Theorem 3.1 of Arslanov, Lempp and Shore [1996]. (We use this result below in Proposition 5.5.) The unusual procedure employed there is allowing nodes to act when they are to the left of the true path. A similar idea would probably work here as well but our construction while unusual in a different way seems simpler in this case.

We discuss some consequences of this unusual construction for a reverse mathematical analysis of our theorem in §5.

2 From the Conjecture to Rigidity

In this section we give a proof based on Conjecture 1.7 of the rigidity of $\mathbf{R}$. If the Conjecture held then by Lemma 1.5 we would know that, for $\mathbf{a}, \mathbf{b} \in \mathbf{H}_2$, $\mathbf{a} = \mathbf{b} \Leftrightarrow \mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$: If $\mathbf{a} \neq \mathbf{b}$, then one would be not below the other and the Conjecture would supply a $\mathbf{c}$ in one of $\mathsf{JB}(\mathbf{a})$ or $\mathsf{JB}(\mathbf{b})$ but not the other.

Now fix any automorphism Φ of $\mathbf{R}$. By Nies, Shore and Slaman [1998], $\mathbf{H}_2$ is definable in $\mathbf{R}$ and so if $\mathbf{a} \in \mathbf{H}_2$ then $\mathbf{\Phi}(\mathbf{a}) \in \mathbf{H}_2$. Now $\mathbf{H}_2$ is an automorphism basis for $\mathbf{R}$. (Indeed, by Lerman [1977] every jump class is one but for $\mathbf{H}_2$ it follows easily from Theorem 1.2: If $\Phi(\mathbf{x}) = \mathbf{y} \neq \mathbf{x}$ then by Theorem 1.2 there is a $\mathbf{z} \in \mathbf{H}_2$ such that $\mathbf{z}$ is above one of $\mathbf{x}$ and $\mathbf{y}$ but not the other for the desired contradiction.) So to establish rigidity (based on our Conjecture) it would suffice to prove that $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\Phi(\mathbf{a}))$ for any $\mathbf{a} \in \mathbf{H}_2$. Assuming the Conjecture, we in fact show that JB is invariant under Φ, i.e. $\mathsf{JB}(\mathbf{x}) = \mathsf{JB}(\Phi(\mathbf{x}))$ for any $\mathbf{x} \in \mathbf{R}$.

We begin with the double jump version of JB: $\mathsf{DJB}(\mathbf{c}) = \{\mathbf{x}'' : \mathbf{x} \leq \mathbf{c}\}$. By using Lemma 1.5 both as stated and relativized, we see that for r.e. $\mathbf{c}$, $\mathsf{DJB}(\mathbf{c}) = \{\mathbf{x}'' : \mathbf{x} \leq \mathbf{c}\ \&\ \mathbf{x}$ is r.e.$\}$.

Claim 2.1. *If* $\mathbf{x}, \mathbf{y} \in \mathbf{H}_2$ *and* $\mathsf{DJB}(\mathbf{x}) = \mathsf{DJB}(\mathbf{y})$*, then* $\mathbf{x}' = \mathbf{y}'$*. Moreover, if* $\mathbf{y} = \Phi(\mathbf{x})$ *and* $\mathbf{x} \in \mathbf{H}_2$ *then then* $\mathbf{x}' = \mathbf{y}'$*.*

Proof. Suppose we have $\mathbf{x}$ and $\mathbf{y}$ as in the hypotheses of the Claim but $\mathbf{x}' \neq \mathbf{y}'$. Without loss of generality we may assume that $\mathbf{x}' \not\leq \mathbf{y}'$. As $\mathbf{x}', \mathbf{y}'$ are high (and so high$_2$) relative to $\mathbf{0}'$, we can apply the Conjecture relativized to $\mathbf{0}'$ to the degrees $\mathbf{x}' \not\leq \mathbf{y}'$. This gives us a $\mathbf{z}$ $REA(\mathbf{0}'')$ which is $REA(\mathbf{x}')$ but not $REA(\mathbf{y}')$. Next apply Theorem 1.2 relative to $\mathbf{x}$ with $\mathbf{0}'$ playing the role of $\mathbf{c}$ to get $\mathbf{f}$ $REA(\mathbf{x})$, $\mathbf{f} > \mathbf{0}'$ and $\mathbf{f}' = \mathbf{z}$. Finally, apply Theorem 1.2 with $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{f}$ playing the roles of $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{z}$, respectively, to get an r.e. $\mathbf{g} < \mathbf{x}$ with $\mathbf{g}' = \mathbf{f}$. Thus $\mathbf{g}'' = \mathbf{z}$ and so $\mathbf{z} \in \mathsf{DJB}(\mathbf{x})$. On the other hand, since $\mathbf{z}$ is not $REA(\mathbf{y}')$, $\mathbf{z} \notin \mathsf{DJB}(\mathbf{y})$ for the desired contradiction.

For the second part of the Claim, we note that by Nies, Shore and Slaman [1998], not only is $\mathbf{y} \in \mathbf{H}_2$ definable in $\mathbf{R}$ but each of the double jump classes (i.e. the sets $\{\mathbf{c} \in \mathbf{R} | \mathbf{c}'' = \mathbf{d}\}$ for any $\mathbf{d}$ $REA(\mathbf{0}')$) are definable in $\mathbf{R}$. Thus $\mathsf{DJB}(\mathbf{c})$ is invariant under Φ and so $\mathsf{DJB}(\mathbf{x}) = \mathsf{DJB}(\mathbf{y})$ and we are done by the first part of the Claim. □

We next wish to prove that the jump is invariant, i.e. for any $\mathbf{x}, \mathbf{y}$ with $\Phi(\mathbf{x}) = \mathbf{y}$, $\mathbf{x}' = \mathbf{y}'$. We consider another operator on $\mathbf{R}$: $\mathsf{JA}(\mathbf{x}) = \{\mathbf{c}' | \mathbf{c} \geq \mathbf{x}\ \&\ \mathbf{c} \in \mathbf{H}_2\}$.

Claim 2.2. *For any* $\mathbf{x}, \mathbf{y} \in \mathbf{R}$*, if* $\mathsf{JA}(\mathbf{x}) = \mathsf{JA}(\mathbf{y})$*, then* $\mathbf{x}' = \mathbf{y}'$*. Moreover, for any* $\mathbf{x} \in \mathbf{R}$ *and* $\mathbf{y} = \Phi(\mathbf{x})$*,* $\mathbf{x}' = \mathbf{y}'$*.*

Proof. If the first assertion fails, assume, without loss of generality, that $\mathbf{y}' \not\leq \mathbf{x}'$. By Theorem 1.2 relative to $\mathbf{0}'$ with $\mathbf{x}'$, $\mathbf{0}''$, $\mathbf{y}'$, and $\mathbf{0}'''$ playing the roles of $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{z}$, respectively, we get an $\mathbf{f} > \mathbf{x}'$ with $\mathbf{f}' = \mathbf{0}'''$ which is $REA(\mathbf{0}')$ but not above $\mathbf{y}'$. By Theorem 1.2 again we have an r.e. $\mathbf{c} > \mathbf{x}$ with $\mathbf{c}' = \mathbf{f}$ and so $\mathbf{c} \in \mathbf{H}_2$. Thus $\mathbf{c}' = \mathbf{f} \in \mathsf{JA}(\mathbf{x})$. On the other hand, for every $\mathbf{v} \in \mathsf{JA}(\mathbf{y})$, $\mathbf{v} \geq \mathbf{y}'$ but $\mathbf{f}$ is not above $\mathbf{y}'$ for the desired contradiction.

