NONLINEAR VIBRATIONS OF A CLAMPED ORTHOTROPIC CIRCULAR PLATE WITH A CONCENTRIC RIGID MASS

BY

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NOMENCLATURE

r, θ, z	cylindrical coordinates used to describe the undeformed configuration of the plate
h	thickness of the plate
a	outer radius of the circular plate
b	outer radius of the mass
ρ	mass density of the plate
t	time variable
u, w	radial and transverse displacements of the middle plane, respectively
$\epsilon_r, \epsilon_\theta$	total strains in the radial and circumferential directions, respectively.
$\epsilon_r^\circ, \epsilon_\theta^\circ$	radial and circumferential strains acting on the middle plane, respectively
$\sigma_r, \sigma_\theta, \sigma_z$	radial, circumferential and normal stresses, respectively
N_r, N_θ	membrane forces per unit length
T_p, T_m	kinetic energy of the plate and of concentric rigid mass, respectively.
U_s, U_b	strain energy due to stretching of the middle plane and due to bending of the plate, respectively
W	work done on the plate by external forces
ψ, ϕ	stress functions
M_r, M_θ	bending moments per unit length
$q(r,t)$	time-variant loading intensity

$E_r, E_\theta, E_{r\theta}$	material constants in the directions indicated by subscripts.
a_{11}, a_{12}, a_{22}	elastic constants (compliance coefficiences)
R	radius ratio, = b/a
γ	mass ratio
α	elastic constant ratio, = $E_{r\theta}/E_r$
β	elasitc constant ratio, = E_θ/E_r
C	elastic constant ratio, = E_r/E_θ
ν	Poisson's ratio, = α/β
D	flexural rigidity of the plate, = $E_r h^3/12$
I	action integral for the vibrating system
δI	first variation of I
η, ζ	admissable variations of w and u, respectively
ϵ	arbitrary infinitesimal constant
ξ, τ	dimensionless space and time variables, respectivley
δW	virtual work of transverse forces
χ	dimensionless transverse displacement
$Q(\xi), Q^*(\xi)$	dimensionless loading densities
$g(\xi), f(\xi)$	shape functions
A, κ	amplitude parameters
λ	nondimensional eigenvalue
ω	nondimensional angular frequency
ω_ℓ	linear angular frequency

$\overline{Y}, \overline{Z}, \overline{H}$	(6×1) vector functions
[M], [N]	coefficient matrices
$\overline{\eta}$	missing initial values

Chapter I INTRODUCTION

Anisotropic materials play an important role in modern technology, because lots of new anisotropic materials, such as reinforced plastics and composite materials, are being used in missiles, aircraft, space vehicles, pressure vessels and parts of structures to meet the special requirements.

In the past, materials, regardless of their composition, were usually considered to be homogeneous and isotropic because such assumptions make calculation simple. However, these simplified assumptions often lead to inadequate or incorrect results. Today, to meet the sophisticated technology requirements, we need to consider the anisotropic property of materials, that is, the differences in elastic properties of materials in various directions.

A circular anisotropic plate whose material symmetry of aeolotropy at a point is in radial and tangential directions is called an orthotropic plate.

The governing differential equations describing the motion of thin circular plates, in general, are nonlinear and coupled. Due to the complex nature of the resulting governing equations, an analytical solution is very difficult to obatain. Thus, approximate methods [1] of analysis must be used.

Large amplitude vibrations of a clamped isotropic circular plate with a concentric rigid mass at the center have been studied by Handelman and Cohen [2], Chiang and Chen [3] and Becker [4].

On the other hand, large amplitude vibrations of a clamped orthotropic circular plate with isotropic core have been studied by Huang [5]. Woo [6] payed attention to the same problem except he limited the vibration to a small amplitude. However, the addition of a concentric rigid mass attached rigidly to the orthotropic circular plate has received limited attention.

In this thesis Hamilton's principle is applied to derive the basic differential equations for the problem of the large amplitude of an orthotropic circular plate carrying a concentric rigid mass at the center. Assuming the system performs the harmonic vibrations, the time variable is eliminated by employing the Ritz-Kantorovich averaging method. Therefore, the basic governing differential equations are reduced to a pair of ordinary differential equations, which form a nonlinear boundary-value problem with the associated boundary conditions. For the purpose of numerical computation, this boundary-value problem is converted to the related initial-value problem.

The relation among natural frequency, stiffness parameters and mass ratio for free vibration as well as for forced vibration are investigated, and presented by figures. These results provide useful information for the design of the nonlinear vibration of an orthotropic circular plate with a concentric rigid mass. The free and forced vibration with variable parameters are also investigated; their effects and the corresponding results are illustrated. Finally, the radial and circumferential stresses in free vibration are also determined. These results of stress distribution also provide useful information for stress analysis in industrial utilization.

Chapter II DERIVATION OF GOVERNING EQUATIONS OF MOTION

Consider the thin circular plate with a concentric rigid mass M_c as shown in Fig. 1. The plate, except for a concentric rigid mass M_c of radius b occupying the central portion, is composed of an elastic, homogeneous and cylindrically orthotropic medium bounded by the planes $z=\pm h/2$. The plate is clamped at the outer edge of radius a. In discussing circular boundaries problems, it is more convenient to use cylindrical coordinates (r, θ, z) and set the origin at the center of the middle plane to describe the motion.

Before deriving governing differential equations, we have made the following assumptions

1. Any straight line normal to the middle plane before deformation remains a straight line normal to the neutral plane during deformation.
2. Normal stress, σ_z is small in comparison with the other stresses, and may be neglected.
3. Circular symmetry with respect to the z-axis is retained during vibration.
4. The effects of stretching the middle plane of the plate are included.

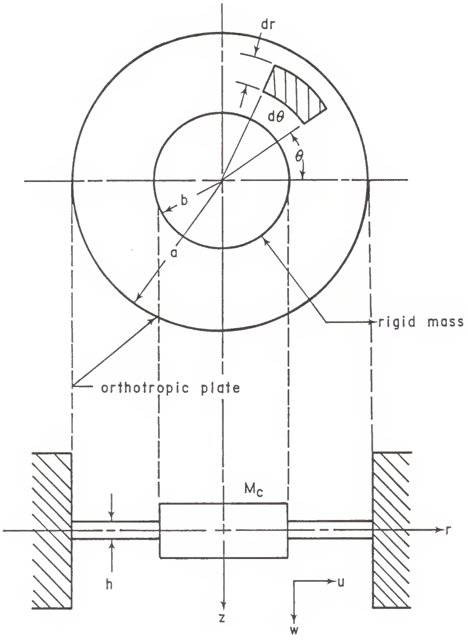

Fig. 1. Circular Plate and Cylindrical Coordinate System

5. The vibration of the plate is not limited to the infinitesimal.

The radial and circumferential strain-displacement relations derived from large deflection theory [8,10] are as follows:

$$\epsilon_r = u_r + \frac{1}{2} w_r^2 - z w_{rr} \tag{1a}$$

$$\epsilon_\theta = \frac{u}{r} - \frac{z}{r} w_r \tag{1b}$$

where u(r,t) and w(r,t) denote the radial and transverse components of the mid-plane displacement, respectively, while the subscripted variables w_r and w_{rr} represent the first and second partial derivatives of w with respect to r.

The strain-stress relations in the polar coordinate system are :

$$\sigma_\theta = E_\theta \epsilon_\theta + E_{r\theta} \epsilon_r \tag{2a}$$

$$\sigma_r = E_{r\theta} \epsilon_\theta + E_r \epsilon_r \tag{2b}$$

where σ_θ and σ_r are normal stresses in the tangential and radial directions respectively, E_θ, $E_{r\theta}$ and E_r are the material constants (Young's modulus), and ϵ_θ and ϵ_r are normal strains in the tangential and radial directions respectively.

The stress-strain relations in equations (2) can also be represented in the form [9]

$$\epsilon_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r \qquad (3a)$$

$$\epsilon_r = a_{12}\sigma_\theta + a_{22}\sigma_r \qquad (3b)$$

where a_{ij} are elastic constants.

Equations (3a) and (3b) can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \sigma_\theta \\ \sigma_r \end{bmatrix} = \begin{bmatrix} \epsilon_\theta \\ \epsilon_r \end{bmatrix} .$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad X = \begin{bmatrix} \sigma_\theta \\ \sigma_r \end{bmatrix}, \quad B = \begin{bmatrix} \epsilon_\theta \\ \epsilon_r \end{bmatrix}$$

the matrix can be expressed symbolically

$$AX = B.$$

then

$$X = A^{-1} B$$

which implies

$$\begin{bmatrix} \sigma_\theta \\ \\ \sigma_r \end{bmatrix} \quad \begin{bmatrix} \dfrac{a_{22}}{a_{11}a_{22} - a_{12}^2} & \dfrac{-a_{12}}{a_{11}a_{22} - a_{12}^2} \\ \\ \dfrac{-a_{12}}{a_{11}a_{22} - a_{12}^2} & \dfrac{a_{11}}{a_{11}a_{22} - a_{12}^2} \end{bmatrix} \begin{bmatrix} \epsilon_\theta \\ \\ \epsilon_r \end{bmatrix}$$

thus σ_θ and σ_r can be represented in terms of a_{ij} , ϵ_θ and ϵ_r by

$$\sigma_\theta = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}\,\epsilon_\theta - \frac{a_{12}}{a_{11}a_{22} - a_{12}^2}\,\epsilon_r \tag{4a}$$

$$\sigma_r = \frac{-a_{12}}{a_{11}a_{22} - a_{12}^2}\,\epsilon_\theta + \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}\,\epsilon_r \tag{4b}$$

and

$$E_\theta = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \tag{5a}$$

$$E_r = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2} \tag{5b}$$

$$E_{r\theta} = \frac{-a_{12}}{a_{11}a_{22} - a_{12}^2} \;. \tag{5c}$$

Here we define

$$\alpha = \frac{E_{r\theta}}{E_r} = -\frac{a_{12}}{a_{11}} \qquad \beta = \frac{E_\theta}{E_r} = \frac{a_{22}}{a_{11}}$$

$$c = \frac{E_r}{E_\theta} = \frac{a_{11}}{a_{22}} = \frac{1}{\beta} \qquad \nu = -\frac{a_{12}}{a_{22}} = \frac{\alpha}{\beta} \;.$$

Integrating the in-plane stresses across the thickness of the plate, radial and circumferential bending moment per unit length, M_r and M_θ, respectively, are found to be

$$M_r = \int_{-h/2}^{+h/2} \sigma_r \, zdz$$

$$= \int_{-h/2}^{+h/2} (E_r\epsilon_r + E_{r\theta}\epsilon_\theta)zdz$$

$$= \int_{-h/2}^{+h/2} [E_r(u_r + \frac{1}{2} w_r^2 - zw_{rr}) + E_{r\theta}(\frac{u}{r} - \frac{z}{r} w_r)]zdz$$

$$= - \frac{h^3}{12} (E_r w_{rr} + \frac{1}{r} E_{r\theta} w_r)$$

$$= - D \, (w_{rr} + \frac{1}{r} \alpha w_r) \qquad (6a)$$

$$M_\theta = \int_{-h/2}^{+h/2} \sigma_\theta \, zdz$$

$$= \int_{-h/2}^{+h/2} (E_{r\theta}\epsilon_r + E_\theta\epsilon_\theta)zdz$$

$$= \int_{-h/2}^{+h/2} [E_{r\theta}(u_r + \frac{1}{2} w_r^2 - zw_{rr}) + E_\theta(\frac{u}{r} - \frac{z}{r} w_r)]zdz$$

$$= - \frac{E_r h^3}{12} \left[\frac{E_{r\theta}}{E_r} w_{rr} + \frac{E_\theta}{E_r}(\frac{w_r}{r})\right]$$

$$= - D \, (\alpha \, w_{rr} + \frac{1}{r} \beta \, w_r) \qquad (6b)$$

in which, $D=E_r h^3/12$ is the flexural rigidity of the plate. In addition, α and β has the following restrictions [Appendix A] :

$$0 < \alpha < 1$$

and $\alpha < \beta$

In the same way, the expressions for the radial and circumferential forces per unit length, N_r and N_θ, are obtained by integrating the respective stresses across the thickness of the plate.

$$N_r = \int_{-h/2}^{+h/2} \sigma_r \, dz$$

$$= \int_{-h/2}^{+h/2} (E_r \epsilon_r + E_{r\theta} \epsilon_\theta) \, dz$$

$$= E_r(u_r h + \frac{1}{2} w_r^2 h) + E_{r\theta}(\frac{u}{r} h)$$

$$= \frac{12D}{h^2} (u_r + \frac{1}{2} w_r^2 + \alpha \frac{u}{r}) \tag{7a}$$

$$N_\theta = \int_{-h/2}^{+h/2} \sigma_\theta \, dz$$

$$= \int_{-h/2}^{+h/2} (E_{r\theta} \epsilon_r + E_\theta \epsilon_\theta) dz$$

$$= E_{r\theta}(u_r h + \frac{1}{2} w_r^2 h) + E_\theta(\frac{u}{r} h)$$

$$= \frac{12D}{h^2} [\alpha(u_r + \frac{1}{2} w_r^2) + \beta \frac{u}{r}]. \tag{7b}$$

<u>The Energy Method</u> :

To formulate the boundary-value problem , we shall use Hamilton's principle. The power of this approach is that it furnishes automatically the correct number of boundary conditions and the differential equation. In this system during the motion, three kinds of energy and the work done by the external forces are considered as follows:

U_b= strain energy due to bending

U_s= strain energy due to stretching of the middle plane

T = kinetic energy

W = work done by the external forces.

