Equivalences between spin models induced by defects

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Abstract

The spectrum of integrable spin chains are shown to be independent of the ordering of their spins. As an application we introduce defects (local spin inhomogeneities in homogenous chains) in two-boundary spin systems and, by changing their locations, we show the spectral equivalence of different boundary conditions. In particular we relate certain nondiagonal boundary conditions to diagonal ones.

1 Introduction

There have been an increasing interest in the recent years in two-boundary spin systems. This is due to theoretical and practical reasons. On one hand Nepomechie [2, 3] and Cao et al. [4] were able to extend the diagonal Bethe Ansatz (BA) solution of the XXZ spin chain [1] to certain nondiagonal cases where a constraint is satisfied between the parameters of the two boundary conditions (BCs). On the other hand these models have interesting applications in recent problems of statistical physics with open boundaries such as the description of the asymmetric exclusion process or the raise and peel models, see for example [6, 7]. There are developments in similar open spin chains as well, see e.g. the result on XYZ and XXX models [8, 9, 10], but the basic example is the XXZ model, from which other interesting models like the lattice sine-Gordon, or the lattice Liouville models can be derived [13].

In the algebraic Bethe Ansatz method a pseudo vacuum vector is needed on which the action of the monodromy matrix is triangular. In the XXX [10], XXZ [4] and XYZ [8] models this requirement resulted in a constraint relating the parameters of the two boundary conditions. The same constraint appeared also in the functional relations approaches in [2, 3] for the XXZ, while in [9] for the XYZ cases. Since they show up also in the Temperly-Lieb formulation of the XXZ spin chains [11, 12] we conclude that they are intrinsically encoded in the system.

It was observed in [12] that certain nondiagonal BCs in the two-boundary XXZ spin chain are equivalent to diagonal ones. This was then extended to other spin chains in [30].

The aim of the paper is to derive these equivalences and show the physical origin of the constraints. In doing so we analyze inhomogeneous spin chains and show, both in the periodic and in the open case, that the spectrum of the transfer matrix/Hamiltonian does not depend on the actual order of the spins merely on the spin content of the chain. We define a different spin in a homogenous chain as a defect, then we demonstrate how the equivalences observed in quasi periodic XXX spin chains [15] can be re-derived by moving defects. In the open case the nondiagonal BC is represented as a diagonal one dressed by a defect. The idea is to move the defect, by performing similarity transformations on the transfer matrix, to the other boundary in order to dress that one instead. In the XXX model dressing the generic nondiagonal BC gives rise to triangular BC if the constraint is satisfied. In the XXZ model dressing the quantum group invariant BC gives diagonal BC. Dressing the nondiagonal BC gives upper triangular type BC, whenever the constraint is satisfied. Satisfying two constraints between the two nondiagonal BCs the spectrum can be described by diagonal BCs on both ends.

The paper is organized as follows: We start by deriving the equivalence in the periodic chain. Having summarized the notations we introduce defects and show that they can be shifted without altering the spectrum of the transfer matrix. As a consequence we derive correspondences between various periodic spin chains, and pedagogically illustrate their usage on the example of the XXX model. Turning to the boundary problem we recall the notations, then show how nondiagonal BCs can be described by dressing diagonal ones with defects. The change of the location of the defect from one side to the other leads to equivalences between different BCs. In order to demonstrate the method we analyze the XXX and the XXZ models in some detail.

2 Equivalences in periodic chains

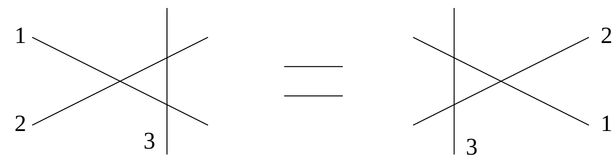
Lets summarize the notations in the periodic case following the paper of Sklyanin [16]. We take a solution $R : \mathbb{C} \mapsto End(V \otimes V)$ of the Yang-Baxter equation (YBE)

$$R_{12}(u_{12})R_{13}(u_{13})R_{23}(u_{23}) \quad = \quad R_{23}(u_{23})R_{13}(u_{13})R_{12}(u_{12}) \tag{1}$$

(with $u_{ij} = u_i - u_j$) which is symmetric $\mathcal{P}_{12}R_{12}\mathcal{P}_{12} = R_{12}^{t_1 t_2} = R_{12}$ and satisfies unitarity and crossing symmetry:

$$R_{12}(u)R_{12}(-u) = \rho(u) \quad ; \quad R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta) = \tilde{\rho}(u) \tag{2}$$

Here ρ and $\tilde{\rho}$ are scalar factors, $\mathcal{P}_{12}$ permutes the factors in the tensor product, $V \otimes V$, of the vector space V, and the index refers to the factor in which the operators act, i.e. the YBE is an equation in $V \otimes V \otimes V$. The transposition in the i-th tensor product factor is denoted by t_i. Graphically the YBE is represented as



