SOME ELEMENTARY CONCEPTS
OF FINITE PLANE PROJECTIVE GEOMETRY

by

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INTRODUCTION

In this report some of the properties of finite plane projective geometry are developed. Hence, the first question to be answered is, "What is a geometry?". Geometry, like many of the other sciences, has undergone considerable evolution since its conception in the "Golden Age of the Greeks". Derived from the Greek words meaning "earth measure", geometry was first considered to be a study of the properties of the physical world in which the Greeks lived. Because the Greeks considered the earth to be flat, it was logical that their geometry was essentially the measurement of line segments, angles, and other figures on a plane (8,1)[1]. This concept of geometry was gradually extended to include the study of higher-dimensional spaces. Coordinates, and the study of spaces based upon systems of coordinates, were introduced (8,1).

A major step in the study of geometry occurred when Felix Klein proposed his revolutionary Erlanger Programm in 1872. In his inaugural address upon assuming a chair at the University of Erlangen, Klein defined a geometry to be "the study of those properties of a set that remain invariant when the elements of that set are subjected to the transformations of some transformation group" (6,145). More recently geometries have been developed which do not satisfy Klein's general definition. Since 1900, geometry has been extended to the consideration of purely logical systems

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[1]This reference will serve as a model for subsequent references with the first number of the ordered pair being the reference number as listed in the references, and the second number being the page number.

based upon undefined elements and relations and the theory of abstract spaces.

These definitions have become so embracive that the boundary lines between geometry and other branches of mathematics have become somewhat blurred. If Klein's definition is considered as a basis, however, one finds that many interesting geometries can be included in the study. These geometries are called "Kleinian geometries" (7,259). In particular, projective geometry is such a "Kleinian geometry", although the group properties are not emphasized in this report.

In this report, initial emphasis is placed on synthetic projective geometry. A set of axioms for finite plane projective geometries is introduced, and some basic theorems are developed. Coordinates are then introduced in the plane and analytic properties are discussed for the finite projective plane. Higher dimensional spaces are not considered in this report. Coxeter has stated, "We soon find that what happens in a single plane is sufficiently exciting to occupy our attention for a long time." A special example, PG (2,5)[2], has been found to be of interest (3,92) (4, 113).

What is a projective geometry? An easily understandable but somewhat misleading intuitive approach to projective geometry can be obtained from Euclidean geometry. Due mainly to the work of Kepler, Desargues, and Poncelet, the projective plane was derived from ordinary Euclidean space by postulating a "line at infinity" and the relationship that "any pair of

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[2]PG (2,5) is the notation used by O. Veblen and W. H. Bussey for the projective geometry of dimension two with coordinates taken from a finite field with 5 elements.

parallel lines intersect somewhere on this line at infinity". Whereas Euclidean geometry makes use of the unmarked straight edge and compass in constructions, all that is allowed in projective geometry is the straight edge. When one disallows the compass, many concepts are sacrificed, such as, circles, distance, angles, "betweenness", and parallelism. However, a consistent geometry is obtained which has the property known as the "principle of duality" (3,4).

Pieri placed projective geometry on an axiomatic foundation in 1899. From this starting point projective geometry has blossomed in many different directions, including the study of finite plane projective geometry (2,230).

SYNTHETIC PLANE PROJECTIVE GEOMETRY

One of the many sets of axioms for the classical projective plane is the following development, due to Bachmann (2,230). The undefined concepts for this development are <u>point</u> and <u>line</u> and the undefined relation is <u>incidence</u>. A <u>projective plane</u> is a set of points with certain subsets called lines such that the five axioms listed below are satisfied. The related words <u>on</u> and <u>through</u>, <u>join</u> and <u>intersect</u>, <u>concurrent</u> and <u>collinear</u> have their usual meanings. Three non-collinear points are the vertices of a <u>triangle</u> whose sides are lines. Line segments are not defined. A <u>complete quadrangle</u>, its four vertices, six sides, and three diagonal points, have the usual projective definition. A <u>hexagon</u> $A_1B_2C_1A_2B_1C_2$ has six vertices A_1, B_2, C_1, A_2, B_1, C_2 and six sides A_1B_2, B_2C_1, C_1A_2, A_2B_1, B_1C_2, C_2A_1. Opposite sides are A_1B_2 and A_2B_1, B_2C_1 and B_1C_2, C_1A_2 and C_2A_1. The following are axioms for a general plane projective geometry.

Bachmann's Axioms

AXIOM 1: Any two distinct points are incident with just one line.

AXIOM 2: Any two lines are incident with at least one point.

AXIOM 3: There exist four points of which no three are collinear.

AXIOM 4: (Fano's Axiom) The three diagonal points of a complete quadrangle are never collinear.

AXIOM 5: (Pappus's Theorem) If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear.

This set of axioms is not categorical, as these axioms do not require anything special about the number of points in the projective plane. By adding an axiom which limits the number of points to be finite, one has essentially the axioms for a finite plane projective geometry.

SYNTHETIC FINITE PLANE PROJECTIVE GEOMETRY

The following synthetic definition of a finite plane projective geometry was formulated by O. Veblen and W. H. Bussey in 1910. It is obtained from their more general definition by deleting the two axioms which postulate higher dimensional spaces (11,241). The primitive concepts are point and line, and the primitive relationship is that of incidence.

A plane ABC (A, B, and C being non-collinear points) is defined as the set of all points collinear with a point A and any point of the line joining B and C and satisfying the five axioms listed below. In this development the line joining the points B and C is denoted by BC (11,241).

Veblen and Bussey's Axioms

AXIOM I: The plane contains a finite number (> 2) of points. It contains at least one line and one point not on that line, and each line contains at least three points.

AXIOM II: If A and B are distinct points there is one and only one line that contains A and B.

AXIOM III: If A, B, C are non-collinear points and if a line m contains a point D of the line AB and a point E of the line BC, but does not contain A, B, or C, then the line m contains a point F of the line CA (Fig. 1).

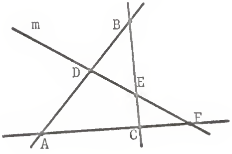

Fig. 1

AXIOM IV: The diagonal points of a complete quadrangle are not collinear.

AXIOM V: Let A_1, B_1, C_1 be three collinear points and let A_2, B_2, C_2 be three other collinear points, not on the same line. If the pairs of lines A_1B_2 and A_2B_1, B_1C_2 and B_2C_1, C_1A_2 and C_2A_1 intersect, the three points of intersection C_3, A_3, B_3 are collinear (Fig. 2).