The strain energy due to bending can be expressed as [10]

$$U_b = -\int_0^{2\pi}\int_b^a \left(\frac{1}{2} M_r w_{rr} + \frac{1}{2r} M_\theta w_r\right) r dr d\theta$$

Substituting equations (6a) and (6b) into this equation, one obtains

$$U_b = \frac{D}{2} \int_0^{2\pi}\int_b^a [(w_{rr} + \frac{\alpha}{r} w_r)w_{rr} + (\alpha w_{rr} + \frac{\beta}{r} w_r)\frac{1}{r} w_r] r dr d\theta$$

$$= \pi D \int_b^a \left(w_{rr}^2 + 2\frac{\alpha}{r} w_{rr} w_r + \frac{\beta}{r^2} w_r^2\right) r dr \qquad (8)$$

Strain energy due to stretching of the middle plane can be expressed as

$$U_s = \int_0^{2\pi}\int_b^a \left(\frac{1}{2} N_r \epsilon_r^\circ + \frac{1}{2} N_\theta \epsilon_\theta^\circ\right) r dr d\theta$$

substituting equations (7a) and (7b) into this equation, one obtains

$$U_s = \pi \int_b^a \left\{\frac{12D}{h^2}[(u_r + \frac{1}{2} w_r^2 + \alpha \frac{u}{r})(u_r + \frac{1}{2} w_r^2)]\right.$$

$$\left. + \frac{12D}{h^2} [\alpha(u_r + \frac{1}{2} w_r^2) + \beta(\frac{u}{r})](\frac{u}{r})\right\} r dr$$

$$= \frac{12\pi D}{h^2}\int_b^a [u_r^2+u_r w_r^2+2\frac{\alpha}{r}u_r u+\frac{\alpha}{r}uw_r^2+\frac{\beta}{r^2}u^2+\frac{1}{4}w_r^4]rdr \tag{9}$$

where ϵ_r° and ϵ_θ° represent the radial and circumferential strain at the mid-plane, respectively.

Kinetic energy due to translation can be divided into two portions, one is the contribution of the plate portion T_p, the other one is due to the concentric rigid mass portion T_m. The radial component of velocity, in this problem, is rather small in comparison with the transverse component of velocity and can be neglected. Thus, only the transverse component of velocity is considered in kinetic energy.

$$T_p = \int_0^{2\pi}\int_b^a \frac{1}{2}(\rho h w_t^2) rdrd\theta = \pi\rho h \int_b^a w_t^2 \ rdr \tag{10a}$$

where ρ is the mass per unit volume of the plate, and the kinetic energy T_m is

$$T_m = \frac{1}{2} M_c w_t^2 \Big|_{r=b} \tag{10b}$$

where M_c is the concentric rigid mass.

The work done by the external forces can be expressed as

$$W = \int_0^{2\pi}\int_b^a q(r,t)w(r,t)rdrd\theta \tag{11}$$

where q(r,t) is the time-dependent external loading intensity, assumed to be symmetric with respect to the z-axis.

Consider the motion of the plate system between two fixed instants t_0 and t_1, and applying Hamilton's principle , the first variation of the action integral δI must be zero.

I is the action integral defined by

$$I= \int_{t_0}^{t_1} (T_p+T_m-U_s-U_b+W)dt \tag{12}$$

substituting equations (8), (9), (10a), (10b) and (11) into equation (12) and taking the first variation of I, one obtains δI.

$$\delta I= -2\pi\rho h\epsilon\int_{t_0}^{t_1}\int_b^a \eta w_{tt} r dr dt-\epsilon M_c\int_{t_0}^{t_1} \eta w_{tt} dt$$

$$-2\pi D\epsilon\int_{t_0}^{t_1} \eta_r(rw_{rr}+\alpha w_r)\Big|_b^a dt + 2\pi D\epsilon\int_{t_0}^{t_1} \eta(rw_{rrr}+w_{rr})\Big|_b^a dt$$

$$-2\pi D\epsilon\int_{t_0}^{t_1} \eta(\frac{\beta}{r}w_r)\Big|_b^a dt - 2\pi D\epsilon\int_{t_0}^{t_1}\int_b^a \eta(rw_{rrrr}+2w_{rrr})drdt$$

$$+2\pi D\epsilon\int_{t_0}^{t_1}\int_b^a \eta(\frac{\beta}{r}w_{rr}- \frac{\beta}{r^2}w_r)drdt$$

$$- \frac{12\pi D\epsilon}{h^2} \int_{t_0}^{t_1} \zeta(2ru_r+rw_r^2+2\alpha u)\Big|_b^a dt$$

$$- \frac{12\pi D\epsilon}{h^2}\int_{t_0}^{t_1} \eta(2ru_r w_r+rw_r^3+2\alpha uw_r)\Big|_b^a dt$$

$$+ \frac{12\pi D\epsilon}{h^2}\int_{t_0}^{t_1}\int_b^a 2\zeta(ru_{rr}+u_r+ \frac{1}{2}w_r^2+rw_r w_{rr}+\alpha u_r)drdt$$

$$- \frac{12\pi D\epsilon}{h^2}\int_{t_0}^{t_1}\int_b^a \zeta(2\alpha u_r+\alpha w_r^2+\frac{2}{r}\beta u)drdt$$

$$+\frac{12\pi D\epsilon}{h^2}\int_{t_0}^{t_1}\int_b^a \eta\left\{2w_r[u_r(1+\alpha)+ru_{rr}]+w_{rr}(2\alpha u+2ru_r+3rw_r^2)+w_r^3\right\}drdt$$

$$+\ 2\pi\epsilon\int_{t_0}^{t_1}\int_b^a \eta q(r,t)rdrdt$$

$$=0$$

The necessary condition to let $\delta I=0$ is that the integrands of the double integrals and the single integrals must vanish separately. Thus, the double integrals provide the governing differential equations of motion,

$$D(w_{rrrr}+\frac{2}{r}w_{rrr}-\frac{\beta}{r^2}w_{rr}+\frac{\beta}{r^3}w_r)+\rho hw_{tt}-q(r,t)$$

$$=\frac{12D}{h^2}(\frac{1}{r}u_rw_r+u_rw_{rr}+u_{rr}w_r+\frac{\alpha}{r}u_rw_r+\frac{\alpha}{r}uw_{rr}+\frac{1}{2r}w_r^3+\frac{3}{2}w_r^2w_{rr}) \qquad (13a)$$

and $$u_{rr}+\frac{1}{r}u_r+\frac{1}{2r}w_r^2+w_rw_{rr}-\frac{\alpha}{2r}w_r^2-\frac{\beta}{r^2}u=0 \qquad (13b)$$

while the single integrals yield the boundary conditions

$$2\pi Dr(w_{rr}+\frac{\alpha}{r}w_r)\eta_r\Big|_b^a=0\ , \qquad (14a)$$

$$\frac{24\pi D}{h^2}r(u_r+\frac{1}{2}w_r^2+\frac{\alpha}{r}u)\zeta\Big|_b^a=0 \qquad (14b)$$

$$2\pi Dr(w_{rrr}+\frac{1}{r}w_{rr}-\frac{\beta}{r^2}w_r)\eta\Big|_b^a-\frac{24\pi D}{h^2}rw_r(u_r+\frac{\alpha}{r}u+\frac{1}{2}w_r^2)\eta\Big|_b^a$$

$$-\ M_cw_{tt}\eta\Big|_b=0 \qquad (14c)$$

The corresponding geometric boundary conditions for a clamped orthotropic circular plate carrying a concentric rigid mass are :

$$w\big|_{r=a}=0 \qquad w_r\big|_{r=b}=0 \qquad w_r\big|_{r=a}=0$$

$$u\big|_{r=a}=0 \qquad u\big|_{r=b}=0 \tag{15a}$$

which means transverse displacement at r=a and slope both at r=a and r=b are equal to zero, and radial displacements at mid-plane are also zero at r=a and r=b.

The geometric and natural boundary conditions are supplemental to each other and add up to the right number of boundary conditions. Their satisfaction ensures that the solution of the differential equation is unique.

Since $W_r\big|_{r=b}=0$ and $W_r\big|_{r=a}=0$, equation (14c) can be reduced to be

$$(w_{rrr}+\frac{1}{r}w_{rr})\big|_b = -\frac{M_c}{2\pi bD}w_{tt}\big|_{r=b} \tag{15b}$$

Now, introduce the stress function $\psi(r,t)$

$$\psi(r,t) = \frac{12D}{h^2}(ru_r+\frac{r}{2}w_r^2+\alpha u)$$

which has the following relations with N_r and N_θ

$$\frac{\psi}{r}=N_r \qquad \frac{\partial\psi}{\partial r}=N_\theta \tag{16a}$$

where N_r and N_θ must satisfy the equilibrium equation [11]

$$\frac{\partial N_r}{\partial r} + \frac{N_r - N_\theta}{r} = 0 \qquad (16b)$$

Using these relations, equation (13a) and (13b) can be transformed as follows :

$$D(w_{rrrr} + \frac{2}{r}w_{rrr} - \frac{\beta}{r^2}w_{rr} + \frac{\beta}{r^3}w_r) + \rho h w_{tt} - q(r,t) = \frac{1}{r}(w_r\psi)_r \qquad (17a)$$

$$\psi_{rr} + \frac{1}{r}\psi_r - \frac{\beta}{r^2}\psi = -\frac{1}{2r}E_r h(\beta - \alpha^2)w_r^2 \qquad (17b)$$

At the mid-plane of the plate, the circumferential strain, $\overset{\circ}{\epsilon}_\theta$ can be expressed as $\overset{\circ}{\epsilon}_\theta = u/r$. Then from equations (2), (7) and (16a) we can obtain

$$u = \frac{rE_r}{E_rE_\theta - E_{r\theta}^2}(\psi_r - \frac{\alpha}{r}\psi) = 0$$

The conditions for u to be vanished at r=a and r=b are

$$(\psi_r - \frac{\alpha}{r}\psi)_{r=a} = 0$$

$$(\psi_r - \frac{\alpha}{r}\psi)_{r=b} = 0$$

For the purpose of simplification in mathematics, the following dimensionless quantities are introduced

$$\chi = w/a , \quad \xi = r/a , \quad R = b/a , \quad \gamma = M_c/\pi b^2\rho h ,$$

$$\tau = t(\beta D/\rho h a^4)^{1/2}, \qquad \phi = a_{22}\psi/ha ,$$

and $$Q = 12a_{22}\left(\frac{a}{h}\right)^3 q \ .$$

Then, the governing differential equations (17a) and (17b) and boundary conditions (15a) and (15b) are converted into the following non-dimensional forms :

$$\chi_{\xi\xi\xi\xi} + \frac{2}{\xi}\chi_{\xi\xi\xi} - \frac{\beta}{\xi^2}\chi_{\xi\xi} + \frac{\beta}{\xi^3}\chi_{\xi} + \beta\chi_{\tau\tau} - \left(1 - \frac{\alpha^2}{\beta}\right)Q$$

$$= \frac{12}{\xi}\left(\frac{a}{h}\right)^2\left(1 - \frac{\alpha^2}{\beta}\right)(\chi_\xi \phi)_\xi \ , \tag{18a}$$

$$\phi_{\xi\xi} + \frac{1}{\xi}\phi_\xi - \frac{\beta}{\xi^2}\phi + \frac{\beta}{2\xi}\chi_\xi^2 = 0 \ , \tag{18b}$$

and
$$\left.\begin{aligned}
&(\ \chi\ )_{\xi=1} = 0 \\
&(\ \chi_\xi)_{\xi=1} = 0 \\
&(\ \chi_\xi)_{\xi=R} = 0 \\
&(\ \phi_\xi - \frac{\alpha}{\xi}\phi\ )_{\xi=1} = 0 \\
&(\ \phi_\xi - \frac{\alpha}{\xi}\phi\ )_{\xi=R} = 0 \\
&(\ \chi_{\xi\xi\xi} + \frac{1}{\xi}\chi_{\xi\xi} + \frac{\xi}{2}\gamma\chi_{\tau\tau})_{\xi=R} = 0
\end{aligned}\right\} \tag{19}$$

Equations (18) and (19) form a set of nonlinear boundary value problems describing the large oscillation of an orthotropic circular plate with a concentric rigid mass at the center of the plate.

Chapter III APPROXIMATE ANALYSIS

The differential equations of motion together with the associated boundary conditions constitute a boundary-value problem. Any solution to the problem defined by the nonlinear differential equations must satisfy the boundary conditions . It is usually very difficult to obtain analytic (closed-form) solutions to the boundary value problems, thus, the approximate method is usually applied. The Ritz-Kantorovich averaging method [7], which has been successfully applied to numerous elasticity problems, is adopted to eliminate the time variable and reduce the governing differential equations to a system of nonlinear ordinary differential equations, which are the only function of the space coordinate.

The Ritz-Kantorovich Method

Assuming the dynamic system performs the harmonic vibrations, then the functions $Q(\xi,\tau)$, $\chi(\xi,\tau)$ and $\phi(\xi,\tau)$ of equations (18a) and (18b) can be expressed as

$$Q(\xi,\tau)= Q^{*}(\xi)(\frac{h}{a})\sin\omega\tau \tag{20a}$$

$$\chi(\xi,\tau)= A\, g(\xi)\sin\omega\tau \tag{20b}$$

$$\phi(\xi,\tau)= A^{2}f(\xi)\sin^{2}\omega\tau \tag{20c}$$

where A denotes a nondimensional amplitude parameter; ω is the nondimensional angular frequency; $g(\xi)$ and $f(\xi)$ are shape functions to be determined corresponding to the functions χ and ϕ, respectively, and $Q^*(\xi)$ is the dimensionless loading function. Since equations (20a),(20b) and (20c) can only satisfy equation (18b) , but not satisfy equation (18a) exactly for all τ, thus the residual may be found and minimized by the use of Ritz-Kantorovich method. For any instant of the dimensionless time τ, the virtual work of all the transverse forces as they move through a virtual displacement, $\delta\chi = A\sin\omega\tau(\delta g)$, is

$$\delta W = \int_R^1 A_1 \delta\chi \, \xi \, d\xi \tag{21}$$

where

$$A_1 = \chi_{\xi\xi\xi\xi} + \frac{2}{\xi}\chi_{\xi\xi\xi} - \frac{\beta}{\xi^2}\chi_{\xi\xi} + \frac{\beta}{\xi^3}\chi_\xi + \beta\chi_{\tau\tau} - (1 - \frac{\alpha^2}{\beta})Q$$

$$- \frac{12}{\xi}(\frac{a}{h})^2(1 - \frac{\alpha^2}{\beta})(\chi_\xi\phi)_\xi \ .$$

It is reasonable to require that the integral of this virtual work over a complete period of vibration be zero [12], that is,

$$\int_0^{2\pi/\omega} (\delta W) d\tau = 0 \ . \tag{22}$$

Substitution of equations (20a), (20b), and (20c) into (21), δW can be represented in terms of A, $g(\xi)$, $f(\xi)$, $Q^*(\xi)$ and $\sin\omega\tau$,

$$\delta W = \sin^2\omega\tau\left\{\int\frac{1}{R}\left[g_{\xi\xi\xi\xi}+\frac{2}{\xi}g_{\xi\xi\xi}-\frac{\beta}{\xi^2}g_{\xi\xi}+\frac{\beta}{\xi^3}g_{\xi}-\beta\omega^2 g-(1-\frac{\alpha^2}{\beta})(h/Aa)Q^*\right]\delta g\xi d\xi\right\}d\tau$$

$$-\sin^4\omega\tau\left\{\int\frac{1}{R}\left[\frac{12}{\xi}(Aa/h)^2(1-\frac{\alpha^2}{\beta})(g_\xi f)_\xi\right]\delta g\xi d\xi\right\}d\tau \ .$$

Then substituting δW into equation (22) again and integrating, one obtains

$$g''''+\frac{2}{\xi}g'''-\frac{\beta}{\xi^2}g''+\frac{\beta}{\xi^3}g'-\lambda g-9(1-\frac{\alpha^2}{\beta})\frac{\kappa}{\xi}(g'f)'$$

$$=(1-\frac{\alpha^2}{\beta})\frac{Q^*}{\sqrt{\kappa}} \qquad (23a)$$

in which $\kappa=(Aa/h)^2$ and $\lambda=\beta\omega^2$ are additional parameters related to the amplitude and the angular frequency, respectively.