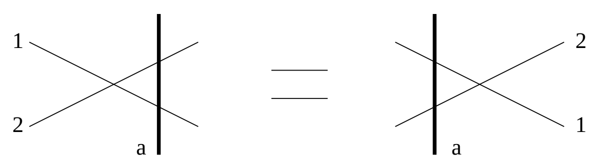
For such a solution of the YBE (1) we define the associative algebra with generators $T_{ij} \quad i,j = 1 \dots dim\, V$ as

$$R_{12}(u_{12})T_1(u_1)T_2(u_2) = T_2(u_2)T_1(u_1)R_{12}(u_{12}) \tag{3}$$

Any representation can be used to define a spin chain, thus we deal with concrete representations. If $T_1(u_1)$ is represented in the space W_a then we denote it by $T_{1a}(u_1)$, and the equation takes the following form

$$R_{12}(u_{12})T_{1a}(u_1)T_{2a}(u_2) = T_{2a}(u_2)T_{1a}(u_1)R_{12}(u_{12}) \tag{4}$$

which is called the RTT equation (RTTE) and can be written graphically as

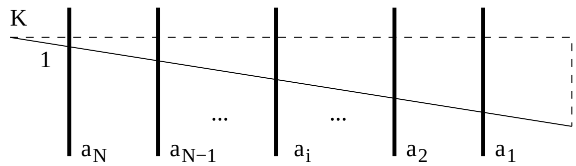


Whenever T_{1a} and T_{1b} are two representations in W_a, W_b, respectively, then $T_{1a}T_{1b}$ is also a representation in $W_a \otimes W_b$. Moreover, if $T_{1a}(u)$ is a solution of the RTTE (4), then $T_{1a}^{-1}(-u)$ and $T_{1a}^{-1}(u)^{t_1}$ are also solutions of the same equation (4).

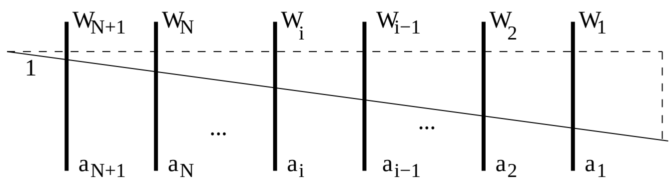
Comparing the two figures (or the YBE (1) to the RTTE (4)) we observe that we can always take $W_a = V$ and choose $T_{ia}(u_i) = R_{ia}(u_i + w)$, which, thanks to the YBE (1), also satisfies the RTTE (4).

In particular if the R-matrix, $R_{12}(u)$ is related to the universal R-matrix, $\mathcal{R}$, of a quasi triangular Hopf algebra as $R_{12}(u_{12}) = (\pi_{u_1} \otimes \pi_{u_2})\mathcal{R}$ then $(\pi_u \otimes I)\mathcal{R}$ provides a solution of the RTT algebra relations (3) and any representation π leads to a concrete realization via $(\pi_u \otimes \pi)\mathcal{R}$.

Suppose that $T_{1\pm}(u)$ are two solutions of the RTTE (4) in the quantum spaces $W_\pm$, then the transfer matrices $t(u) = \mathrm{Tr}_1(T_{1+}(u)T_{1-}(u))$ form a commuting family of matrices, i.e. they can be considered as the generating functionals of conserved quantities for the quantum system on $W_+ \otimes W_-$. The typical example is as follows: take a scalar representation on $W_+ = \mathbb{C}$ with $T_{1+}(u) = K \in End(V)$ and another one on the quantum space $W_- = W \otimes \dots \otimes W$ of the form $T_{1-}(u) = L_{1a_N}(u) \dots L_{1a_i}(u) \dots L_{1a_1}(u)$ where the L_{1a_i}-s are the same representations. The transfer matrix corresponds to a closed integrable quantum spin chain with N sites subject to quasi periodic boundary condition specified by the matrix K, which can be represented graphically as:



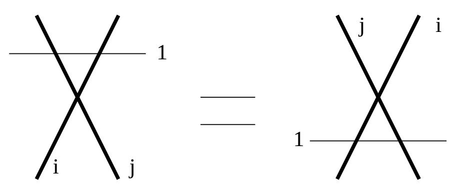
Note that $K = id$ is always a solution, which correspond to periodic boundary condition. If at one position the representation is changed from $L_{1a_i}(u)$ to $L^{'}_{1a_i}(u)$ acting on the space W' instead of W then we can interpret it as a *defect* in the spin chain. We can change all the representations to have a chain of the form $W_- = W_{N+1} \otimes \ldots \otimes W_i \otimes \ldots \otimes W_1$ with the corresponding solutions $L^i_{1a_i}$, where for $N+1$ we can take $a_{N+1} = \mathbb{C}$ and $L^{N+1}_{1a_{N+1}} = K$ and choose $T_{1+}(u) = id$ to simplify the presentation. The transfer matrix can be drawn as