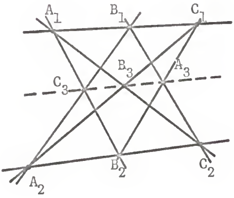

Fig. 2

It has been shown by Coxeter (2,230ff.) that by employing Bachmann's Axioms (1-5) one can develop many theorems of classical projective geometry. Thus, by showing that Veblen and Bussey's Axioms imply Axioms 1-5, all of the theorems of classical projective geometry which do not deal with a specific number of points are deducible in the system of Axioms I-V.

Axiom 1 is easily verified for the finite plane as Axiom II of Veblen and Bussey and Axiom 1 are equivalent. Likewise Axiom 4 is equivalent to Axiom IV. Axiom V postulates the hexagon of Axiom 5 and with minor notational changes they are equivalent statements. To show that Axioms 2 and 3 are deducible from Axioms I-V requires a more complex argument.

<u>THEOREM I</u>: In a plane projective geometry satisfying Axioms I-V any two lines are incident with at least one point.

Proof: Given two lines FG and RS either the lines are distinct or the lines are the same. In the latter case the proof is trivial. If the lines are distinct, they lie on a plane, which is determined by a line AB and a point C not on AB. If one of the lines, say RS, coincides with AB the proof is as follows (Fig. 3).

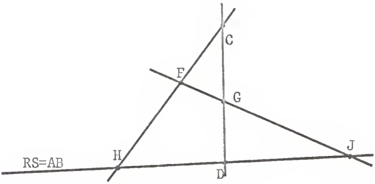

Fig. 3

Either FG passes through C or it does not. If FG passes through C, it must intersect AB=RS by the definition of a plane. If FG does not pass through C, this line is determined by the points F and G. Again by the definition of a plane two lines drawn from C through F and G must intersect RS, say at the distinct points H and D. Applying Axiom III to the non-collinear points C, F, and G it is clear that line HD must intersect FG at some point, say J. By Axiom II lines RS and HD must coincide, so point J is on RS. Therefore, J is the required point of intersection.

Thus, FG and RS intersect.

If neither line coincides with AB then both lines FG and RS have at most one point in common with AB. Since RS is determined by any two of its points and contains at least three points, one may assume that the points R and S are not on AB. Likewise F and G are not on line AB (Fig. 4).

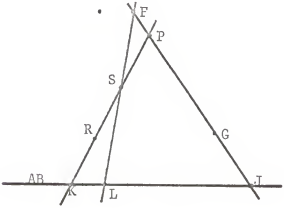

Fig. 4

Designating the intersection of FG and AB by J, the intersection of RS and AB by K, and the intersection of FS and AB by L, one finds that if J=K, the given lines intersect at K; if J=L, they intersect at S; if L=K, they intersect at F. If J, K, and L are distinct points on the line AB, then by applying Axiom III to the non-collinear points J, L, and F, the line KS must intersect FJ at a point P. Since FJ=FG and KS=RS then RS intersects FG at the required point P (8,30).

THEOREM II: In a plane projective geometry satisfying Axioms I-V there exist four points of which no three are collinear.

Proof: One is assured of the existence of a line AB and a point C not on that line. Also, by Axiom I, there are at least three points on AB. Calling the third point D one is assured of a line CD by Axiom II. This line contains another point E by Axiom I, which cannot be collinear with A and B. The points A, B, C, and E exist and satisfy Axiom 3.

PRINCIPLE OF DUALITY

One of the most elegant properties of projective geometry, making it "symmetrical", is the principle of duality. In a projective plane this principle asserts that every definition is significant and every theorem is valid when the words point and line and the relationships lie on and pass through, join and intersect, concurrent and collinear are interchanged (2,231).

To establish this principle for the finite projective plane it will suffice to verify that the axioms imply their duals. Then, given a theorem and its proof, the dual theorem can be asserted without proof; as a proof of the latter consists of merely dualizing every step in the proof of the original theorem (5,419).

THEOREM III: (Dual of Axiom I) In a plane projective geometry satisfying Axioms I-V the plane contains a finite number (> 2) of lines. It contains at least one point and one line not through that point, and each point lies on at least three lines.

Proof: The statement that the plane contains at least one point and one line not through that point is self-dual. By Axiom I there exist $n > 2$ points. By Axiom II any two of these n points determine one and only one line. Taking all possible ways of joining two points at a time to form lines, there are at most ${}_nC_2 = \frac{n(n-1)}{2}$ distinct lines. One is assured of a line AB and a point C not on AB. This line must contain at least three points by Axiom I. By Axiom II there are three distinct lines joining C and the three points on the line AB. Therefore the plane contains a finite number (> 2) of lines (Fig. 5).

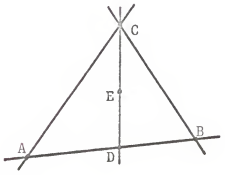

Fig. 5

Theorem II states that there exist four points, say P, Q, R, S, of which no three are collinear. For an arbitrary point T there exists a line which does not pass through T, whether T coincides with one of the four points P, Q, R, S or not. This line must contain n > 2 points by Axiom I. Since there exists one and only one line joining T with each of the n > 2 points on this line, there are at least n > 2 lines through T. Since T was chosen to be arbitrary, there are at least 3 lines through every point (9,86).

THEOREM IV: (Dual of Axiom II) In a plane projective geometry satisfying Axioms I-V if a and b are distinct lines, there is one and only one point contained in a and b.

Proof: By Theorem I, lines a and b are incident with at least one point. Assuming that a and b are incident with two points, it follows that the two points are incident with the two distinct lines a and b. But by

Axiom II, this is impossible. Therefore, a and b are incident with one and only one point.

THEOREM V: (Dual of Axiom III) In a plane projective geometry satisfying Axioms I-V if a, b, and c are non-concurrent lines and if a point M lies on a line d through the point of intersection of lines a and b (denoted by a·b) and on a line e through b·c but does not lie on a, b, or c, then the point M lies on a line f through the point c·a.

Proof: By Axiom II the points a·c and M determine one and only one line, so the dual is proved (Fig. 6).

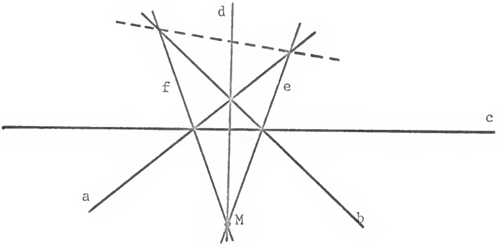

Fig. 6

If points a·e and b·f of Fig. 6 are joined, a complete quadrilateral results with sides a, b, f, and e and vertices b·f, a·b, a·e, M, b·e, a·f. The diagonals are the lines c, d, and the join of a·e and b·f. This is the dual of the complete quadrangle. The existence of a complete quadrilateral is closely tied to the existence of four points of which no three are collinear. By joining these four points so that every point lies on two lines one has the four sides of the complete quadrilateral. These four sides intersect in six points or vertices when extended. These four sides, six vertices, and three diagonals drawn joining opposite vertices form the desired quadrilateral.