Substitution of equations (20a), (20b) and (20c) into the governing equation (18b) yields

$$f''+\frac{1}{\xi}f'-\frac{\beta}{\xi^2}f+\frac{\beta}{2\xi}(g')^2=0 \qquad (23b)$$

where the prime as the superscript denotes differentiation with respect to ξ, i.e., $f' = df/d\xi$. Based on this operation, the time variable is eliminated from the governing differential equations with an average minimum error over a complete cycle of the motion. The assumed harmonic motions are governed by the pair of nonlinear ordinary nonlinear differential equations (23a) and (23b).

In the same way, substituting equations (20a),(20b) and (20c) into equation (19), the boundary conditions can be converted to be

$$\left.\begin{aligned}
&(g)_{\xi=1} = 0 \\
&(g')_{\xi=1} = 0 \\
&(g')_{\xi=R} = 0 \\
&\left(f' - \frac{\alpha}{\xi} f\right)_{\xi=1} = 0 \\
&\left(f' - \frac{\alpha}{\xi} f\right)_{\xi=R} = 0 \\
&\left(g''' + \frac{1}{\xi} g'' - \frac{\xi}{2}\gamma\lambda g\right)_{\xi=R} = 0
\end{aligned}\right\} \quad (24a)$$

In order to obtain a unique relationship between amplitude, κ, and frequency, λ, an additional restriction, which supplements the boundary conditions, must be set. This requirement is fulfilled by introducing a norminalization condition

$$(g)_{\xi=R} = 1 \quad (24b)$$

Finally, The equations (23a), (23b) and boundary conditions (24a), (24b) which constitute a nonlinear eigenvalue problem describing the finite amplitude harmonic response of an orthotropic circular plate, are expressed in terms of shape functions f, g and their derivatives.

Chapter IV NUMERICAL ANALYSIS

Boundary value problems, as the name implies, have the property that conditions are specified at two sets of values of the independent variables. In constrast to the boundary value problem, the initial value problems are specified by one set of values of the independent variables at the initial point. Consequently, the theory of the existance and uniqueness of solutions to boundary value problem is in a less satisfactory condition than the corresponding theory for initial value problem. It should be expected that numerical solution of a boundary value problem for a given ordinary differential equation will in general be a more difficult matter than the numerical solution of the corresponding initial value problem. [13]

The nonlinear eigenvalue problems formed by nonlinear ordinary differential equations (23a) and (23b), associated with the boundary conditions (24a) and (24b), however, are complicated. Hence it is more convenient to solve the problem through the application of numerical integration to the associated initial value problem.

For computational purposes, the associated initial value problem is started by converting the nth-order differential equation into n first order differential equations. Then, apply the Runge-Kutta-Gill integration method [14] to integrate the differential equation from initial point, with the guessed missing and the given initial values, to final point. The value obtained at the final point, in general, can not satisfy the given conditions. This solution is then used to adjust the initial missing values by the Newton-Raphson scheme in an attempt to make the next solution come "closer" to satisfying all of the necessary conditions. If these steps are repeated and the iterative procedure converges, the boundary conditions (24a) and (24b) will eventually be satisfied.

a. General formulation.

Any nth-order differential equation, linear or non-linear, may be reduced to a set of n-simultaneous first-order differential equations [15]. In the same way, Equations (23a) and (23b) can be reformulated to an equivalent system of six first order equations,

$$\frac{d\bar{Y}}{d\xi} = \bar{H}(\xi, \bar{Y}; \lambda, \kappa, \gamma, Q^*) \qquad R < \xi < 1 \qquad (25a)$$

where

$$\bar{Y}(\xi) = \begin{bmatrix} g \\ g' \\ g'' \\ g''' \\ f \\ f' \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} . \qquad (25b)$$

Again, the prime as the superscript donates differentiation, i.e., $g'=dg/d\xi$, and $\bar{H}$ is the appropriate (6×1) vector function defined as :

$$\bar{H} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ -\frac{2}{\xi}y_4+\frac{\beta}{\xi^2}y_3-\frac{\beta}{\xi^3}y_2+\lambda y_1+9(1-\frac{\alpha^2}{\beta})\frac{\kappa}{\xi}(y_2y_6+y_3y_5) \\ \qquad +(1-\frac{\alpha^2}{\beta})\frac{Q^*}{\sqrt{\kappa}} \\ y_6 \\ -\frac{1}{\xi}y_6+\frac{\beta}{\xi^2}y_5-\frac{\beta}{2\xi}(y_2)^2 \end{bmatrix} . \tag{26}$$

The boundary conditions (24a) and the normalization condition, $y_1(R) = 1$, can be expressed in the following forms :

$$[M]\bar{Y}(R) = \begin{bmatrix} g \\ g' \\ g'''+\frac{1}{\xi}g''-\frac{\xi}{2}\gamma\lambda g \\ f'-\frac{\alpha}{\xi}f \end{bmatrix}_{\xi=R} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27a}$$

and

$$[N]\bar{Y}(1) = \begin{bmatrix} g \\ g' \\ f'-\frac{\alpha}{\xi}f \end{bmatrix}_{\xi=1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{27b}$$

where [M] and [N] are the matrices

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -R\lambda\gamma/2 & 0 & 1/R & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha/R & 1 \end{bmatrix}_{\xi=R} \quad (28a)$$

and

$$[N] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1 \end{bmatrix}_{\xi=1} . \quad (28b)$$

The first row of [M] is the normalized condition, $y_1(R)=1$, at point $\xi=R$ which is the junction of the plate and rigid mass. The physical meanings of the remaining rows of [M], referring to the boundary conditions (24a), are the conditions of zero slope, the shear force at the inner boundary caused by inertial force of the rigid mass and zero radial displacement. Likewise, the rows of [N] define the boundary conditions of zero transverse displacement, slope, and radial displacement at the outer boundary of the circular plate, $\xi=1$.

b. Initial-Value Problem

To obtain a unique solution of equations (25a), six boundary conditions must be specified at the boundary points $\xi=R$ and $\xi=1$. However, for this boundary value problem six given conditions are not specified at one boundary point. In order to integrate numerically the six first-order nonlinear differential equations by computer, the missing conditions must be specified with certain guessing values. These values are called missing initial values. If the unknown conditions at one boundary $\xi=R$ are assigned for certain values, step by step integration of equation (25a) from $\xi=R$ to $\xi=1$ would become possible. Obviously, the values of $\bar{Y}(\xi)$ at $\xi=1$ thus obtained would not, in general, satisfy the given conditions at the other end. The problem is therefore to determine the correct initial values at $\xi=R$ so that all the given conditions at $\xi=1$ can be satisfied. Thus, the corresponding initial-value problem may be expressed as

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi,\bar{Z};\lambda,\kappa,\gamma,Q^*) \qquad R < \xi < 1 \qquad (29a)$$

where

$$\bar{Z}(R) = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix}_{\xi=R} = \begin{bmatrix} 1 \\ 0 \\ \eta_1 \\ R\lambda\gamma/2 - \eta_1/R \\ \eta_2 \\ \alpha\eta_2/R \end{bmatrix}_{\xi=R} \qquad (29b)$$

The $\bar{Z}_i(R)$, initial-value vector constructed from the boundary and normalization conditions at $\xi=R$, is identical with $\bar{Y}_i(R)$ in equation (25b), where $i=1,2,\ldots6,$. The parameters η_1,η_2 and λ in equation (29b) are unknown missing initial values and the eigenvalue. Substitution of the initial values, $\bar{Z}(R)$, and equation (28a) into equation (27a) yields a system of four equations.

$$[M]\bar{Z}(R) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\xi=R} . \tag{30}$$

which satisfy the boundary conditions at $\xi=R$.

A solution of the initial value problem thus formulated by equation (29) is symbolically denoted by

$$\bar{Z}(\xi) = \bar{Z}(R)+\int_R^\xi \bar{H}(\xi,\bar{Z};\eta_1,\eta_2,\lambda,\kappa,\gamma,Q^*)d\xi$$

$$= \bar{Z}(R)+\int_R^\xi \bar{H}(\xi,\bar{Z};\bar{\eta},\kappa,\gamma,Q^*)d\xi \tag{31a}$$

where

$$\bar{\eta} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \lambda \end{Bmatrix} \tag{31b}$$

is an unknown vector to be determined. For any set of the given parameters κ, γ and Q^*, the corresponding values η_1 $,\eta_2$ and λ can be solved through the initial-value problem such that the solution of equation (29) satisfies the three given boundary conditions (27b)

$$[N]\bar{Z}(1;\eta,\kappa) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\xi=1} \tag{32}$$

Assume that $\bar{Z}(1;\eta,\kappa)$ is continuously differentiable with respect to η and κ. Then, by a theorem in matrix theory, for the system of equations (32) to have a unique solution, a necessary and sufficient condition is that the determinant of the Jacobian matrix, $J = \frac{\partial}{\partial \bar{\eta}} \left\{ [N]\bar{Z}(1;\bar{\eta},\kappa) \right\}$, is not zero,

i.e., $$\det \left| [N] \frac{\partial}{\partial \bar{\eta}} \bar{Z}(1;\eta,\kappa) \right| \neq 0 \ .$$

Hence, for any given values of κ, γ and Q^*

$$\bar{Y}(\xi) = \bar{Z}(\xi;\bar{\eta}^*,\kappa,\gamma,Q^*)$$

symbolically denotes a solution to the eigenvalue problems, where

$$\bar{\eta}^* = \begin{Bmatrix} \eta_1^* \\ \eta_2^* \\ \lambda^* \end{Bmatrix} \quad . \tag{33}$$

Thus, it is seen that solving the boundary value problem (23) and (24) is equilvalent to obtaining a continuous set of solutions to the related initial value problem (29) which satisfy equation (32).

A root $\bar{\eta}^*$ may be found by applying Newton-Raphson iteration method [16] to solve equation (32). Starting with an initial guess $\bar{\eta}=\bar{\eta}_0$ and given parameters κ, γ, and Q^*, the iterative covergent sequence

$$\bar{\eta}_{k+1} = \bar{\eta}_k + \Delta\bar{\eta}_k \qquad k=0,1,2,3,\ldots.n \qquad (34)$$

is generated, where $\Delta\bar{\eta}_k$ is the correcting vector and the subscript denotes k-th time of iteration.

Neglecting higher-order terms in the Taylor's series expansion about $\bar{\eta}_k$, one obtains

$$[N]\bar{Z}(1;\bar{\eta}_k+\Delta\bar{\eta}_k,\kappa) = [N]\bar{Z}(1;\bar{\eta}_k,\kappa)+\left\{[N]\frac{\partial}{\partial\bar{\eta}_k}\bar{Z}(1;\bar{\eta}_k,\kappa)\right\}\cdot\Delta\bar{\eta}_k = 0$$

thus

$$\Delta\bar{\eta}_k = -\left\{[N]\frac{\partial}{\partial\bar{\eta}_k}\bar{Z}(1;\bar{\eta}_k,\kappa)\right\}^{-1}\cdot[N]\bar{Z}(1;\bar{\eta}_k,\kappa)$$

$$= -\left\{[N](J_1)_k\right\}^{-1}\cdot[N]\bar{Z}(1;\bar{\eta}_k,\kappa) \qquad (35)$$

where, at the k-th step, the (6×3) Jacobian matrix (J_1) is defined as

$$(J_1)_k = \left[\frac{\partial \bar{Z}}{\partial \bar{\eta}} \right]_{\xi=1} = \begin{bmatrix} \frac{\partial Z_1}{\partial \eta_1} & \frac{\partial Z_1}{\partial \eta_2} & \frac{\partial Z_1}{\partial \lambda} \\ \frac{\partial Z_2}{\partial \eta_1} & \frac{\partial Z_2}{\partial \eta_2} & \frac{\partial Z_2}{\partial \lambda} \\ \frac{\partial Z_3}{\partial \eta_1} & \frac{\partial Z_3}{\partial \eta_2} & \frac{\partial Z_3}{\partial \lambda} \\ \frac{\partial Z_4}{\partial \eta_1} & \frac{\partial Z_4}{\partial \eta_2} & \frac{\partial Z_4}{\partial \lambda} \\ \frac{\partial Z_5}{\partial \eta_1} & \frac{\partial Z_5}{\partial \eta_2} & \frac{\partial Z_5}{\partial \lambda} \\ \frac{\partial Z_6}{\partial \eta_1} & \frac{\partial Z_6}{\partial \eta_2} & \frac{\partial Z_6}{\partial \lambda} \end{bmatrix}_{\xi=1} \qquad (36)$$

which represents a change of final values with respect to a change in the initial value $\bar{\eta}$, while $[N]\bar{Z}(1;\bar{\eta}_k,\kappa)$ represents the k-th error vector expressed as

$$[N]\bar{Z}(1;\bar{\eta}_k,\kappa) = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_6-\alpha Z_5 \end{bmatrix}_{\xi=1}. \qquad (37)$$