For the second part of the Claim, note that by the previous Claim, $\mathsf{JA}(\mathbf{x})$ is invariant under Φ: If $\mathbf{z} \in \mathsf{JA}(\mathbf{x})$ then $\mathbf{z} = \mathbf{c}'$ for some $\mathbf{c} \in \mathbf{H}_2$ with $\mathbf{c} \geq \mathbf{x}$. By the previous Claim, $\mathbf{c}' = \Phi(\mathbf{c})'$ (and so, in particular $\Phi(\mathbf{c}) \in \mathbf{H}_2$). As $\Phi(\mathbf{c}) \geq \Phi(\mathbf{x})$, $\mathbf{z} = \mathbf{c}' = \Phi(\mathbf{c})' \in \mathsf{JA}(\Phi(\mathbf{x}))$. Similarly, if $\mathbf{z} \in \mathsf{JA}(\Phi(\mathbf{x}))$ then $\mathbf{z} = \mathbf{c}'$ for some $\mathbf{c} \in \mathbf{H}_2$ with $\mathbf{c} \geq \Phi(\mathbf{x})$. There is then a $\mathbf{y}$ with $\Phi(\mathbf{y}) = \mathbf{c}$ and so $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \in \mathbf{H}_2$. Again, by the previous Claim, $\mathbf{y}' = \Phi(\mathbf{y})' = \mathbf{c}' = \mathbf{z}$ and so $\mathbf{z} \in \mathsf{JA}(\mathbf{x})$ as desired. $\square$

We can now complete our proof of the rigidity of $\mathbf{R}$ from the conjecture by noting that by this last Claim and the definition of JB, JB is invariant under Φ: If $\mathbf{c} \leq \mathbf{x}$ then $\Phi(\mathbf{c}) \leq \Phi(\mathbf{x})$ and by the last Claim $\mathbf{c}' = \Phi(\mathbf{c})'$. Thus $\mathsf{JB}(\mathbf{x}) \subseteq \mathsf{JB}(\Phi(\mathbf{x}))$. Similarly, if $\mathbf{d} \leq \Phi(\mathbf{x})$ then $\mathbf{d} = \Phi(\mathbf{c})$ for some $\mathbf{c} \leq \mathbf{x}$ and so $\mathbf{c}' = \mathbf{d}'$ and $\mathsf{JB}(\Phi(\mathbf{x})) \subseteq \mathsf{JB}(\mathbf{x})$ giving rigidity as in the second paragraph of this section.

3 Proving part 1 of the Main Theorem

Suppose $\mathbf{c}$ is $REA(\mathbf{0}')$ and choose a representative $C \in \mathbf{c}$. We may fix an r.e. set D such that, for all n, the n-th column $D^{[n]} = \{x : \langle n, x \rangle \in D\}$ is an initial segment of ω, finite if $n \in C$, and equal to ω if $n \notin C$. We assume that no element s enters D before stage s. We will use D to build a pair A_0, A_1 of sets such that $\mathbf{a}_0 = \deg(A_0)$ and $\mathbf{a}_1 = \deg(A_1)$ are as required by the theorem. Our argument closely follows the usual pattern of a $0''$ tree construction, with the peculiar feature that the assignment of requirements to nodes is allowed to vary from stage to stage.

Recall from the discussion following Lemma 1.5 that $\mathsf{JB}(\mathbf{a}) = \{\deg(W_e^A \oplus 0') : e \in \omega\}$, where W_e^A is as in Definition 1.4. It is this characterization that we use to frame our requirements.

3.1 Requirements

We begin by listing four basic goals for the construction.

- $A_0 \not\leq_T A_1$ and $A_1 \not\leq_T A_0$

- $C \leq_T A_0', A_1'$
- $A_0', A_1' \leq_T C$
- For every $e \in \omega$ and $i \in \{0,1\}$, there is an r.e. set $V_{e,i}$ such that $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$.

The first three goals are self-explanatory, while the fourth guarantees through Lemma 1.5 that $\mathsf{JB}(\mathbf{a}_0) = \mathsf{JB}(\mathbf{a}_1)$. Hence a construction of any $A_i, V_{e,i}$, $e \in \omega$, $i \in \{0,1\}$ meeting these goals will constitute a proof of the Main Theorem, part 1. We represent the first, second and fourth goals as requirements named with the letters $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{R}$, respectively. The third we do not capture directly as a requirement, although in the end it is satisfied by the $\mathcal{R}$ strategy as well (Proposition 3.8 below).

The $\mathcal{N}$ and $\mathcal{P}$ requirements are as in the usual proofs of the Friedberg-Muchnik and Sacks jump theorems, respectively. Namely, we ensure that $A_i \not\leq_T A_{1-i}$ by using infinitely many diagonalization requirements:

$\mathcal{N}_{e,i} : A_i \neq \Phi_e^{A_{1-i}}$, for all $e \in \omega$, $i \in \{0,1\}$.

We ensure that $C \leq_T A_0', A_1'$ by using infinitely many thickness requirements:

$\mathcal{P}_e : A_0^{[e]} =^* D^{[e]}$ and $A_1^{[e]} =^* D^{[e]}$. (Here $X =^* Y$ means that X, Y differ only by a finite set.)

We attack the fourth goal by breaking it up, for each $e \in \omega$ and each $i \in \{0,1\}$, into infinitely many requirements $\mathcal{R}_{e,i,n}$, with n ranging over ω. The idea is to construct a single r.e. set $V_{e,i}$ so that $W_e^{A_{1-i}}$ contains a number n if and only if $V_{e,i}^{A_i}$ contains $\langle n, m\rangle$ for some $m = m_{e,i,n}$ which can be computed by $0'$ (so that $W_e^{A_{1-i}} \leq_T V_{e,i}^{A_i} \oplus 0'$), while keeping the question of whether $\langle n, k\rangle$ is in $V_{e,i}^{A_i}$ easy to answer (given m) for all $k \neq m$ (so that $V_{e,i}^{A_i} \leq_T W_e^{A_{1-i}} \oplus 0'$). The formal requirements, and a lemma showing that they suffice to guarantee that $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$, are as follows:

$\mathcal{R}_{e,i,n}$: There is an r.e. set $V_{e,i}$, which does not depend on n, and a number $m_{e,i,n}$ computable uniformly in n from $0'$ such that $n \in W_e^{A_{1-i}} \Leftrightarrow \langle n, m_{e,i,n}\rangle \in V_{e,i}^{A_i}$. Furthermore, $k < m_{e,i,n}$ implies $\langle n, k\rangle \in V_{e,i}^{A_i}$, and $k > m_{e,i,n}$ implies $\langle n, k\rangle \notin V_{e,i}^{A_i}$.