THEOREM VI: (Dual of Axiom IV) In a plane projective geometry satisfying Axioms I-V the diagonals of a complete quadrilateral are not concurrent.

Proof: Complete quadrilateral pqrs has vertices p·q, p·s, r·s, q·r, p·r, q·s and diagonals u, v, and w (Fig. 7).

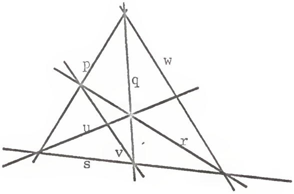

Fig. 7

The points r·s, p·q, p·s, and q·r form a complete quadrangle with sides p, q, r, s, w, and u. Assuming that the diagonals u, v, and w of quadrilateral pqrs are concurrent then point w·u must lie on v. Therefore the points p·r, q·s and w·u are collinear. Since p·r, q·s and w·u are the diagonal points of a quadrangle with vertices r·s, p·q, p·s, q·r, they cannot be collinear. Therefore the diagonals cannot be concurrent (2,232).

THEOREM VII: (Dual of Axiom V) Assume Axioms I-V are satisfied and let a_1, b_1, c_1 be three concurrent lines and a_2, b_2, c_2 be three other concurrent lines not through the same point. If the pairs of points $a_1 \cdot b_2$ and $a_2 \cdot b_1$, $b_1 \cdot c_2$ and $b_2 \cdot c_1$, $c_1 \cdot a_2$ and $c_2 \cdot a_1$ are joined, the three lines joining them are concurrent.

Another of the many possible figures (along with Fig. 2) representing Axiom V is Fig. 8.

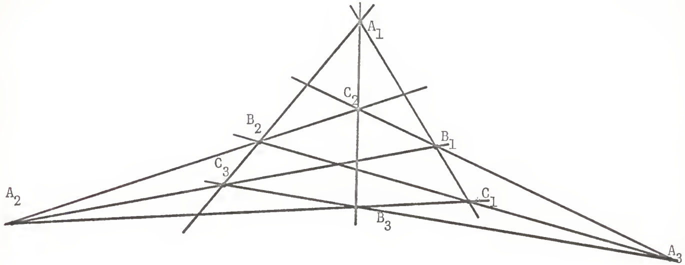

Fig. 8

Proof: Axiom V asserts that A_3, B_3, and C_3 are collinear. The notation has been arranged that the three points A_i, B_j, C_k are collinear if and only if $i+j+k \equiv 0 \pmod 3$. Expressed in matrix form one has

$$\left\| \begin{matrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{matrix} \right\| .$$

If this were a determinant to be evaluated, the elements would be multiplied in triads. These six diagonal triads, as well as the first two rows of the matrix, indicate triads of collinear points. Axiom V asserts that the points in the bottom row are likewise collinear. These lines can be denoted in many ways, one of which is

$$a_1 = A_3B_1C_2,\ b_1 = A_1B_3C_2,\ c_1 = A_2B_2C_2,$$
$$a_2 = A_2B_3C_1,\ b_2 = A_3B_2C_1,\ c_2 = A_1B_1C_1,$$
$$a_3 = A_1B_2C_3,\ b_3 = A_2B_1C_3,\ c_3 = A_3B_3C_3 \quad (2,233).$$

These lines satisfy the hypotheses of Theorem IV, and the inherent self-duality is seen from an analogous matrix of lines

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix} .$$

DESARGUES'S THEOREM

A theorem of special significance in finite projective geometry is the two-triangle theorem of Desargues. Two triangles are said to be perspective from a point if their three pairs of corresponding vertices are joined by concurrent lines. For instance, in Fig. 2 and Fig. 8, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are perspective from C_1. Dually two triangles are perspective from a line if their three pairs of corresponding sides meet in collinear points (2,238).

THEOREM VIII: (Desargues's Theorem) In a plane projective geometry satisfying Axioms I-V if two triangles are perspective from a point, they are perspective from a line, and conversely.

Proof: Let two triangles PQR and P'Q'R' be perspective from a point O, and let their corresponding sides meet in points D=QR·Q'R', E=RP·R'P', F=PQ·P'Q' (Fig. 9).

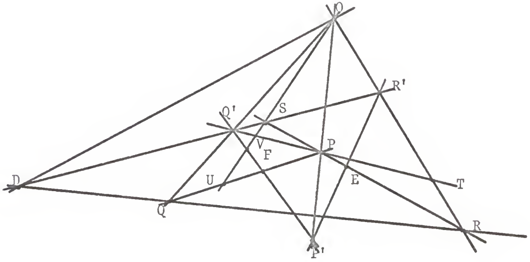

Fig. 9

After defining four other points S=PR·Q'R', T=PQ'·OR, U=PQ·OS, V=P'Q'·OS, Axiom V is applied three times in the matrix form. These three applications are

$$\left\| \begin{matrix} O & Q & Q' \\ P & S & R \\ D & T & U \end{matrix} \right\|, \quad \left\| \begin{matrix} O & P & P' \\ Q' & R' & S \\ E & V & T \end{matrix} \right\|, \quad \left\| \begin{matrix} P & Q' & T \\ V & U & S \\ D & E & F \end{matrix} \right\| .$$

The first matrix shows that the points D, T, and U are collinear. The second matrix shows that E, V, and T are collinear. The final matrix uses these two lines along with the given lines to show that D, E, and F are collinear, and the theorem is proved. The converse follows by the principle of duality (2,238). Thus all the plane projective geometries satisfying Axioms I-V have Desargues's Theorem as a logical deduction; these geometries are called "Desarguesian" (5,423).

SOME NUMERICAL RESULTS

Using only the axioms of finite plane projective geometry it is possible to derive some numerical results. These include the number of points on each line, the number of lines through each point, and the number of lines and points in the plane.

THEOREM IX: In a finite plane projective geometry satisfying Axioms I-V with s+1 points ($s > 2$) on a particular line RS=m there are s+1 points on every line, s+1 lines through every point, s^2+s+1 points in a plane, and s^2+s+1 lines in a plane.

Proof: Let lines PQ and m meet in U. Then PS and RQ meet in a further point T which is not on m and not on PQ (Fig. 10). If V is any point not

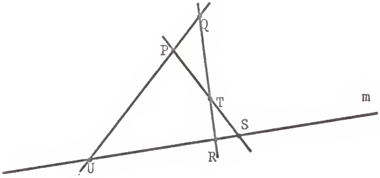

Fig. 10

on m, joining V to the s+1 points of m gives s+1 lines through V. There are no further lines through V, since every line through V must intersect m and there are only s+1 points on m. In particular, there are s+1 lines through each of P, Q, and T. Now any line k of the plane cannot contain more than two of the points P, Q, and T. Line k is therefore intersected by the s+1 lines through one of these points in s+1 points, and these are all the points on k, since any additional points would give rise to additional lines through one of the points P, Q, T. There are s+1 lines through any point V, these being obtained by joining the point to the s+1 points of any line not containing V (9,86).