We claim that if there exist a matrix $S_{a_ia_j}(u_{ij}) \in End(W_i \otimes W_j)$ such that

$$S_{a_ia_j}(u_{ij}) L^i_{1a_i}(u_i) L^j_{1a_j}(u_j) \quad = \quad L^j_{1a_j}(u_j) L^i_{1a_i}(u_i) S_{a_ia_j}(u_{ij}) \tag{5}$$

which we call the SLL equation (SLLE) drawn graphically as



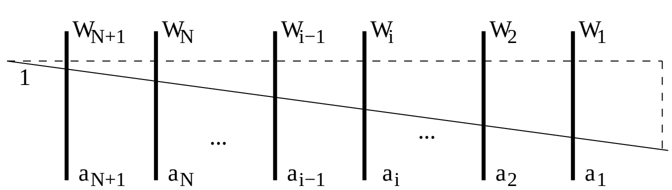
then the spectrum of the transfer matrix does not depend on the actual positions of the representations, merely on the representation content of the chain. For this we show that we can change any two neighboring operators. Take

$$t(u) = \mathrm{Tr}_1(L^{N+1}_{1a_{N+1}}(u) \ldots L^{i+1}_{1a_{i+1}}(u) L^i_{1a_i}(u) L^{i-1}_{1a_{i-1}}(u) L^{i-2}_{1a_{i-2}}(u) \ldots L^1_{1a_1}(u)) \tag{6}$$

and use the SLLE (5) to replace $L^i_{1a_i}(u) L^{i-1}_{1a_{i-1}}(u)$ with $S^{-1}_{a_ia_{i-1}}(0) L^{i-1}_{1a_{i-1}}(u) L^i_{1a_i}(u) S_{a_ia_{i-1}}(0)$, then commute the operator $S = S_{a_ia_{i-1}}(0)$ through the other L-s since they act in different quantum spaces to show that

$$S\, t(u)\, S^{-1} = \mathrm{Tr}_1(L^{N+1}_{1a_{N+1}}(u) \ldots L^{i+1}_{1a_{i+1}}(u) L^{i-1}_{1a_{i-1}}(u) L^i_{1a_i}(u) L^{i-2}_{1a_{i-2}}(u) \ldots L^1_{1a_1}(u))$$

which can be drawn graphically as



As a consequence if we have two defects then the actual position of the defects does not matter, i.e., they do not interact. Moreover, an alternating spin chain is equivalent to two equal length homogenous chains coupled at two points together. Thus the results on alternating spin chains [17, 18, 19, 20] can be reinterpreted in this sense.

The SLLE (5) looks very nontrivial, but it is almost always satisfied in the spin chains considered sofar. As an example we mention, that if $W_i = V$ and $W_j = W$ then the RTTE (4) is equivalent to the SLLE (5) by making the $R_{12} \to L_{12}$, $T_{1a} \to L_{1a}$, and $T_{2a}^{-1} \to S_{2a}$ identification. In general if a universal $\mathcal{R}$ matrix is given, then we can always choose $S_{a_i a_j}(u_{ij}) = (\pi_{u_i}^{a_i} \otimes \pi_{u_j}^{a_j})\mathcal{R}$ which provides a solution of the SLLE (5). Lets see some concrete examples.

2.1 Defects in the XXX spin chain

Here for pedagogical reasons we reinterpret the results of [15] for the $SU(2)$ invariant spin chain in our language. The R matrix is taken to be

$$R_{12}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}$$

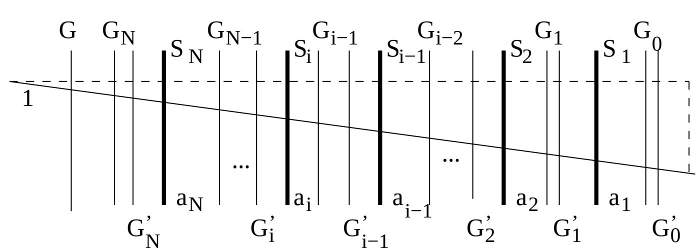
while for describing the spin S_j at site j we take the operator

$$L_{1a_j}^{S_j}(u) = \begin{pmatrix} u + \frac{\eta}{2} + \eta S_{a_j}^z & \eta S_{a_j}^- \\ \eta S_{a_j}^+ & u + \frac{\eta}{2} - \eta S_{a_j}^z \end{pmatrix}$$