Lemma 3.1. *Fix e and i. If $\mathcal{R}_{e,i,n}$ is met for all n, then $W_e^{A_{1-i}} \oplus 0' \equiv_T V_{e,i}^{A_i} \oplus 0'$.*

Proof. To compute whether n is in $W_e^{A_{1-i}}$, first use $0'$ to find $m_{e,i,n}$, and then check whether $\langle n, m_{e,i,n}\rangle$ is in $V_{e,i}^{A_i}$. This shows that $W_e^{A_{1-i}} \leq_T V_{e,i}^{A_i} \oplus 0'$. To compute whether $\langle n, k\rangle$ is in $V_e^{A_i}$, first use $0'$ to find $m_{e,i,n}$, and compare it with k: if $k < m_{e,i,n}$ then the answer is yes, if $k > m_{e,i,n}$ then the answer is no, and if $k = m_{e,i,n}$ then it is enough to check whether n is in $W_e^{A_{1-i}}$. Hence $V_{e,i}^{A_i} \leq_T W_e^{A_{1-i}} \oplus 0'$. $\square$

3.2 Notation and bookkeeping.

The tree of nodes. We assign requirements to nodes on a tree as in a typical $\mathbf{0}''$ priority argument, except that the assignment of requirements to nodes is allowed to change from stage to stage. The tree itself is $\{\mathtt{0}, \mathtt{1}\}^{<\omega}$, ordered lexicographically with $\mathtt{0} <_{\mathrm{lex}} \mathtt{1}$ as usual. The $\mathcal{P}$ requirements have two possible outcomes, $\mathtt{0}$ and $\mathtt{1}$, representing an infinitary and a finitary action, respectively. The $\mathcal{R}$ and $\mathcal{N}$ requirements have only one outcome, $\mathtt{0}$.

The accessible path δ_s. At each stage s, we specify a node $\delta_s \in \{\mathtt{0}, \mathtt{1}\}^s$. (The precise construction of δ_s is presented in §3.3 below.) We say that δ_s and each of its initial segments $\alpha \subseteq \delta_s$ are *accessible at stage s*. No other node is accessible at stage s.

Restraints $r_{\alpha,s}$ and $r_{<\alpha,s}$. At each stage s, each node $\alpha \in \{\mathtt{0}, \mathtt{1}\}^{<\omega}$ places a *restraint* $r_{\alpha,s}$ limiting the possible actions of nodes that are lexicographically greater than α. For each α, the initial value is $r_{\alpha,0} = 0$. If α is not accessible, then $r_{\alpha,s} = r_{\alpha,s-1}$. Otherwise, $r_{\alpha,s}$ is as specified in §3.3 below. For each α and s, we use $r_{<\alpha,s}$ to denote $\max\{r_{\beta,s} : \beta <_{\mathrm{lex}} \alpha\}$. Notice that at every stage s cofinitely many $r_{\alpha,s}$ are equal to zero, and every $r_{<\alpha,s}$ is finite.

Assigning requirements to nodes. The $\mathcal{R}$ requirements are sensitive to injury because a single set $V_{e,i}$ is shared across all nodes assigned an $\mathcal{R}_{e,i,n}$ requirement, and distinct $\mathcal{R}_{e,i,n}$ with the same e and i may each be adding elements to $V_{e,i}$. These elements cannot subsequently be removed, as $V_{e,i}$ is r.e. Thus a version of an $\mathcal{R}_{e,i,n}$ requirement can act far to the right of the true path (which is defined as usual in §3.4) and we may later have to react to some injury from a node to its left (but still to the right of the true path) by making the only correction we can, i.e. changing the value of $m_{e,i,n}$. Allowing this to happen infinitely often will send $m_{e,i,n}$ to infinity and ruin our coding procedure for computing $W_e^{A_{1-i}}(n)$. One appropriate response is to increase the priority of the $\mathcal{R}_{e,i,n}$ requirements, relative to those that injured them, each time this happens; the countervailing constraint is the obvious one that we must eventually deal with all the requirements (on the true path) and so cannot increase the priority of all the $\mathcal{R}_{e,i,n}$ arbitrarily. The solution is to increase the priority of the $\mathcal{R}_{e,i,n}$ requirements in a controlled way that allows other requirements to act as well along the true path. We do this by assigning the requirements dynamically, i.e. by a scheme that depends on the stage s. We could define an assignment simultaneously with the full construction that depends directly on the nodes accessible at s and the actions taken (injuries sustained) at stages less than or equal to s. While that might produce a more intuitive definition (given that one already understood the construction), we instead give a simple (if uninformative) definition that is independent of the construction's details and that uses a counting argument to allow the priority of an $\mathcal{R}_{e,i,n}$ requirement to increase while still leaving room for the other requirements on the true path. This makes both the assignment of requirements to nodes on the tree and the eventual verifications significantly simpler.

The precise assignment scheme is as follows. First, fix some recursive list of all the $\mathcal{R}$ requirements and a second recursive list of all the $\mathcal{P}$ and $\mathcal{N}$ requirements. At stage s, we assign a requirement to each node $\alpha \in \{\mathtt{0}, \mathtt{1}\}^{<\omega}$ by recursion on its initial segments. Let u be the number of proper initial segments of α assigned an $\mathcal{R}$ requirement at stage s, and let v be the number of nodes $\beta \leq_{\text{lex}} \alpha$ which have been accessible at any stage $t \leq s$. If $u < v/2$, then assign to α the $(u+1)$-th $\mathcal{R}$ requirement; otherwise, assign to α the next unused (i.e. the $(|\alpha| - u + 1)$-th) requirement from the $\mathcal{P}, \mathcal{N}$ list.

Conventions for V^A, *use*, and $\langle \cdot, \cdot \rangle$. The use of a convergent computation $\Phi_e^A(x)$ or $\Phi_{e,s}^A(x)$ is the least $u \leq s$ such that $\Phi_{e,u}^{A \restriction u}(x) \downarrow$. If V^A is as in Definition 1.4, we identify V^A with its characteristic function as usual; if $V^A(n) = 1$, the *use* of $V^A(n)$ is the shortest $\sigma \subseteq A$ such that $\langle n, \sigma \rangle \in V$. We do not define a use for $V^A(n) = 0$. We follow the convention that $\Phi_{e,s}(x) \downarrow$ only if $x < s$, and a V_s in an r.e. approximation $(V_s)_s$ may contain n only if $n < s$. The pairing function $\langle x, y \rangle$ is recursive and increasing in each coordinate. Each binary string σ is naturally identified with a natural number through its binary expansion; this number grows monotonically with the length-lexicographic ordering. The pairing function is left-associative, so we may write $\langle n, m, \sigma \rangle$ for $\langle \langle n, m \rangle, \sigma \rangle$.