To obtain the number of points in a plane consider the same line m and a point P not on m. P and each of the s+1 points on m determine a line, with each of these lines containing s points in addition to the point P. Since every point of the plane is on one of these s+1 lines containing P, the number of points in the plane must be $s\cdot(s+1)+1=s^2+s+1$. By duality there are s^2+s+1 lines in the plane (11,243).

INCIDENCE MATRICES

In order to clarify the relationships between points and lines in these geometries, it is helpful to use "incidence matrices" (4,89). These incidence matrices serve as a convenient method of describing which points lie on each line and which lines pass through each point. Let $N=s^2+s+1$ and call the points of the plane P_1, P_2, ..., P_N. Let the lines be called m_1, m_2, ..., m_N and define "incidence numbers" a_{ij} where

$$a_{ij} = 1 \text{ if } P_i \epsilon m_j$$

$$a_{ij} = 0 \text{ if } P_i \notin m_j$$

and (i, j=1, 2, ..., N).

A simple example to illustrate incidence matrices is the "triple system", which consists of $s^2+s+1=N=7$ points arranged in $s^2+s+1=N=7$ lines. There are $s+1=3$ points on each line and $s+1=3$ lines through each point. The seven points, P_1, P_2, P_3, ..., P_7, are arranged into seven lines in the following manner (11,243).

m_1: $(P_1P_2P_3)$ m_2: $(P_1P_6P_4)$

m_3: $(P_2P_6P_5)$ m_4: $(P_3P_4P_5)$

m_5: $(P_3P_6P_7)$ m_6: $(P_2P_4P_7)$

m_7: $(P_1P_7P_5)$

These are not ordinary lines in the Euclidean sense. The incidence matrix of this geometry with the given notation is

$$A = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix} .$$

Considering $P_2P_3P_4P_6$ as a complete quadrangle the diagonal points are P_1, P_5, and P_7 (Fig. 11).

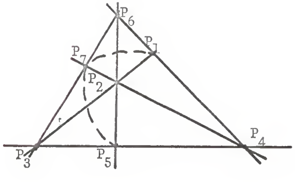

Fig. 11

By observing the incidence matrix for this geometry, one finds that points P_1, P_5, and P_7 all lie on line m_7. This violates Axiom IV, so this system is not a finite projective geometry by the definition used in this report. This is a specific example of a more general theorem which is proved later in the analytic development of finite plane projective geometries.

The most simple example satisfying Axioms I-V is the system composed of s+1=4 points on every line, s+1=4 lines through every point, $s^2+s+1=13$ points in the plane, and $s^2+s+1=13$ lines in the plane (11,243). An incidence matrix for this finite projective geometry is

$$A = \left\| \begin{matrix} 1&0&0&0&1&0&0&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0&0&0&0&0&1&0 \\ 0&1&1&0&0&0&1&0&0&0&0&0&1 \\ 1&0&1&1&0&0&0&1&0&0&0&0&0 \\ 0&1&0&1&1&0&0&0&1&0&0&0&0 \\ 0&0&1&0&1&1&0&0&0&1&0&0&0 \\ 0&0&0&1&0&1&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&1&1&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1&1&0&0&0&1 \\ 1&0&0&0&0&0&1&0&1&1&0&0&0 \\ 0&1&0&0&0&0&0&1&0&1&1&0&0 \\ 0&0&1&0&0&0&0&0&1&0&1&1&0 \\ 0&0&0&1&0&0&0&0&0&1&0&1&1 \end{matrix} \right\| .$$

Let A' denote the transpose of A. By matrix multiplication it may be verified that

$$AA'=A'A=\begin{Vmatrix} 4&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&4&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&4&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&4&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&4&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&4&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&4&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&4&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&4&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&4&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&4&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&4&1\\ 1&1&1&1&1&1&1&1&1&1&1&1&4 \end{Vmatrix}.$$

This case is a special example of the following theorem.

THEOREM X: $AA'=A'A=sI+U$ with I being the unit matrix, s+1 being the number of points on a line and U being the square matrix with every element equal to one.

<u>Proof</u>: Let $AA'=C$, where $C=(c_{rt})$, and

$$c_{rt} = \sum_{j=1}^{N} a_{rj}a_{tj}$$

by the definition of matrix multiplication and transpose of a matrix. The rth row of A gives the incidences of the point P_r. This point is on s+1 lines, so that the rth row of A contains s+1 entries equal to 1, the remainder being zero. Since c_{rr} arises from the multiplication of corresponding elements, $c_{rr}=s+1$. If $r=t$, we see that $c_{rt}=1$ since $a_{rj}a_{tj}=0$ unless both $a_{rj}=1$ and $a_{tj}=1$. But $a_{rj}=a_{tj}=1$ indicates that the line m_j contains both P_r and P_t. Given P_r and P_t there is exactly one line containing both points. Hence $c_{rt}=1$ for $r=t$ and since $c_{rr}=s+1$, it is true that $AA'=sI+U$. A dual argument shows that $A'A=sI+U$ (9,90).

COLLINEATIONS

The points and lines of these projective planes are subjected to many transformations. One of these transformations of special interest in finite projective planes is the collineation. A collineation in a finite projective plane π (satisfying Axioms I-V) is a one-to-one transformation of the points of π into the points of π, which is denoted by α, so that point P maps into P^{α} and the property of points being collinear is preserved. This implies P^{α}, Q^{α}, and R^{α} are collinear if and only if P, Q, and R are collinear.

A fixed point (or united point) of a collineation α in π is a point P such that $P=P^{\alpha}$. A fixed line is a line m such that $m=m^{\alpha}$. Although a line of fixed points is necessarily a fixed line, a fixed line does not necessarily consist of only fixed points. However two theorems concerning collineations may be stated (9,99).

THEOREM XI: In a finite projective plane satisfying Axioms I-V the line determined by two fixed points is a fixed line.

THEOREM XII: In a finite projective plane satisfying Axioms I-V the point (of intersection) determined by two fixed lines is a fixed point.

Proof: By definition a collineation must preserve incidence properties. Thus if $P=P^{\alpha}$ and $Q=Q^{\alpha}$, then $PQ=P^{\alpha}Q^{\alpha}$. Also if lines $m=m^{\alpha}$ and $p=p^{\alpha}$ then $m \cdot p=m^{\alpha} \cdot p^{\alpha}$. With these statements the theorems are proved.