Therefore, the correcting vector can be expressed in matrix form :

$$
\Delta\bar{\eta} = \begin{bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{bmatrix} = - \left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial Z_1}{\partial \eta_1} & \frac{\partial Z_1}{\partial \eta_2} & \frac{\partial Z_1}{\partial \lambda} \\ \frac{\partial Z_2}{\partial \eta_1} & \frac{\partial Z_2}{\partial \eta_2} & \frac{\partial Z_2}{\partial \lambda} \\ \frac{\partial Z_3}{\partial \eta_1} & \frac{\partial Z_3}{\partial \eta_2} & \frac{\partial Z_3}{\partial \lambda} \\ \frac{\partial Z_4}{\partial \eta_1} & \frac{\partial Z_4}{\partial \eta_2} & \frac{\partial Z_4}{\partial \lambda} \\ \frac{\partial Z_5}{\partial \eta_1} & \frac{\partial Z_5}{\partial \eta_2} & \frac{\partial Z_5}{\partial \lambda} \\ \frac{\partial Z_6}{\partial \eta_1} & \frac{\partial Z_6}{\partial \eta_2} & \frac{\partial Z_6}{\partial \lambda} \end{bmatrix}_{\xi=1} \right]^{-1}
$$

$$
\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix}_{\xi=1}
$$

i.e.,

$$
\begin{bmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \frac{\partial Z_1}{\partial \eta_1} & \frac{\partial Z_1}{\partial \eta_2} & \frac{\partial Z_1}{\partial \lambda} \\ \frac{\partial Z_2}{\partial \eta_1} & \frac{\partial Z_2}{\partial \eta_2} & \frac{\partial Z_2}{\partial \lambda} \\ \frac{\partial Z_6}{\partial \eta_1} - \alpha\frac{\partial Z_5}{\partial \eta_1} & \frac{\partial Z_6}{\partial \eta_2} - \alpha\frac{\partial Z_5}{\partial \eta_2} & \frac{\partial Z_6}{\partial \lambda} - \alpha\frac{\partial Z_5}{\partial \lambda} \end{bmatrix}_{\xi=1}^{-1} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_6 - \alpha Z_5 \end{bmatrix}_{\xi=1} \tag{38}
$$

which provides the linear correction of the estimated data η_1, η_2 and λ. If the initial estimate of $\bar{\eta}$ is chosen in a sufficiently small neighborhood of the root $\bar{\eta}^*$, the sequence (34) will converge to the root $\bar{\eta}^*$. The matrix $(J_1)_k$ must be evaluated at each step of the iteration process in order to generate the correction vector $\Delta\bar{\eta}_k$ and then to find the solutions. Thus, the associated variational equations are introduced :

$$\frac{d}{d\xi}\left(\frac{\partial \bar{Z}}{\partial \eta_1}\right) = \frac{\partial}{\partial \eta_1}\left(\frac{d\bar{Z}}{d\xi}\right) = \left(\frac{\partial \bar{H}}{\partial \bar{Z}}\right)\left(\frac{\partial \bar{Z}}{\partial \eta_1}\right)$$

$$\frac{d}{d\xi}\left(\frac{\partial \bar{Z}}{\partial \eta_2}\right) = \frac{\partial}{\partial \eta_2}\left(\frac{d\bar{Z}}{d\xi}\right) = \left(\frac{\partial \bar{H}}{\partial \bar{Z}}\right)\left(\frac{\partial \bar{Z}}{\partial \eta_2}\right) \qquad (39)$$

$$\frac{d}{d\xi}\left(\frac{\partial \bar{Z}}{\partial \lambda}\right) = \frac{\partial}{\partial \eta}\left(\frac{d\bar{Z}}{d\xi}\right) = \left(\frac{\partial \bar{H}}{\partial \bar{Z}}\right)\left(\frac{\partial \bar{Z}}{\partial \lambda}\right)+\frac{\partial \bar{H}}{\partial \lambda} \quad .$$

The specific expressions, which result from performing the operations indicated in equation (39) can be presented as follows:

$$\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial \eta_1}\right) = \frac{\partial Z_2}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial \eta_1}\right) = \frac{\partial Z_3}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial \eta_1}\right) = \frac{\partial Z_4}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial \eta_1}\right) = -\frac{2}{\xi}\left(\frac{\partial Z_4}{\partial \eta_1}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_3}{\partial \eta_1}\right)-\frac{\beta}{\xi^3}\left(\frac{\partial Z_2}{\partial \eta_1}\right)+\lambda\left(\frac{\partial Z_1}{\partial \eta_1}\right)$$

$$+9\left(1-\frac{\alpha^2}{\beta}\right)\frac{\kappa}{\xi}\left[Z_3\left(\frac{\partial Z_5}{\partial \eta_1}\right)+Z_5\left(\frac{\partial Z_3}{\partial \eta_1}\right)+Z_2\left(\frac{\partial Z_6}{\partial \eta_1}\right)+Z_6\left(\frac{\partial Z_2}{\partial \eta_1}\right)\right] \qquad (40a)$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial \eta_1}\right) = \frac{\partial Z_6}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial \eta_1}\right) = -\frac{1}{\xi}\left(\frac{\partial Z_6}{\partial \eta_1}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_5}{\partial \eta_1}\right)-\frac{\beta}{\xi}(Z_2)\left(\frac{\partial Z_2}{\partial \eta_1}\right)$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial \eta_2}\right) = \frac{\partial Z_2}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial \eta_2}\right) = \frac{\partial Z_3}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial \eta_2}\right) = \frac{\partial Z_4}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial \eta_2}\right) = -\frac{2}{\xi}\left(\frac{\partial Z_4}{\partial \eta_2}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_3}{\partial \eta_2}\right)-\frac{\beta}{\xi^3}\left(\frac{\partial Z_2}{\partial \eta_2}\right)+\lambda\left(\frac{\partial Z_1}{\partial \eta_2}\right)$$

$$+9\left(1-\frac{\alpha^2}{\beta}\right)\frac{\kappa}{\xi}\left[Z_3\left(\frac{\partial Z_5}{\partial \eta_2}\right)+Z_5\left(\frac{\partial Z_3}{\partial \eta_2}\right)+Z_2\left(\frac{\partial Z_6}{\partial \eta_2}\right)+Z_6\left(\frac{\partial Z_2}{\partial \eta_2}\right)\right] \qquad (40b)$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial \eta_2}\right) = \frac{\partial Z_6}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial \eta_2}\right) = -\frac{1}{\xi}\left(\frac{\partial Z_6}{\partial \eta_2}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_5}{\partial \eta_2}\right)-\frac{\beta}{\xi}(Z_2)\left(\frac{\partial Z_2}{\partial \eta_2}\right)$$

$$
\left.
\begin{aligned}
\frac{d}{d\xi}\left(\frac{\partial Z_1}{\partial\lambda}\right) &= \frac{\partial Z_2}{\partial\lambda} \\
\frac{d}{d\xi}\left(\frac{\partial Z_2}{\partial\lambda}\right) &= \frac{\partial Z_3}{\partial\lambda} \\
\frac{d}{d\xi}\left(\frac{\partial Z_3}{\partial\lambda}\right) &= \frac{\partial Z_4}{\partial\lambda} \\
\frac{d}{d\xi}\left(\frac{\partial Z_4}{\partial\lambda}\right) &= -\frac{2}{\xi}\left(\frac{\partial Z_4}{\partial\lambda}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_3}{\partial\lambda}\right)-\frac{\beta}{\xi^3}\left(\frac{\partial Z_2}{\partial\lambda}\right)+\lambda\left(\frac{\partial Z_1}{\partial\lambda}\right)+Z_1 \\
&\quad +9\left(1-\frac{\alpha^2}{\beta}\right)\frac{\kappa}{\xi}\left[Z_3\left(\frac{\partial Z_5}{\partial\lambda}\right)+Z_5\left(\frac{\partial Z_3}{\partial\lambda}\right)+Z_2\left(\frac{\partial Z_6}{\partial\lambda}\right)+Z_6\left(\frac{\partial Z_2}{\partial\lambda}\right)\right] \\
\frac{d}{d\xi}\left(\frac{\partial Z_5}{\partial\lambda}\right) &= \frac{\partial Z_6}{\partial\lambda} \\
\frac{d}{d\xi}\left(\frac{\partial Z_6}{\partial\lambda}\right) &= -\frac{1}{\xi}\left(\frac{\partial Z_6}{\partial\lambda}\right)+\frac{\beta}{\xi^2}\left(\frac{\partial Z_5}{\partial\lambda}\right)-\frac{\beta}{\xi}(Z_2)\left(\frac{\partial Z_2}{\partial\lambda}\right)
\end{aligned}
\right]
\qquad (40c)
$$

Also, variational equations at $\xi=R$ may be obtained by taking derivatives of $\overline{Z}$ with respect to η_1, η_2 and λ seperately, namely

$$
\left[\frac{\partial\overline{Z}}{\partial\eta_1}\right]_{\xi=R} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1/R \\ 0 \\ 0 \end{bmatrix} \qquad (41a)
$$

$$\left[\frac{\partial \bar{Z}}{\partial \eta_2}\right]_{\xi=R} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \alpha/R \end{bmatrix} \tag{41b}$$

$$\left[\frac{\partial \bar{Z}}{\partial \lambda}\right]_{\xi=R} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma R/2 \\ 0 \\ 0 \end{bmatrix} . \tag{41c}$$

Equations (40a), (40b) and (40c), 18 first-order equations, and equations (41a), (41b) and (41c), the initial values, constitute the variational problem. For a given vector $\bar{\eta}$ and given values of κ and γ, this derived problem and the initial value problem (29) may be numerically integrated simultaneously over the interval from $\xi=R$ to $\xi=1$ with a Runge-Kutta-Gill method. Evaluation of the resulting solution to the variational problems at $\xi=1$ provides the values for each entry of the Jacobian matrix $(J_1)_1$ corresponding to the given values of $\bar{\eta}$, κ and γ. Recall equation (38), which may be rewritten symbolically as

$$[\Delta\bar{\eta}_k] = (J_1)^{-1}[Z] .$$

Premultiply (J_1) on each side of the equation, we obtain

$$(J_1)\ [\Delta\bar{\eta}_k] = [Z] .$$

Then, applying Gaussian Elimination method, we can solve the matric equation easily, and obtain a set of correcting vectors $\Delta\bar{\eta}_k$ for this step k. Substitution of this correcting vector into equation (34), we obtain a new set of trial values for the vector $\bar{\eta}$. Repeating the same procedure until the results converges within a specified error bound which is usually related to the accuracy of the integration method used.

Having obtained a root $\bar{\eta}^{*(j)}$ corresponding to $\kappa=\kappa^{(j)}$, with a fixed γ, the solutions corresponding to the value of amplitude parameter $\kappa^{(j)}$ are obtained. Next, in order to obtain a solution for a higher value of κ than $\kappa^{(j)}$, the method of continuation is applied. The value of amplitude parameter $\kappa^{(j)}$ is perturbed to be

$$\kappa^{(j+1)}= \kappa^{(j)}+ \Delta\kappa^{(j)}, \qquad j=0,1,2,3,...n .$$

For every new κ, iteration is restarted from the previously obtained values of $\bar{\eta}=\bar{\eta}^{*(j)}$. If $\Delta\kappa^{(j)}$ is small enough for $\kappa^{(j)}$ to be within the new contraction domain of the Newton-Raphson method, iteration will converge to the new root $\bar{\eta}^{*(j+1)}$, corresponding to the new $\kappa^{(j+1)}$. Successive this repetition until κ reaches the reasonable large value.

Chapter V Numerical Computation

The application of a numerical integration technique is suggested for the analysis presented previously. Thus, by integrating the initial-value problem (29) with a fourth-order Runge-Kutta-Gill method, and performing the successive iterations by Newton-Raphson method, approximate numerical solutions to the boundary value problem, (23) and (24), can be obtained. A FORTRAN 77 computer program listed in appendix B for numerical calculation is available for this particular problem.

The following procedure of numerical computation is suggested and used in the investegation of the problem :

First, the problem of free linear vibration is considered . Thus, the loading parameter Q^* and amplitude parameter κ are set equal to zero; the elastic constant ratios, α and β, for the orthotropic plate are set equal to a specific value. In this case the equation (23a), which governs the transverse displacement, becomes linear. Then the initial-value problem (29) and the associated variational problem (40) and (41) can be integrated numerically over the interval [R,1] with a set of estimated initial values $\bar{\eta}_1$. By the Newton-Raphson method, successive correction and integration are carried out until the error norm satisfies the inequality :

$$\mathrm{Max}\left|[N]\bar{Z}(1)\right| < 0.1\times10^{-5}$$

where Max| |, the maximum-element norm of the error vector, is consistent with the forth-order Runge-Kutta-Gill integration method with step size μ=1/40. Therefore, the solution for the linear free vibration of the circular plate is obtained and the corresponding values of $\bar{\eta}^*$ are stored. From this solution of the linear free vibration the solutions of finite amplitudes can be examined with the concept of neighboring solutions, which is called the method of continuation. The discrete representations of the resonance curves and accompanying solutions are found by successively increasing the value of amplitude, κ, and re-starting the correction and integration procedure from those values of $(\eta_1, \eta_2, \lambda)$ obtained in the solution corresponding to the previous value of κ. This process is terminated when κ reaches a reasonable large value, κ^m, and the corresponding initial values of $(\eta_1^m, \eta_2^m, \lambda^m)$ are stored. At this stage the steady state response due to forced vibration can be determined by a perturbation technique . On setting $\kappa=\kappa^m$, $(\eta_1, \eta_2, \lambda) = (\eta_1^m, \eta_2^m, \lambda^m)$, and applying a small load, $Q^*(\xi)$, to the plate, the response is easily determined with a slight modification of the above mentioned process. Namely, the type of loading, $Q^*(\xi)$ is held fixed while the value of κ is gradually decremented from κ^m as the integration and correction procedure of the initial-value method yields successive solutions to the forced problem. Hence, distinct response curves are obtained by merely changing the sign of $Q^*(\xi)$. The totality of this approach is illustrated schematically in figure 2.

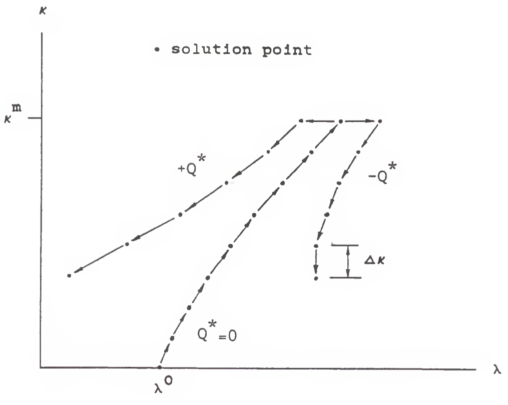

Fig. 2.