α-believable computations. Fix any $\alpha \in \{\mathtt{0}, \mathtt{1}\}^{<\omega}$, and suppose $\Phi_{e,s}^{A_{1-i,s}}(x) \downarrow$ with use u. We say this computation is an *α-believable computation at stage s* if for every $\mathcal{P}_j$ which is assigned to an initial segment $\beta \subseteq \alpha$ at stage s with outcome $\alpha(|\beta|) = \mathtt{0}$, we have

$$\{k \in A_{1-i,s}^{[j]} : r_{<\alpha,s} \leq \langle j, k \rangle \leq u - 1\} = \{k : r_{<\alpha,s} \leq \langle j, k \rangle \leq u - 1\}$$

where $[x, y]$ denotes a closed interval in ω.

Now fix $\alpha \in \{\mathtt{0}, \mathtt{1}\}^{<\omega}$ and suppose $n \in W_{e,s}^{A_{1-i,s}}$ by $\langle n, \sigma \rangle \in W_{e,s}$ with $\sigma \subseteq A_{1-i,s}$. We call this enumeration an *α-believable computation at stage s* if, for all j as above,

$$\{k \in A_{1-i,s}^{[j]} : r_{<\alpha,s} \leq \langle j, k \rangle \leq |\sigma| - 1\} = \{k : r_{<\alpha,s} \leq \langle j, k \rangle \leq |\sigma| - 1\}$$

3.3 The basic strategies and outcomes; defining δ_s.

Suppose $k < s$ is fixed, $\alpha = \delta_s \restriction k$, and α is assigned the requirement $\mathcal{Q}$ at stage s. Our strategy for α determines any changes made by α to A_0, A_1, $V_{e,0}$, $V_{e,1}$, or $m_{e,i,n}$ at stage s, the restraint $r_{\alpha,s}$, and the outcome $\delta_s(k)$, and with it, if $k < s - 1$, the next accessible node $\alpha^\frown \delta_s(k)$. We say that a node α acts at stage s if and only if its strategy changes one of A_0, A_1, $V_{e,0}$, $V_{e,1}$, $m_{e,i,n}$, or $r_{\alpha,s} \neq r_{\alpha,s-1}$ at stage s. If the construction does not explicitly change one of these sets or variables at stage s, then it takes the same value as at stage $s - 1$. Here are the strategies:

If $\mathcal{Q} = \mathcal{P}_e$: If this is the first time α has been accessible or if no new element has entered $D_s^{[e]}$ since last time α was accessible, do nothing; the outcome is the finitary outcome $\mathtt{1}$. Otherwise, add to $A_0^{[e]}$ and $A_1^{[e]}$ all k such that $r_{<\alpha,s} \leq \langle e, k \rangle$ and $k \leq s$. In this case, the outcome is $\mathtt{0}$. [The intention is, as usual, that $A_i^{[e]}$ will be finite if $D^{[e]}$ is finite, and cofinite if $D^{[e]}$ is ω. We add whole intervals at once to make it easier to determine when, and in what way, this action injures lower-priority requirements.]

If $\mathcal{Q} = \mathcal{N}_{e,i}$: Check whether there is an x in the interval $r_{<\alpha,s} < x < s$ such that

i. $x \neq \langle j, k\rangle$ for all $j < |\alpha|$ and all k; and

ii. $\Phi_{e,s}^{A_{1-i,s}}(x) \downarrow= y$ by an α-believable computation, where either $y = 0$, or $y \neq A_{i,s}(x)$.

If there is no such x, do nothing. Otherwise, take the least x which minimizes the use of the convergent computation $\Phi_{e,s}^{A_{1-i,s}}(x)$, and consider the value of y from condition (ii). If $y = 0$ and x is not in $A_{i,s}$, add x to A_i; otherwise, do not change A_i. In either case, set the restraint $r_{\alpha,s}$ to equal the use of the computation $\Phi_{e,s}^{A_{1-i,s}}(x)$. [The minimization is to guarantee that after the requirement has been "permanently" satisfied, it will not act again. Condition (i) helps ensure that the $\mathcal{N}$ strategy doesn't interfere too often with a $\mathcal{P}_j$ requirement.]

If $\mathcal{Q} = \mathcal{R}_{e,i,n}$: Let $m = m_{e,i,n,s-1}$, or $m = 0$ if $s = 0$. Check whether $n \in W_{e,s}^{A_{1-i,s}}$ by an α-believable computation.

Case 1: n is not in $W_{e,s}^{A_{1-i,s}}$ by an α-believable computation. Check whether $\langle n, m\rangle \in V_{e,i,s}^{A_{i,s}}$. If not, let $m_{e,i,n,s} = m$. Otherwise, let $m_{e,i,n,s} = m + 1$ and add all $\{\langle n, m, \sigma\rangle : \sigma \in 2^{<\omega}\}$ to $V_{e,i}$. [This is to meet the part of the requirement involving $k < m_{e,i,n}$.]

Case 2: n is in $W_{e,s}^{A_{1-i,s}}$ by an α-believable computation. In this case, we leave $m_{e,i,n,s} = m$. Check whether $\langle n, m\rangle$ is already in $V_{e,i,s}^{A_{i,s}}$. If so, let $r_{\alpha,s}$ be either the use of $V_{e,i,s}^{A_{i,s}}(\langle n, m\rangle)$ or the use of $W_{e,s}^{A_{1-i,s}}(n)$, whichever is larger. If, on the other hand, $\langle n, m\rangle \notin V_{e,i,s}^{A_{i,s}}$, let σ be the shortest initial segment of $A_{i,s}$ satisfying:

i. $r_{<\alpha,s} < |\sigma|$;

ii. $|\sigma|$ is greater than the use of $W_{e,s}^{A_{1-i,s}}$; and

iii. for each proper initial segment $\beta \subsetneq \alpha$ assigned a requirement $\mathcal{P}_j$ with outcome $\alpha(|\beta|) = \mathtt{1}$, there exists an x of the form $x = \langle j, k\rangle$ for some k such that $r_{\beta,s} < x < |\sigma|$ and $\sigma(x) = 0$.

Add $\langle n, m, \sigma\rangle$ to $V_{e,i}$, and set $r_{\alpha,s}$ to equal $|\sigma|$. [The intuition behind condition (iii) is that the existence of such an x with $\sigma(x) = 0$ protects against the possibility that α's belief about the outcome of β – that $D^{[j]}$ is finite – might be wrong. If it is wrong, the computation based on σ will automatically be injured by the action of β or some other node assigned $\mathcal{P}_j$, and $\langle n, m\rangle$ will be removed automatically from $V_{e,i,s}^{A_{i,s}}$.]

3.4 Verification.

Define the *true path* $\mathtt{tr} \in \{\mathtt{0}, \mathtt{1}\}^\omega$ as the leftmost path which is visited infinitely often. That is, for all n, the initial segment $\mathtt{tr} \restriction n$ is the $\leq_{\text{lex}}$-least node of length n that is accessible infinitely often. We call $\mathtt{tr}(n)$ the *true outcome* of $\mathtt{tr} \restriction n$. A node α is *on the true path* if α is an initial segment of $\mathtt{tr}$. If a node α is assigned a particular requirement $\mathcal{Q}$ at all but finitely many stages, we say that α is *eventually assigned* $\mathcal{Q}$ and write $\mathsf{ev}(\alpha) = \mathcal{Q}$. If there is no such $\mathcal{Q}$, we leave $\mathsf{ev}(\alpha)$ undefined. We begin with two straightforward lemmas. The first of these is, in fact, independent of the construction in §3.3.