Many of the theorems about collineations follow directly from Axioms I-V just as in general projective geometry. The two theorems above are especially important as they lead to the relationship between the number of fixed points and fixed lines in a finite projective plane. It has previously been proved in this report that if a line contains s+1 points, then every line contains s+1 points, s+1 lines pass through every point, and the number of lines and points in the plane is s^2+s+1. The number of fixed points under a collineation α is therefore finite, say $N(\alpha)$ (9,114).

<u>THEOREM XIII</u>: In a finite projective plane satisfying Axioms I-V if the number of fixed points is $N(\alpha)$, then the number of fixed lines is also $N(\alpha)$.

<u>Proof</u>: The point P and the line m are said to form a <u>pair</u> with respect to α if P is on m and also on m^α (Fig. 12a). Denoting by $M(\alpha)$ the number of distinct pairs with respect to α, and by $N'(\alpha)$ the number of fixed lines of α, one finds that if m is a fixed line of α, then (P, m) is a pair if, and only if, P is on m. Since every line of π contains s+1 points, the number of pairs (P, m), where m is a fixed line of α, is $(s+1)N'(\alpha)$. If m is not a fixed line of α, then m and m^α meet in a unique point $m \cdot m^\alpha$, and $(m \cdot m^\alpha, m)$ is the only pair containing m. Hence, since there are s^2+s+1 lines in π, there are $s^2+s+1-N'(\alpha)$ pairs (P, m) where m is not a fixed line. Combining these results the number of distinct pairs $M(\alpha)$ is given by:

$$M(\alpha) = (s+1)N'(\alpha)+1+s+s^2-N'(\alpha) = sN'(\alpha)+1+s+s^2.$$

The point P and the line m are said to form a <u>couple</u> with respect to

α if m contains both P and P^α (Fig. 12b).

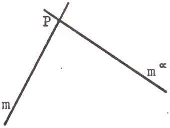

Fig. 12a

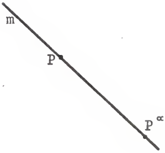

Fig. 12b

Denoting by $M'(\alpha)$ the number of distinct couples with respect to α, one finds that since couple is the dual of pair, the dual of the proof above gives

$$M'(\alpha) = sN(\alpha)+1+s+s^2.$$

Since α is a one-to-one mapping of the points of π onto π, the inverse mapping, α^{-1}, exists. The mapping α^{-1} is a collineation since P^α, Q^α, and R^α are collinear if and only if P, Q, and R are collinear (9,102). If the transformation α^{-1} is applied to a pair, a couple with respect to α^{-1} is obtained (Fig. 13). If the transformation α is applied to a couple under α^{-1}, also shown in Fig. 13, a pair is obtained with respect to α.

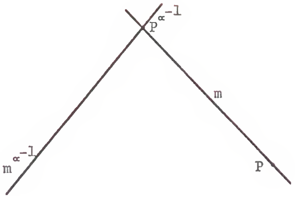

Fig. 13

Hence the point P and the line m form a pair with respect to α if and only if they form a couple with respect to α^{-1}. Therefore,

$$M(\alpha) = M'(\alpha^{-1}).$$

By the definition of α^{-1} the collineations α and α^{-1} have the same fixed elements so that

$$N(\alpha) = N(\alpha^{-1}).$$

Combining these results

$$\begin{aligned} M(\alpha) &= sN'(\alpha)+1+s+s^2 = M'(\alpha^{-1}) \\ &= sN(\alpha^{-1})+1+s+s^2 \\ &= sN(\alpha)+1+s+s^2 \end{aligned}$$

so that $N(\alpha)=N'(\alpha)$. Therefore the number of fixed points is equal to the number of fixed lines (9,116).

COORDINATES FOR THE PLANE

It often occurs in projective geometry that "pure geometric" proofs are difficult to obtain. This has led to the introduction of coordinates for the plane and the utilization of these coordinates to develop analytic properties. The important idea is to take an ordered set of numbers (x_1, x_2, x_3) and call it a point. The numbers that are used can be the elements of any commutative field in which $1+1\neq 0$ (3,112).

To coordinatize the finite projective plane the numbers are taken from Galois (finite) fields (with Galois fields of modulus two not considered). In the following development x_1, x_2, x_3, u_1, u_2, u_3, and k are elements of a Galois field (11,244). Then any point in the plane is simply a set of three numbers (x_1, x_2, x_3) with the understanding that (x_1, x_2, x_3) and (kx_1, kx_2, kx_3) represent the same point for all values of k in the finite field provided $k\neq 0$ and provided that (0, 0, 0) is excluded from consideration. Any line in the plane is the set of all these points which satisfy an equation of the form

$$u_1x_1+u_2x_2+u_3x_3 = 0$$

with u_1, u_2, and u_3 being given explicitly and x_1, x_2, and x_3 being variables (12,201). The set of all lines is obtained by giving the coordinates (u_1, u_2, u_3) all possible values in the field (except (0, 0, 0)). It follows that (u_1, u_2, u_3) and (ku_1, ku_2, ku_3), $(k\neq 0)$, represent the same line (12,201).

By eliminating u_1, u_2, u_3 from the equations

$$u_1a_1+u_2a_2+u_3a_3 = 0$$

$$u_1b_1+u_2b_2+u_3b_3 = 0$$

$$u_1c_1+u_2c_2+u_3c_3 = 0$$

of three given points $A=(a_1, a_2, a_3)$, $B=(b_1, b_2, b_3)$, and $C=(c_1, c_2, c_3)$, one finds

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

to be a necessary and sufficient condition for the three points to be collinear. This condition is equivalent to the existence of numbers λ, μ, γ, not all zero such that

$$\lambda a_i+\mu b_i+\gamma c_i = 0 \qquad (i = 1, 2, 3).$$

If B and C are distinct points, $\lambda \neq 0$. Hence the general point collinear with B and C is $(\mu b_1+\gamma c_1, \mu b_2+\gamma c_2, \mu b_3+\gamma c_3)$ or, briefly,

$$\mu B+\gamma C$$

where μ and γ are not both zero. When $\gamma=0$, this is the point B itself. For any other position, since γC is the same point as C, we can allow the coordinates of C to "absorb" the γ, and the point collinear with B and C is simply $\mu B+C$ (3,114).

If one is concerned with only one such point, one may allow the μ to be "absorbed" too; thus three distinct collinear points may be expressed as

B, C, and B+C. The symbol μB+C can be made to include every point on the line BC if we accept the convention that the point B is μB+C with $\mu=\infty$ (3,115).

A Galois field with s elements is said to be of order s and is denoted by the symbol GF(s). The geometry with coordinates taken from GF(s) is denoted by the symbol PG(2,s), and the plane is said to be of order s. However Galois fields of order s where $s=2^n$ are excluded, as it is impossible for Axiom IV to be satisfied by a plane geometry with coordinates taken from $GF(2^n)$.

THEOREM XIV: There can be no PG(2,s) with $s=2^n$ which satisfies Axioms I-V (11,245).