Figures 3 through 6 represent the graphs of the dimensionless angular frequency vs the elastic constant ratio, c. The results point out that the frequency parameter ω for an axisymmetric free vibration of an orthotropic circular plate carrying a concentric mass increases as the stiffness parameter, c, increases, while the radius ratio is held constant. These data provide the starting frequency of nonlinear free vibration for different materials which possess different stiffness parameters. If the stiffness constant becomes unity, that is c=1.0, then the orthotropic plate case is simplied to a isotropic plate case. The results of linear free vibration of isotropic circular plate with different mass ratio are illustrated in Table 1. Close agreement with the results found by Becker [5] and Chiang and Chen [4] is observed directly.

The results in tables 2 through 5 list some of the data set in figures 3 through 6, which illustrate the influence of variations of stiffness parameters c on linear angular frequency. The higher stiffness parameter causes a higher angular frequency as the mass ratio is held fixed.

Figures 7 and 8 present the results determined for nonlinear free vibration with various values of stiffness parameter (c=0.5, 1.0, and 2.0), radius ratio (R=0.1 and 0.3). At the same amplitude, the materials with a higher stiffness parameter oscillate at higher resonant frequency, while the materials with a low stiffness parameter oscillate at a lower resonant frequency. In addition, the lower stiffness material has a steeper resonant curve than higher stiffness materials. The physical meaning of this phenomenon is that the resonant frequency of higher stiffness materials increases faster than that of lower stiffness materials while the amplitude increases at the same amount.

In figures 9-14, forced nonlinear vibrations areindicated with various values of stiffness parameters (c=0.5, 1.0, and 2.0), radius ratios (R=0.1 and 0.3) and acting force parameters (Q^*=0.0, -20.0, 20.0, -40.0 and 40.0). In practice it is the custom to plot $|w/h|$ against ω . It is noted also that the quantity of amplitude is negative on the response curves to the right of the free oscillation curve for Q^*=0 and positive to the left of it, which means that the motion is in phase with the external force when Q^* is postiive and 180° out of phase with the external force when Q^* is negative. This behavior of nonlinear resonance is similar to that found for the free and forced oscillation of Duffing's hard spring system for a single degree of freedom [17].

Chapter VI Stress Analysis

Stress distribution on the circular plate can be obtained from the stress-strain relation and strain-displacement relation [7]. The following symbols and expressions are used to denote the stresses

σ_r^b: radial bending stress

σ_θ^b: circumferential bending stress

σ_r^m: radial membrane stress

σ_θ^m: circumferential membrane stress

where

$$\sigma_r^b = \pm \frac{\beta h}{2a_{22}(\beta-\alpha^2)a}(\chi_{\xi\xi} + \frac{\alpha}{\xi}\chi_\xi)$$

$$\sigma_\theta^b = \pm \frac{\beta h}{2a_{22}(\beta-\alpha^2)a}(\alpha\chi_{\xi\xi} + \frac{\beta}{\xi}\chi_\xi)$$

$$\sigma_r^m = \pm \frac{1}{a_{22}}(\frac{\phi}{\xi})$$

and $$\sigma_\theta^m = \pm \frac{1}{a_{22}}(\phi_\xi) \quad ,$$

which are derived from equations (1) and (2), relate the bending and membrane stresses to the dimensionless deflection, χ, and stress function, ϕ, respectively.

In terms of the previous assumptions,

$$\chi(\xi,\tau) = A\, g(\xi)\sin\omega\tau$$

$$\phi(\xi,\tau) = A^2 f(\xi)\sin^2\omega\tau$$

and when time, τ, is equal to an odd multiple of $\pi/2\omega$ the maximum excursions occur, thus stresses become

$$\frac{\sigma_r^b a_{22} a^2}{h^2} = \pm \frac{\beta\sqrt{\kappa}}{2(\beta-\alpha^2)}\left(g'' + \frac{\alpha}{\xi}g'\right)$$

$$\frac{\sigma_\theta^b a_{22} a^2}{h^2} = \pm \frac{\beta\sqrt{\kappa}}{2(\beta-\alpha^2)}\left(\alpha g'' + \frac{\beta}{\xi}g'\right) \qquad (44)$$

$$\frac{\sigma_r^m a_{22} a^2}{h^2} = \pm \kappa\left(\frac{f}{\xi}\right)$$

and

$$\frac{\sigma_\theta^m a_{22} a^2}{h^2} = \pm \kappa f_\xi \quad .$$

Figures 15 through 17 represent the radial bending stresses; figures 18 through 20 represent the radial membrane stresses; figures 21 through 23 represent the circumferential bending stresses, while figures 24 through 26 represent the circumferential membrane stresses. All of these stresses are calculated with radius ratio, R=0.3, mass ratio, γ=1.0, in the condition of free vibration, i.e., Q^*=0.0. Also, the stress distribution at the inner edge and at the outer edge of radial membrane stress and radial bending stress are shown in figures 27 through 30

From equations (44), it is noted the amplitude has a great influence upon the distrubution of bending stress. The system behaves as a linear property when the amplitude is small, say, κ=0.2. On the contrary, a higher amplitude causes stress to be larger and has a nonlinear distribution. Also, elastic constant ratios α and β, shape functions f and g and their derivatives and location ξ, play an important role on the stress distribution.

Table 1

Dimensionless Angular Frequency for Linear Free Vibration

R \ γ	0.1			0.3		
	present investigation	Becker	Chiang and Chen	present investigation	Becker	Chiang and Chen
1.0	10.4410	10.4413	10.4400	12.3858	12.3862	12.3855
2.0	10.1994	10.1997	10.1984	10.9334	10.9336	10.9329
8.0	9.0084	9.0084	9.0072	7.1365	7.1364	7.1364
16.0	7.8944	7.8944	7.8933	5.3642	5.3643	5.3639

Table 2

Dimensionless Angular Frequency ω_l

$\gamma = 1.0$				
c \ R	0.1	0.2	0.3	0.4
0.5	8.1811	8.5052	9.2413	10.5576
1.0	10.4410	11.1341	12.3858	14.4140
1.5	12.2449	13.2373	14.8760	17.4362
2.0	13.8016	15.0459	17.0044	20.0067
2.5	15.1942	16.6580	18.8941	22.2826
3.0	16.4667	18.1267	20.6112	24.3465

Table 3

Dimensionless Angular Frequency ω_l

$\gamma = 2.0$				
c \ R	0.1	0.2	0.3	0.4
0.5	8.0053	7.9246	8.1674	8.8931
1.0	10.1994	10.3559	10.9334	12.1348
1.5	11.9530	12.3031	13.1261	14.6764
2.0	13.4673	13.9787	15.0009	16.8385
2.5	14.8224	15.4728	16.6659	18.7528
3.0	16.0610	16.8343	18.1789	20.4891

Table 4

Dimensionless Angular Frequency ω_ℓ

$\gamma = 8.0$				
c \ R	0.1	0.2	0.3	0.4
0.5	7.1239	5.8871	5.3415	5.3524
1.0	9.0084	7.6526	7.1365	7.2987
1.5	10.5241	9.0746	8.5618	8.8254
2.0	11.8730	10.3007	9.7813	10.1243
2.5	13.0139	11.3950	10.8646	11.2746
3.0	14.0908	12.3928	11.8493	12.3179

Table 5

Dimensionless Angular Frequency ω_ℓ

$\gamma = 16.0$				
c \ R	0.1	0.2	0.3	0.4
0.5	6.2795	4.6424	4.0171	3.9324
1.0	7.8941	6.0229	5.3642	5.3615
1.5	9.2006	7.1371	6.4345	6.4827
2.0	10.3351	8.0985	7.3503	7.4367
2.5	11.3536	8.9569	8.1639	8.2815
3.0	12.2864	9.7398	8.9035	9.0477