Lemma 3.2. *If α is a node and $\alpha \leq_{\text{lex}} \mathsf{tr}$, then $\mathsf{ev}(\alpha)$ is defined.*

Proof. By induction on the length of α. Choose a stage s_0 large enough that for each $s \geq s_0$ we have $\alpha \leq_{\text{lex}} \delta_s$, and each strict initial segment of α is assigned the same requirement at stage s as at stage s_0. The assignment scheme in §3.2 gives the same requirement to α at each stage $s \geq s_0$. □

The second lemma relies on the construction in §3.3 only in that an $\mathcal{R}$ requirement always has $\mathsf{0}$ as its outcome.

Lemma 3.3. *The function $n \mapsto \mathsf{ev}(\mathsf{tr}\upharpoonright n)$ is a bijection between ω and the set $\{\mathcal{P}_e, \mathcal{N}_{e,i}, \mathcal{R}_{e,i,n} : e, n \in \omega,\ i < 2\}$ of all requirements.*

Proof. For each n, let u_n be the number of $\ell < n$ for which $\mathsf{ev}(\mathsf{tr} \upharpoonright \ell)$ is an $\mathcal{R}$ requirement, and let v_n be the total number of nodes $\beta \leq_{\text{lex}} \mathsf{tr}\upharpoonright n$ that are ever accessible (which is finite by the definition of tr). From the assignment scheme in §3.2 we know that $\mathsf{ev}(\mathsf{tr}\upharpoonright n)$ is an $\mathcal{R}$ requirement if $u_n < v_n/2$, and a $\mathcal{P}$ or $\mathcal{N}$ requirement if $u_n \geq v_n/2$.

It suffices to check that $\mathsf{ev}(\mathsf{tr}\upharpoonright n)$ is an $\mathcal{R}$ requirement infinitely often, and a $\mathcal{P}$ or $\mathcal{N}$ requirement infinitely often. A few observations: (i) if $\mathsf{ev}(\mathsf{tr}\upharpoonright n)$ is a $\mathcal{P}$ or $\mathcal{N}$ requirement, then $u_{n+1} = u_n$ and $v_{n+1} > v_n$; and (ii) if $\mathsf{ev}(\mathsf{tr}\upharpoonright n)$ is an $\mathcal{R}$ requirement, then $u_{n+1} = 1+u_n$ and $v_{n+1} = 1 + v_n$ (since the outcome must be $\mathsf{0}$, and $\beta <_{lex} \alpha^\frown 0$ implies either $\beta <_{lex} \alpha$ or $\beta = \alpha$). If cofinitely many $\mathsf{ev}(\mathsf{tr} \upharpoonright n)$ were $\mathcal{P}$ or $\mathcal{N}$ requirements, then by definition $u_n \geq v_n/2$ cofinitely often, eventually contradicting (i); while if cofinitely many $\mathsf{ev}(\mathsf{tr}\upharpoonright n)$ were $\mathcal{R}$ requirements, then by definition $u_n < v_n/2$ for cofinitely many n, eventually contradicting (ii). This completes the proof. □

Now we check that each requirement is met. We do this in two steps: first, in Propositions 3.4, we argue that nodes along the true path act infinitely often if and only if they are eventually assigned a $\mathcal{P}$ requirement with the infinitary $\mathsf{0}$ as their true outcome; and then in Proposition 3.6 we argue that these nodes' actions satisfy their respective requirements. For $\mathcal{P}$ and $\mathcal{N}$ requirements, the verification similar to the usual proof of the Sacks jump inversion; the method for $\mathcal{R}$ is new but straightforward. The remainder of this section makes full use of the construction in §3.3.

Proposition 3.4. *If $\alpha \leq_{\text{lex}} \mathsf{tr}$, then α acts infinitely often if and only if $\alpha \subseteq \mathsf{tr}$, $\mathsf{ev}(\alpha)$ is a $\mathcal{P}$ requirement, and its true outcome $\mathsf{tr}(|\alpha|)$ is $\mathsf{0}$.*

Proof. Since $\{\alpha : \alpha \leq_{\text{lex}} \mathsf{tr}\}$ is well-ordered by $\leq_{\text{lex}}$, we may work by induction on $\leq_{\text{lex}}$. Fix α. If α is strictly to the left of the true path, then the result is immediate, so assume that $\alpha = \mathsf{tr}\upharpoonright n$ for some n. Fix s_0 such that α is accessible at stage s_0, and large enough that every $\beta <_{\text{lex}} \alpha$ meets the inductive hypothesis before stage s_0 and $\alpha <_{\text{lex}} \delta_s$ for all $s \geq s_0$. In particular, the restraint $r_{<\alpha,s}$ is constant at stages $s \geq s_0$, and α is assigned the same requirement $\mathcal{Q} = \mathsf{ev}(\alpha)$ at all stages $s \geq s_0$. Consider the possible values of $\mathcal{Q}$.

Case 1: $\mathcal{Q} = \mathcal{P}_e$. By the definition of an action for a $\mathcal{P}$ requirement, α acts infinitely often if and only if it has the infinitary outcome $\mathsf{0}$ infinitely often, which happens if and only if its true outcome is $\mathsf{0}$.

Case 2: $\mathcal{Q} = \mathcal{N}_{e,i}$. If α does not act after stage s_0, there is nothing to prove; so let $s \geq s_0$ be least such that α acts, using the restraint $r_{\alpha,s}$ to preserve an α-believable computation $\Phi_{e,s}^{A_{1-i,s}}(x) \downarrow= y$ with $y \neq A_{i,s}(x)$. Because the computation is α-believable, and because, by choice of s_0, the only higher-priority nodes acting after stage s are initial segments $\beta \subseteq \alpha$ assigned a $\mathcal{P}_j$ requirement with true outcome $\mathsf{0}$, the computation $\Phi_{e,t}^{A_{1-i,t}}(x) = y$ continues to be α-believable as long as $r_{\alpha,t}$ does not decrease. Furthermore, since the $\mathcal{N}$-action of α stipulates as point (i) x is not of the form $\langle j, k\rangle$ for any such $\mathcal{P}_j$, the disagreement $y \neq A_{i,s}(x)$ is also preserved. Although α may act again after stage s to preserve some other computation with lesser use or lesser x, this happens at most finitely often, as x and the use are chosen to be minimal. Therefore α acts at most finitely many times.