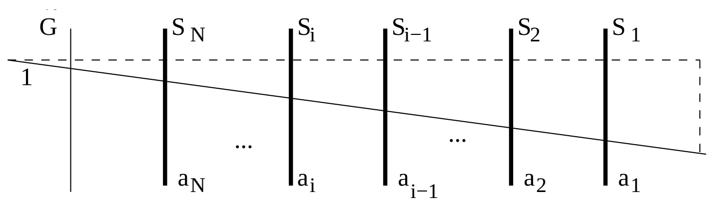
where S_a represents $SU(2)$ with spin S. For $S = \frac{1}{2}$ we can recover the R-matrix itself $L_{1a}(u) = R_{1a}(u)$. Solving the RTTE (4) we can realize that we have also scalar but u independent solutions:

$$T_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad ; \qquad AD - BC = 1 \tag{7}$$

and they correspond to the global $SU(2)$ symmetry of the model. (Here we supposed invertibility of this matrix and normalized its determinant to one). It can be used either as defects in the spin chain or as the $L_{1N+1} = K$ matrix specifying quasi periodic boundary conditions in (6) as follows: Lets consider a chain depicted on the next figure



At site i a spin S_i representation is introduced on the quantum space a_i via the operator $L^{S_j}_{1a_j}$ denoted by a thick line. Between sites i and $i-1$ two defects with matrices of the form (7), denoted by G_{i-1} and $G^{'}_{i-1}$, are inserted. Finally quasi periodic boundary condition is introduced by G of the form (7). Thin lines correspond to scalar solution (7) of the RTTE (4). The $G^{'}_i L^{S_i}_{1a_i} G_{i-1}$ triple can be interpreted as the dressing of the spin S_i with defects and this transformation can be used to bring $L^{S_i}_{1a_i}$ to a triangular form. As was shown in Appendix B of [15] the matrix S exists in (5) so thick and thin lines can be changed by similarity transformation. As a consequence we can move the defects to the left and describe the spectrum by the following transfer matrix:



where $\tilde{G} = G \prod_{i=N}^{0} G_i G^{'}_i$ or any of its cyclic permutation. Lets consider two applications. Take first $G_i = id$ for $i > 0$ and $G^{'}_0 = G^{-1}_0$ and use cyclicity to show that the spin chain with quasi periodic BCs specified by G and $\tilde{G} = G^{-1}_0 G G_0$ are equivalent. This transformation can be used to diagonalize the matrix G. If we take now $G^{'}_i = id$ for all i then we realize that we can 'collect' the defects into $\tilde{G} = G \prod_i G_i$ and using the previous argument we conclude that only the eigenvalues of the product of the defect matrices determine the spectrum. All these statements can be checked on the explicit BA solution of the model in [15], where the independence of the orders of the spins S_i is also obvious. The generalization of these results for the $SU(n)$ case following the lines in [15] is straightforward.

2.2 Defects in the XXZ spin chain

Consider the XXZ spin chain, that is take $U_q(\hat{sl}_2)$ to be the quasi triangular Hopf algebra. The $R_{12}(u_{12})$ matrix is the universal $\mathcal{R}$ matrix taken in the $\pi^{(1)}_{u_1} \otimes \pi^{(1)}_{u_2}$ representation:

$$R_{12}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix} \tag{8}$$

For $L_{1a}(u)$ we take the same representation written as

$$L_{1a_i}(u) = \begin{pmatrix} \sinh(u + \frac{\eta}{2}(1+\sigma^z_i)) & \sinh\eta\, \sigma^-_i \\ \sinh\eta\, \sigma^+_i & \sinh(u + \frac{\eta}{2}(1-\sigma^z_i)) \end{pmatrix} \tag{9}$$

where the operators σ_i are the standard Pauli matrices. The matrix structure always refers to the representation space, $\pi^{(1)}_{u_1}$ labeled by 1. This defines a spin $\frac{1}{2}$ chain via the transfer matrix (6) as shown in [16].

For the defect, which is relevant in the boundary problem, we take the $\pi_u^{(1)} \otimes \pi_u^q$ representation, where π_u^q is the q-oscillator ($q = e^{-\eta}$) representation of the Hopf algebra, [21]. This gives the following defect operator

$$T_{1a}(u, \beta, \mu_1, \mu_2) = \Gamma_1 \begin{pmatrix} e^{u+\beta} q^{-J_0} & J_- q^{J_0} \\ -J_+ q^{-J_0} & e^{u+\beta} q^{J_0} \end{pmatrix} \Gamma_2^{-1} \tag{10}$$

where $\Gamma_i = diag(e^{\mu_i/2}, e^{-\mu_i/2})$ and the infinite dimensional matrices, $J_\pm$, J_0 can be written in terms of the matrix units as

$$J_0 = \sum_{j=-\infty}^{\infty} j e_{jj} \quad ; \quad J_\pm = \sum_{j=-\infty}^{\infty} e_{jj\mp 1}$$

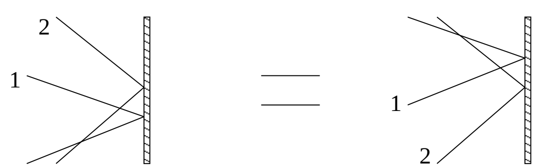
We have slightly changed the basis compared to [21]. The matrices Γ_i are the constant solutions of the RTTE (4) and are related to the global symmetries of the model. They also can be used to define quasi periodic boundary conditions as we have already seen in the example of the XXX model. The analogous solution for the defect equation in the sine-Gordon theory was analyzed in [24, 25]. It is clear from the previous considerations that if we have more than one defect only their number, and not their locations, influences the spectrum. The solution of the model even with one defect is an open and interesting problem. Here we would like to concentrate on how it can be used to derive equivalences in the open case and leave this analysis for a future work.