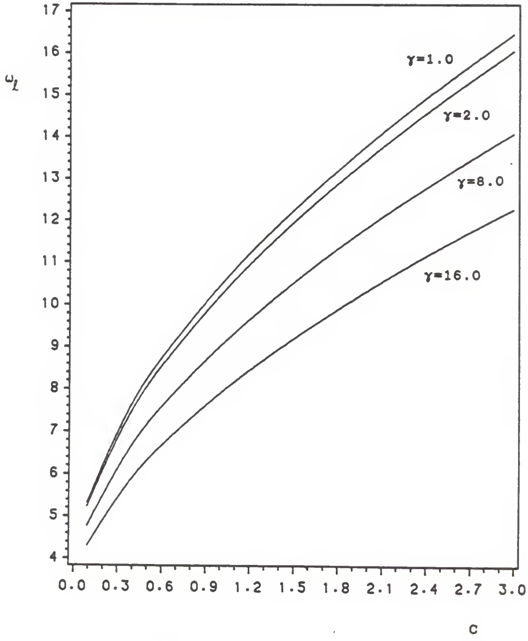

Fig. 3. Dimensionless Angular Frequency with R=0.1

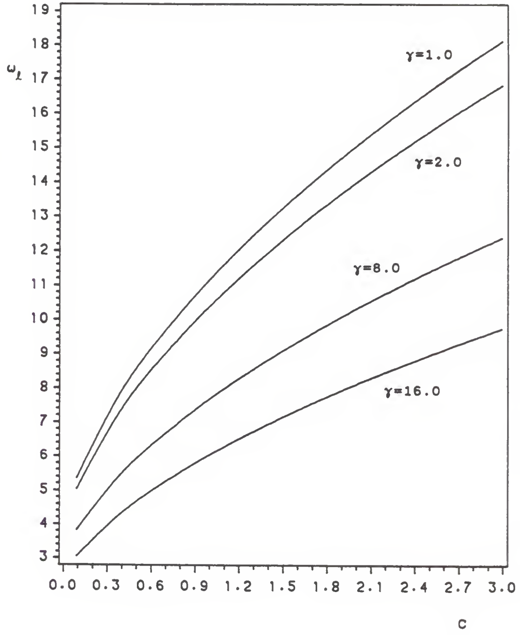

Fig. 4. Dimensionless Angular Frequency with R=0.2

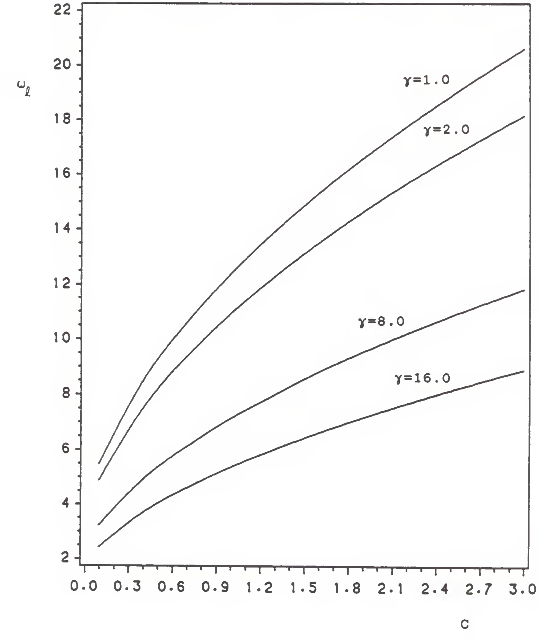

Fig. 5. Dimensionless Angular Frequency with R=0.3

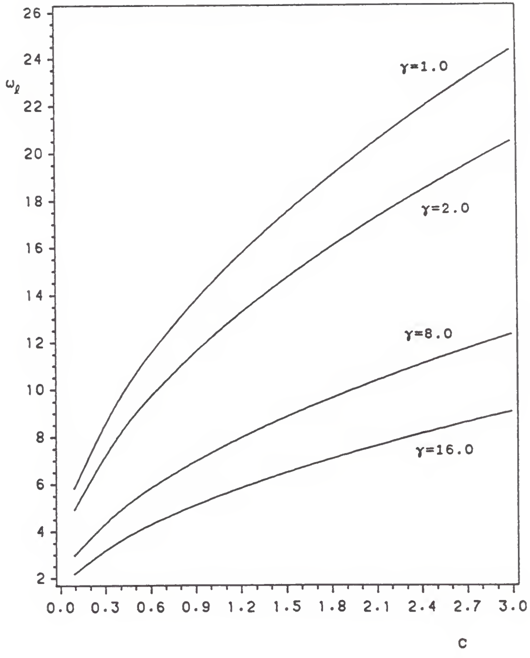

Fig. 6. Dimensionless Angular Frequency with R=0.4

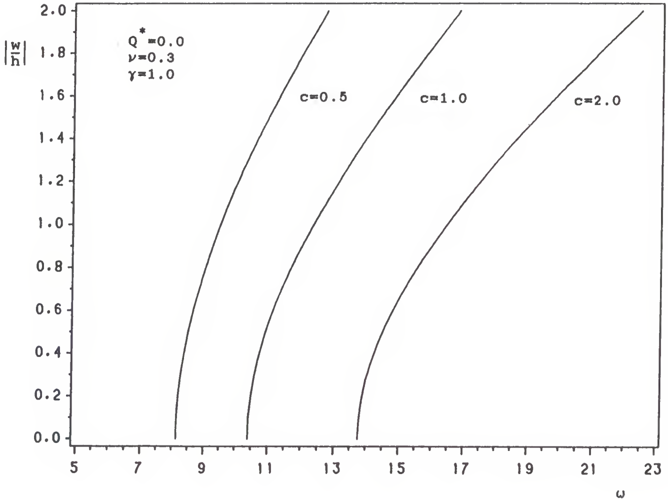

Fig. 7. Harmonic Response of Free Vibration with R=0.1

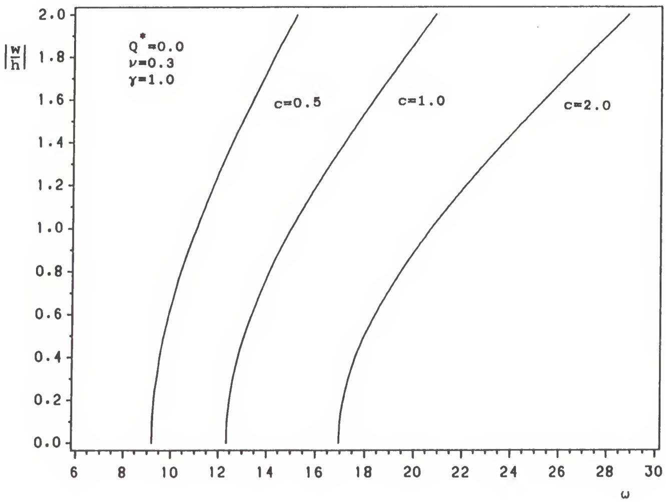

Fig. 8. Harmonic Response of Free Vibration with R=0.3

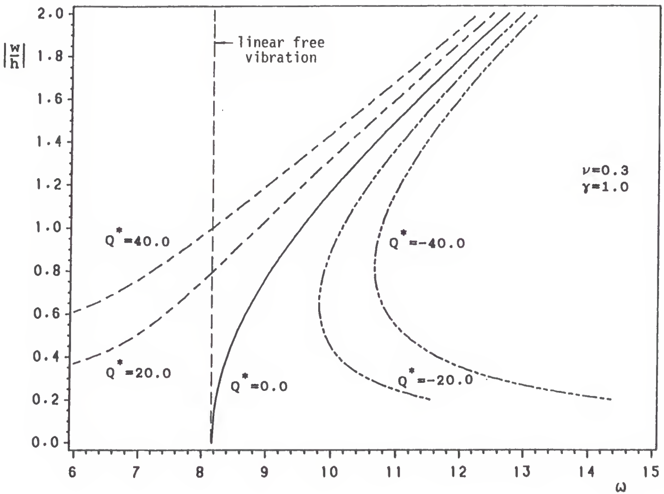

Fig. 9. Harmonic Response of Forced Vibration with R=0.1 and c=0.5

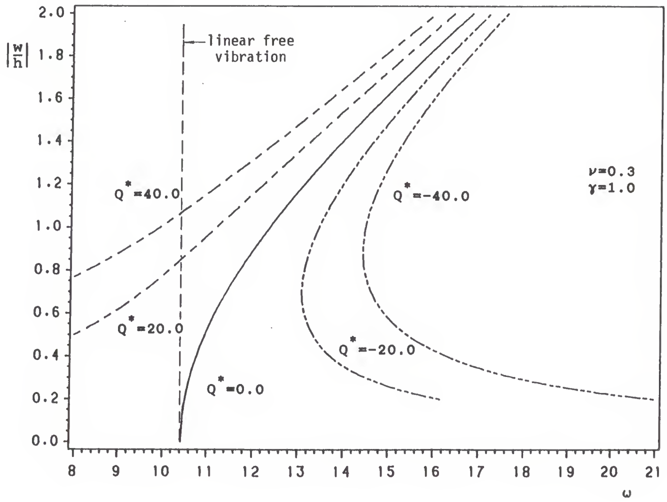

Fig. 10. Harmonic Response of Forced Vibration with R=0.1 and c=1.0

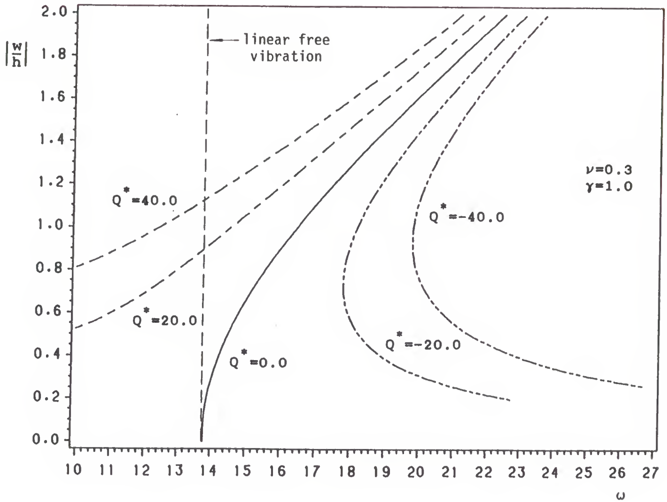

Fig. 11. Harmonic Response of Forced Vibration with R=0.1 and c=2.0

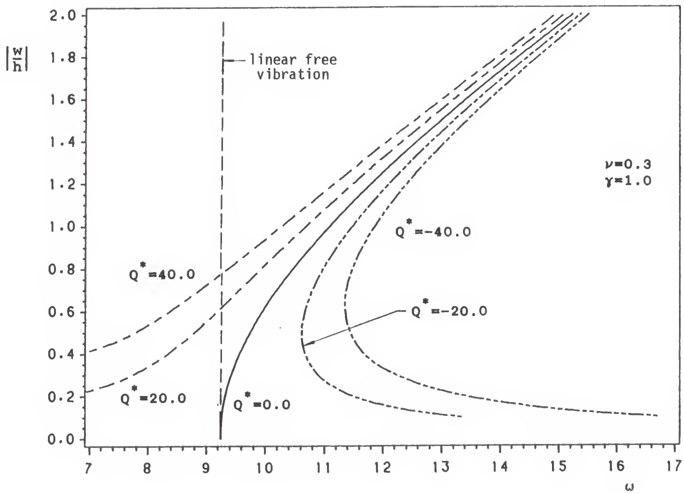

Fig. 12. Harmonic Response of Forced Vibration with R=0.3 and c=0.5

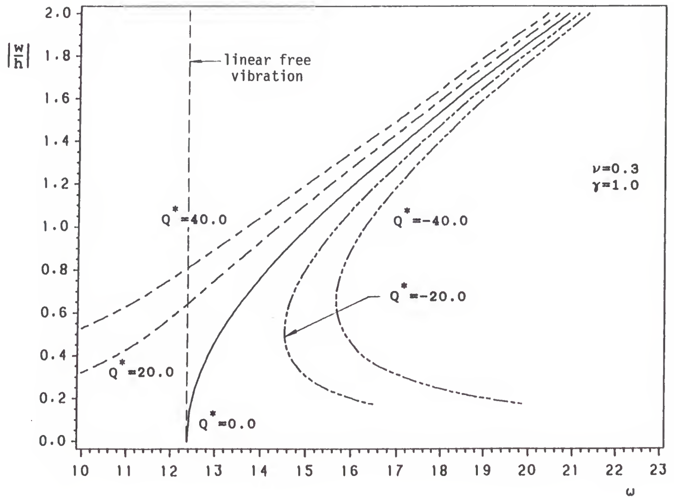

Fig. 13. Harmonic Response of Forced Vibration with R=0.3 and c=1.0

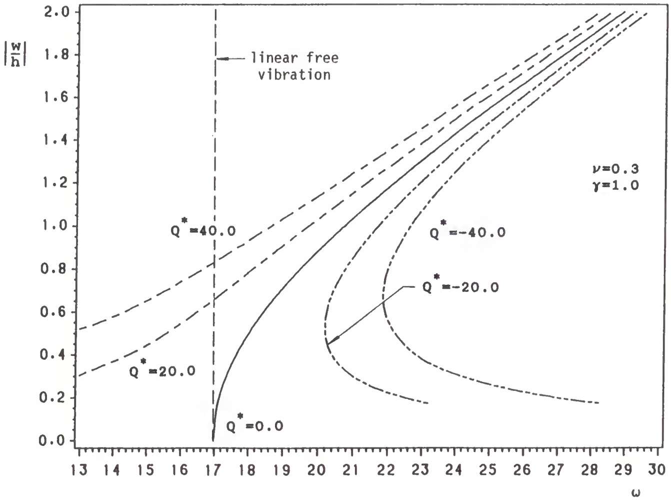

Fig. 14. Harmonic Response of Forced Vibration with R=0.3 and c=2.0

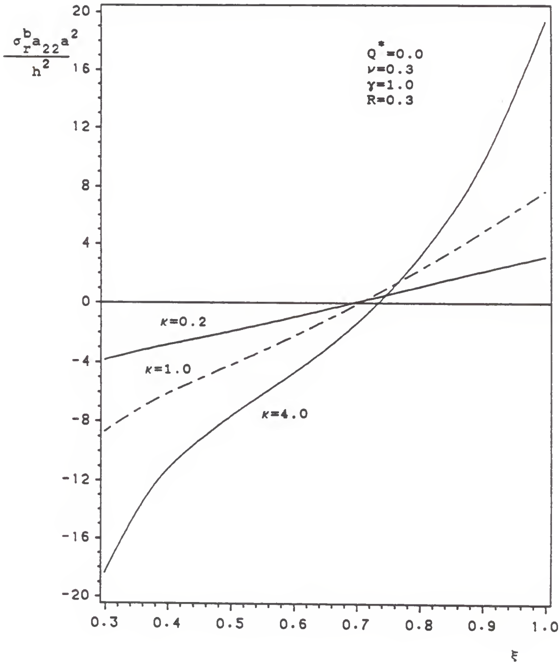

Fig. 15. Radial Bending Stress with c=0.5

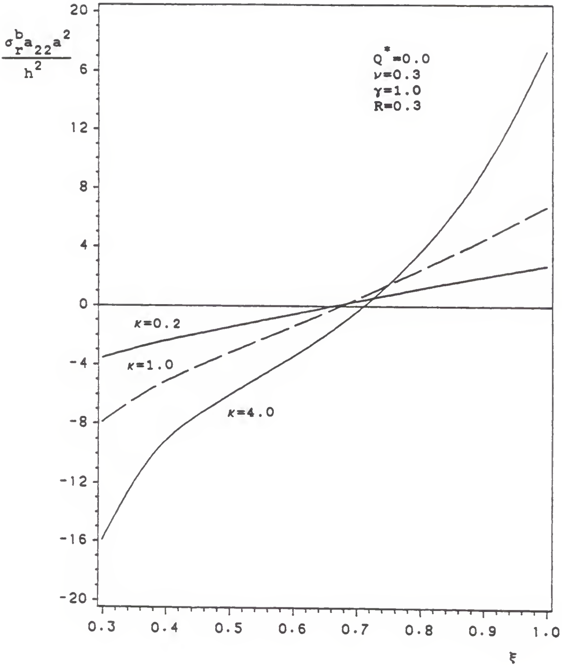

Fig. 16. Radial Bending Stress with c=1.0

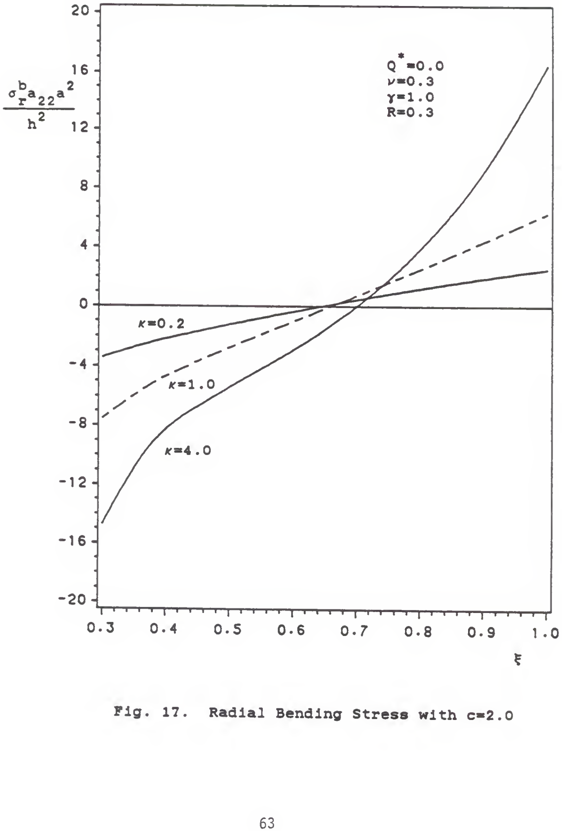

Fig. 17. Radial Bending Stress with c=2.0

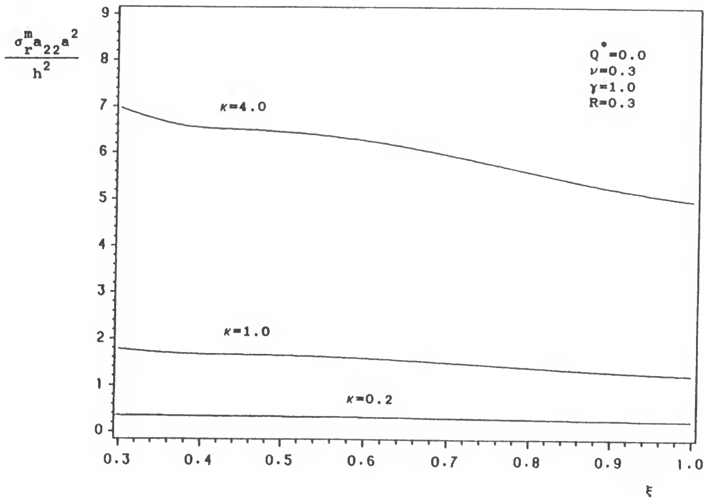

Fig. 18. Radial Membrane Stress with c=0.5

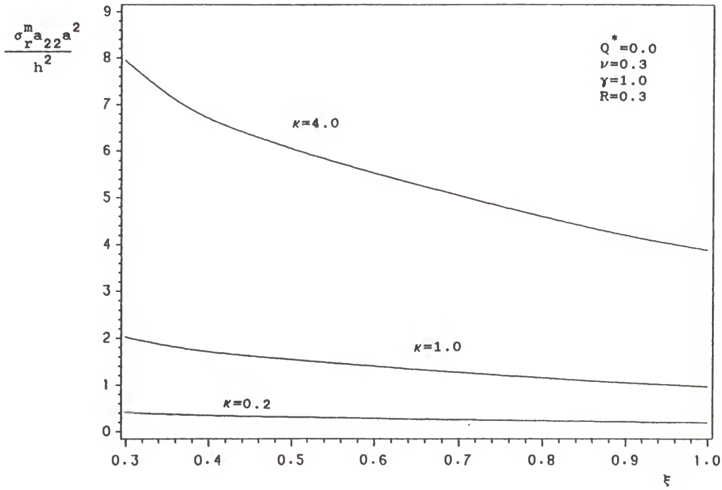

Fig. 19. Radial Membrane Stress with c=1.0

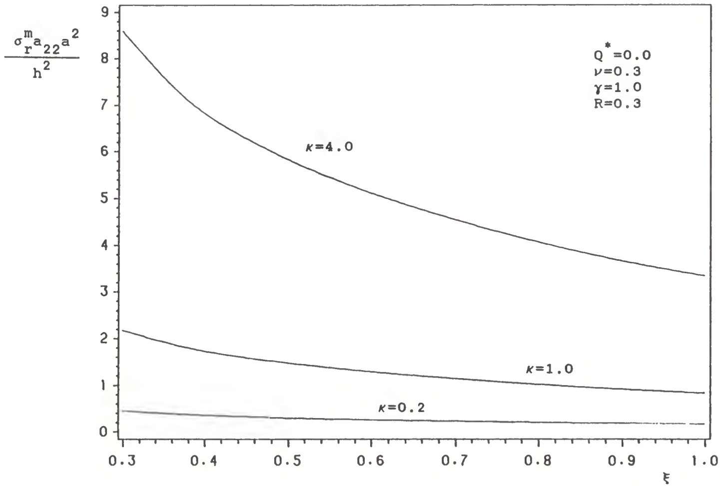

Fig. 20. Radial Membrane Stress with c=2.0

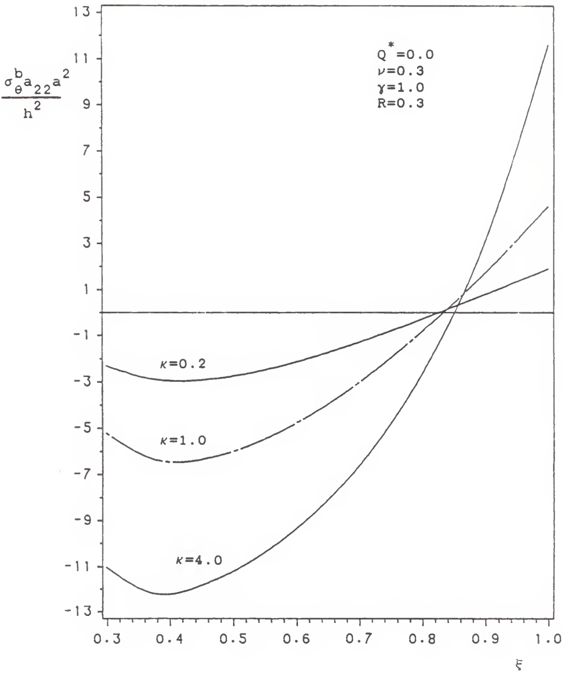

Fig. 21. Circumferential Bending Stress with c=0.5

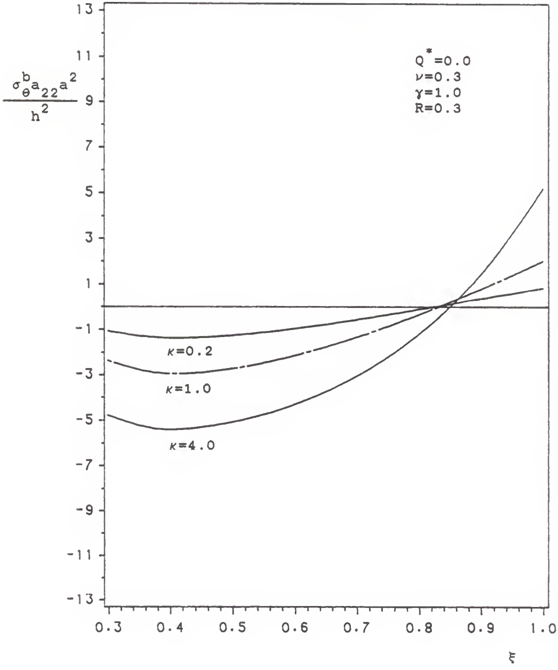

Fig. 22. Circumferential Bending Stress with c=1.0

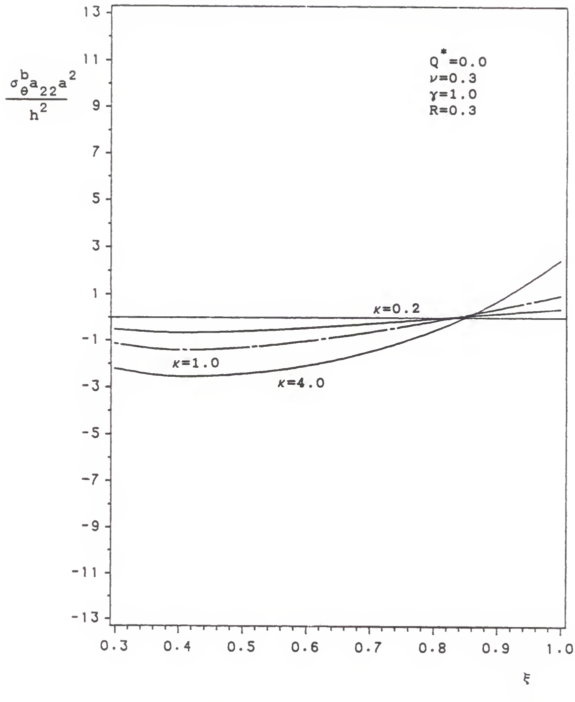

Fig. 23. Circumferential Bending Stress with c=2.0

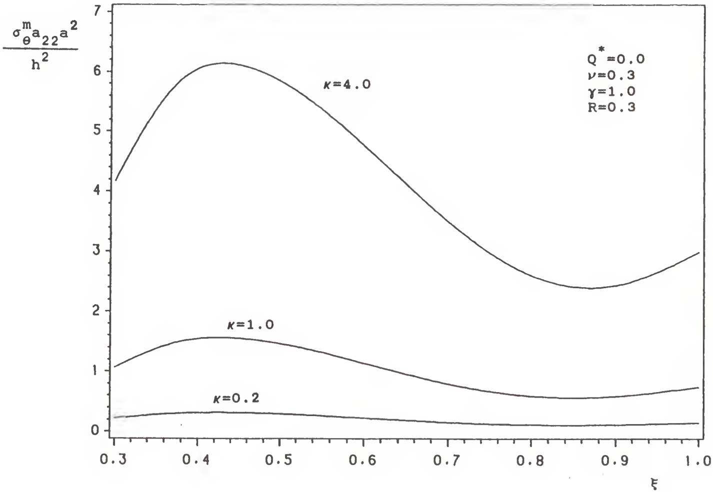

Fig. 24. Circumferential Membrane Stress with c=0.5

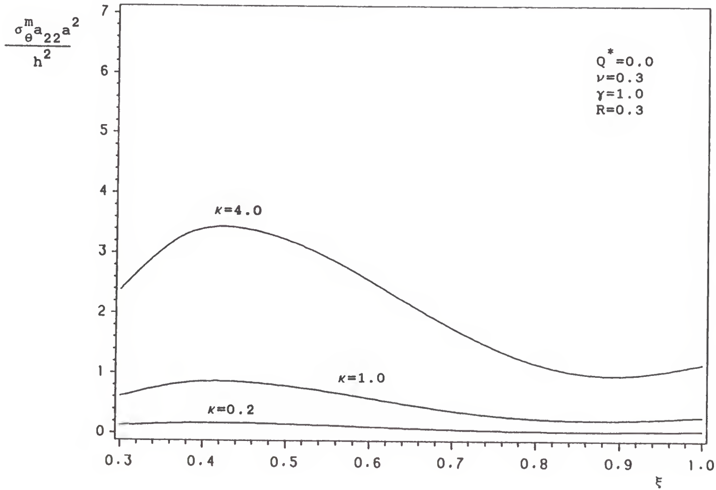