Case 3: $\mathcal{Q} = \mathcal{R}_{e,i,n}$. We claim that $m_{e,i,n,s}$ is constant for $s > s_0$. Suppose for a contradiction that $s > s_0 + 1$ is the least stage at which some node assigned $\mathcal{R}_{e,i,n}$ acts by setting $m_{e,i,n,s} = m_{e,i,n,s-1} + 1$. This action is in response to $\langle n, m_{e,i,n,s-1}\rangle$ being in $V_{e,i,s}^{A_{i,s}}$ but n not being in $W_{e,s}^{A_{1-i,s}}$. Let β be the node that had placed the element in $V_{e,i}$ in response to a β-believable computation, and let γ be the node that had injured this computation by placing an element into A_{1-i} below the use. Then $\gamma \leq_{lex} \beta$, i.e. γ has higher priority, and γ acted after stage s_0, since otherwise α would already have dealt with this disagreement, or set up a restraint to prevent it, at stage s_0. Furthermore, γ does not extend α, or again α would have set up a restraint to prevent its action. Hence by choice of s_0, γ and β are strictly to the right of α. Since all initial segments of β which are not initial segments of α are assigned an $\mathcal{R}$ requirement (this is clear from the assignment scheme), and γ does not have an $\mathcal{R}$ (as it changes A_{1-i}), γ is strictly to the left of β, that is, they have a common initial segment δ with $\gamma(|\delta|) = \mathsf{0}$ and $\beta(|\delta|) = \mathsf{1}$. But then at the stage at which γ was accessible, δ was assigned a $\mathcal{P}$ requirement which acted by adding elements to A_i below the use of β's coding, and so (by condition (iii) in the $\mathcal{R}$-action of β) γ itself removed $\langle n, m_{e,i,n,s-1}\rangle$ from $V_{e,i}^{A_i}$ before γ had a chance to act. This is the desired contradiction. □

We are ready to begin checking that requirements are satisfied. We begin with the $\mathcal{P}$ requirements, as they will be useful in checking the others.

Lemma 3.5. *Every $\mathcal{P}$ requirement is satisfied.*

Proof. Fix a requirement $\mathcal{P}_e$, and let $\alpha \subseteq \mathsf{tr}$ be such that $\mathsf{ev}(\alpha) = \mathcal{P}_e$. Let s_0 be as in the proof of Proposition 3.4 and let $r = r_{<\alpha,s_0}$. If $D^{[e]} = \omega$ then there are infinitely many stages s at which new elements enter $D_s^{[e]}$, and so there are infinitely many stages at which α acts by adding elements to $A_0^{[e]}$ and $A_1^{[e]}$. In the limit, $A_0^{[e]}$ and $A_1^{[e]}$ contain all $k \geq r$, and so $A_0^{[e]} =^* D^{[e]} =^* A_1^{[e]}$, as required.

If, on the other hand, $D^{[e]}$ is finite, it is easy to see that after some stage s no node assigned $\mathcal{P}_e$ ever again adds elements to $A_i^{[e]}$. We claim in addition that there is a stage s after which no node assigned an $\mathcal{N}$ requirement adds elements to $A_i^{[e]}$. By condition (i) of the $\mathcal{N}$ strategy, we need only consider nodes β of length $\leq e$. If β is strictly left of the true path, it eventually stops acting by definition of the true path; if β is strictly to the right of the true path, then eventually it is assigned an $\mathcal{R}$ requirement instead of an $\mathcal{N}$ requirement; and if β is on the true path, then by the previous Proposition, β either stops acting or is assigned a $\mathcal{P}$ requirement. $\square$

Proposition 3.6. *Every requirement is satisfied.*

Proof. We have already dealt with the $\mathcal{P}$ requirements in the previous Lemma. Fix a requirement $\mathcal{Q}$ of the form $\mathcal{N}_{e,i}$ or $\mathcal{R}_{e,i,n}$, and let $\alpha \subseteq \mathsf{tr}$ such that $\mathsf{ev}(\alpha) = \mathcal{Q}$. Let s_0 be as in the proof of Proposition 3.4, and let $r = r_{<\alpha,s_0}$. Assume by induction that the requirements assigned to each proper initial segment of α are eventually satisfied. Of course, our methods depend on whether $\mathcal{Q}$ is an $\mathcal{N}$ or an $\mathcal{R}$ requirement.

Case 1: $\mathcal{Q} = \mathcal{N}_{e,i}$. Suppose for a contradiction that $\Phi_e^{A_{1-i}} = A_i$. Choose a j such that $D^{[j]}$ is finite, j is larger than the restraint r, and no $\beta \leq_{\text{lex}} \alpha$ is ever assigned $\mathcal{P}_j$. (Such a j exists because C is nonrecursive, and hence not cofinite.) Let $\alpha^* \subseteq \mathsf{tr}$ be such that $\mathsf{ev}(\alpha^*) = \mathcal{P}_j$, and notice that $\alpha \subsetneq \alpha^*$. As $A_i^{[j]}$ is finite by Lemma 3.5, there is an $x = \langle j, k \rangle$ such that $k \notin A_i^{[j]}$, so that $\Phi_e^{A_{1-i}}(x) = 0 = A_i(x)$. Since by Lemma 3.5 every $\mathcal{P}$ requirement assigned to an initial segment of α^* is satisfied, there is an $s \geq s_0$ with $\alpha \subseteq \delta_s$ such that $\Phi_{e,t}^{A_{1-i,t}}(x) = 0$ is α-believable for all $t \geq s$. But then α acts at or before this stage s and preserves a disagreement – a contradiction.

Case 2: $\mathcal{Q} = \mathcal{R}_{e,i,n}$. We saw in the proof of Proposition 3.4 that $m = m_{e,i,n,s}$ is constant when $s > s_0$. If $n \in W_e^{A_{1-i}}$, then, again appealing to the Lemma, $n \in W_{e,s}^{A_{1-i,s}}$ by an α-believable computation for large enough s, and so α eventually sets a restraint (and possibly adds to $V_{e,i}$) to preserve a computation $\langle n, m \rangle \in V_{e,i}^{A_i}$ as desired. If $n \notin W_e^{A_{1-i}}$, suppose for a contradiction that $\langle n, m \rangle \in V_{e,i}^{A_i}$. Then there is an $s > s_0$ such that $\langle n, m \rangle \in V_{e,i}^{A_i}$ and $n \notin W_e^{A_{1-i}}$. Since $\alpha \leq_{\text{lex}} \delta_s$ by choice of s_0, there is a $\beta \subseteq \delta_s$ assigned the requirement $\mathcal{R}_{e,i,n}$ at stage s [this is immediate from the assignment scheme and the fact that $\delta_s > |\alpha|$]. But then β should act at stage s by incrementing m, a contradiction. $\square$

It remains only to verify that $A_0', A_1' \leq_T C$. We use the following:

Lemma 3.7. *The true path* tr *is recursive in* C.