Proof: Consider quadrangle PQRS with diagonal points A, B, and C (Fig. 14).

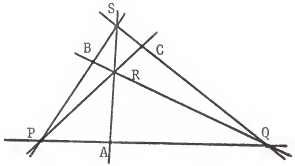

Fig. 14

The vertices $P=(p_1, p_2, p_3)$, $Q=(q_1, q_2, q_3)$, and $R=(r_1, r_2, r_3)$ must satisfy

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0$$

Since the side PS joins P to the diagonal point A=QP·PS, it is possible to take A(on QR, but distinct from Q and R) to be Q+R. Likewise S(on PA, but distinct from P and A) may be taken to be P+Q+R, meaning

$$(p_1+q_1+r_1,\ p_2+q_2+r_2,\ p_3+q_3+r_3).$$

Then B, on both RP and QS, must be R+P, and C, on both PQ and RS, must be P+Q. In general, the diagonal points A, B, C are noncollinear since

$$\begin{vmatrix} q_1+r_1 & q_2+r_2 & q_3+r_3 \\ r_1+p_1 & r_2+p_2 & r_3+p_3 \\ p_1+q_1 & p_2+q_2 & p_3+q_3 \end{vmatrix} = 2 \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0.$$

However, in a Galois field of order $s=2^n$ it happens that

$$2 \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

is always equal to zero; therefore the diagonal points are collinear and Axiom IV is violated (3,116 and 127).

In a Galois field of order $s \neq 2^n$ Axiom V is not violated since $2 \neq 0$. Using only the projective coordinates and analytic properties based on the real number system, Coxeter (2,235) has proved that Bachmann's Axioms 1-5 are logical deductions. In an analogous manner it can be verified that Axioms I-V can be deduced using projective coordinates taken from Galois fields of order $s \neq 2^n$. The projective coordinates serve as a model for the finite plane projective geometry defined by Axioms I-V, and using these coordinates one can arrive at some numerical results analogous to Theorem IX.

SOME NUMERICAL RESULTS

In the Galois field of order s, there are $s \cdot s \cdot s = s^3$ ways of forming triples (x_1, x_2, x_3). However, since the triple (0, 0, 0) is excluded and the triples (kx_1, kx_2, kx_3) and (x_1, x_2, x_3) are equivalent whenever k is an element of the Galois field not equal to zero, the number of possible points in the PG(2,s) is $\frac{s^3-1}{s-1} = s^2+s+1$. Using the coordinates (u_1, u_2, u_3) and analogous reasoning there are s^2+s+1 lines in the plane (11,244).

The coordinates of all points on the line BC may be given by $\mu B+C$ where μ ranges over the Galois field plus the value ∞. Since GF(s) has s elements there are s+1 allowable coordinates for points on a line. Dually there are s+1 lines through each point (3,115). These results agree with the numerical results of Theorem IX.

ORDER OF SUBPLANES

It is possible to determine the possible orders of subplanes of a finite projective plane π. Let $\{P_i\}$ be a set of points in π with $\{m_j\}$ a subset of lines. Considering the possible joins and intersections of these points and lines, further points and lines are obtained. These further points and lines are said to be generated by $\{P_i\}$ and $\{m_j\}$. This gives a projective plane π^* which is contained in (and may coincide with)

π. The trivial case when $\pi=\pi^*$ is excluded. It is said that π^* is a subplane of π.

THEOREM XV: If a finite projective plane π of order s has a subplane π^* of order k, then $s=k^2$ or $s\geq k^2+k$.

Proof: Let m be a line of π^* and P be a point of m not belonging to π^*. There are k+1 points of π^* on m and k^2 points of π^* not on m. Joining P to each of these k^2 points of π^* not on m, each of the k^2 lines obtained must be different. If two were the same, such a line would contain at least two points of π^* and would therefore be a line of π^*. The point P being the intersection of two lines of π^*, would be in π^*, contrary to hypothesis. Hence through P there are at least k^2+1 distinct lines, namely m and the joins of P to the k^2 points of π^* not on m. Since there are s+1 lines of π through P, one must have $s\geq k^2$.

If $s>k^2$, there will be at least one further line through P not containing any point of π^*. Calling this line m' and considering the intersection of m' with the k^2+k+1 lines of π^*, one finds that these intersections are all distinct. This is true, because if two were coincident, the intersection of two lines of π^* would be on m'. But two lines of π^* intersect in a point of π^*, and by hypothesis m' contains no point of π^*. It follows that m' contains at least k^2+k+1 points and therefore $s\geq k^2+k$ (9,91).

HARMONIC PROPERTIES

Four collinear points A, B, C, D are said to form a harmonic set if there is a quadrangle of which two opposite sides pass through A and two other opposite sides pass through B while the remaining sides pass through C and D, respectively. C and D are said to be harmonic conjugates with respect to A and B or H(AB, CD). The Mobius (harmonic) net is the smallest set of points that contains, for every three of its members, the harmonic conjugate of each with respect to the other two. Some interesting results occur when the coordinates are taken from finite fields.

Any one of the family of lines with third coordinate equal to zero may be denoted $x_3=0$, and any line with first coordinate equal to zero may be denoted $x_1=0$. On the line $x_1=0$ any point except (0, 1, 0) may be represented by a single coordinate, the value of x_2 when $x_3=1$. The point (0, 1, 0) may be represented by the coordinate ∞. Let α and β be the coordinates of two points on the line. The harmonic conjugate of α with respect to β and ∞ is constructed as follows. Let S and T be any two points on a line through the point ∞. Let the lines $S\beta$ and $T\alpha$ meet in a point R. If the lines $T\beta$ and $R\infty$ meet in a point Q, the line SQ meets the line $x_1=0$ in the required harmonic conjugate point γ.

Choosing the line $ST\infty$ to be the line $x_3=0$ and S and T to be the points $(\lambda, 1, 0)$ and (1, 0, 0) respectively, the line coordinates for line $S\beta$ are $[1, -\lambda, \lambda B]$ and for $T\alpha$ are $[0, 1, -\alpha]$. Thus the coordinates of R are found to be $(\lambda\beta-\alpha\lambda, -\alpha, -1)$. Line $R\infty$ is then $[-1, 0, \lambda(\alpha-\beta)]$. Line $T\beta$ has coordinates $[0, -1, \beta]$, so the point Q has coordinates $(\lambda(\beta-\alpha), -\beta, -1)$. Thus line SQ is described by the equation $x_1-\lambda x_2+\lambda(2\beta-\alpha)x_3=0$. The lines

SQ and $x_1=0$ intersect in the point $\gamma=2\beta-\alpha$ (11,248) (Fig. 15).