3 Equivalences in open spin chains

In describing an open chain, additionally to the RTT algebra (3), one also introduces two algebras spanned by $\mathcal{T}_{ij}^\pm$, $i, j = 1...dimV$. $\mathcal{T}^-$ corresponds to the right boundary and satisfies

$$\begin{aligned} R_{12}(u_1 - u_2)\mathcal{T}_1^-(u_1)R_{12}(u_1 + u_2)\mathcal{T}_2^-(u_2) = \\ \mathcal{T}_2^-(u_2)R_{12}(u_1 + u_2)\mathcal{T}_1^-(u_1)R_{12}(u_1 - u_2) \end{aligned} \tag{11}$$

which is called the boundary YBE (BYBE) and is represented graphically as

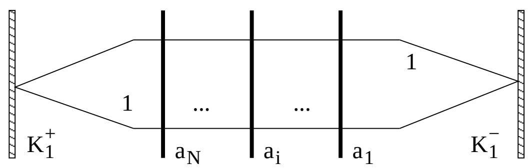


There is an analogous equation for $\mathcal{T}^+$ (cf. [16]). What we actually need is that the two algebras are isomorphic: for any solution $\mathcal{T}^-(u)$ of the right BYBE (11) $\mathcal{T}^+(u) = \mathcal{T}^-(-u-\eta)^t$ solves the analogous left BYBE. In dealing with spin chains we are interested in representations of these algebras $\mathcal{T}_{1a}^\pm$. It is shown in [16] that taken two solutions of the

BYBE (11), $\mathcal{T}_1^{\pm}$, $t(u) = \mathrm{Tr}_1(\mathcal{T}_1^+(u)\mathcal{T}_1^-(u))$ forms a commuting family of matrices and is the generating functional for the integrals of motions of an open quantum system. It was also shown in [16] that taking two solutions $K_1^{\mp}(u)$ of the BYBE (11) and a solution T_{1a} of the RTTE (4) then $\mathcal{T}_{1a}^-(u) = T_{1a}(u)K_1^-(u)T_{1a}^{-1}(-u)$ solves the left (11), while $\mathcal{T}_{1a}^+(u) = (T_{1a}(u)^{t_1}K_1^+(u)^{t_1}T_{1a}^{-1}(-u)^{t_1})^{t_1}$ the right BYBE. Choosing any of these *dressed* solutions, $\mathcal{T}_{1a}^{\mp}$, with the other undressed one, $K_1^{\pm}$, leads to the same transfer matrix:

$$t(u) = \mathrm{Tr}_1(K_1^+(u)T_{1a}(u)K_1^-(u)T_{1a}^{-1}(-u)) \tag{12}$$

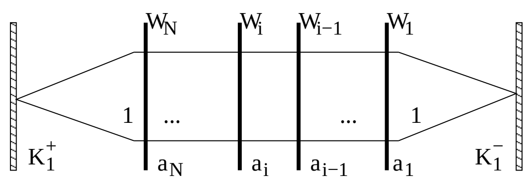
Taking for $T_{1a}(u)$ the one used in the periodic chain $T_{1a}(u) = L_{1a_N}(u)\dots L_{1a_i}(u)\dots L_{1a_1}(u)$ the transfer matrix, $t(u)$, generates the conserved quantities for an open spin chain with BCs specified by K_1^+ and K_1^-, written graphically as:



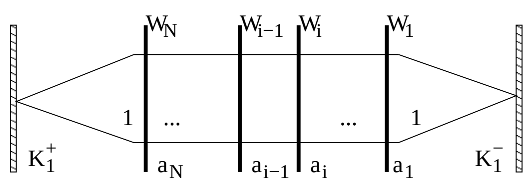
We can introduce defects by changing a representation at one site, or, similarly to the periodic case, we can change the representations on each site to have a more general chain with,

$$T_{1-} = L_{1a_N}^N(u)...L_{1a_{i+1}}^{i+1}(u)L_{1a_i}^i(u)L_{1a_{i-1}}^{i-1}(u)L_{1a_{i-2}}^{i-2}(u)...L_{1a_1}^1(u)$$

whose transfer matrix , $t(u)$, can be written graphically as



Now manipulations similar to those performed in the periodic case show that any two neighboring representations can be changed. This leads to a spectrally equivalent description by the transfer matrix:



We conclude again that only the representation content matters and not the actual order of the representations. In the typical applications we take a chain with $N+1$ sites and interpret the first site with operator $L_{1a_1}^1 = T_{1a}(u)$ as a dressing of the boundary:

$$K_{1a}^-(u) = T_{1a}(u)K_1^-(u)T_{1a}^{-1}(-u) \tag{13}$$

(For algebraic analysis of this type of dynamical BCs see [22, 23].) The defect can be moved to the other boundary to dress that one

$$K_{1a}^{+}(u)^{t_1} = T_{1a}(u)^{t_1} K_1^{+}(u)^{t_1} T_{1a}^{-1}(-u)^{t_1} \tag{14}$$

giving equivalences between different BCs, namely the system with BCs K_1^- and K_{1a}^+ is equivalent to the system with different BCs described by K_{1a}^- and K_1^+, moreover, the equivalence is independent of the bulk spin content of the chain. This isomorphism can map nondiagonal BCs to diagonal ones as we can see in the next examples.

3.1 Defects in the open XXX model

Here we follow the presentation of the open XXX model of [10] but rewrite the results to our language. The simplest solution of the BYBE (11) has a diagonal form and contains one parameter

$$K_1^-(u,\bar{\xi}_-)^{diag} = \begin{pmatrix} \bar{\xi}_- + u & 0 \\ 0 & \bar{\xi}_- - u \end{pmatrix}$$

This solution can be dressed (13) by the defect T_1 in (7) to obtain the most general nondiagonal one

$$K_1^-(u,\xi_-,c_-,d_-) = T_1 K_1^-(u,\bar{\xi}_-)^{diag} T_1^{-1} = \begin{pmatrix} \xi_- + u & c_- u \\ d_- u & \xi_- - u \end{pmatrix}$$

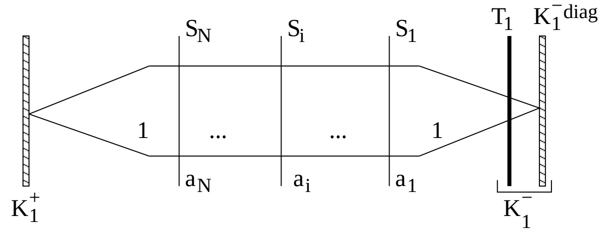
where the parameters of the defect T_1 can be calculated from c_-, d_- and ξ_- which is the dressed version of $\bar{\xi}_-$. For the ratios we have two solutions with $\epsilon = \pm 1$ as

$$\frac{A}{B} = \frac{1+\epsilon\sqrt{1+c_- d_-}}{c_-} \quad ; \quad \frac{C}{D} = \frac{1-\epsilon\sqrt{1+c_- d_-}}{c_-}$$

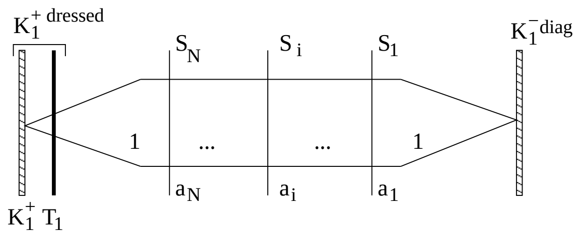
Consider an open spin chain with bulk spins S_i and boundary conditions specified by $K_1^-(u,\xi_-,c_-,d_-)$ on the right and

$$K_1^+(u,\xi_+,c_+,d_+) = \begin{pmatrix} \xi_+ - u - \eta & -d_+(u+\eta) \\ -c_+(u+\eta) & \xi_+ + u + \eta \end{pmatrix}$$

on the left, which can be drawn graphically as



Just as in the bulk case we can move the defect to the other boundary and then the transfer matrix is equivalently described as



where now the left boundary is dressed (14) as $K_1^{+dressed} = T_1^{-1} K_1^+ T_1$. Demanding the upper/lower triangularity of the matrix, which is needed to find a pseudo vacuum in the BA formulation, we obtain the

$$\frac{1 \pm \epsilon\sqrt{1 + c_- d_-}}{c_-} = \frac{1 + \epsilon\sqrt{1 + c_+ d_+}}{c_+}$$

constraint. Under this condition BA equations for the eigenvalues of the transfer matrix can be derived [10]. If both conditions are satisfied then the spin chain with two nondiagonal BCs is equivalent to a chain with diagonal boundary conditions on both sides.