Proof. Using an oracle for C, we construct tr by recursion by building finite initial segments $\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \mathsf{tr}$. Begin with $\alpha_0 = \emptyset$, the empty string. For the recursive step, suppose we have defined $\alpha_n = \mathsf{tr}\restriction n$. Use $0'$ (which is recursive in C) to find out exactly how many s there are such that $\delta_s \leq_{\text{lex}} \alpha_n$, and hence to compute $\mathsf{ev}(\alpha_n)$. If $\mathsf{ev}(\alpha_n)$ is an $\mathcal{N}$ or $\mathcal{R}$ requirement, then the true outcome is $\mathtt{0}$, so we let $\alpha_{n+1} = \alpha_n{}^\frown\mathtt{0}$. On the other hand, if $\mathsf{ev}(\alpha_n)$ is $\mathcal{P}_e$, then by Proposition 3.6 the true outcome $\mathsf{tr}(n)$ is $\mathtt{0}$ if $D^{[e]}$ is infinite, and $\mathtt{1}$ otherwise. In other words, $\mathsf{tr}(n)$ is $\mathtt{0}$ if $e \notin C$, and $\mathtt{1}$ otherwise. Use the C oracle to define $\alpha_{n+1} = \alpha^\frown\mathtt{0}$ or $\alpha_{n+1} = \alpha^\frown\mathtt{1}$, as appropriate. □

Proposition 3.8. *$A_0', A_1' \leq_T C$.*

Proof. We show that C can compute $\{e : e \in W_e^{A_i}\}$ for either value of i. Using the method of Lemma 3.7, use C to find, uniformly in e, a node α on the true path such that $\mathsf{ev}(\alpha) = \mathcal{R}_{e,i,e}$. As in the proof of Proposition 3.6, if $e \in W_e^{A_{1-i}}$, then eventually this computation is α-believable, so α acts (and succeeds) in preserving a coding $\langle e, m\rangle \in V_{e,i,s}^{A_{i,s}}$; on the other hand, if α acts at a stage after s_0 (defined as in the proof of Proposition 3.4) to preserve a such coding, then it also preserves $e \in W_e^{A_{1-i}}$. An oracle C can decide, using a query to $0'$, whether α acts in this way, and hence whether $e \in W_e^{A_{1-i}}$. □

This completes the proof of Theorem 1.11, part 1.

4 Proving part 2 of the Main Theorem

Here we give a quick summary of the alterations needed to convert the proof of Theorem 1.11 part 1 into a proof of part 2. Fix a $\mathbf{c}$, C, and D as in the beginning of Section 4. Build two sets A_0, A_1 meeting the following requirements, for all e, i, n:

$\mathcal{P}_e : A_0^{[e]} =^* D^{[e]}$.

$\mathcal{N}_e : A_1 \neq \Phi_e^{A_0}$.

$\mathcal{R}_{e,n}$: There is an r.e. set V_e, which does not depend on n, and a number $m_{e,n}$ computable uniformly in n from $0'$ such that $n \in W_e^{A_0 \oplus A_1} \Leftrightarrow \langle n, m_{e,n}\rangle \in V_e^{A_0}$. Furthermore, $k < m_{e,n}$ implies $\langle n, k\rangle \in V_e^{A_0\oplus A_1}$, and $k > m_{e,n}$ implies $\langle n, k\rangle \notin V_e^{A_0\oplus A_1}$.

Then the required degrees are $\mathbf{b}_0 = \deg(A_0)$, and $\mathbf{b}_1 = \deg(A_0 \oplus A_1)$. The differences to keep in mind when adapting the construction are:

- $\mathcal{P}_e$ alters A_0 but never A_1.
- $\mathcal{N}_e$ alters A_1 but never A_0.

- $\mathcal{R}_{e,n}$ uses $A_0 \oplus A_1$ as an oracle instead of A_0 or A_1.
- The notion of α-believable computations $W_e^{A_0 \oplus A_1}(n)$ is adapted to allow for the fact that $\mathcal{P}$ requirements add elements to columns of A_0 but not to A_1.

From here the the proof proceeds by a sequence of lemmas analogous to that in Subsection 3.4.

5 Questions and Observations

In this section, we pose a number of questions that naturally extend Proposition 1.8, Corollary 1.10 and Theorem 1.11 and make some observations which impede or even restrict such possibilities.

The first set of questions deal with the issue of when r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ in the same jump class, given say by $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$, have $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ by extending Corollary 1.10.

Question 5.1. When (for $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$ and $\mathbf{c}'' \neq \mathbf{0}'''$) do we have r.e. $\mathbf{a} \neq \mathbf{b}$ with jump $\mathbf{c}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$? Of course, we must here at least have $\mathbf{c} > \mathbf{0}'$.

The noninversion theorem of Shore [1988] gives some examples of degrees distinguished by the JB operator along these lines beyond those given by Corollary 1.10. Indeed, it supplies two upward cones in $\mathbf{R}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ for any incomplete $\mathbf{a}$ and $\mathbf{b}$ in each cone and a cone of jump classes all realized by degrees in each of these two cones.

Proposition 5.2. *There are incomparable r.e.* $\mathbf{c}$ *and* $\mathbf{d}$ *such that for* $\mathbf{0}' > \mathbf{a} \geq \mathbf{c}$ *and* $\mathbf{0}' > \mathbf{b} \geq \mathbf{d}$, $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$*. Moreover, there is a* $\mathbf{w} \in \mathbf{REA}(\mathbf{0}')$ *with* $\mathbf{w} < \mathbf{0}''$ *such that for any* $\mathbf{z} \geq \mathbf{w}$ *with* $\mathbf{z} \in \mathbf{REA}(\mathbf{0}')$*, there are* $\mathbf{a} \geq \mathbf{c}$ *and* $\mathbf{b} \geq \mathbf{d}$ *with* $\mathbf{a}' = \mathbf{z} = \mathbf{b}'$*.*

Proof. By Shore [1988, Theorem 1.1], there are $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{REA}(\mathbf{0}')$ such that $\mathbf{a}_0 \vee \mathbf{a}_1 < \mathbf{0}'$ and if $\mathbf{u} < \mathbf{0}'$ then not both $\mathbf{a}_0$ and $\mathbf{a}_1$ are $REA(\mathbf{u})$. Now take r.e. $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{c}' = \mathbf{a}_0$ and $\mathbf{d}' = \mathbf{a}_1$. Consider now any incomplete $\mathbf{a} \geq \mathbf{c}$ and $\mathbf{b} \geq \mathbf{d}$. It is clear that $\mathbf{a}_0 \in \mathsf{JB}(\mathbf{a})$ and $\mathbf{a}_1 \in \mathsf{JB}(\mathbf{b})$. On the other hand, if $\mathbf{a}_1 \in \mathsf{JB}(\mathbf{a})$ then $\mathbf{a}$ would be complete and so $\mathbf{a}_1 \notin \mathsf{JB}(\mathbf{a})$. Similarly $\mathbf{a}_0 \notin \mathsf{JB}(\mathbf{b})$ as required. Of course, $\mathbf{c}$ and $\mathbf{d}$ are incomparable as otherwise the larger of the two would be an incomplete $\mathbf{u}$ in which both $\mathbf{a}_0$ and $\mathbf{a}_1$ would be r.e.

Finally, by Theorem 1.2, we may take $\mathbf{a}_0 \vee \mathbf{a}_1$ as the $\mathbf{w}$ required in the Proposition. $\square$

All the examples of pairs of incomplete r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$ that we have seen provide, as far as we have specifically determined, only incomparable $\mathbf{a}$ and $\mathbf{b}$. As we know of no others, we ask for comparable such pairs.