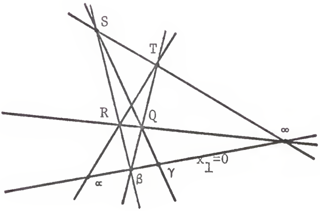

Fig. 15

A harmonic net is determined by any three points of a line. Therefore if the three points are A_1, A_2, ∞ of the line $x_1=0$ (Fig. 16), the fourth point A_3 is determined as the harmonic conjugate of A_1 with respect to A_2 and ∞; A_4 is the harmonic conjugate of A_2 with respect to A_3 and ∞; A_k is the harmonic conjugate of A_{k-2} with respect to A_{k-1} and ∞. The harmonic net is the set of points A_1, A_2, A_3, ..., ∞ and not the whole figure.

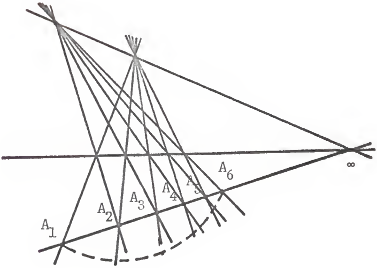

Fig. 16

From the above, the harmonic conjugate of a point α with respect to the points β and ∞ is the point $2\beta-\alpha$. Letting the coordinate of A_1 be α and that of A_2 be $\alpha+1$, one finds that the points of the harmonic net are α, $\alpha+1$, $\alpha+2$, ..., ∞. Since the modulus of the field is s, the series α, $\alpha+1$, $\alpha+2$, ..., $\alpha+(s-1)$ consists of s elements. The (s+1)-st element is again α, and the series repeats periodically. Fig. 16 is drawn for the case s=5 with the point A_6 considered as being coincident with the point A_1. This illustrates the following theorem.

THEOREM XVI: A finite projective geometry cannot be represented by a figure in ordinary geometry in which a line of the finite geometry consists of a finite set of points on a line of ordinary geometry (11,249).

An Example: PG(2,5)

As stated earlier the points of these geometries can have coordinates chosen from any Galois field of order $s \neq 2^n$. The resulting finite plane projective geometry is denoted PG(2,s). The simplest of these are the modular fields, in which the order is an odd prime number p. An example is the field of residue classes, modulo 5, composed of the familiar symbols 0, 1, 2, 3, 4 with the well-known rules for addition and multiplication (12,201).

This finite plane projective geometry is denoted as PG(2,5) and has 5+1=6 points on each line, 6 lines through each point and $5^2+5+1=31$ points in the plane and lines in the plane. The incidence matrix for this geometry is (2,94)

$$A=\begin{Vmatrix}
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
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0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{Vmatrix}$$

if the points are denoted by the symbols P_1, P_2, ..., P_{31} and the lines by m_1, m_2, ..., m_{31}. It is also possible to condense this matrix to the simple statement that point P_r and line m_s are incident if and only if

$$r+s \equiv 0,\ 1,\ 3,\ 8,\ 15,\ \text{or}\ 18\ (\text{mod}\ 31).$$

The consistency of the geometry defined by Axioms I-V can be verified by showing that PG(2,5) serves as a model. The plane certainly contains a finite number of points (31) and $31 > 2$. It obviously contains one line and one point not on that line, and each line contains at least three (in fact six) points. If A and B are distinct points, there is one and only one line that contains A and B. There is at least one line, as verified by the fact that given any two rows of the incidence matrix, say row i and row k, there exist the elements $a_{ij}=1$ and $a_{ik}=1$ for some column j. There is only one line for, taking the two elements above, one finds that there are not elements $a_{im}=1$ and $a_{km}=1$ simultaneously for any column $m \neq j$. One case is used to illustrate Axiom III. The points P_1, P_2, and P_3 are certainly not collinear as $a_{1j}=1$, $a_{2j}=1$, and $a_{3j}=1$ does not occur for any column j. Line m_4 (using the notation m_1, m_2, ..., m_{31}) contains P_4, which is also contained in the line $P_1 \cdot P_2 = m_{30}$ by the matrix, and P_{14}, which is also contained in $P_2 \cdot P_3 = m_{29}$. Also P_1, P_2, and $P_3 \notin m_4$. It is then true that $P_8 \varepsilon m_4$ and also $P_8 \varepsilon m_{31}$.

Axiom IV could also be verified for all complete quadrangles but only the complete quadrangle $P_1P_2P_5P_{31}$ is used as an example. The diagonal points are $m_{29} \cdot m_{31} = P_3$, $m_1 \cdot m_7 = P_{11}$, $m_3 \cdot m_{30} = P_9$; which are certainly not collinear by the matrix (3,95). Choosing A_1, B_1, C_1 to be the points

P_1, P_2, P_4 all contained in m_{30} and A_2, B_2, C_2 to be the points P_6, P_7, P_{14}, one finds the lines $A_1B_2=m_{11}$, $A_2B_1=m_6$, $B_1C_2=m_{29}$ and $B_2C_1=m_{27}$, $C_1A_2=m_{28}$ and $C_2A_1=m_{17}$. These lines intersect, in pairs as above, in the points $P_{28}=m_{11}\cdot m_6$, $P_5=m_{29}\cdot m_7$, $P_{15}=m_{28}\cdot m_{17}$. Points P_5, P_{15} and P_{28} are all elements of line m_3.

In PG(2,5) it is possible to determine some numerical results concerning figures analogous to figures in Euclidean geometry. To determine the number of triangles one needs to consider a triangle as being determined by any three non-collinear points. In PG(2,5) there are 31 possibilities for the first vertex, 30 possibilities for the second vertex and 25 possibilities for the third vertex as the six points on the first line obtained must be omitted from consideration. Since the triangles with merely their vertices permuted are the same there are

$$\frac{31\cdot 30\cdot 25}{3!} = 3875 \text{ triangles.}$$

Since there are again 25 points not on the join of the first line and 31-15=16 points not on a given triangle and 4! ways of permuting the vertices of a quadrangle, there are

$$\frac{31\cdot 30\cdot 25\cdot 16}{4!} = 15{,}500 \text{ quadrangles.}$$

Using the number of quadrangles possible, one can determine the number of pentagons. Taking one of the quadrangles obtained above it is evident that the four vertices and three diagonal points cannot be the remaining vertex; neither can any of the other three points on any of the six sides of the quadrangle. The number of eligible vertices is therefore 31-(4+3+18)=6 so there are possible

$$\frac{15500 \cdot 6}{5} = 18{,}600 \text{ pentagons.}$$

By the same reasoning it is possible to add a vertex to one of the pentagons from above. The five vertices, fifteen diagonal points, and any of the single points on each of the ten sides are eliminated so the number of possible vertices is 31-(5+15+10)=1. Therefore the number of hexagons possible is

$$\frac{18600 \cdot 1}{6} = 3100 \text{ hexagons } (4,116).$$

One can also determine the number of collineations possible. By a theorem of classical projective geometry (3,97) it is possible to determine every collineation by its effect on a particular quadrangle, such as $P_1P_2P_5P_{31}$. The collineation may transform P_1 into any one of 31 points, P_2 into any one of the remaining 30, P_5 into any of the 25 points not on $P_1 \cdot P_2$, and P_{31} into any one of the 31-15=16 points not on the triangle $P_1P_2P_5$. Thus there are possible $31 \cdot 30 \cdot 25 \cdot 16 = 372{,}000$ collineations.