Fig. 25. Circumferential Membrane Stress with c=1.0

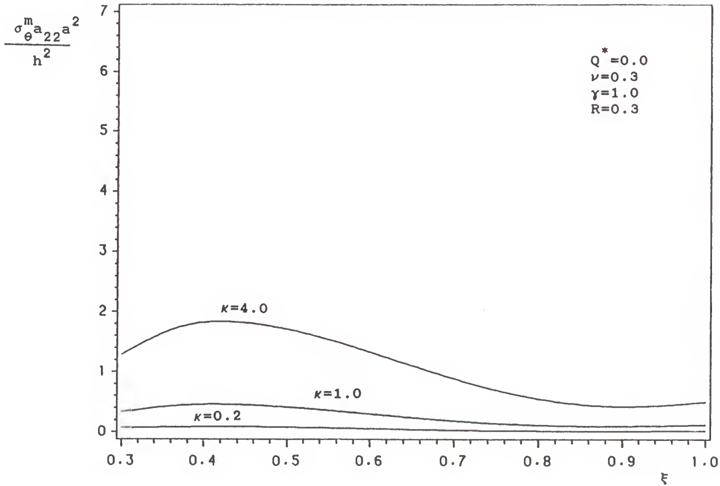

Fig. 26. Circumferential Membrane Stress with c=2.0

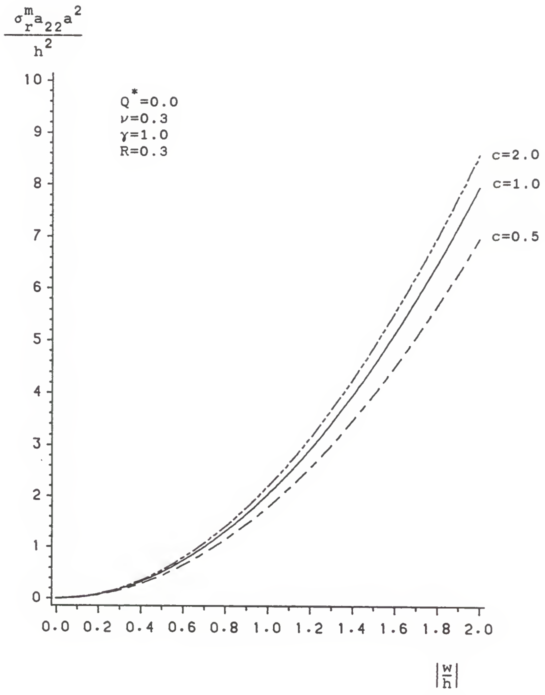

Fig. 27. Radial Membrane Stress at the inner edge

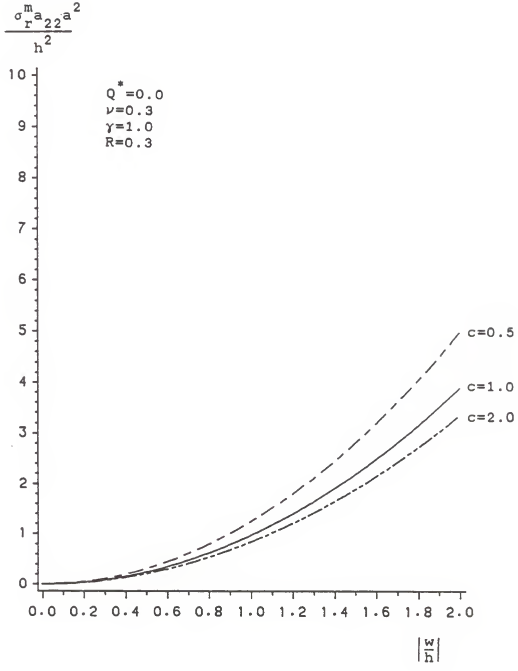

Fig. 28. Radial Membrane Stress at the outer edge

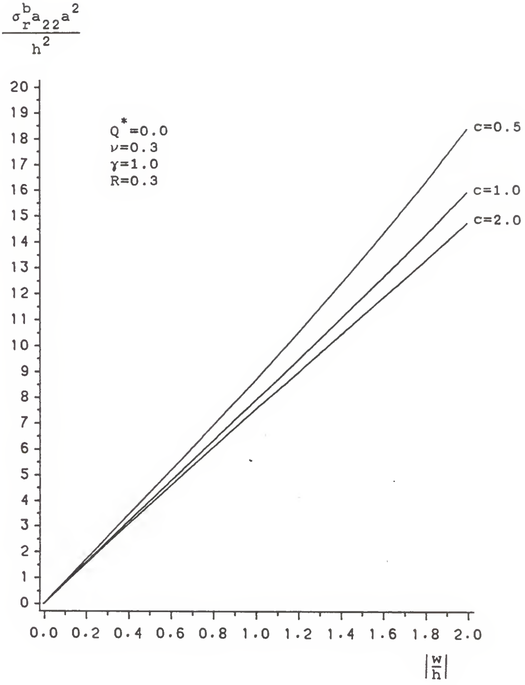

Fig. 29. Radial Bending Stress at the inner edge

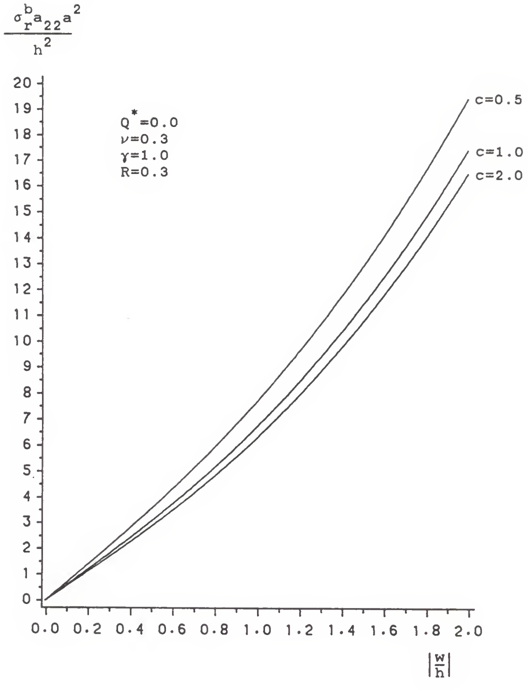

Fig. 30. Radial Bending Stress at the outer edge

CONCLUSION

The energy method for deriving basic governing differential equations has been presented in this thesis. The merit of this approach is that it provides automatically the governing differential equations and the correct number of boundary conditions and their expressions.

An exact solution for the non-linear boundary-value problem is very complicated. Thus, a approximate method must be used. Based on the assumption of the existence of harmonic oscillations, the Ritz-Kantorovich averaging method is introduced to convert the partial differential equations to ordinary differential equations by elimination of the time variable, which makes the problem easy to solve. Then, approximate numerical results can be obtained by the initial-value problem method, the related Runge-Kutta-Gill integration method, and the Newton-Raphson iteration method. The use of numerical integration leads to a discrete approximation of the continuous system.

It has been noted that the elastic constant ratio, c, and mass ratio, , have great influence on the vibration. At the same value of amplitude, the higher frequency accompanies the higher stiffness and lower mass ratio ; on the contrary, the lower frequency is related to the lower stiffness and higher mass ratio.

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APPENDIX A

The generalized Hooke's law for the case of cylinderical orthotropy can be writen in the form:

$$\epsilon_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r$$

$$\epsilon_r = a_{12}\sigma_\theta + a_{22}\sigma_r \qquad \text{(A1)}$$

$$\epsilon_z = a_{13}\sigma_\theta + a_{23}\sigma_r$$

where a_{ij} are elastic constants (compliance coefficiences).