3.2 Defects in the open XXZ model

One of the simplest solutions of the BYBE (11) is

$$K_1^-(u)^{QGI} = diag(e^u, e^{-u}) \tag{15}$$

which corresponds to the quantum group invariant (QGI) chain. By dressing it (13) with the defect (10) with parameters $T_{1a}(u) = T_{1a}(u, \alpha_-, \mu_1, \mu_2)$ and taking the $\mu_1 = \rightarrow -\infty$ limit we can obtain the most general diagonal solution

$$K_1^-(u,\alpha)^{diag} = \frac{n(u,\alpha)}{2} T_{1a}(u) K_1^{-QGI} T_{1a}^{-1}(-u) = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} \tag{16}$$

where $P_\pm = \cosh(u \pm \alpha_-)$ and $n(u,\alpha) = e^{\alpha - 2u} + e^{-\alpha}$. Dressing it again (13) with the defect (10)

$$T_{1a}(u) = T_{1a}(u, \beta_-, \mu_1 = \gamma_- - \alpha_-, \mu_2) \tag{17}$$

we can obtain the following nondiagonal solution

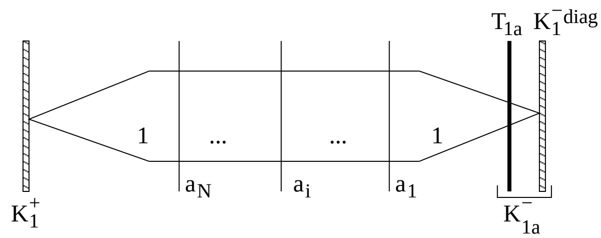
$$K_{1a}^-(u, \alpha_-, \beta_-, \gamma_-) = n(u, \beta_-) T_{1a}(u) K_1^-(u,\alpha)^{diag} T_{1a}^{-1}(-u) = \begin{pmatrix} P_+^- & J_- Q_+^- \\ J_+ Q_-^- & P_-^- \end{pmatrix} \tag{18}$$

where $P_\pm^- = e^{\beta_-} P_\pm + e^{-\beta_-} P_\mp$ and $Q_\pm^- = \mp e^{\pm\gamma_-} \sinh(2u)$. The operators $J_\pm$ are the inverses of each other and can be diagonalized on the basis $|\theta\rangle = \sum_{j=-\infty}^{\infty} e^{i\theta j} |j\rangle$ as $J_\pm |\theta\rangle = e^{\mp i\theta} |\theta\rangle$. Thus each $|\theta\rangle$ subspace is invariant under the action of K_{1a}^- on which it takes the most general nondiagonal form

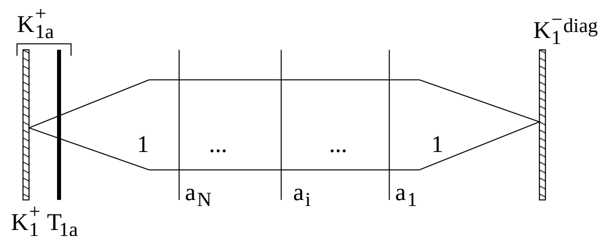
$$K_1^-(u, \alpha_-, \beta_-, \gamma_-) = \begin{pmatrix} P_+^- & Q_+^- \\ Q_-^- & P_-^- \end{pmatrix} \tag{19}$$

where the effect of the defect is the shift in the parameter $\gamma_- \to \gamma_- + i\theta$. Our parameters are in spirit close to the parameterization used in the sine-Gordon model [24, 25]. They can be related to the $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ parameters used in [3] as $\hat{\alpha} = \alpha - \frac{i\pi}{2}, \hat{\beta} = \beta, \hat{\theta} = \gamma + \frac{i\pi}{2}$.

Lets consider the two-boundary spin $\frac{1}{2}$ XXZ chain with nondiagonal BCs on the right end (19) specified by $K^-_{1a}(u, \alpha_-, \beta_-, \gamma_-)$ and $K^+_1(u)$ on the left end. The transfer matrix can be represented as



Moving the defect to the other boundary the transfer matrix is equivalently described as



where now the left boundary is dressed.

We start by deriving the equivalence found in [12]. In doing so we take the QGI (15) $K^+_1(u) = K^-_1(-u-\eta)^{QGI}$ BC on the left and the dressed diagonal $K^-_{1a}(u, \alpha_-, \beta_-, \gamma_-)$ on the right (18). Although the defect depends on μ_2 the dressed boundary and correspondingly the transfer matrix does not. The dressed left boundary (14) takes the form

$$K^+_{1a} = \frac{1}{2}\begin{pmatrix} e^{\beta_- - u - \eta} + e^{-\beta_- + u + \eta} & e^{\mu_2}(e^{-2u-2\eta} - e^{2u+2\eta})q^{J_0}J_-q^{J_0} \\ 0 & e^{\beta_- + u + \eta} + e^{-\beta_- - u - \eta} \end{pmatrix}$$

Since the spectrum does not depend on μ_2 we can take the limit $\mu_2 \to -\infty$ and now the dressed left boundary, K^+_{1a} is diagonal $K^+_{1a} = K^-_1(-u-\eta)^{diag}$ with $\alpha_- \to \alpha_+ = \beta_-$, making equivalences between, nondiagonal BC $K^-_1(\alpha_-, \beta_-, \gamma_-)$ on one end and QGI on the other, with diagonal BCs on both sides $K^-_1(\alpha_-)^{diag}$ and $K^+_1(\beta_-)^{diag}$ as was observed in [12].