By considering the possible triads of coordinates, Coxeter (3,128) devised a systematic method of assigning coordinates to the points and lines of PG(2,5). These coordinates are

$P_{31}=(0, 0, 1)$,	$P_{16}=(1, 3, 1)$,	$m_{31}=(1, 0, 0)$,	$m_{16}=(0, 1, 2)$,
$P_1=(1, 1, 0)$,	$P_{17}=(2, 0, 1)$,	$m_1=(0, 1, 0)$,	$m_{17}=(2, 0, 1)$,
$P_2=(1, 0, 0)$,	$P_{18}=(1, 1, 2)$,	$m_2=(4, 0, 1)$,	$m_{18}=(2, 1, 0)$,
$P_3=(0, 1, 1)$,	$P_{19}=(1, 2, 0)$,	$m_3=(1, 1, 0)$,	$m_{19}=(2, 1, 3)$,
$P_4=(1, 1, 0)$,	$P_{20}=(3, 1, 1)$,	$m_4=(4, 1, 1)$,	$m_{20}=(1, 2, 1)$,
$P_5=(4, 1, 1)$,	$P_{21}=(1, 3, 2)$,	$m_5=(1, 3, 2)$,	$m_{21}=(1, 3, 1)$,
$P_6=(1, 2, 1)$,	$P_{22}=(1, 2, 3)$,	$m_6=(0, 2, 1)$,	$m_{22}=(1, 1, 3)$,
$P_7=(1, 0, 2)$,	$P_{23}=(1, 1, 2)$,	$m_7=(1, 0, 1)$,	$m_{23}=(1, 1, 1)$,
$P_8=(0, 1, 4)$,	$P_{24}=(1, 1, 4)$,	$m_8=(1, 4, 0)$,	$m_{24}=(3, 1, 1)$,
$P_9=(1, 4, 0)$,	$P_{25}=(2, 1, 3)$,	$m_9=(1, 1, 4)$,	$m_{25}=(1, 1, 2)$,
$P_{10}=(1, 1, 1)$,	$P_{26}=(1, 1, 3)$,	$m_{10}=(0, 1, 0)$,	$m_{26}=(2, 1, 1)$,
$P_{11}=(4, 0, 1)$,	$P_{27}=(3, 1, 2)$,	$m_{11}=(1, 0, 2)$,	$m_{27}=(3, 2, 1)$,
$P_{12}=(0, 2, 1)$,	$P_{28}=(3, 1, 2)$,	$m_{12}=(1, 2, 0)$,	$m_{28}=(1, 4, 1)$,
$P_{13}=(2, 1, 0)$,	$P_{29}=(1, 4, 1)$,	$m_{13}=(3, 1, 2)$,	$m_{29}=(0, 1, 4)$,
$P_{14}=(2, 1, 1)$,	$P_{30}=(1, 0, 1)$,	$m_{14}=(2, 3, 1)$,	$m_{30}=(0, 0, 1)$.
$P_{15}=(2, 3, 1)$,		$m_{15}=(1, 2, 3)$,	

According to Edge, however, the most significant feature of PG(2,5) is the fact that every set of four distinct points on a line forms a harmonic set if taken in a suitable order. In fact, if any two of the six points on a line are denoted A and B, then the remaining four points are separated into pairs C, D and E, F each of which is harmonic to A, B. Not only is this true, but C, D and E, F are harmonic to each other, for the harmonic conjugate of C with respect to E and F is distinct from C, E, and F and cannot be A or B (each of which has the other for its harmonic conjugate). It must, therefore, be D (4,114).

There are many different sets of axioms which could be used and theorems developed for finite plane projective geometries. Axioms I-V, Theorems I-XVI and example PG(2,5) serve as an introduction to the fundamental concepts of finite plane projective geometry.

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SOME ELEMENTARY CONCEPTS

OF FINITE PLANE PROJECTIVE GEOMETRY

by

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This report contains a discussion of some of the properties of finite plane projective geometries. Very little geometric sophistication is demanded of the reader. Some knowledge of projective coordinates, finite fields, and transformation groups is helpful.

Using a point and a line as the undefined elements and incidence as the relation, a synthetic definition of a general plane projective geometry is given. This is done by stating the definitions and five axioms of Bachmann. When the number of points in the plane is restricted to be finite, a finite plane projective geometry results. The synthetic development of a finite plane projective geometry is based on the axiom system of Veblen and Bussey.

The axioms for the general projective plane are then shown to be deducible from the axioms of Veblen and Bussey. The principle of duality is proved for the finite projective plane by verifying that the set of axioms imply their duals. One of the classical theorems of projective geometry, the two triangle theorem of Desargues, also is proved for the finite projective plane. Finite projective planes are "Desarguesian".

From the fact that a particular line in a finite projective plane contains a finite number (> 3) of points, say $s+1$, it is shown that each line contains $s+1$ points, each point is on $s+1$ lines, and the plane contains s^2+s+1 points and s^2+s+1 lines. The incidence relationships between the points and lines of these geometries are stated in the form of "incidence matrices". An incidence matrix for a particular finite projective plane with given notation describes which lines pass through which points and which points are on which lines. Collineations are introduced, and some basic theorems about fixed points and fixed lines are proved. The main theorem states that for a given collineation the number of fixed points in a finite projective plane is equal to the number of fixed lines.

Coordinates are introduced for the points and lines of the plane. These coordinates are taken from Galois fields of order s, and the corresponding finite plane projective geometry is denoted by the symbol PG(2,s). A proof is then given that for $s=2^n$ the diagonal points of a complete quadrangle must be collinear, so there exists no projective plane geometry of order $s=2^n$. A theorem about the possible orders of subplanes of a given finite projective plane is proved. Using only the coordinates and analytic properties it is verified that there are s+1 points on each line, s+1 lines through each point, and s^2+s+1 points and lines in the plane. The construction of a harmonic set of points is given and, using these coordinates, a harmonic net is found. A line of a finite projective plane cannot be represented, in general, by an ordinary line containing only a finite number of points.

As an example of a finite plane projective geometry, PG(2,5) is used. This is the finite plane projective geometry with s+1=6 points on each line and $s^2+s+1=31$ points in the plane. A significant property of PG(2,5) is the fact that the six points on each line form a harmonic net.