In general, the strain in a direction normal to an applied tensile stress will be negative and the change in the volume of an element subjected to a tensile stress will be positive. Thus, under tensile stress, the change of volume V_0 in an element will obey the inequality

$$V_0(\epsilon_r + \epsilon_\theta + \epsilon_z) > 0 \qquad \text{(A2)}$$

when there is only a radial tensile stress σ_r the volume change can be expressed as:

$$V_0\sigma_r(a_{12} + a_{22} + a_{23}) > 0 \qquad \text{(A3)}$$

and for a circumferential tensile stress σ_θ

$$V_0\sigma_\theta(a_{11} + a_{12} + a_{13}) > 0 \qquad \text{(A4)}$$

equations (A3) and (A4) can be rewritten as follow:

$$-\frac{a_{12}}{a_{22}} - \frac{a_{23}}{a_{22}} < 1 \qquad \text{(A5)}$$

$$- \frac{a_{12}}{a_{11}} - \frac{a_{13}}{a_{11}} < 1 \quad \text{(A6)}$$

But, a_{12}, a_{13}, and a_{23} are negative. Thus $-a_{12}/a_{22}$ and $-a_{12}/a_{11}$ are restricted to values between 0 and 1; i.e.,

$$0 < - \frac{a_{12}}{a_{22}} < 1 \quad \text{(A7)}$$

$$0 < - \frac{a_{12}}{a_{11}} < 1 \quad \text{(A8)}$$

from previous definitions on page 7, $\alpha = -a_{12}/a_{11}$ and $\beta = a_{22}/a_{11}$, the restrictions in equations (A7) and (A8) can be rewritten as:

$$0 < \alpha < 1 \quad \text{(A9)}$$

Also, Poisson's ratio $\nu = - a_{12}/a_{22}$, defined on page 7, must be less than one. Hence

$$\nu = (-a_{12}/a_{11})/(a_{22}/a_{11}) = \alpha/\beta < 1$$

or

$$\alpha < \beta \quad \text{(A10)}$$

APPENDIX B

```
C-----------------------------------------------------------------*
C                                                                 *
C     FREE VIBRATION OF A CLAMPED ORTHOTROPIC CIRCULAR PLATE      *
C                                                                 *
C     WITH CONCENTRIC RIGID MASS AT THE CENTER                    *
C                                                                 *
C     R=0.3  VV=0.3  GAMMA=1.0  BETA= 'VARIABLE'                  *
C                                                                 *
C-----------------------------------------------------------------*
C
C     VARIABLES DESCRIPTION:
C
C     A: AMPLITUDE PARAMETER
C     DA: INCREMENT IN AMPLITUDE
C     R: RADIUS RATIO
C     DR: INCREMENT IN RADIUS RATIO
C     H: STEP-SIZE FOR NUMERICAL INTEGRATION
C     VV: POISSON'S RATIO
C     GAMMA: MASS RATIO
C     DG: INCREMENT IN MASS RATIO
C     ALPHA: ELASTIC CONSTANT RATIO
C     BETA: ELASTIC CONSTANT RATIO
C     C: ELASTIC CONSTANT RATIO (INVERSE OF BETA)
C     QSTART: LOADING DENSITY
C   ****************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-Z),INTEGER(I-N)
      INTRINSIC ABS,SQRT
      DIMENSION Y(24),Q(24),TP(3,4),C(3),ER(3),D(6,41)
      DIMENSION SP(50),SRA(50),FRE(50),AMP(50)
      CHARACTER *5 RR
 112  FORMAT(5X,I2,3X,'FREQ=',F8.4,3X,'AMP=',F8.4)
 113  FORMAT(/,10X,'QSTAR=',F6.2,/)
 114  FORMAT(/,10X,'BETA=',F5.2,/)
 119  FORMAT(/,10X,'RADIUS RATIO=',F6.2,I5,/)
 120  FORMAT(/,10X,'MASS RATIO=',F6.2,/)
      QSTAR=0.0D-0
      A=0.0D-0
      P=10.2158D-0**2
      BETA=2.0D-0
      GAMMA=0.0D-0
      ALPHA=0.3D-0*BETA
      DR=0.0D-0
      DG=1.0D-0
      VV=0.3D-0
      R=0.3D-0
      H=2.5D-2
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```
      LL=29
      WRITE(6,113)QSTAR
      WRITE(6,114)BETA
      DO 510 IR=1,1
      LL=LL-0
      R=R+DR
      WRITE(6,119)R,LL
      DO 520 IG=1,1
      IK=1
      GAMMA=GAMMA+DG
      WRITE(6,117)
      WRITE(6,120)GAMMA
      WRITE(6,117)
      WRITE(6,117)
C *** CONSTRUCT INITIAL VALUES
500   DO 10 I=1,24
10    Y(I)=0.0D-0
      Y(1)=1.0D-0
      Y(3)=-4.67D-0
      Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
      Y(5)=0.82D-0
      Y(6)=(ALPHA*Y(5))/R
      Y(9)=1.0D-0
      Y(10)=-1.0D-0/R
      Y(17)=1.0D-0
      Y(18)=ALPHA/R
      Y(22)=0.5D-0*R*GAMMA
      IF(IK.EQ.1) GO TO 600
      DO 15 I=1,6
15    Y(I)=D(I,1)
C *** X=INDEPENDENT VARIABLE
600   X=R
      DO 20 I=1,24
20    Q(I)=0.0D-0
      DO 21 I=1,6
21    D(I,1)=Y(I)
C *** PERFORMANCE "RUNGE-KUTTA-GILL" INTEGRATION
      DO 25 I=2,LL
      CALL RKGPL(X,H,Y,Q,P,A,ALPHA,BETA,QSTAR)
      DO 30 J=1,6
30    D(J,I)=Y(J)
25    CONTINUE
C *** ER(I)=ERROR VECTOR FOR BOUNDARY CONDITION AT X=1.0
      ER(1)=D(1,LL)
      ER(2)=D(2,LL)
      ER(3)=D(6,LL)-ALPHA*D(5,LL)
      DO 35 I=1,3
      DER=DABS(ER(I))
      IF(DER.GT.0.1D-5) GO TO 36
35    CONTINUE
```

```
      GO TO 900
36    CONTINUE
C *** TP(I,J) IS THE JACOBIAN OF THE MAPPING OF INTERAL VALUES TO
C     FINAL VALUES
      TP(1,1)=Y(7)
      TP(2,1)=Y(8)
      TP(3,1)=Y(12)-ALPHA*Y(11)
      TP(1,2)=Y(13)
      TP(2,2)=Y(14)
      TP(3,2)=Y(18)-ALPHA*Y(17)
      TP(1,3)=Y(19)
      TP(2,3)=Y(20)
      TP(3,3)=Y(24)-ALPHA*Y(23)
      DO 40 I=1,3
40    TP(I,4)=ER(I)
      CALL GAUSSX(TP,C,3,4)
C *** C(I)=CORRECTION VECTOR
      DO 76 I=1,6
76    Y(I)=D(I,1)
      Y(3)=Y(3)-C(1)
      Y(5)=Y(5)-C(2)
      P=P-C(3)
      DO 80 I=7,24
80    Y(i)=0.0d-0
      Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
      Y(6)=(ALPHA*Y(5))/R
      Y(9)=1.0D-0
      Y(10)=-1.0D-0/R
      Y(17)=1.0D-0
      Y(18)=ALPHA/R
      Y(22)=0.5D-0*R*GAMMA
      GO TO 600
900   SRA(IK)=DSQRT(A)
      PP=ABS(P)/BETA
      SP(IK)=DSQRT(PP)
      WRITE(6,112)IK,SP(IK),SRA(IK)
      IF (A.LE.0.09D-0)THEN
      DA=0.01D-0
      ELSE
      DA=0.1D-0
      ENDIF
      A=A+DA
      IK=IK+1
      IF(IK.GT.50)GO TO 520
      GO TO 500
520   CONTINUE
      OPEN(UNIT=10,FILE='RR')
      REWIND 10
      DO 44 IK=1,50
      FRE(IK)=SP(IK)
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```
      AMP(IK)=SRA(IK)
      WRITE(10,45)FRE(IK),AMP(IK)
45    FORMAT(5X,2F10.4)
44    CONTINUE
      CLOSE (10)
510   continue
550   CONTINUE
      STOP
      END
c-----------------------------------------------------------------
      SUBROUTINE RKGPL(X,H,Y,Q,P,AP,ALPHA,BETA,QSTAR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),Q(24),DY(24),A(2)
      A(1)=.2928932188134524
      A(2)=1.707106781186547
      H2=0.5D-0*H
      CALL DERIVL(X,H,Y,DY,P,AP,ALPHA,BETA,QSTAR)
      DO 13 I=1,24
      B=H2*DY(I)-Q(I)
      Y(I)=Y(I)+B
      Q(I)=Q(I)+3.0D-0*B-H2*DY(I)
13    CONTINUE
      X=X+H2
      DO 60 J=1,2
      CALL DERIVL(X,H,Y,DY,P,AP,ALPHA,BETA,QSTAR)
      DO 20 I=1,24
      B=A(J)*(H*DY(I)-Q(I))
      Y(I)=Y(I)+B
      Q(I)=Q(I)+3.0D-0*B-A(J)*H*DY(I)
20    CONTINUE
60    CONTINUE
      X=X+H2
      CALL DERIVL(X,H,Y,DY,P,AP,ALPHA,BETA,QSTAR)
      DO 26 I=1,24
      B=.166666666666666*(H*DY(I)-2.0D-0*Q(I))
      Y(I)=Y(I)+B
      Q(I)=Q(I)+3.0D-0*B-H2*DY(I)
26    CONTINUE
      RETURN
      END
c-----------------------------------------------------------------
      SUBROUTINE DERIVL(X,H,Y,DY,P,AP,ALPHA,BETA,QSTAR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),DY(24)
      C1=1.0D-0-ALPHA**2/BETA
      DO 10 I=1,3
10    DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 15 I=7,9
15    DY(I)=Y(I+1)
```

```
      DY(11)=Y(12)
      DO 20 I=13,15
20    DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 25 I=19,21
25    DY(I)=Y(I+1)
      DY(23)=Y(24)
      IF (AP.EQ.0.0D-0)THEN
      DY(4)=-2.0D-0*(Y(4)/X)+BETA*Y(3)/(X*X)-Y(2)*BETA/(X*X*X)+P*Y(1)
     &+9.0D-0*AP*C1*(Y(3)*Y(5)+Y(2)*Y(6))/X
      GO TO 90
      ENDIF
      DY(4)=-2.0D-0*(Y(4)/X)+BETA*Y(3)/(X*X)-Y(2)*BETA/(X*X*X)+P*Y(1)
     &+9.0D-0*AP*C1*(Y(3)*Y(5)+Y(2)*Y(6))/X+QSTAR*C1/DSQRT(AP)
90    DY(6)=-Y(6)/X+BETA*Y(5)/(X*X)-(Y(2)*Y(2)*BETA)/(2.0D-0*X)
      DY(10)=-2.0d-0*(Y(10)/X)+BETA*Y(9)/(X*X)-Y(8)*BETA/(X*X*X)+P*Y(7)
     &+9.0D-0*AP*C1*(Y(5)*Y(9)+Y(3)*Y(11)+Y(2)*Y(12)+Y(6)*Y(8))/X
      DY(12)=-Y(12)/X+BETA*Y(11)/(X*X)-(Y(2)*Y(8)*BETA)/X
      DY(16)=-2.0D-0*(Y(16)/X)+BETA*Y(15)/(X*X)-Y(14)*BETA/(X*X*X)
     &+9.0D-0*AP*C1*(Y(3)*Y(17)+Y(5)*Y(15)+Y(2)*Y(18)+Y(6)*Y(14))/X
     &+P*Y(13)
      DY(18)=-Y(18)/X+BETA*Y(17)/(X*X)-(Y(2)*Y(14)*BETA)/X
      DY(22)=-2.0D-0*(Y(22)/X)+BETA*Y(21)/(X*X)-Y(20)*BETA/(X*X*X)
     &+Y(1)+9.0D-0*AP*C1*(Y(3)*Y(23)+yY5)*Y(21)+Y(2)*Y(24)+yY6)*Y(20))/X
     &+P*Y(19)
      DY(24)=-Y(24)/X+BETA*Y(23)/(X*X)-(Y(2)*Y(20)*BETA)/X
70    RETURN
      END
c-----------------------------------------------------------------------
      SUBROUTINE GAUSSX(A,X,N,N1)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z),INTEGER(I-N)
      DIMENSION A(N,N1),X(N)
      DO 200 J=1,N
      J1=J+1
      IF (J1.GT.N) GO TO 980
      BIG=DABS(A(J,J))
      M=J
      DO 900 L=J1,N
      IF (DABS(A(L,J)).LE.BIG) GO TO 900
      M=L
      BIG=DABS(A(L,J))
900   CONTINUE
      DO 990 JJ=J,N1
      DUMMY=A(M,JJ)
      A(M,JJ)=A(J,JJ)
990   A(J,JJ)=DUMMY
980   CONTINUE
      S=1.0D-0/A(J,J)
      DO 201 K=J,N1
201   A(J,K)=A(J,K)*S
```

```
      DO 202 I=1,N
      IF(I.EQ.J) GO TO 202
      AIJ=-A(I,J)
      DO 204 K=J,N1
204   A(I,K)=A(I,K)+AIJ*A(J,K)
202   CONTINUE
200   CONTINUE
      DO 300 I=1,N
300   X(I)=A(I,N1)
      RETURN
      END
```

NONLINEAR VIBRATIONS OF A CLAMPED ORTHOTROPIC CIRCULAR PLATE WITH A CONCENTRIC RIGID MASS

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ABSTRACT

The problem of free and forced nonlinear large amplitude vibrations of a clamped orthotropic circular plate with a concentric rigid mass is studied. Hamilton's principle is applied to constitute the governing differential equations and boundary conditions. Assuming the existence of harmonic vibration, the time variable can be eliminated by employing a Kantorovich averaging method. Then, the governing differential equations for the problem are reduced from nonlinear partial differential equations to non-linear ordinary differential equations. The related approximate numerical results can be obtained by using the initial-value problem in conjunction with the Newton-Raphson iteration scheme. The results reveal the effects of finite amplitude and orthotropy of materials upon the dynamic responses.

The characteristics of nonlinear vibration as well as radial and circumferential stresses are studied for various mass ratios, radius ratios and elastic constant ratios. For the purpose of comparison, the present orthotropis circular plate problem can be reduced to a flat isotropic circular plate problem by setting both elastic constant ratio and mass ratio equal to 1. The results are in agreement with prior work. [3,4]