As another application we take the most general nondiagonal $K^+_1(u) = K^-_1(-u-\eta)$ BC (19) with the $(\alpha_+, \beta_+, \gamma_+)$ parameters: $P^+_\pm(u) = P^-_\pm(-u-\eta)$ and $Q^+_\pm = Q^-_\pm(-u-\eta)$.

The dressed transfer matrix takes the form

$$K^+_{1a} = \begin{pmatrix} P^+_{+a} & q^{-2J_0}e^{-\mu_- - \eta}Q^+_{+a} \\ q^{2J_0}e^{\mu_- - \eta}Q^+_{-a} & P^+_{-a} \end{pmatrix}$$

$$
\begin{aligned}
P^+_{\pm a} &= \sum_{\epsilon=\pm}(e^{\pm\epsilon\beta_-}P^+_\epsilon \mp e^{\pm\epsilon(u\pm\mu_-)}Q^+_\epsilon J_{-\epsilon}) \\
Q^+_{\pm a} &= \sum_{\epsilon=\pm}\epsilon(e^{\pm\epsilon(u+\eta)}P^+_\epsilon \pm e^{\pm\epsilon(\beta_-+\eta\pm\mu_-)}Q^+_\epsilon J_{-\epsilon})J_\pm
\end{aligned}
$$

We see that the $|\theta\rangle$ subspace is not invariant under the action of K^+_{1a} since $q^{\pm J_0}|\theta\rangle = |\theta+i\eta\rangle$. (In most of the cases η is purely imaginary). If we demand a highest/lowest weight property of the vector $|\theta\rangle$, or some other words, the lower/upper triangularity of the matrix we obtain the following constraint

$$
\beta_+ \mp (\alpha_+ - \gamma_+) = \beta_- \mp (\alpha_- - \gamma_- - i\theta) + \eta \tag{20}
$$

which, using the discrete symmetries of the model, is the analog of Nepomechie's constraint [2, 3, 29]. This condition is sufficient to find a reference state in the BA [4, 13]. The two choices of signs are related as $\beta_\pm \leftrightarrow -\beta_\pm$ and $\eta \leftrightarrow -\eta$. This transformation does not change the Hamiltonian whose spectrum we are describing (see [3]) merely its realization in terms of the transfer matrix. Similarly the same Hamiltonian can be described in terms of different Temperly-Lieb algebras [26] and the two constraints correspond to exceptional representations of the two algebras, respectively. If we demand both constraints

$$
\beta_+ = \beta_- + \eta \quad ; \qquad \alpha_+ - \gamma_+ = \alpha_- - \gamma_- - i\theta
$$

the $|\theta\rangle$ subspace becomes invariant on which the dressed left boundary matrix (14) takes the form $K^+_{1a} = f(u)K^-_1(-u-\eta,\alpha_+)^{diag}$ with $f(u) = 4\cosh(u+\beta_+)\cosh(u-\beta_-)$. This shows that the two-boundary spin chain with nondiagonal BCs specified by K^+_1 and K^-_{1a} can be described by a spin chain with diagonal BCs specified by K^{-diag}_1 and K^+_{1a} on the two ends. One can check that under these circumstances the nondiagonal BA equations [3] are equivalent to the diagonal ones [2].

We note that the equivalences derived here are valid for any value of η. For special values, however, the defect admits a finite dimensional representation, which might lead to other equivalences observed in [26], or help to understand the derivation of BA equations for special cases in [27, 28].

4 Conclusion

To sum up we have established an equivalence between different spin chains: we have shown that under quite general (integrable) circumstances the spectrum of the chain depends only on its representation (spin) content and not on the actual order of the representations. We have successfully applied this equivalence to make correspondence between different BCs. We demonstrated the machinery on the examples of the XXX and XXZ spin $\frac{1}{2}$ chain, where we mapped the system with special nondiagonal BCs to diagonal ones. This equivalence is quite general, however, and can be extended to other two-boundary spin systems, like to the higher representation of $SU(2)$ or higher rank algebras [10, 15, 30], in order for making equivalences between different BCs. An advantage of the method is, that it is necessary to solve one model from each equivalence class only.

Finally we note that the analogue of dressing the boundary with defects was already analised in conformal field theories [31], in integrable quantum field theories [32] and in classical field theories [33]. The straightforward generalization of our ideas to these theories leads to equivalences between various BCs in their two-boundary versions.

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