Question 5.3. When (for $\mathbf{c} \in \mathbf{REA}(\mathbf{0}')$) do we have r.e. $\mathbf{a} < \mathbf{b} < \mathbf{0}'$ with $\mathbf{a}' = \mathbf{b}' = \mathbf{c}$ such that $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$?

Of course, we must here also at least have $\mathbf{c} > \mathbf{0}'$ but Proposition 1.8 and Corollary 1.10 do not supply an answer even for $\mathbf{c} = \mathbf{0}''$ or $\mathbf{c}' = \mathbf{0}'''$. We can ask for even more along the lines of distinguishing r.e. degrees.

Question 5.4. Is there, for every nonlow r.e. $\mathbf{a}$, an r.e. $\mathbf{b} \neq \mathbf{a}$ with $\mathbf{a}' = \mathbf{b}'$ and $\mathsf{JB}(\mathbf{a}) \neq \mathsf{JB}(\mathbf{b})$? For which such $\mathbf{a}$ can we, in addition, choose $\mathbf{b}$ so that we have $\mathbf{a} < \mathbf{b}$, $\mathbf{b} < \mathbf{a}$ or $\mathbf{a}|\mathbf{b}$?

Using a result of Arslanov, Lempp and Shore [1996], we can show that the strongest possible version of such a statement does not hold.

Proposition 5.5. *There is a nonlow r.e.* $\mathbf{c}$ *such that every incomplete r.e.* $\mathbf{b} \geq \mathbf{c}$ *with* $\mathbf{c}' = \mathbf{b}'$ *has* $\mathsf{JB}(\mathbf{c}) = \mathsf{JB}(\mathbf{b})$.

Proof. Arslanov, Lempp and Shore [1996, Theorem 3.1] states that there is an incomplete nonrecursive r.e. A such that every set $REA(A)$ and recursive in $\mathbf{0}'$ is of r.e. degree. By the uniformity inherent in the proof of this result, we can apply the pseudojump inversion theorem of Jockusch and Shore [1983] to get an r.e. C such that some set $A \in \mathbf{0}'$ has this property relative to C. That is, $C <_T A < C'$ and every set $REA(A)$ and recursive in C' is of degree r.e. in C. Thus if the degree $\mathbf{x} \in \mathbf{REA}(\mathbf{0}')$ (which, of course, is the same as $\mathbf{REA}(\mathbf{a})$) and $\mathbf{x} \leq \mathbf{c}'$ then some $X \in \mathbf{x}$ is $REA(A)$ and so $\mathbf{x} \in \mathbf{REA}(\mathbf{c})$. Thus, in particular, if $\mathbf{b} \geq \mathbf{c}$ and $\mathbf{b}' = \mathbf{c}'$ then then $\mathsf{JB}(\mathbf{c}) = \mathsf{JB}(\mathbf{b})$. □

We note that, as Arslanov, Lempp and Shore point out, the A they construct cannot be either low$_2$ or high. This translates into fact that the $\mathbf{c}$ produced in the above Proposition is neither high$_2$ nor low$_2$. We do not know any more about the possibilities for such $\mathbf{c}$ but there are several tempting possibilities. For example, could such a $\mathbf{c}$ also be least in its jump class with this maximal value of $\mathsf{JB}(\mathbf{c})$, i.e. could it be that for $\mathbf{x} \not\geq \mathbf{c}$, $\mathsf{JB}(\mathbf{x}) \neq \mathsf{JB}(\mathbf{c})$? If so, this would in a different way characterize $\mathbf{c}$ in terms of the JB operator. If not then, perhaps it might be minimal, i.e. for $\mathbf{x} < \mathbf{c}$, $\mathsf{JB}(\mathbf{x}) \subsetneq \mathsf{JB}(\mathbf{c})$ and so one would have characterized at least an antichain of degrees as the ones with this value of JB.

Moving in the other direction, i.e. towards stronger versions of Theorem 1.11 and Question 1.13, we can ask the following:

Question 5.6. Is there, for every nonrecursive, incomplete r.e. $\mathbf{a}$ an r.e. $\mathbf{b}$ with $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$ for which we can also guarantee that $\mathbf{b} > \mathbf{a}$, $\mathbf{b} < \mathbf{a}$ or $\mathbf{b}|\mathbf{a}$?

Note that as we pointed out after Definition 1.3, Theorem 1.1 shows that even to get a $\mathbf{b} < \mathbf{a}$ with $\mathsf{JB}(\mathbf{a}) = \mathsf{JB}(\mathbf{b})$ we must assume that $\mathbf{a}$ is incomplete. It is hard to see how the assumption of the incompleteness of $\mathbf{a}$ can be used in the construction of a $\mathbf{b} < \mathbf{a}$.

We conclude with some methodological remarks about the construction that highlights a reverse mathematical issue. First, we note that despite the unusual type of argument about the assignment of requirements along the true path, the construction is still, by the usual criterion, a $0''$ one: In particular, our proof shows that $0''$ can compute the true path. Once one knows the fact that each requirement is eventually assigned to a fixed node along the true path, $0''$ can compute where and when this happens and so the precise way in which each requirement is satisfied.

Now, it is generally the case that $0''$ constructions can be carried out in $I\Sigma_2$. The anomaly here is that the proof that each requirement is eventually assigned to a fixed node along the true path (Lemma 3.3) and so of the fact that $0''$ can calculate all the outcomes of the construction seems to require $I\Sigma_3$. The point here is that in order to prove Lemma 3.3, we use an instance of the principle that any finite iterate $f_m = \underbrace{f \circ \cdots \circ f}_{m \text{ times}}$ of a total Π_2-definable function f is itself a total function. (In our case, $f(n)$ is the number of elements which are ever accessible, and which are $\leq_{\text{lex}} \mathsf{tr} \restriction n$, where tr denotes the true path. This f comes from the scheme for assigning requirements to nodes, which can be found in §3.) This principle is known as Π_2 *recursion*. It is denoted by $\mathrm{T}\Pi_2$ in Hajek and Pudlak [1993] in the setting of first order arithmetic and PREC_3 in Hirschfeldt and Shore [2007] in the setting of second order arithmetic. In each setting, it is shown equivalent to $I\Sigma_3$. Thus our proof, unlike previous examples, uses $I\Sigma_3$ and so more induction than one would expect.

The solution to this problem is to use Shore blocking to assign blocks of requirements along the paths of the construction (and so along the true path). Thus, for example, instead of assigning at stage s a single requirement of some type ($\mathcal{R}_{e,i,n}$, $\mathcal{P}_e$ or $\mathcal{N}_{e,i}$) at a node α as in our construction, one assigns the block of the next requirements of the same type of size s (i.e. ones not yet on the path of the form $\mathcal{R}_{k,j,l}$ for $k, j, l < s$, $\mathcal{P}_j$ for $j < s$ or $\mathcal{N}_{j,i}$ for $j < s$, respectively). We do not know of another $0''$ construction that requires blocking along the paths of the priority tree to carry out the argument that the requirements are satisfied in $I\Sigma_2$.

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